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Subgame-Perfection in Quitting Games with Perfect Information and Differential Equations

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Abstract

We introduce a new approach to study subgame-perfect equilibrium payoffs in stochastic games: the differential equations approach.

We apply our approach to quitting games with perfect information. Those are sequential game in which at every stage one of n players is chosen; each player is chosen with probability $1/n$. The chosen player i decides whether he quits, in which case the game terminates and the terminal payoff is some vector $a_i \in \mathbf{R}^n$, or whether he continues, in which case the game continues to the next stage. If no player ever quits, the payoff is some vector $a_* \in \mathbf{R}^n$.

We define a certain differential inclusion, prove that it has at least one solution, and prove that every vector on a solution of this differential inclusion is a subgame-perfect equilibrium payoff.

Keywords: Stochastic games, Dynkin games, subgame-perfect equilibrium, differential inclusions.

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1 Introduction

Existence of an equilibrium payoff in multi-player stochastic games is still an open problem. The traditional approach to proving existence is by using the limit of stationary discounted equilibria. Namely, one takes for every discount factor a stationary discounted equilibria, and considers the stationary profile which is the limit of the stationary discounted equilibria, as the discount factor goes to zero. Depending on the exact class of games that is studied, one constructs a non-stationary ϵ -equilibrium in which players play mainly the limit stationary strategy profile, and perturb to other actions with small probability, while monitoring the actions of their opponents to detect deviations.

This approach, which was initiated by Mertens and Neyman (1981) to prove the existence of the uniform value in two-player zero-sum stochastic games, was later exploited in numerous studies, see, e.g., Vrieze and Thuijsman (1989), Thuijsman and Raghavan (1997) and Vieille (2000a).

In some cases variants of this approach were used; namely, instead of approximating the so called undiscounted game by the discounted games, one considers other approximations. Flesch et al. (1996), Vieille (2000b) and Solan (2000) approximate the strategy spaces of the players instead of the payoff functions, and Solan (1999) and Solan and Vohra (2002) consider a sequence of stationary discounted equilibria of a modified game, rather than of the original game.

This approach proved to be useful not only for the basic model of stochastic games, but also for other models, such as stochastic games with incomplete information (Rosenberg and Vieille (2000)), stochastic games with imperfect monitoring (Coulomb (2002) and Rosenberg et al. (2002)) and stopping games (Rosenberg et al. (2001)).

The limit of this approach was exhibited by Solan and Vieille (2001), who constructed a four-player quitting game¹ in which the simplest equilibrium strategy profile is periodic with period two. Moreover, for ϵ sufficiently small, there is no ϵ -equilibrium in which players play mainly some stationary strategy profile, and perturb to other actions with small probability.

Once the traditional approach fails, a need for new approaches arises. Solan and Vieille (2001) study equilibrium payoffs in quitting games. Motivated by dynamical systems, they define some set-valued function, and prove that every infinite orbit of this function corresponds to an ϵ -equilibrium.

Simon (2002) introduced tools from topology to the study of stochastic games. He showed that if a certain topological conjecture holds, then every quitting game admits an equilibrium payoff. However, it is yet unknown whether his conjecture is valid or not.

Shmaya et al. (2002) and Shmaya and Solan (2002) use Ramsey Theorem² as a substitute for a fixed point theorem to prove existence of an equilibrium payoff in two-player non-zero-sum stopping games.

Here we present a new approach to study equilibrium payoffs in multi-player stochastic games: a differential equations approach. Since differential equations and dynamical systems are closely related (see, e.g., Hubbard and West (1997)), it still remains to explore the connection between the approach we introduce here, and the one used by Solan and Vieille (2001).

The class of games we study is quitting games with perfect information: at every stage one of n players is chosen at random, independently of past play; each player i is chosen with probability

¹*Quitting games* are sequential games in which at every stage each player chooses whether to continue or to quit. The game terminates once at least one player quits, and the terminal payoff vector depends on the subset of players that choose to quit at the terminal stage. If everyone always continues, the payoff is 0 to all players.

²Ramsey Theorem states that for every coloring of a complete infinite graph by finitely many colors there is an infinite complete monochromatic subgraph.

$1/n$.³ The chosen player i may decide either (a) to quit, in which case the game terminates, and the terminal payoff is some vector $a_i \in \mathbf{R}^n$, which depends only on the identity of the chosen player, or (b) to continue, in which case the game continues to the next stage. If no player ever quits, the payoff is some vector $a_* \in \mathbf{R}^n$. Observe that this game is a simple multi-player Dynkin game (see Dynkin (1969)).

Since players do not play simultaneously, this game is a game with perfect information. It is well known that games with perfect information admit ϵ -equilibria in pure strategy profiles (see Mertens (1987) for a general argument in Borel games, or Thuijsman and Raghavan (1997), where this argument is adapted to stochastic games). Unfortunately, the ϵ -equilibrium strategy profiles Mertens (1987) and Thuijsman and Raghavan (1997) constructed use threats of punishment, which might be non-credible.

Here we study *subgame-perfect* ϵ -equilibria in this model, namely, strategy profiles which are an ϵ -equilibrium after any possible history.

Roughly speaking, our approach is as follows. Let $W \subset \mathbf{R}^n$ be the compact set that contains all the vectors w in the convex hull of $\{a_1, \dots, a_n\}$ such that $w^i \leq a_i^i$ for at least one player i . We define a certain set-valued function $F : W \rightarrow \mathbf{R}^n$, characterize the set of equilibrium payoffs that are supported by stationary strategies in terms of F , and prove that the differential inclusion $\dot{w} \in F(w)$ has a solution; namely, there is a continuous function $w : [0, +\infty) \rightarrow W$ such that $\dot{w}_t \in F(w_t)$ for almost every t . We then prove that any vector on a solution of the differential inclusion is a subgame-perfect equilibrium payoff. In particular, we prove that every quitting game with perfect information admits either an equilibrium payoff that is supported by stationary strategies, or (continuum of) subgame-perfect 0-equilibrium payoffs.

There are several motivations for our study. First, we try to find new approaches to study equilibrium payoffs in multi-player stochastic games and multi-player Dynkin games. Second, subgame-perfect equilibria are more useful than (Nash) equilibria in applications. Third, there are games, like quitting games and stopping games, in which conditioned on the stage of the game, there is only one possible history, so that deviations from a completely mixed strategy cannot be detected immediately. The study of subgame-perfect equilibria in our model may help us understand (Nash) equilibria in those models.

Our approach is somewhat related to the approach taken by Vieille (1992), and Laraki (2002), who use differential games to study repeated games with vector payoffs and repeated games with incomplete information on one side respectively. The dynamics of the differential game they use is $\dot{z}_t = -x_t A y_t$, where x_t is the control vector at time t of player 1, y_t is the control vector at time t of player 2, A is a payoff matrix, and z_t is the parameter at time t . Since in multi-player games there is a multiplicity of equilibria, the dynamics we study is a differential inclusion.

The paper is arranged as follows. The model is presented in section 2. In Section 3 we provide several examples that illustrate some features of the model. In Section 4 we define the notion of dummy players, and argue that w.l.o.g. one can assume there are no dummy players. In Section 5 we define the set-valued function F . We provide sufficient conditions for the existence of a subgame-perfect equilibrium payoff, and characterize the set of equilibrium payoffs that are supported by stationary strategies in terms of F in Section 6, and prove that the differential inclusion $\dot{w} \in F(w)$ has a solution in Section 7. After presenting a preliminary result in Section 8, we classify in Section 9 the solutions of the differential inclusion into two types. It is then shown that solutions of one type

³The case where the choice is not uniform is discussed in Section 10.

correspond to equilibrium payoffs that are supported by stationary strategies, and every vector on a solution of the other type correspond to a subgame-perfect 0-equilibrium. Extensions and open problems are discussed in Section 10.

2 The Model and the Main Result

A *quitting game with perfect information* Γ is given by

- A finite set $I = \{1, \dots, n\}$ of players.
- $n + 1$ vectors $a_1, \dots, a_n, a_*$ in $\mathbf{R}^n$.

The game is played as follows. At every stage $k \geq 1$ one of the players is chosen at random; each player is chosen with probability $1/n$. The chosen player i decides whether to quit, in which case the game terminates and the terminal payoff vector is a_i , or whether to continue, in which case the game continues to the next stage. If no player ever quits, the payoff is a_* .

We assume throughout that $\|a_*\| \leq 1$,⁴ and for every $i \in I$ $\|a_i\| \leq 1$ and $a_i^i = 0$.

For technical reasons, it will be more convenient to assume that players choose actions even if the game has already terminated. Setting $A = \{\text{Continue}, \text{Quit}\}$, the set of *histories* of length k is $H_k = (I \times A)^k$, the set of finite histories is $H = \cup_{k \geq 0} H_k$, and the set of infinite histories is $H_\infty = (I \times A)^{+\infty}$. Then H_∞ , equipped with the σ -algebra of cylinder sets, is a measurable space. We denote by $\mathcal{H}_k$ the sub- σ -algebra induced by the cylinder sets defined by H_k . Whenever $h_\infty \in H_\infty$ and $k \geq 0$, h_k is the unique finite history in H_k which is a prefix of h_∞ .

A (behavior) *strategy* of player i is a function $\sigma^i : H \rightarrow [0, 1]$; for every $h \in H_k$, $\sigma^i(h)$ is the probability that player i quits if the history h occurs and player i is chosen at stage $k + 1$.

A *stationary strategy* is a strategy in which $\sigma^i(h)$ is independent of h ; namely, a strategy in which player i quits whenever he is chosen with some fixed probability. We denote by 1^i (resp 0^i) the strategy of player i in which he quits with probability 1 (quits with probability 0) whenever he is chosen.

A *strategy profile*, or simply a *profile*, is a vector $\sigma = (\sigma^i)_{i \in I}$ of strategies, one for each players. A stationary profile, which is a vector of stationary strategies, is identified with a vector $\rho \in [0, 1]^n$; ρ^i is the probability that player i quits whenever he is chosen.

We denote by $\mathbf{i}_k$ and $\mathbf{a}_k$ the player chosen at stage k and the action he chooses respectively. Those are random variables.

Let $\theta = \min\{k \in \mathbf{N} \mid \mathbf{a}_k = \text{Quit}\}$ be the first stage in which the chosen player decides to quit. If no player ever quits, $\theta = +\infty$. Every profile σ induces a probability distribution $\mathbf{P}_\sigma$ over H_∞ . We denote by $\mathbf{E}_\sigma$ the corresponding expectation operator.

A strategy profile σ is *terminating* if $\mathbf{P}_\sigma(\theta < +\infty) = 1$, that is, under σ the game terminates a.s. Observe that a stationary profile ρ is terminating if and only if $\sum_{i \in I} \rho^i > 0$.

The *expected payoff* of player i that corresponds to a profile σ is

$$\gamma^i(\sigma) = \mathbf{E}_\sigma[\mathbf{1}_{\{\theta < +\infty\}} \mathbf{a}_{\mathbf{i}_\theta}^i + \mathbf{1}_{\{\theta = +\infty\}} a_*^i].$$

Definition 1 Let $\epsilon \geq 0$. A profile σ is an ϵ -equilibrium if for every player $i \in I$ and every strategy σ^i of player i ,

$$\gamma^i(\sigma) \geq \gamma^i(\sigma^{-i}, \sigma^i) - \epsilon.$$

⁴The norm we use along the paper is the maximum norm.

The expected payoff that corresponds to an ϵ -equilibrium is an ϵ -equilibrium payoff, and any accumulation point of ϵ -equilibrium payoffs, as ϵ goes to 0, is an equilibrium payoff. Since payoffs are bounded, an equilibrium payoff exists once for every $\epsilon > 0$ an ϵ -equilibrium exists.

It is well known (see, e.g., Mertens (1987) or Thuijsman and Raghavan (1997)) that in every quitting game with perfect information, and more generally, in every Borel game, an ϵ -equilibrium exists.

Given a strategy σ^i of player i and a finite history $h = (i_1, \dots, i_k) \in H$, the strategy σ_h^i is given by:

$$\sigma_h^i(h') = \sigma^i(h, h'),$$

for every finite history $h' = (i'_1, \dots, i'_{k'})$, where $(h, h') = (i_1, \dots, i_k, i'_1, \dots, i'_{k'})$. This is the continuation strategy given the history h occurs.

Given a profile σ and a finite history $h \in H$, we denote $\sigma_h = (\sigma_h^i)_{h \in H}$.

Definition 2 Let $\epsilon \geq 0$. A profile σ is a subgame-perfect ϵ -equilibrium if for every finite history $h \in H$, the profile σ_h is an ϵ -equilibrium.

Clearly any ϵ -equilibrium in stationary strategies is a subgame-perfect ϵ -equilibrium.

In the present paper we study subgame-perfect equilibrium payoffs. Our approach is to define a certain differential inclusion, and to relate its solutions to subgame-perfect equilibrium payoffs in the game.

3 Examples

We provide here few examples that illustrate some features of the model. In the first two examples, a_* may be arbitrary.

Example 1: Take $n = 4$, $a_1 = (0, 3, -1, -1)$, $a_2 = (3, 0, -1, -1)$, $a_3 = (-1, -1, 0, 3)$ and $a_4 = (-1, -1, 3, 0)$. This is an adaptation of the game studied by Solan and Vieille (2001).

This game admits a 0-equilibrium in pure stationary strategies: players 1 and 3 quit whenever chosen, and players 2 and 4 continue whenever chosen. The equilibrium payoff is $\frac{1}{2}(0, 3, -1, -1) + \frac{1}{2}(-1, -1, 0, 3) = (-\frac{1}{2}, 2, -\frac{1}{2}, 2)$, so indeed only players 1 and 3 have any incentive to quit.

Example 2: Take $n = 3$, $a_1 = (0, 2, -1)$, $a_2 = (-1, 0, 2)$ and $a_3 = (2, -1, 0)$. This is an adaptation of the game studied by Flesch et al. (1997) to our setup. We present here two subgame-perfect 0-equilibria in Markovian strategies. The two equilibria are periodic, one is pure with period 9, and the other is mixed with period 6.

Stage	Profile	Payoffs	Profile	Payoffs
1	1, 0, 1	-0.178, 1.381, -0.202	1, 0, 0	0, $\frac{1}{2}$, $\frac{1}{2}$
2	1, 0, 0	-0.267, 1.072, 0.196	$\frac{3}{4}$, 0, 0	0, 0, 1
3	1, 0, 0	-0.401, 0.607, 0.794	0, 1, 0	$\frac{1}{2}$, 0, $\frac{1}{2}$
4	1, 1, 0	-0.202, -0.178, 1.381	0, $\frac{3}{4}$, 0	1, 0, 0
5	0, 1, 0	0.196, -0.267, 1.072	0, 0, 1	$\frac{1}{2}$, $\frac{1}{2}$, 0
6	0, 1, 0	0.794, -0.401, 0.607	0, 0, $\frac{3}{4}$	0, 1, 0
7	0, 1, 1	1.381, -0.202, -0.178		
8	0, 0, 1	1.072, 0.196, -0.267		
9	0, 0, 1	0.607, 0.794, -0.401		

In every row appear the probabilities by which the players quit if they are chosen, and the continuation payoff.

For example, in the first equilibrium, the equilibrium payoff is $(0.607, 0.794, -0.401)$ (the payoff at the end of the period). The continuation payoff (=expected payoff if stage 2 is reached) is $(-0.178, 1.381, -0.202)$, so that players 1 and 3 want to quit at stage 1. And indeed,

$$(0.607, 0.794, -0.401) = \frac{1}{3}(0, 2, -1) + \frac{1}{3}(-0.178, 1.381, -0.202) + \frac{1}{3}(2, -1, 0).$$

Similarly, the expected payoff if stage 3 is reached is $(-0.267, 1.072, 0.196)$, so that at stage 2 only player 1 wants to quit. And indeed we have

$$(-0.178, 1.381, -0.202) = \frac{1}{3}(0, 2, -1) + \frac{2}{3}(-0.267, 1.072, 0.196).$$

The second equilibrium corresponds to the one identified by Flesch et al. (1997) for their example. Indeed, in every period, the probability that player 1 quits is $\frac{1}{3} + \frac{2}{3} \times \frac{1}{3} \times \frac{3}{4} = \frac{1}{2}$. Similarly, the probability that player 2 quits in a given period provided player 1 did not quit in that period is $1/2$, and the probability that player 3 quit in a given period provided players 1 and 2 did not quit is $1/2$.

Observe that one can construct more equilibria. Since $(0, 0, 1)$ (and by symmetry $(0, 1, 0)$ and $(1, 0, 0)$) is an equilibrium payoff (see the second equilibrium), $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ is an equilibrium payoff as well: at the first stage the chosen player continues, while from the second stage on the players implement the equilibrium that corresponds to $(0, 0, 1)$ (resp. $(0, 1, 0)$, $(1, 0, 0)$) if player 1 (resp. 2, 3) was chosen at the first stage.

In fact, one can show that for this example every feasible and individually rational payoff vector (that is, every vector in the set $\text{conv}\{a_1, \dots, a_n\} \cap \{x \in \mathbf{R}^3 \mid x^i \geq -1/2 \ \forall i\}$) is a subgame-perfect 0-equilibrium payoff. This observation does not hold in general.

We end this example by describing another subgame-perfect 0-equilibrium in Markovian strategies, which gives the basic idea of the equilibria we construct in the general case. In this equilibrium the players use a parameter, which is the expected continuation payoff; each player's mixed action depends solely on the expected continuation payoff of all players. There are six possible continuation payoffs: $(1, 0, 0)$, $(0, 1, 0)$, $(0, 0, 1)$, $(0, \frac{1}{2}, \frac{1}{2})$, $(\frac{1}{2}, 0, \frac{1}{2})$, and $(\frac{1}{2}, \frac{1}{2}, 0)$. We will use the following identities:

$$(0, 1, 0) = \frac{1}{3}(1 \times (0, 2, -1) + 0 \times (0, \frac{1}{2}, \frac{1}{2})) + \frac{1}{3}(0 \times (-1, 0, 2) + 1 \times (0, \frac{1}{2}, \frac{1}{2})) + \frac{1}{3}(0 \times (2, -1, 0) + 1 \times (0, \frac{1}{2}, \frac{1}{2})),$$

and

$$(0, \frac{1}{2}, \frac{1}{2}) = \frac{1}{3}(\frac{3}{4} \times (0, 2, -1) + \frac{1}{4} \times (0, 0, 1)) + \frac{1}{3}(1 \times (-1, 0, 2) + 0 \times (0, 0, 1)) + \frac{1}{3}(0 \times (2, -1, 0) + 1 \times (1, 0, 0)).$$

We describe the subgame-perfect 0-equilibrium when the continuation payoff is $(0, 1, 0)$ and $(0, \frac{1}{2}, \frac{1}{2})$. The behavior after the other four vectors is symmetric.

Assume the continuation payoff is $(0, 1, 0)$. If player 1 (resp. 2, 3) is chosen, he quits with probability 1 (resp. 0, 0). If the chosen player does not quit, the continuation payoff is $(0, \frac{1}{2}, \frac{1}{2})$.

Assume the continuation payoff is $(0, \frac{1}{2}, \frac{1}{2})$. If player 1 (resp. 2, 3) is chosen, he quits with probability $\frac{3}{4}$ (resp. 1, 0). If he does not quit, the continuation payoff is $(0, 0, 1)$ (resp. $(0, 0, 1)$, $(1, 0, 0)$).

The two identities given above imply that this is indeed a subgame-perfect 0-equilibrium.

In the following example, there is no subgame-perfect 0-equilibrium.

Example 3: Take $n = 2$, $a_1 = (0, 1)$, $a_2 = (-1, 0)$, and $a_* = (1, -1)$. This is an adaptation of Example 3 in Solan and Vieille (2002).

Here there is a subgame-perfect ϵ -equilibrium in mixed stationary strategies: player 1 quits whenever chosen with probability 1, and player 2 quits whenever chosen with probability ϵ . The expected payoff is $\frac{1}{1+\epsilon}(0, 1) + \frac{\epsilon}{1+\epsilon}(-1, 0) = (-\frac{\epsilon}{1+\epsilon}, \frac{1}{1+\epsilon})$. One can verify that player 1 cannot profit by deviating, while player 2 cannot profit more than $\frac{\epsilon}{1+\epsilon}$ by deviating.

The same analysis that was performed by Solan and Vieille (2002, Example 3) shows that this game admits no subgame-perfect 0-equilibrium. The basic idea is the following. Consider the three events: A = the game is terminated by player 1, B = the game is terminated by player 2, and C = the game continues indefinitely. Player 1 prefers C to A and A to B , while player 2 prefers A to B and B to C . To achieve event A , which is controlled by player 1, player 2 must threaten player 1 by event B , which is suboptimal for both players. However, since player 1 prefers event C , without such a threat event C will be realized, but event C is worse than B to player 2, so a suboptimal threat is necessary.

4 Dummy Players

In this section we define the notion of dummy players, and we see that dummy players essentially never participate in the game: they never quit. One can then eliminate those players from the game. Recall that $a_i^i = 0$ for every $i \in I$.

Definition 3 A player i is dummy if (i) $a_*^i > 0$, and (ii) $a_j^i > 0$ for every $j \neq i$.

A dummy player never wants to quit: whether the game is going to continue indefinitely, or whether some other player is going to quit, he himself does not want to quit. It is no surprise then that one can eliminate dummy players.

Lemma 4 Let $i \in I$ be a dummy player in Γ , and let $\epsilon > 0$. Let Γ' be the $n - 1$ -player game in which we eliminate player i . Then any ϵ -equilibrium in Γ' can be extended to an ϵ -equilibrium in Γ , by instructing player i to continue whenever he is chosen. Moreover, in every ϵ -equilibrium in Γ the overall probability that the game is terminated by player i is at most ϵ/A , where $A = \min\{a_*^i, a_j^i, j \neq i\}$.

In particular, every ϵ -equilibrium in Γ can be turned into an $(1 + 1/A)\epsilon$ -equilibrium in which player i never quits, simply by modifying the profile so that player i never quits.

The proof is straightforward and omitted.

Observe that a player may not be a dummy player, but, after eliminating some other dummy player, may become dummy in the $n - 1$ -player game.

From now on our games contain no dummy players.

5 A Differential Inclusion

A differential inclusion is an equation of the form $\dot{w} \in F(w)$, where F is a set-valued function. This is a generalization of standard differential equations. A solution of a differential inclusion is a continuous function $t \mapsto w_t$ such that $\dot{w}_t \in F(w_t)$ for almost every t . Differential inclusions have been extensively studied, see, e.g., Aubin and Cellina (1984) or Fillipov (1988).

In the present section we construct a differential inclusion from that data of the game. In subsequent sections we show how it is related to the study of subgame-perfect equilibrium payoffs in the game.

Set

$$W = \{w \in \text{conv}\{a_1, \dots, a_n\} \mid w^i \leq 0 \text{ for some } i \in I\}.$$

The set W is non-empty as it contains a_i , $i \in I$. It is also compact, but not necessarily convex or even connected (e.g., if $n = 2$, $a_1 = (0, 1)$ and $a_2 = (1, 0)$ then $W = \{(0, 1), (1, 0)\}$).

For every $w \in W$ define

$$\begin{aligned} I_N(w) &= \{i \in I \mid w^i < 0\}, \\ I_Z(w) &= \{i \in I \mid w^i = 0\}, \text{ and} \\ I_P(w) &= \{i \in I \mid w^i > 0\}. \end{aligned}$$

Observe that for every $w \in W$, $I_N(w) \cup I_Z(w) \neq \emptyset$.

Set⁵

$$\Delta(w) = \left\{ \rho \in [0, 1]^n \mid i \in I_P(w) \Rightarrow \rho^i = 0, i \in I_N(w) \Rightarrow \rho^i = 1, \sum_{i=1}^n \rho^i \geq 1 \right\}.$$

This definition captures the following idea. If w is the continuation payoff and player i is chosen, i will quit with probability 1 if $w^i < 0$, and he will quit with probability 0 if $w^i > 0$. Thus, any $\rho \in \Delta(w)$ is a possible description of the behavior of rational players at a given stage, when the continuation payoff is w .

Observe that since $I_N(w) \cup I_Z(w) \neq \emptyset$ for every $w \in W$, Δ has non-empty values.

Define

$$F(w) = \left\{ \sum_{i=1}^n \rho^i (w - a_i) \mid \rho \in \Delta(w) \right\} \subset \mathbf{R}^n.$$

We will be interested in solutions of the equation

$$\dot{w} \in F(w). \tag{1}$$

The reader may wonder why the differential inclusion (1) is of interest. Consider a variation of the game, in which at every stage each player is chosen with probability ϵ , and with probability $1 - n\epsilon$ no player is chosen. If the continuation payoff at stage k is w_{k+1} , then any vector $\rho \in \Delta(w_{k+1})$ may describe the behavior of the players at that stage. Fix $\rho \in \Delta(w_{k+1})$. The expected payoff at stage k is, then,

$$w_k = \sum_{i \in I} \epsilon \rho^i a_i + (1 - \sum_{i \in I} \epsilon \rho^i) w_{k+1}.$$

⁵In the definition of $\Delta(w)$, one can take $\sum_{i=1}^n \rho^i \geq c$ for any fixed $0 < c \leq 1$.

This implies that

$$\frac{w_{k+1} - w_k}{\epsilon} = \sum_{i \in I} \rho^i (w_{k+1} - a_i). \quad (2)$$

The differential inclusion (1) is simply the limit of (2) as ϵ goes to 0.

6 Conditions for Existence of Subgame-Perfect Equilibria

In the present section we present several sufficient conditions for the existence of a subgame-perfect equilibrium payoffs. Furthermore, we characterize the set of equilibrium payoffs that are supported by stationary strategies in terms of the set-valued function F .

The following Lemma provides a condition that ensures that a subgame-perfect 0-equilibrium exists.

Lemma 5 *Let $Y \subseteq W$. Assume that there exists $\eta > 0$ such that for every $y \in Y$ there exist $\rho \in [0, 1]^n$ and $y_1, \dots, y_n \in Y$ which satisfy:*

$$C.1 \quad y = \frac{1}{n} \sum_{i=1}^n (\rho^i a_i + (1 - \rho^i) y_i).$$

$$C.2 \quad y_i^i > 0 \text{ implies that } \rho^i = 0.$$

$$C.3 \quad y_i^i < 0 \text{ implies that } \rho^i = 1.$$

$$C.4 \quad \max_{i=1}^n \rho^i \geq \eta.$$

Assume moreover that

$$C.5 \quad \text{For every } i \in I, \text{ if } a_i \text{ is in the closure of } Y \text{ then } a_i^j < 0 \text{ for some } j \in I.$$

Then every $y \in Y$ is a subgame-perfect 0-equilibrium payoff.

The Lemma is an adaptation of a well known result in the context of discounted stochastic game (see, e.g., Solan (1998, Lemma 5.1)).

Proof. Choose an arbitrary $y \in Y$. We define simultaneously a profile σ and a function $u : H \rightarrow Y$.

Set $u(\emptyset) = y$. Assume we have already defined $u(h)$ for some finite history h . By assumption, there exist $\rho \in [0, 1]^n$ and $y_1, \dots, y_n \in Y$ that satisfy $u(h) = \frac{1}{n} \sum_{i=1}^n (\rho^i a_i + (1 - \rho^i) y_i)$ and (C.2)-(C.4).

Thus, if y_i is the continuation payoff if player i is chosen, then by (C.2) and (C.3) ρ^i is an optimal response of player i , and $u(h)$ is the expected payoff conditioned that h is realized.

Set $\sigma^i(h) = \rho^i$ and $u(h, i) = y_i$ for every $i \in I$.

By (C.4), under σ the game eventually terminates, hence for every finite history h , $u(h)$ is indeed the expected payoff under σ_h .

Condition (C.5) implies that for every player $i \in I$, the profile $(\sigma^{-i}, 0^i)$ in which all players but i follow σ and player i never quits is terminating with probability 1. Indeed, otherwise, for every $\delta > 0$ there is a finite history h and a player i such that the probability the game terminates under $(\sigma_h^{-i}, 0^i)$ is at most δ . But then for every $j \in I$, the probability the game terminates under $(\sigma_{h,j}^{-i}, 0^i)$

is at most $n\delta$. Since under $\sigma_{h,j}$ termination occurs with probability 1, the expected payoff under $\sigma_{h,j}$ is within $n\delta$ of a_i .

If a_i is not in the closure of Y , then there is $\delta' > 0$ such that the distance between a_i and the closure of Y is at least δ' . Since $u(h, j) = \gamma(\sigma_{h,j})$ is in Y , this leads to a contradiction if $\delta < \delta'/n$.

If, on the other hand, there is $j \in I$ such that $a_i^j < 0$, by choosing $\delta < 1/n$ sufficiently small such that $a_i^j < -n\delta$, (C.3) implies that $\sigma^j(h) = 1$. Therefore the probability of termination under $(\sigma_h^{-i}, 0^i)$ is at least $1/n$, a contradiction.

Finally we prove that no player $i \in I$ can profit by deviating from σ . The same proof holds for any subgame, hence σ is a subgame-perfect 0-equilibrium. Fix a player $i \in I$ and a strategy σ'^i of player i .

For every $k \in \mathbf{N}$ define a r.v. X_k as follows. $X_k = \mathbf{a}_{\mathbf{i}_\theta}^i$ if $\theta < k$, and $X_k = \gamma^i(\sigma_k)$ otherwise, where σ_k is the random strategy induced from stage k on (that is, σ_k is a strategy-valued r.v.)

By (C.2)-(C.3) and the definition of σ ,

$$\mathbf{E}_{\sigma^{-i}, \sigma'^i}[X_{k+1} \mid \mathcal{H}_k] \leq X_k \quad \forall k \geq 0. \quad (3)$$

Since the profile $(\sigma^{-i}, 0^i)$ is terminating, so is the profile (σ^{-i}, σ'^i) . By taking expectations over (3) and summing up to k we obtain

$$\gamma^i(\sigma^{-i}, \sigma'^i) = \mathbf{E}_{\sigma^{-i}, \sigma'^i}[\mathbf{a}_{\mathbf{i}_\theta}^i \mathbf{1}_{\{\theta < +\infty\}}] = \lim_{k \rightarrow +\infty} \mathbf{E}_{\sigma^{-i}, \sigma'^i}[X_k] \leq X(\emptyset) = \gamma^i(\sigma),$$

as desired. ■

Remark 1: Actually, every y in $\bar{Y}$, the closure of Y , is a subgame-perfect 0-equilibrium payoff. Indeed, if a set Y satisfies the conditions of Lemma 5, then by a limiting argument the set $\bar{Y}$ satisfies these conditions as well.

The following Lemma is a variation of Lemma 5, in which we replace condition (C.5) by the condition that there are no dummy players. The conclusion is weakened as well: every $y \in Y$ is a subgame-perfect ϵ -equilibrium payoff rather than a subgame-perfect 0-equilibrium payoff.

Lemma 6 *Let $Y \subseteq W$. Assume there exists $\eta > 0$ such that for every $y \in Y$ there exist $\rho \in [0, 1]^n$ and $y_1, \dots, y_n \in Y$ which satisfy (C.1)-(C.4) of Lemma 5. Assume moreover that there are no dummy players. Then every $y \in \bar{Y}$ is a subgame-perfect ϵ -equilibrium payoff, for every $\epsilon > 0$.*

Proof. Fix $\epsilon \in (0, 1/2)$. We prove that every $y \in Y$ is a subgame-perfect ϵ -equilibrium payoff. The proof that every $y \in \bar{Y}$ is a subgame-perfect ϵ -equilibrium is done as in Remark 1.

Define the profile σ as in the proof of Lemma 5. In contrast to the proof of Lemma 5, $(\sigma^{-i}, 0^i)$ may not be terminating for some $i \in I$. Our goal is to augment σ so that the same idea can still be applied.

Since there are no dummy players, for every player $i \in I$ either $a_*^i \leq 0$, or there is $j_i \neq i$ such that $a_{j_i}^i \leq 0$ (j_i is a “punisher” of i). For every $j \in I$ set $I_j = \{i \in I \mid j = j_i\}$, the set of players that j punishes. Since for every player we choose at most one punisher, $(I_j)_{j \in I}$ are disjoint sets.

Define a profile τ as follows.

$$\tau^j(h) = \min \left\{ \sigma^j(h) + \frac{\epsilon}{n} \sum_{i \in I_j} \sigma^i(h), 1 \right\}.$$

In words, for every player i that j punishes, the probability that j quits is increased by ϵ times the probability that i quits.

We prove that no player can profit more than 4ϵ by deviating from τ . The same proof holds for any subgame, hence τ is a subgame-perfect 4ϵ -equilibrium. Fix a player $i \in I$ and a strategy τ^i of player i .

Define the sequence $(X_k)_{k \in \mathbb{N}}$ as in the proof of Lemma 5. By the definition of τ and (3) one obtains

$$\begin{aligned} \mathbf{E}_{\tau^{-i}, \tau^i}[X_{k+1} \mid \mathcal{H}_k] &\leq \mathbf{E}_{\sigma^{-i}, \tau^i}[X_{k+1} \mid \mathcal{H}_k] + \epsilon \mathbf{P}_{\sigma^{-i}, \tau^i}(\theta = k+1 \mid \mathcal{H}_k) \\ &\leq X_k + 2\epsilon \mathbf{P}_{\tau^{-i}, \tau^i}(\theta = k+1 \mid \mathcal{H}_k). \end{aligned} \quad (4)$$

Taking expectation in (4), and summing over $k \geq 0$, one obtains $\gamma^i(\tau^{-i}, \tau^i) \leq \gamma^i(\tau) + 2\epsilon$, provided (τ^{-i}, τ^i) is terminating.

Assume now that (τ^{-i}, τ^i) is not terminating, that is, $\mathbf{P}_{\tau^{-i}, \tau^i}(\theta < +\infty) < 1$. Then there exists k_0 such that $\mathbf{P}_{\tau^{-i}, \tau^i}(\theta < +\infty \mid \tau \geq k_0) < \epsilon$. For every $h \in H_{k_0}$ σ_h is terminating, and therefore so is τ_h . Hence, the expected payoff under τ_h is within ϵ of a_i , while the expected payoff under (τ_h^{-i}, τ_h^i) is within ϵ of a_* . Since $\mathbf{P}_{\tau^{-i}, \tau^i}(\theta < +\infty) \leq \mathbf{P}_{\tau^{-i}, \tau^i}(\theta < +\infty) < 1$, the definition of τ implies that i has no punisher, so that $a_*^i \leq 0$. Therefore

$$\mathbf{E}_{\tau^{-i}, \tau^i}[\mathbf{1}_{\theta < +\infty} \mathbf{a}_{i_\theta}^i \mid \theta \geq k_0] \leq a_*^i + \epsilon \leq \epsilon = a_i^i + \epsilon \leq \mathbf{E}_\tau[\mathbf{1}_{\theta < +\infty} \mathbf{a}_{i_\theta}^i \mid \theta \geq k_0] + 2\epsilon. \quad (5)$$

By the definition of τ , and since ϵ is sufficiently small,

$$\mathbf{P}_{\sigma^{-i}, \tau^i}(\theta = k+1 \mid \mathcal{H}_k) \leq \mathbf{P}_{\tau^{-i}, \tau^i}(\theta = k+1 \mid \mathcal{H}_k) \leq (1 + \epsilon) \mathbf{P}_{\sigma^{-i}, \tau^i}(\theta = k+1 \mid \mathcal{H}_k).$$

It follows that

$$\|\mathbf{P}_{\sigma^{-i}, \tau^i}(\cdot \mid \mathcal{H}_k) - \mathbf{P}_{\tau^{-i}, \tau^i}(\cdot \mid \mathcal{H}_k)\| < \epsilon/(1 - \epsilon) \leq 2\epsilon.$$

This implies, together with (5), that $\gamma^i(\tau^{-i}, \tau^i) \leq \gamma^i(\tau) + 2\epsilon(1 + 2\epsilon) \leq \gamma^i(\tau) + 4\epsilon$, as desired. ■

Remark 2: One can weaken condition (C.4) (both in Lemma 5 and in Lemma 6.) All that is needed is that for every $h \in H$, the profile σ_h that is defined in the proof of Lemma 5 is terminating.

As a consequence of Lemma 5 we obtain the following.

Corollary 7 *Assume there are no dummy players, and let $w \in W$. If $\vec{0} \in F(w)$ then w is an equilibrium payoff that is supported by terminating stationary profiles.*

Remark 3: If i is a dummy player, then it might happen that $\vec{0} \in F(a_i)$ but a_i is not an equilibrium payoff (e.g., $n = 2$, $a_1 = (0, 2)$, $a_2 = (2, 0)$ and $a_* = (1, 1)$.)

The converse of Corollary 7 is given by the following Lemma.

Lemma 8 *Let $w \in W$. If w is an equilibrium payoff that is supported by terminating stationary profiles then $\vec{0} \in F(w)$.*

Proof. Assume that w is an equilibrium payoff that is supported by terminating stationary profiles. That is, for every $\epsilon > 0$ there is a terminating stationary ϵ -equilibrium ρ_ϵ with expected payoff w_ϵ such that $\lim_{\epsilon \rightarrow 0} w_\epsilon = w$. In particular, $\frac{\sum_{i \in I} \rho_\epsilon^i a_i}{\sum_{i \in I} \rho_\epsilon^i} = w_\epsilon$. Since by multiplying all the coordinates of ρ_ϵ by a constant larger than 1 one still obtains an ϵ -equilibrium, we can assume w.l.o.g. that $\max_{i=1, \dots, n} \rho_\epsilon^i = 1$ for every $\epsilon > 0$.

By taking a subsequence, we assume w.l.o.g. that the support of $(\rho_\epsilon)_{\epsilon > 0}$, that is, the set of players that quit under ρ_ϵ with positive probability whenever chosen, is independent of ϵ .

We will show that any accumulation point ρ of the sequence $(\rho_\epsilon)_{\epsilon > 0}$ as ϵ goes to 0 is in $\Delta(w)$. Observe that any such accumulation point satisfies $\sum_{i \in I} \rho^i \geq 1$ and $\frac{\sum_{i \in I} \rho^i a_i}{\sum_{i \in I} \rho^i} = w$, so that one would have $\vec{0} \in F(w)$, as desired.

Case 1: The support of ρ_ϵ contains a single player i for every $\epsilon > 0$.

For every $\epsilon > 0$ $\max_{j=1, \dots, n} \rho_\epsilon^j = 1$, hence $\rho_\epsilon^i = 1$ and $\rho_\epsilon^j = 0$ for every $j \neq i$. Therefore, $w_\epsilon = a_i$ for every $\epsilon > 0$, so that $w = a_i$. Since ρ_ϵ is an ϵ -equilibrium, if player $k \neq i$ quits with probability 1 then he gains at most ϵ . Since $a_k^k = 0$, one obtains:

$$\frac{1}{2}a_i^k = \gamma^k(\rho_\epsilon^{-k}, 1^k) \leq w_\epsilon + \epsilon = a_i^k + \epsilon,$$

which implies that $a_i^k \geq -2\epsilon$. Since ϵ is arbitrary, we get $a_i^k \geq 0$ for every $k \neq i$, and by assumption $a_i^i = 0$. This means that the vector ρ that is defined by $\rho^i = 1$ and $\rho^k = 0$ for every $k \neq i$ is in $\Delta(a_i) = \Delta(w)$, as desired.

Case 2: The support of ρ_ϵ contains at least two players for every $\epsilon > 0$.

Fix $\epsilon > 0$. Since $a_j^j = 0$, one has for every $j \in I$,

$$\sum_{i \neq j} \rho_\epsilon^i a_i^j = \sum_{i \in I} \rho_\epsilon^i w_\epsilon^j. \quad (6)$$

Since ρ_ϵ is an ϵ -equilibrium, if j quits with probability 1 whenever he is chosen he profits at most ϵ :

$$\frac{\sum_{i \neq j} \rho_\epsilon^i a_i^j}{1 + \sum_{i \neq j} \rho_\epsilon^i} = \gamma^j(\rho_\epsilon^{-j}, 1^j) \leq w_\epsilon^j + \epsilon.$$

Incorporating (6), and since $\rho_\epsilon^i \in [0, 1]$ for every $i \in I$, this yields $-\epsilon n \leq w_\epsilon^j(1 - \rho^j)$, so that

$$w^j < 0 \Rightarrow w_\epsilon^j < 0 \text{ for every } \epsilon \text{ sufficiently small} \Rightarrow \rho^j = 1. \quad (7)$$

Similarly, player j cannot profit more than ϵ if he continues whenever he is chosen:

$$\frac{\sum_{i \neq j} \rho_\epsilon^i a_i^j}{\sum_{i \neq j} \rho_\epsilon^i} = \gamma^j(\rho_\epsilon^{-j}, 0^j) \leq w_\epsilon^j + \epsilon.$$

Since the support of ρ_ϵ contains at least two players, $\sum_{i \neq j} \rho_\epsilon^i > 0$ for every $\epsilon > 0$. Incorporating (6), and since $\rho_\epsilon^i \in [0, 1]$ for every $i \in I$, this yields $\rho_\epsilon^j w_\epsilon^j \leq \epsilon n$, so that

$$w^j > 0 \Rightarrow w_\epsilon^j > 0 \text{ for every } \epsilon \text{ sufficiently small} \Rightarrow \rho^j = 0. \quad (8)$$

Since $\max_{i=1,\dots,n} \rho_\epsilon^i = 1$, we have $\max_{i=1,\dots,n} \rho^i = 1$, and $\rho \in \Delta(w)$, as desired. ■

The next lemma, together with Corollary 7, implies that if there are no dummy player and $\vec{0} \in W$ then $\vec{0}$ is an equilibrium payoff that is supported by stationary strategies.

Lemma 9 *If $\vec{0} \in W$ then $\vec{0} \in F(\vec{0})$.*

Proof. Assume that $\vec{0} \in W$. Then there is $\rho \in [0, 1]^n$ that satisfies (i) $\sum_{i \in I} \rho^i = 1$, and (ii) $\sum_{i \in I} \rho^i a_i = \vec{0}$. Since $I_Z(\vec{0}) = I$, it follows that $\rho \in \Delta(\vec{0})$, and the result follows. ■

Lemma 10 *If for some $i \in I$, $a_i^j \geq 0$ for every $j \in I$, then $\vec{0} \in F(a_i)$.*

Indeed, under the assumptions, the vector ρ that is defined by $\rho^i = 1$ and $\rho^j = 0$ for every $j \neq i$, is in $\Delta(a_i)$, so that $\vec{0} \in F(a_i)$.

By Corollary 7 a_i is an equilibrium payoff that is supported by stationary strategies.

The next Lemma states that if every player prefers the game to continue indefinitely rather than to quit, then “no one ever quits” is a 0-equilibrium. Such a situation occurs, for example, if all players are dummy players. Its proof is omitted.

Lemma 11 *If $a_*^i \geq 0$ for every player $i \in I$, then the profile σ that is defined by*

$$\sigma^i(h) = 0 \quad \forall i \in I, h \in H,$$

that is, no player ever quits, is a stationary 0-equilibrium.

7 Existence of a Solution to the Differential Inclusion

In the present section we prove that the differential inclusion $\dot{w} \in F(w)$ has at least one solution.

Definition 12 *A function $g : \mathbf{R} \rightarrow \mathbf{R}^n$ is absolutely continuous if for every $\epsilon > 0$ there is $\delta > 0$ such that for every $m \in \mathbf{N}$ and every collection $(x_i, y_i)_{i=1}^m$ of real numbers, if $\sum_{i=1}^m |x_i - y_i| < \delta$ then $\sum_{i=1}^m \|g(x_i) - g(y_i)\| < \epsilon$.*

Observe that if g is absolutely continuous it is in particular uniformly continuous.⁶

The main result of this section is the following.

Proposition 13 *There is an absolutely continuous function $w : \mathbf{R} \rightarrow W$ that satisfies $\dot{w}_t \in F(w_t)$ for almost every $t \in \mathbf{R}$.*

The rest of the section is devoted to the proof of Proposition 13.

The proof of the following Lemma follows from the definitions.

Lemma 14 *For every $w \in W$, $\Delta(w)$ and $F(w)$ are compact, convex and non-empty subsets of $\mathbf{R}^n$.*

Lemma 15 *The set-valued functions $w \mapsto \Delta(w)$ and $w \mapsto F(w)$ are upper-semi-continuous; that is, their graphs are closed sets in $\mathbf{R}^{2n}$.*

⁶A function g is *uniformly continuous* if the condition in Definition 12 holds for $m = 1$.

Proof. We first prove that $w \mapsto \Delta(w)$ is upper-semi-continuous. Let $(w(k))_{k \in \mathbf{N}}$ be a sequence of elements in W that converge to w , and let $(\rho(k))_{k \in \mathbf{N}}$ be a sequence of elements in $[0, 1]^n$ that converge to ρ , such that $\rho(k) \in \Delta(w(k))$ for every $k \in \mathbf{N}$. We show that $\rho \in \Delta(w)$.

As $\rho(k) \in \Delta(w(k))$ for every $k \in \mathbf{N}$, $\sum_{i \in I} \rho^i(k) \geq 1$ for every $k \in \mathbf{N}$. Hence $\sum_{i \in I} \rho^i \geq 1$.

Fix $i \in I$. If $i \in I_N(w)$ then $w^i < 0$. Hence $i \in I_N(w(k))$ for every k sufficiently large. In particular, $\rho^i(k) = 1$ for every k sufficiently large, so that $\rho^i = 1$. If $i \in I_P(w)$ then $w^i > 0$. Hence $i \in I_P(w(k))$ for every k sufficiently large. In particular, $\rho^i(k) = 0$ for every k sufficiently large, so that $\rho^i = 0$.

We now prove that $w \mapsto F(w)$ is upper-semi-continuous as well. Let $(w(k))_{k \in \mathbf{N}}$ and $(y(k))_{k \in \mathbf{N}}$ be two converging sequences of elements in W and $\mathbf{R}^n$ respectively. Denote their limits by w and y respectively. We assume that $y(k) \in F(w(k))$ for every $k \in \mathbf{N}$, and prove that $y \in F(w)$.

For every $k \in \mathbf{N}$ there is $\rho(k) \in \Delta(w(k))$ such that $y(k) = \sum_{i=1}^n \rho^i(k)(w(k) - a_i)$. By taking a subsequence we assume w.l.o.g. that $\rho(k) \rightarrow \rho$ for some $\rho \in [0, 1]^n$. Then $y = \sum_{i=1}^n \rho^i(w - a_i)$. By upper-semi-continuity of Δ , $\rho \in \Delta(w)$. The result follows. ■

Lemma 16 *For every $w \in W$ there is $y \in F(w)$ such that $w - \lambda y \in W$ for every $\lambda > 0$ sufficiently small.*

Geometrically the lemma claims that for every vector $w \in W$ there is a vector $y \in F(w)$ such that $-y$ “points into W ”.

Proof. Fix $w \in W$ and $y \in F(w)$. Then $y = (\sum_{i=1}^n \rho^i)w - \sum_{i=1}^n \rho^i a_i$ for some vector $\rho \in [0, 1]^n$. In particular, $w - \lambda y = (1 - \sum_{i=1}^n \lambda \rho^i)w + \sum_{i=1}^n \lambda \rho^i a_i$. Since w and y are in the convex hull of $\{a_1, \dots, a_n\}$, so is $w - \lambda y$, provided $\lambda \leq 1/n$.

It remains to show that there is $y \in F(w)$ such that $w^i - \lambda y^i \leq 0$ for some $i \in I$.

If $I_N(w) \neq \emptyset$ then $w^i < 0$ for some $i \in I$, and any $y \in F(w)$ satisfies this requirement. If $I_N(w) = \emptyset$ then, since $w \in W$, there is $i \in I_Z(w)$ such that $w^i = 0$. In particular, setting $y = w - a_i \in F(w)$, one has $w^i - \lambda y^i = (1 - \lambda)w^i + \lambda a_i^i = 0$. ■

We will use the following Lemma.

Lemma 17 *Let $-\infty < a < b < +\infty$. For every $k \in \mathbf{N}$ let $w(k) : (a, b) \rightarrow W$ be an absolutely continuous function such that $\dot{w}_t(k) \in F(w_t(k))$ for almost every $t \in [a, b]$. Assume there is a function $w : (a, b) \rightarrow W$ such that $w_t = \lim_{k \rightarrow \infty} w_t(k)$ for almost every $t \in (a, b)$. Then w is absolutely continuous, and $\dot{w}_t \in F(w_t)$ for almost every $t \in (a, b)$.*

Proof. Fillipov (1988, Chapter 2, Lemma 13) proves the lemma when $F(w)$ is a convex, compact and non-empty set which is independent of $w \in W$. However, his proof is also valid in the case that F is an upper-semi-continuous set-valued function with convex, compact and non-empty values. ■

Proof of Proposition 13: By Lemmas 14, 15 and 16, and since W is compact, one can apply Theorem 1 in Deimling (1988) or Theorem 2.2.1 in Kunze (2000). One concludes that for every $y_0 \in W$ there is an absolutely continuous function $w : [0, +\infty) \rightarrow W$ that satisfies (i) $w_0 = y_0$, and (ii) $\dot{w}_t \in -F(w_t)$ for almost every $t \in [0, +\infty)$.

By reversing the direction of time, for every $k \in \mathbf{N}$ there is an absolutely continuous function $w(k) : (-\infty, k] \rightarrow W$ that satisfies $\dot{w}_t(k) \in F(w_t(k))$ for almost every $t \in (-\infty, k]$.

As a consequence of Ascoli-Arzelà Theorem (see Aubin and Cellina, 1984, Theorem 0.3.4), there is a subsequence $(k_j)_{j \in \mathbf{N}}$ and a function $w : \mathbf{R} \rightarrow W$ such that $\lim_{j \rightarrow \infty} w_{k_j}(t) = w(t)$ for every $t \in \mathbf{R}$. Indeed, the functions $w(k)$ are uniformly bounded (as their values are in the compact set W), and their derivatives are also uniformly bounded (as the derivatives are a.e. in the compact set $F(W)$).

By Lemma 17 w is absolutely continuous over every open and bounded interval, and $\dot{w}_t \in F(w_t)$ for almost every t in this interval. It follows that w satisfies these two properties over $\mathbf{R}$ as well. ■

8 A Representation Result

Let w be a solution of $\dot{w} \in F(w)$. Since $\dot{w}$ and F are measurable functions, there are measurable functions $(\rho^i)_{i \in I}$ such that (i) $\rho_t^i \in [0, 1]$ for every $i \in I$ and every $t \geq 0$, (ii) $\sum_{i \in I} \rho_t^i \geq 1$ for every $t \geq 0$, and (iii) the following equality holds:

$$\dot{w}_t = \sum_{i \in I} \rho_t^i (w_t - a_i) \quad \text{for almost every } t. \quad (9)$$

The following Lemma states that for every t_0 and every $t \geq t_0$, w_{t_0} is a convex combination of $(a_i)_{i \in I}$ and w_t .

Lemma 18 *Fix $t_0 \in \mathbf{R}$. For every $i \in I$ there is a continuous and weakly monotonic increasing functions $\delta^i : [t_0, +\infty) \rightarrow [0, 1]$ such that for every $t \geq t_0$,*

$$A.1 \quad 1 - \exp(-(t - t_0)) \leq \sum_{i \in I} \delta_t^i \leq 1 - \exp(-n(t - t_0)), \text{ and}$$

$$A.2 \quad w_{t_0} = \sum_{i \in I} \delta_t^i a_i + (1 - \sum_{i \in I} \delta_t^i) w_t.$$

In particular, one has

$$A.3 \quad w_{t_0} = \sum_{i \in I} (\lim_{t \rightarrow +\infty} \delta_t^i) a_i.$$

Moreover,

$$A.4 \quad \text{if } \rho_s^i \geq \rho_s^j \text{ for every } s \in [t_0, t) \text{ then } \delta_t^i \geq \delta_t^j, \text{ and}$$

$$A.5 \quad \text{if } \rho_s^i = 0 \text{ for every } s \in [t_0, t) \text{ then } \delta_t^i = 0.$$

Proof. Assume w.l.o.g. that $t_0 = 0$. Let $(\delta^i)_{i \in I}$ be the unique solution of the following system of differential equations.

$$\begin{aligned} \dot{\delta}_0^i &= 0 & \forall i \in I, \\ \dot{\delta}_t^i &= (1 - \sum_{j \in I} \delta_t^j) \rho_t^i & \forall i \in I, t > 0. \end{aligned} \quad (10)$$

Summing (10) over $i \in I$ we obtain

$$\sum_{i \in I} \dot{\delta}_t^i = (1 - \sum_{j \in I} \delta_t^j) (\sum_{i \in I} \rho_t^i) \quad \forall t \geq 0. \quad (11)$$

We first prove that (A.2) holds, that is, $w_0 = \sum_{i \in I} \delta_t^i a_i + (1 - \sum_{i \in I} \delta_t^i) w_t$ for every $t \geq 0$. It is enough to show that the derivative of the right-hand side vanishes a.e. This derivative is equal to

$$\sum_{i \in I} \dot{\delta}_t^i a_i - \left(\sum_{i \in I} \dot{\delta}_t^i \right) w_t + \left(1 - \sum_{i \in I} \delta_t^i \right) \dot{w}_t.$$

Using (9) this derivative is equal a.e. to

$$\sum_{i \in I} \dot{\delta}_t^i a_i - \left(\sum_{i \in I} \dot{\delta}_t^i \right) w_t + \left(1 - \sum_{i \in I} \delta_t^i \right) \left(\sum_{i \in I} \rho_t^i \right) w_t - \left(1 - \sum_{i \in I} \delta_t^i \right) \sum_{i \in I} \rho_t^i a_i.$$

Reordering the terms, the derivative is equal a.e. to

$$\sum_{i \in I} a_i \left(\dot{\delta}_t^i - \left(1 - \sum_{j \in I} \delta_t^j \right) \rho_t^i \right) - w_t \left(\sum_{i \in I} \dot{\delta}_t^i - \left(1 - \sum_{j \in I} \delta_t^j \right) \left(\sum_{i \in I} \rho_t^i \right) \right).$$

Finally, the two terms vanish by (10) and (11).

Since $1 \leq \sum_{i \in I} \rho_t^i \leq n$ for every $t \geq 0$, one has by (11)

$$1 - \sum_{i \in I} \delta_t^i \leq \sum_{i \in I} \dot{\delta}_t^i \leq n \left(1 - \sum_{i \in I} \delta_t^i \right) \quad \forall t \geq 0.$$

Since the solution of the equation $\dot{x} = 1 - x$ with initial condition $x_0 = 0$ is $x_t = 1 - \exp(-t)$, while the solution of the equation $\dot{x} = n(1 - x)$ with the initial condition $x_0 = 0$ is $x_t = 1 - \exp(-nt)$, it follows that

$$1 - \exp(-t) \leq \sum_{i \in I} \delta_t^i \leq 1 - \exp(-nt) \quad \forall t \geq 0.$$

Therefore (A.1) holds as well. This implies by (10) that for every $i \in I$, $\dot{\delta}_t^i \geq 0$ for every $t \geq 0$, hence δ^i is weakly monotonic increasing.

Finally, we show that (A.4) and (A.5) hold as well. If $\rho_s^i \geq \rho_s^j$ for every $s \in [0, t)$ then by (10) $\dot{\delta}_s^i \geq \dot{\delta}_s^j$ for every $s \in [0, t)$, so that $\delta_t^i \geq \delta_t^j$. If $\rho_s^i = 0$ for every $s \in [0, t)$ then by (10) $\dot{\delta}_s^i = 0$ for every $s \in [0, t)$, so that $\delta_t^i = 0$. ■

The following lemma states that for every t_0 and every collection $s_1, s_2, \dots, s_n \in [t_0, +\infty]$ (some of the s_i 's may be equal to $+\infty$), one can represent w_{t_0} as a proper convex combination of $(a_i)_{i \in I}$, $(w_{s_i})_{i \in I}$ and w_t , provided t is not too large.

Lemma 19 *For every $t_0 \in \mathbf{R}$ and every collection $s_1, s_2, \dots, s_n \in [t_0, +\infty]$ there exist $T > t_0$ and for every $i \in I$ a function $\alpha^i : [t_0, T] \rightarrow [0, 1]$ that satisfy*

B.1 α^i is continuous and weakly monotonic increasing, $\alpha_{t_0}^i = 0$ and $\alpha_T^i \leq 1$ for every $i \in I$.

B.2 $w_{t_0} = \frac{1}{n} \left(\sum_{i \in I} \alpha_t^i a_i + \sum_{i | s_i \leq t} (1 - \alpha_t^i) w_{s_i} + \sum_{i | s_i > t} (1 - \alpha_t^i) w_t \right)$ for every $t \in [t_0, T]$.

B.3 Either (a) $T = \max\{s_i, i \in I\}$ and there is $i \in I$ such that $\alpha_T^i \geq (1 - \exp(-T))/n$, or (b) there is $i \in I$ with $\alpha_T^i = 1$.

B.4 $\sum_{i \in I} \alpha_t^i \geq 1 - \exp(-(t - t_0))$ for every $t \geq t_0$, and $\sum_{i \in I} \alpha_t^i \leq n - \exp(-2n(t - t_0))$ provided $0 \leq t - t_0 \leq -n/\ln(1 - 1/2n)$.

Moreover,

B.5 If $\rho_s^i \geq \rho_s^j$ for every $s \in [t_0, t)$ then $\alpha_t^i \geq \alpha_t^j$.

B.6 If $\rho_s^i = 0$ for every $s \in [t_0, t)$ then $\alpha_t^i = 0$.

B.7 If $w_t^i > 0$ for every $t \in [t_0, T]$ then $\alpha^i = 0$.

B.8 If $w_t^i < 0$ for every $t \in [t_0, T]$, and either $w_{s_i}^i = 0$ or $s_i = +\infty$ then $\alpha^i = 1$.

Proof. Assume w.l.o.g. that $t_0 = 0$. For every $i \in I$, applying Lemma 18 to $t_0 = s_i$, one obtains weakly monotonic increasing functions $\delta^{i,j} : [s_i, \infty) \rightarrow [0, 1)$, $j \in I$, that satisfy for every $t \geq 0$

$$w_{s_i} = \sum_{j \in I} \delta_t^{i,j} a_j + (1 - \sum_{j \in I} \delta_t^{i,j}) w_t, \text{ and} \quad (12)$$

$$\sum_{j \in I} \delta_t^{i,j} < 1.$$

For every $t \geq t_0$, set $J_t = \{i \in I \mid s_i \leq t\}$ and $K_t = \{i \in I \mid s_i > t\}$. Then $t \mapsto J_t$ and $t \mapsto K_t$ are piecewise constant, and $J_t \cup K_t = I$ for every $t \geq 0$.

Let $(\alpha^i)_{i \in I}$ be the unique solution of the following system of differential equations.

$$\begin{aligned} \alpha_0^i &= 0, & \forall i \in I, \\ \dot{\alpha}_t^i &= \rho_t^i \sum_{k \in K_t} (1 - \alpha_t^k) + \sum_{j \in J_t} \dot{\alpha}_t^j \delta_t^{i,j} & \forall i \in I, t > 0. \end{aligned} \quad (13)$$

A solution to this system exists as long as $\alpha_t^i \leq 1$ for every $i \in I$. Indeed, the system of linear equations $x_i = b_i + \sum_{j \in I} p_{j,i} x_j$, $i \in I$, where for every $i \in I$ $b_i \geq 0$ and $\sum_{j \in I} p_{i,j} < 1$ has a unique solution. This system represents the following situation. There are n queues, initially at each queue i there are b_i people, and, after being served at queue i , a person goes to queue j with probability $p_{i,j}$, and with probability $1 - \sum_{j \in I} p_{i,j}$ he goes home. x_i is the expected number of services provided by queue i .

Set $T = \min\{\max\{s_1, \dots, s_n\}, \min\{t \geq 0, \alpha_t^i = 1 \text{ for some } i \in I\}\}$. Then (B.1) is satisfied.

By (13) $\sum_{i \in K_t} \dot{\alpha}_t^i \geq 1 - \sum_{i \in K_t} \alpha_t^i$ for every $t \in [0, T]$. This implies that $\sum_{i \in I} \alpha_T^i \geq 1 - \exp(-T)$, and therefore there is $i \in I$ such that $\alpha_T^i \geq (1 - \exp(-T))/n$. Condition (B.3) holds as well.

Eq. (13) implies that $\sum_{i \in I} \dot{\alpha}_t^i \geq 1 - \sum_{i \in I} \alpha_t^i$. Since the solution of the equation $\dot{x} = 1 - x$ with initial condition $x_0 = 0$ is $x = 1 - \exp(-t)$, the first claim in (B.4) follows.

Fix $0 \leq t \leq -n/\ln(1 - 1/2n)$. Then $n(1 - \exp(-nt)) \leq 1/2$. By (A.1) in Lemma 18 $\sum_{j \in I} \delta_t^{j,i} \leq 1/2$. By (13),

$$\frac{1}{2} \dot{\alpha}_t^i \leq \sum_{i \in I} \dot{\alpha}_t^i (1 - \sum_{j \in I} \delta_t^{j,i}) \leq n \sum_{i \in I} (1 - \alpha_t^i).$$

Since the solution of the equation $\dot{x} = 2n^2 - 2nx$ with the initial condition $x_0 = 0$ is $x = n - \exp(-2nt)$, the second claim in (B.4) follows.

Summing (13) over $i \in I$ gives us

$$\sum_{i \in I} \dot{\alpha}_t^i = \left(\sum_{i \in I} \rho_t^i \right) \left(\sum_{i \in K_t} (1 - \alpha_t^i) \right) + \sum_{i \in I} \sum_{j \in J_t} \dot{\alpha}_t^j \delta_t^{i,j}. \quad (14)$$

So that condition (B.2) is satisfied we need the derivative of the right-hand side in (B.2) to vanish for almost every $t \geq 0$. We show that the derivative vanishes for every t such that $t \notin \{s_1, \dots, s_n\}$. The derivative of the right-hand side in (B.2), multiplied by n , is

$$\sum_{i \in I} \dot{\alpha}_t^i a_i - \sum_{i \in J_t} \sum_{j \in I} \dot{\alpha}_t^j w_{s_i} - \sum_{i \in K_t} \dot{\alpha}_t^i w_t + \left(\sum_{i \in K_t} (1 - \alpha_t^i) \right) \dot{w}_t.$$

By incorporating (9) and (12), and since $J_t \cup K_t = I$, reordering the terms yields

$$\begin{aligned} \sum_{i \in I} a_i \left(\dot{\alpha}_t^i - \sum_{j \in I} \dot{\alpha}_t^j \delta_t^{i,j} - \sum_{j \in K_t} (1 - \alpha_t^j) \rho_t^i \right) \\ - w_t \left(\sum_{i \in I} \dot{\alpha}_t^i - \sum_{j \in I} \sum_{i \in I_t} \dot{\alpha}_t^j \delta_t^{i,j} - \left(\sum_{i \in K_t} (1 - \alpha_t^i) \right) \left(\sum_{i \in I} \rho_t^i \right) \right). \end{aligned}$$

This sum is zero by (13) and (14).

We now show that (B.5) and (B.6) hold as well. If $\rho_s^i \geq \rho_s^j$ for every $s \in [0, t]$ then by Lemma 18 $\delta_s^{k,i} \geq \delta_s^{k,j}$ for every $j \in I$ and every $s \in [s_j, t]$. By (13) $\dot{\alpha}_s^i \geq \dot{\alpha}_s^j$ for every $s \in [0, t]$, so that $\alpha_t^i \geq \alpha_t^j$.

If $\rho_s^i = 0$ for every $s \in [0, t]$ then by Lemma 18 $\delta_s^{j,i} = 0$ for every $j \in I$ and every $s \in [s_j, t]$. By (13) $\dot{\alpha}_s^i = 0$ for every $s \in [0, t]$, so that $\alpha_t^i = 0$.

Finally we show that (B.7) and (B.8) hold. If $w_t^i > 0$ for every $t \in [0, T]$ then $\rho_t^i = 0$ in this range, and (B.7) follows from (B.6).

If $w_t^i < 0$ for every $t \in [0, T]$ then $\rho_t^i = 1$ in this range. Moreover, since either $w_{s_i}^i = 0$ or $s_i = +\infty$, $T < s_i$. By (B.3) there is $j \in I$ such that $\alpha_T^j = 1$. Since $\rho_t^i = 1 \geq \rho_t^j$ for every $t \in [0, T]$, (B.5) implies that $\alpha_T^i = 1$. ■

9 From Solutions of $\dot{w} \in F(w)$ to Subgame-Perfect Equilibria

We first classify solutions of the equation $\dot{w} \in F(w)$ into two types. For every solution w of $\dot{w} \in F(w)$, denote by Y_w the range of w :

$$Y_w = \{w_t \mid t \in \mathbf{R}\} \subseteq W,$$

and by $\overline{Y}_w$ the closure of Y_w .

Definition 20 *A solution w of $\dot{w} \in F(w)$ has type 0 if $\vec{0} \in F(y)$ for some $y \in \overline{Y}_w$. It has type 1 otherwise.*

By Lemma 8, if w is a solution of type 0 and there are no dummy players then the game admits an equilibrium payoff that is supported by stationary strategies.

Here we prove the following two propositions.

Proposition 21 *Assume there are no dummy players, and let w be a solution of $\dot{w} \in F(w)$ of type 1. Then every $y \in \bar{Y}_w$ is a subgame-perfect 0-equilibrium payoff.*

More generally, our proof shows that if there are no solutions of type 0, then every vector in the closure of the range of all solutions of type 1 is a subgame-perfect 0-equilibrium payoff.

Proposition 22 *Assume there are no dummy players, and let w be a solution of $\dot{w} \in F(w)$ of type 0. Then every $y \in \bar{Y}_w$ is a subgame-perfect ϵ -equilibrium payoff, for every $\epsilon > 0$.*

Remark 4: The range of all solutions of $\dot{w} \in F(w)$ does *not* necessarily coincide with the set of subgame-perfect equilibrium payoffs. Indeed, the former set is a subset of W , whereas there are subgame-perfect 0-equilibrium payoffs that are not in W (see, e.g., Example 2.)

Choose once and for all two constants $0 < \delta_1 < \delta_2 < -n/\ln(1 - 1/2n)$ that satisfy the following.

$$\text{D.1 } n - n \exp(1 - \exp(n\delta_2)) < 1.$$

$$\text{D.2 } 2\delta_1 < \delta_2.$$

Fix for a moment a solution w of $\dot{w} \in F(w)$. For every player $i \in I$ define

$$U_w^i = \{t \in \mathbf{R} \mid w_t^i = 0\} \subseteq \mathbf{R}.$$

Since $t \mapsto w_t$ is continuous, U_w^i is closed.

Define for every $i \in I$ a function $s_w^i : [0, \infty) \rightarrow [0, \infty]$ by:⁷

$$s_w^i(t) = \begin{cases} \min(U_w^i \cap (t, +\infty)) & w_t^i \neq 0 \\ \min(U_w^i \cap [t + \delta_1, t + \delta_2]) & w_t^i = 0, U_w^i \cap [t + \delta_1, t + \delta_2] \neq \emptyset, \\ \max(U_w^i \cap (t, t + \delta_1]) & w_t^i = 0, U_w^i \cap [t + \delta_1, t + \delta_2] = \emptyset, U_w^i \cap (t, t + \delta_1] \neq \emptyset, \\ \min(U_w^i \cap [t + \delta_2, +\infty)) & w_t^i = 0, U_w^i \cap (t, t + \delta_2] = \emptyset. \end{cases} \quad (15)$$

Observe that $s_w^i(t) > t$, and $s_w^i(t) < +\infty$ as soon as there is $u > t$ such that $w_u^i = 0$. Moreover, if $s_w^i(t) < +\infty$ then $w_{s_w^i(t)}^i = 0$. Set

$$M_w(t) = \max_{i \in I} s_w^i(t) - t.$$

Lemma 23 *Let w be a solution of $\dot{w} \in F(w)$ of type 1. Then $\inf_{t \in \mathbf{R}} M_w(t) > 0$.*

Proof. Assume to the contrary that $\inf_{t \in \mathbf{R}} M_w(t) = 0$. Then there is a sequence $(t(k))_{k \in \mathbf{N}}$ such that $\lim_{k \rightarrow \infty} M_w(t(k)) = 0$. We will prove that $w_{t(k)} \rightarrow \vec{0}$, which implies that $\vec{0} \in \bar{Y}_w \subseteq W$. By Lemma 9 $\vec{0} \in F(\vec{0})$, so that w has type 0, a contradiction.

Fix $\epsilon > 0$. Since w is uniformly continuous, there is $\delta < \delta_1$ such that $|u - t| < \delta$ implies $\|w_u - w_t\| < \epsilon$. Let k be sufficiently large such that $M_w(t(k)) < \delta$. For every $i \in I$, $w_u^i = 0$ for some $u \in (t(k), t(k) + M_w(t(k)))$. This implies that $\|w_{t(k)}\| < \epsilon$, and the claim follows. ■

⁷By convention, the minimum of an empty set is $+\infty$.

Lemma 24 *Let w be a solution of $\dot{w} \in F(w)$ of type 0. For every $\eta \in (0, \delta_1)$ and every $t \in \mathbf{R}$ at least one of the following statements hold.*

1. $M_w(t) \geq \eta$,
2. $M_w(s_w^i(t)) \geq \eta$ for some $i \in I$, or
3. $M_w(s_w^i(s_w^i(t))) \geq \eta$ for every $i \in I$.

Proof. Assume that the first statement does not hold, that is, $M_w(t) < \eta < \delta_1$.

We first assume that $w_t^i = 0$ for some $i \in I$. Since $s_w^i(t) \leq t + M_w(t) < t + \delta_1$ it follows that $U_w^i \cap [t + \delta_1, t + \delta_2] = \emptyset$ and $U_w^i \cap (t, t + \delta_1] \neq \emptyset$. Since $2\delta_1 < \delta_2$, $U_w^i \cap (t + s_w^i(t), t + s_w^i(t) + \delta_1] = \emptyset$, so that $s_w^i(s_w^i(t)) \geq s_w^i(t) + \delta_1$, and the second statement holds.

Assume now that $w_t^i \neq 0$ for every $i \in I$. If the second statement does not hold, then for every $i \in I$ $s_w^i(s_w^i(t)) < t + \eta$, so that $w_{s_w^i(t)}^i = 0$. Applying the second paragraph to $s_w^i(t)$ rather than to t , one deduces that the third statement holds. ■

Proof of Proposition 21: Let w be a solution of $\dot{w} \in F(w)$ of type 1. We show that the set $\bar{Y}$ satisfies the conditions of Lemma 5.

Since w has type 1, condition (C.5) is satisfied. Indeed, if condition (C.5) is not satisfied then for some $i \in I$ $a_i \in \bar{Y}_w$ and $a_i^j \geq 0$ for every $j \in I$. By Lemma 10 $\vec{0} \in F(a_i)$, so that w has type 0, a contradiction.

Let $y \in Y_w$. Then $y = w_t$ for some $t \in \mathbf{R}$. By Lemma 19 there are $T > t$ and $\alpha^1, \dots, \alpha^n \in [0, 1]$ such that

$$y = w_t = \frac{1}{n} \sum_{i=1}^n (\alpha^i a_i + \mathbf{1}_{\{s_w^i(t) \leq T\}} (1 - \alpha^i) w_{s_w^i(t)} + \mathbf{1}_{\{s_w^i(t) > T\}} (1 - \alpha^i) w_T).$$

Set $y_i = w_{\min\{s_w^i(t), T\}}$ and $\rho^i = \alpha^i$ for every $i \in I$. Condition (C.1) then holds.

We now show that conditions (C.2) and (C.3) hold as well. Fix $i \in I$. Assume first that $s_w^i(t) > T$. By Lemma 19(B.3) $\alpha^j = 1$ for some $j \in I$. By (D.1) $\delta_2 < T$, so that $\delta_2 < s_w^i(t)$. By the definition of $s_w^i(t)$, $w_u^i \neq 0$ for every $u \in (t, s_w^i(t))$. Lemma 19(B.7, B.8) either (a) $w_u^i > 0$ for every $u \in (t, T]$, in which case $y_i^i = w_T^i > 0$ and $\alpha^i = 0$, or (b) $w_u^i < 0$ for every $u \in (t, T]$, in which case $y_i^i = w_T^i < 0$ and $\alpha^i = 1$. Assume now that $s_w^i(t) \leq T$. Then $y_i^i = w_{s_w^i(t)}^i = 0$, and (C.2) and (C.3) hold trivially.

By Lemma 23, $\max_{i=1, \dots, n} \alpha^i \geq \eta$ for some $\eta > 0$ that depends only of w , and condition (C.4) holds as well. ■

Proof of Proposition 22: The proof is similar to the proof of Proposition 21, but instead of applying Lemma 5 we apply Lemma 6. Recall that in the proof of Lemma 5, (C.4) followed from Lemma 23. Unfortunately, when w has type 0, Lemma 24 does not give us (C.4). However, by Remark 2, to apply Lemma 6 it is sufficient to prove that the profile σ we constructed in the proof of Lemma 6 is terminating. This fact follows from Lemma 24. ■

Example 2, Continued: A graphic representation of the differential inclusion $\dot{w} \in F(w)$ shows that it has a unique periodic solution (up to time shifts), and the range of this solution coincides

with the edges of the triangle that is defined by $\{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}$. Observe that this set coincides with the set of equilibrium payoffs in the game studied by Flesch et al. (1997).

One subgame-perfect 0-equilibrium that is generated by the procedure we used in the proof is the last one we described in Section 3. In fact, all subgame-perfect 0-equilibria that are generated by the procedure we used in the proof coincide with that equilibrium from stage 2 and on.

If one modifies the definition of $s_w^i(t)$ in the last case to t (rather than $\min(U_w^i \cap (t, +\infty))$), Lemma 23, and therefore Proposition 21, is still valid, and the generated subgame-perfect 0-equilibrium is the periodic profile with period 6 that was presented in Section 3.

10 Extensions and Open Problems

The proof we provided here is valid with minor modifications when the probability distribution over I according to which players are chosen at every stage is not the uniform distribution but any distribution $p = (p^i)_{i \in I}$. Indeed, the definition of F becomes

$$F(w) = \left\{ \sum_{i=1}^n p^i \rho^i (w - a_i) \mid \rho \in \Delta(w) \right\},$$

and from that point on, every appearance of ρ^i is changed to $p^i \rho^i$.

One can consider a model in which once chosen, every player has several terminating actions; that is, he can choose one of finitely many terminal payoffs. The model is easily reduced to the model we studied, by choosing at the outset for each player a single terminating action: one that gives him the highest payoff.

Another generalization may be to have $n \times K$ vectors $(a_{i,k})_{i \in I, k=1, \dots, K}$ in $\mathbf{R}^n$, and a probability distribution p over $\{(1, 1), (1, 2), \dots, (n, K)\}$. At every stage a pair (i, k) is chosen according to p , and player i decides whether to continue, or to terminate the game with terminal payoff $a_{i,k}$.

Our approach works in this more general model as well.

We have proven here the existence of a stationary ϵ -equilibrium or a subgame-perfect 0-equilibrium. However, in all the examples the author analyzed in which there is a subgame-perfect 0-equilibrium, this equilibrium is supported by *pure Markovian* profiles. If this observation is true in general, this might have significant implications on the study of stochastic games and Dynkin games.

The model we have studied is stationary, in the sense that the probability by which a player is chosen and the terminal payoffs are fixed throughout the game. What happens when this is not the case is not known. The two simplest cases which we do not know how to analyze are (a) players are chosen by the uniform distribution, at odd stages there is one set of terminal payoffs and at even stages there is another set of terminal payoffs, and (b) there is one set of terminal payoffs, at odd stages a player is chosen by the uniform distribution, and at even stages a player is chosen by another distribution.

Another generalization of the model we studied is to allow players to quit simultaneously. This class of games, termed *quitting games*, have been studied by Solan and Vieille (2001), where partial results are reported.

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