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Discussion Paper, No. 1296

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Suggested Citation: de Vries, Sven; Vohra, Rakesh (2000) : Combinatorial Auctions: A Survey, Discussion Paper, No. 1296, Northwestern University, Kellogg School of Management, Center for Mathematical Studies in Economics and Management Science, Evanston, IL

This Version is available at:
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Combinatorial Auctions: A Survey

Sven de Vries* & Rakesh Vohra†

May 9, 2000

Abstract

Many auctions involve the sale of a variety of distinct assets. Examples are airport time slots, delivery routes and furniture. Because of complementarities (or substitution effects) between the different assets, bidders have preferences not just for particular items but for sets or bundles of items. For this reason, economic efficiency is enhanced if bidders are allowed to bid on bundles or combinations of different assets. This paper surveys the state of knowledge about the design of combinatorial auctions. Second, it uses this subject as a vehicle to convey the aspects of integer programming that are relevant for the design of such auctions and combinatorial markets in general.

*Zentrum Mathematik, TU München, D-80290 München, Germany.
e-mail: devries@ma.tum.de

†Department of Managerial Economics and Decision Sciences, Kellogg School of Management, Northwestern University, Evanston IL 60208. e-mail: r-vohra@nwu.edu

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1 Introduction

Many auctions involve the sale of a variety of distinct assets. Examples are the FCC spectrum auction and auctions for airport time slots, railroad segments (Brewer (1999)) and delivery routes (Caplice (1996)). Because of complementarities (or substitution effects) between different assets, bidders have preferences not just for particular items but for sets or bundles of items. For this reason, economic efficiency is enhanced if bidders are allowed to bid on *combinations* of different assets.

Auctions where bidders submit bids on combinations have recently received much attention. See for example Caplice (1996), Rothkopf *et. al.* (1998), Fujishima *et. al.* (1999), and Sandholm (1999). However such auctions were proposed as early as 1982 (Rassenti *et. al.* (1982)) to allocate airport time slots. Increases in computing power have made them more attractive to implement. In fact a number of logistics consulting firms tout software to implement combinatorial auctions. SAITECH-INC, for example, offers a software product called SBIDS that allows trucking companies to bid on "bundles" of lanes. Logistics.com's system is called OptiBid$^{\text{TM}}$. Logistics.com claims that more than $5 billion in transportation contracts have been bid to date (January 2000) using OptiBid$^{\text{TM}}$ by Ford Motor Company, Wal-Mart and K-Mart.[1]

The most obvious problem that bids on combinations of items impose is in selecting the winning set of bids. Call this the Combinatorial Auction Problem (CAP).[2] CAP can be formulated as an Integer Program. This paper will survey what is known about the CAP. It assumes a knowledge of linear programming and familiarity with basic graph theoretic terminology. The penultimate section is devoted to incentive issues in the design of combinatorial auctions.

2 The CAP

The first and most obvious difficulty an auction which allows bidders to bid on combinations faces is that each bidder must submit a bid for *every* subset of objects he is interested in. The second problem is how to transmit this bidding function in a succinct way to the auctioneer. The only resolution

[1] Yet another firm called InterTrans Logistics Solutions, offers a software product called Carrier Bid Optimizer that allows trucking companies to bid on "bundles" of lanes over several bidding rounds. They appear to have been acquired by $i2$ and we have not been able to find anything more about them.

[2] We assume that the auctioneer is a seller and bidders are buyers.

of these two problems is to restrict the kinds of combinations that bidders may bid on.

A discussion of various ways in which bids can be restricted and their consequences can be found in Nisan (1999). In that paper Nisan asks, given a language for expressing bids, what preferences over subsets of objects can be correctly represented by the language. An alternative approach, not much explored, is to rely on an 'oracle'. An oracle is a program (black box) that, for example, given a bidder and a subset computes the bid for it. Thus bidders submit oracles rather than bids. The auctioneer can simply invoke the relevant oracle at any stage to determine the bid for a particular subset.[3] Even if this problem is resolved (in a non-trivial way) to the satisfaction of the parties involved, it still leaves open the problem of deciding which collection of bids to accept. This is the problem we consider.

Let N be the set of bidders and M the set of m distinct objects. For every subset S of M let $b^j(S)$ be the bid that agent $j \in N$ has announced he is willing to pay for S. Let $b(S) = \max_{j \in N} b^j(S)$. Then problem CAP can be formulated as:

$$\max \sum_{S \subset M} b(S) x_S$$

$$\text{s.t.} \quad \sum_{S \ni i} x_S \leq 1 \ \forall i \in M$$

$$x_S = 0, 1 \ \forall S \subset M$$

Here $x_S = 1$ is interpreted to mean that the highest bid on the set S is to be accepted, whereas, $x_S = 0$ means that no bid on the set S are accepted. The constraint

$$\sum_{S \ni i} x_S \leq 1 \ \forall i \in M$$

ensures that no object in M is assigned to more than one bidder. Call this formulation CAP1.

The formulation above correctly models the CAP when the bid functions b^i are all superadditive, i.e. $b^j(A) + b^j(B) \leq b^j(A \cup B)$ for all $j \in N$ and $A, B \subset M$ such that $A \cap B = \emptyset$. This corresponds to the idea that the goods complement each other. When goods are substitutes, $b^j(A) + b^j(B) > b^j(A \cup B)$ for some $j \in N$ and $A, B \subset M$, formulation CAP1 is incorrect. An optimal solution to CAP1 may assign sets A and B to bidder j and *incorrectly* record a revenue of $b^j(A) + b^j(B)$ rather than $b^j(A \cup B)$ to

[3] Sandholm (1999) points out that another advantage of oracles is that bidders need not be present. Their application does rely on the probity of the auctioneer.

that allocation. This difficulty can be obviated through the introduction of dummy goods, g, say. The bidder is then instructed to replace the bids $b^j(A)$, $b^j(B)$ and $b^j(A \cup B)$ with $b^j(A \cup g)$, $b^j(B \cup g)$ and $b^j(A \cup B)$ and to replace M by $M \cup g$. Notice that by the constraints of the integer programming formulation, if the set A is assigned to j then so is g and thus B cannot be assigned to j.

Problem CAP as formulated here is an instance of what is known as the Set Packing Problem (SPP) which is described below. Another way to write the problem for (not necessarily superadditive) bids without explicitly involving dummy items is the following integer program (which we call CAP2). Let $y(S, j) = 1$ if the bundle $S \subseteq M$ is allocated to $j \in N$ and zero otherwise.

$$\max \sum_{j \in N} \sum_{S \subseteq M} b^j(S) y(S, j)$$

$$\text{s.t.} \sum_{S \ni i} \sum_{j \in N} y(S, j) \leq 1 \ \forall i \in M$$

$$\sum_{S \subseteq M} y(S, j) \leq 1 \ \forall j \in N$$

$$y(S, j) = 0, 1 \ \forall S \subseteq M, \ j \in N$$

The first constraint ensures that overlapping sets of goods are never assigned. The second ensures that no bidder receives more than one subset. This formulation is also an instance of the SPP.[4]

There is another interpretation of the CAP possible. If we interpret the bids submitted as the true values that bidders have for various combinations, then the solution to the CAP is the efficient allocation of indivisible objects in an exchange economy.

We have formulated CAP1 under the assumption that there is at most one copy of each object. It is an easy matter to extend the formulation to the case when there are multiple copies of the same object and each bidder wants at most one copy of each object. All that happens is that the right hand side of the constraints in CAP1 take on values larger than 1.

[4] As an aside, an important lesson of integer programming is that parsimony in formulation is not always to be desired; important information is sometimes lost. Two distinct formulations of the same problem as an integer program will share the same set of feasible integer solutions but not necessarily the same set of *non-integer* solutions. For solution purposes the integer program with the smaller set of feasible non-integer solutions may be preferred. Strictly speaking, smallness is not what matters. Rather one would like the set of feasible non-integer solutions to have sufficient structure that they can be identified and eliminated.

In the case when there are multiple units and bidders may want more than one copy of the same unit, bids are column vectors of the form $\{a_{jk}\}_{k\geq 1}$ where a_{jk} is the number of items of object k requested by bidder j. Again it is easy to see that the problem of determining the winning set of bids can be formulated as an integer program. Multi-unit combinatorial auctions are investigated in Leyton-Brown *et. al.* (2000).[5]

2.1 The Set Packing Problem

The SPP is a well studied integer program. Given a ground set M of elements and a collection V of subsets with non-negative weights, find the largest weight collection of subsets that are pairwise disjoint. To formulate this problem as an integer program, let $x_j = 1$ if the j^{th} set in V with weight c_j is selected and $x_j = 0$, otherwise. Define a_{ij} to be 1 if the j^{th} set in V contains element $i \in M$. Given this, the SPP can be formulated as:

$$\max \sum_{j \in V} c_j x_j$$

$$\text{s.t.} \quad \sum_{j \in V} a_{ij} x_j \leq 1 \ \forall i \in M$$

$$x_j = 0, 1 \ \forall j \in V$$

It is easy to see that the CAP is an instance of the SPP. Just take M to be the set of objects and V the set of all subsets of M.

Before continuing with a discussion of the SPP it is worth mentioning two of its close relatives. The first is the **set partitioning problem** (SPA) and the second is the **set covering problem** (SCP). Both would be relevant had we cast the auction problem in procurement rather than selling terms. The auctions used in the transport industry are of this type. In that setting, objects are origin-destination pairs, called lanes. Bidders submit bids on bundles of lanes that represent how much they must be offered to undertake the deliveries on the specified lanes. The auctioneer wishes to choose a collection of bids of lowest cost such that all lanes are served.[6]

The SPA is:

$$\min \sum_{j \in V} c_j x_j$$

[5] If the number of units of each type is large, then one could approximate the problem of selecting the winning set of bids using a linear program. The relevant decision variables would be the percentage of each type allocated to a bidder.

[6] In fact, one must specify not only lanes but volume as well, so this problem constitutes an instance of a multi-unit combinatorial auction.

$$\text{s.t.} \quad \sum_{j \in V} a_{ij} x_j = 1 \ \forall i \in M$$
$$x_j = 0, 1 \ \forall j \in V$$

It can arise in an auction setting where the bidders are sellers rather than buyers. For example, trucking companies bidding for the opportunity to ship goods from a particular warehouse to retail outlet. Any instance of SPP can be rewritten as an instance of SPA by using slack variables and negating the objective function coefficients.

The SCP is:

$$\min \sum_{j \in V} c_j x_j$$
$$\text{s.t.} \quad \sum_{j \in V} a_{ij} x_j \geq 1 \ \forall i \in M$$
$$x_j = 0, 1 \ \forall j \in V$$

A prominent application of the SCP is the scheduling of crews for railways. Both the SPA and SCP have been extensively investigated.

Other applications of the SPP include switching theory, the testing of VLSI circuits, line balancing and scheduling problems where one wishes to satisfy as much demand as possible, without creating conflicts. The survey by Balas and Padberg [1976] contains a bibliography on applications of the SPP, SCP and SPA. The instances of SPP that have received the most attention are those that that stem from relaxations of SPAs.

2.2 Complexity of the SPP

How hard is the SPP to solve? By enumerating all possible 0-1 solutions we can find an optimal solution in a finite number of steps. If $|V|$ is the number of variables, then the number of solutions to check would be $2^{|V|}$, clearly impractical for all but small values of $|V|$. For the instances of SPP that arise in the CAP, the cardinality of V is the number of subsets of M; a large number.

Is there an efficient algorithm for solving SPP? The answer depends on the definition of efficiency. In complexity theory, efficiency is measured by the number of elementary operations (addition, subtraction, multiplication and rounding) needed to determine the solution. An algorithm for a problem is said to be **efficient** or **polynomial** if for *all instances* of the input problem the number of elementary operations needed grows as a polynomial function of the size of the instance. Of course the definition allows the

number of operations executed to grow with the size of the problem. To ensure that the number of instructions grows only modestly in the size of the problem, usually polynomiality is required. By focusing on the number of operations rather than time, the definition is machine independent. Efficiency should not hinge on the particular technology being used. The size of a problem instance is measured by the number of binary bits needed to encode the instance. In the case of the SPP the proxy for size of an instance is $\max\{|M|, |V|, \ln c_{\max}\}$ where $c_{\max} = \max_{j \in V} c_j$.

Returning to the efficiency question, no polynomial time algorithm for the SPP is known and there are strong reasons for believing none exists. The SPP belongs to an equivalence class of problems called NP-hard.[7] No problem in this class is currently known to admit a polynomial time solution method. It is conjectured that no member in this class admits a polynomial time solution algorithm. The conjecture is sometimes called the P≠NP conjecture and its resolution is the holy grail of theoretical computer science.

Two points about this notion of efficiency are in order. First, polynomial time algorithms while theoretically efficient may still be impractical. For example, an algorithm whose complexity is a 23^{rd} order polynomial of problem size would still take a long time to run on even the fastest known machines. Generally, a third order polynomial seems to be the upper limit of what is viewed as practical to implement. Second, the fact that a problem is NP-hard in general does not mean there is no hope for solving it rapidly. NP-hard does not mean 'hard *all* the time', just 'hard some times'. An algorithm that is polynomial on *some* rather than all instances may be good enough for the purpose at hand.

For the CAP, this discussion of complexity may have little relevance. Any algorithm for the CAP, that uses directly the bids for the sets, must scan, in the worst case, the bids and the number of such bids could be exponential in $|M|$. Thus effective solution procedures for the CAP must rely on two things. The first is that the number of distinct bids is not large or is structured in computationally useful ways. The second is that the underlying SPP can be solved reasonably quickly.

2.3 Solvable Instances of the SPP

The usual way in which instances of the SPP can be solved by a polynomial algorithm is when the extreme points of the polyhedron $P(A) = \{x : \sum_{j \in V} a_{ij} x_j \leq 1 \ \forall i \in M;\ x_j \geq 0 \ \forall j \in V\}$ are all integral, i.e. 0-1. In these

[7] More precisely, the recognition version of SPP is NP-complete.

cases we can simply drop the integrality requirement from the SPP and solve it as a linear program. Linear programs can be solved in polynomial time. It turns out that in most of these cases, because of the special structure of these problems, algorithms more efficient than linear programming ones exist.

A polyhedron with all integral extreme points is called **integral**. Identifying sufficient conditions for when a polyhedron is integral has been a cottage industry in integer programming. These sufficient conditions involve restrictions on the constraint matrix, which in this case amount to restrictions on the kinds of subsets for which bids are submitted. We list the most important ones here.

Rothkopf *et. al.* (1998) covers the same ground but organizes the solvable instances differently as well as suggesting auction contexts in which they may be salient. An example of one such context is given below.

2.3.1 Total Unimodularity

The most well known of these sufficient conditions is **total unimodularity**, sometimes abbreviated to 'TU'. A matrix is said to be TU if the determinant of every square submatrix is 0, 1 or -1. Notice also that if a matrix is TU so is its transpose.

If the matrix $A = \{a_{ij}\}_{i \in M, j \in V}$ is TU then all extreme points of the polyhedron $P(A)$ are integral. It is easy to see why. Every extreme point of $P(A)$ corresponds to a basis or square submatrix of A. Now apply Cramer's rule to the system of equations associated with the basis.

The problem of characterizing the class of TU matrices was solved in 1981 by Paul Seymour who gave a polynomial (in the number of rows and columns of the matrix) time algorithm to decide whether a matrix was TU. A nice description of the algorithm can be found in Chapter 20 of Schrijver (1986). However, for most applications, in particular for the CAP, the following characterization of TU for restricted classes of matrices due to Ghouila-Houri seems to be the tool of choice:

THEOREM 2.1 *Let B be a matrix each of whose entries is 0, 1 or -1. Suppose each subset S of columns of B can be divided into two sets L and R such that*

$$\left| \sum_{j \in S \cap L} b_{ij} - \sum_{j \in S \cap R} b_{ij} \right| = 0, 1 \ \forall i$$

then B is TU. The converse is also true.

Other characterizations for matrices each of whose entries are 0, 1 or -1 can be found in Chapter 19 of Schrijver (1986).

The most important class of TU matrices are called network matrices. A matrix is a **network matrix** if each column contains at most two non-zero entries of opposite sign and absolute value 1. One of the usual tricks for establishing that a matrix is TU is to use a *restricted* set of row and column operations to convert it into a network matrix. These operations are negating a row (or column) or adding one row (column) to another. Notice that if a matrix is TU before the operation it must be TU after the operation.

A 0-1 matrix has the *consecutive ones property* if the non-zero entries in each column occur consecutively.

THEOREM 2.2 *All 0-1 matrices with the consecutive ones property are TU.*

Whether any of the results identified above apply to the CAP depends upon the context. Rothkopf *et. al.* (1998) offer the following to motivate the consecutive ones property. Suppose the objects to be auctioned are parcels of land along a shore line. The shore line is important as it imposes a linear order on the parcels. In this case it is easy to imagine that the most interesting combinations (in the bidders eyes) would be contiguous. If this were true it would have two computational consequences. The first is that the number of distinct bids would be limited (to intervals of various length) by a polynomial in the number of objects. Second, the constraint matrix A of the CAP would have the consecutive ones property in the columns.

2.3.2 Balanced Matrices

A 0-1 matrix B is **balanced** if it has no square submatrix of odd order with exactly two 1's in each row and column. The usefulness of the balanced condition comes from:

THEOREM 2.3 *Let B be a balanced 0-1 matrix. Then the following linear program:*

$$\max \left\{ \sum_j c_j x_j : \sum_j b_{ij} x_j \leq 1 \ \forall i, \ x_j \geq 0 \ \forall j \right\}$$

has an integral optimal solution whenever the the c_j's are integral.

Notice that balancedness does not guarantee integrality of the polyhedron, but only that there will be an integral optimal solution to the linear program. However, it is true that when B is balanced then the polyhedron $\{x : \sum_j b_{ij} x_j = 1 \ \forall i, \ x_j \geq 0 \ \forall j\}$ is integral.

We now describe one instance of balancedness that may be relevant to the CAP. Consider a tree T with a distance function d. For each vertex v in T let $N(v, r)$ denote the set of all vertices in T that are within distance r of v. If you like, the vertices represent parcels of land connected by a road network with no cycles. Bidders can bid for subsets of parcels but the subsets are constrained to be of the form $N(v, r)$ for some vertex v and some number r. Now the constraint matrix of the corresponding SPP will have one column for each set of the form $N(v, r)$ and one row for each vertex of T. This constraint matrix is balanced. See Nemhauser and Wolsey (1988) for a proof as well as efficient algorithms. In the case when the underlying tree T is a path the constraint matrix reduces to having the consecutive ones property. If the underlying network were not a tree then the corresponding version of SPP becomes NP-hard.

Characterizing the class of balanced matrices is an outstanding open problem.

2.3.3 Perfect Matrices

More generally, if the constraint matrix A can be identified with the vertex-clique adjacency matrix of what is known as a perfect graph, then SPP can be solved in polynomial time. The interested reader should consult Chapter 9 of Grötschel *et. al.* (1988) for more details. The algorithm, while polynomial, is impractical.

We now describe one instance of perfection that may be relevant to the CAP. It is related to the example on balancedness. Consider a tree T. As before imagine the vertices represent parcels of land connected by a road network with no cycles. Bidders can bid for *any* connected subset of parcels. Now the constraint matrix of the corresponding SPP will have one column for each connected subset of T and one row for each vertex of T. This constraint matrix is perfect.

2.3.4 Graph Theoretic Methods

There are situations where $P(A)$ is not integral yet the SPP can be solved in polynomial time because the constraint matrix of A admits a graph theoretic interpretation in terms of an *easy* problem. The most well known instance of this is when each column of the matrix A contains at most two 1's. In this case the SPP becomes an instance of the maximum weight matching problem in a graph which can be solved in polynomial time.

Each row (object) corresponds to a vertex in a graph. Each column

(bid) corresponds to an edge. The identification of columns of A with edges comes from the fact that each column contains two non-zero entries. It is well known that $P(A)$ contains fractional extreme points. Consider for example a graph which is a cycle on three vertices. A comprehensive discussion of the matching problem can be found in the book by Lovász and Plummer (1986). Instances of SPP where each column has at most $K \geq 3$ non-zero entries are NP-hard.

It is natural to ask what happens if one restricts the number of 1's in each row rather than column. Instances of SPP with at most two non-zero entries per row of A are NP-hard. These instances correspond to what is called the stable set problem in graphs, a notoriously difficult problem.

Another case is when the matrix A has the circular ones property. A 0-1 matrix has the **circular ones** property if the non-zero entries in each column (row) are consecutive; first and last entries in each column (row) are treated consecutively. Notice the resemblance to the consecutive ones property. In this case the constraint matrix can be identified with what is known as the vertex-clique adjacency matrix of a circular arc graph.[8] The SPP then becomes the maximum weight independent set problem for a circular arc graph. This problem can also be solved in polynomial time, see Golumbic *et. al.* (1988). Following the parcels of land on the seashore example, the circular ones structure makes sense when the land parcels lie on the shores of an island or lake.

2.3.5 Using Preferences

The solvable instances above work by restricting the sets of objects over which preferences can expressed. Another approach would be to study the implications of restrictions in the preference orderings of the bidders themselves. This can be accomplished using formulation CAP1 but is more transparent with formulation CAP2. Recall CAP2:

$$\max \sum_{j \in N} \sum_{S \subseteq M} b^j(S) y(S,j)$$

$$\text{s.t.} \sum_{S \ni i} \sum_{j \in N} y(S,j) \leq 1 \ \forall i \in M$$

[8] Take a circle and a collection of arcs of the circle. To each arc associate a vertex. Two vertices will be adjacent if the corresponding arcs overlap. The consecutive ones property also bears a graph theoretic interpretation. Take intervals of the real line and associate them with vertices. Two vertices are adjacent if the corresponding intervals overlap. Such graphs are called interval graphs.

$$\sum_{S \subseteq M} y(S,j) \leq 1 \ \forall j \in N$$

$$y(S,j) = 0,1 \ \forall S \subseteq M, \ j \in N$$

One common restriction that is placed on $b^j(\cdot)$ is that it be non-decreasing and supermodular. Suppose now that bidders come in two types. The type one bidders have $b^j(\cdot) = g^1(\cdot)$ and those of type two have $b^j(\cdot) = g^2(\cdot)$ where $g^r(\cdot)$ are non-decreasing, integer valued supermodular functions. Let N^r be the set of type r bidders. Now the dual to the linear programming relaxation of CAP2 is:

$$\min \sum_{i \in M} p_i + \sum_{j \in N} q_j$$

$$\text{s.t.} \ \sum_{i \in S} p_i + q_j \geq g^1(S) \ \forall S \subseteq M, \ j \in N^1$$

$$\sum_{i \in S} p_i + q_j \geq g^2(S) \ \forall S \subseteq M, \ j \in N^2$$

$$p_i, \ q_j \geq 0 \ \forall i \in M, \ j \in N$$

This problem is an instance of the polymatroid intersection problem and is polynomially solvable; see Theorem 10.1.13 in Grötschel *et. al.* (1988). More importantly it has the property of being **totally dual integral**, which means that its linear programming dual, the linear relaxation of the original primal problem, has an integer optimal solution. This last observation is used in Bikhchandani and Mamer (1997) to establish the existence of competitive equilibria in exchange economies with indivisibilities. Utilizing the method to solve problems with three or more types of bidders is not possible because it is known in those cases that the dual problem above admits fractional extreme points. In fact the problem of finding an integer optimal solution for the intersection of three or more polymatroids is NP-hard.

In the case when each of the $b^j(\cdot)$ have the gross substitutes property (Kelso and Crawford (1982)), CAP2 reduces to a sequence of matroid partition problems (see Nemhauser and Wolsey (1988)), each of which can be solved in polynomial time. Gul and Stachetti (1997) describe the reduction as well as provide a 'Walrasian' auctioneer interpretation of it.

2.4 Exact Methods

Exact approaches to solving the SPP require algorithms that generate both good lower and upper bounds on the maximum objective function value of the instance. In general, the upper bound on the optimal solution value is

obtained by solving a **relaxation** of the optimization problem. That is, one solves a related optimization problem whose set of feasible solutions properly contains all feasible solutions of the original problem and whose objective function value is at least as large as the true objective function value for points feasible to the original problem. Thus, we replace the "true" problem by one with a larger feasible region that is more easily solved. There are two standard relaxations for SPP: Lagrangean relaxation (where the feasible set is usually required to maintain 0-1 feasibility, but many if not all of the constraints are moved to the objective function with a penalty term) and the linear programming relaxation (where only the integrality constraints are relaxed—the objective function remains the original function). Lagrangean relaxation will be discussed in greater detail in Section 3 on decentralization.

Exact methods come in three varieties: branch and bound, cutting planes and a hybrid called branch and cut. The basic idea of branch and bound can be described as intelligent enumeration. At each stage, after solving the LP, a fractional variable, x_j, is selected and two subproblems are set up (this is the branching phase) one where x_j is set to 1 and the other where x_j is set to 0. The linear programming relaxation of the two subproblems are solved to identify an upper bound on the objective function value for each subproblem. From each subproblem with a nonintegral solution we branch again to generate two subproblems and so on. In the worst case we generate a binary tree that includes all feasible solutions. However by comparing the linear programming bound across nodes in different branches of the tree, one can prune some branches in advance without the need to explore them further. That is solutions with some variable x_k, say, set to 1 (or zero) can never be optimal. This is the 'bound' in the name of the method.

Cutting plane methods find linear inequalities that are violated by a solution of a given relaxation but are satisfied by all feasible zero-one solutions. These inequalities are called **cuts**. If one adds enough cuts, one is left with integral extreme points. Later in this paper (see Subsection 3.5) we show how cuts can be used to develop prices for various subsets of objects. The most successful cutting plane approaches are based on polyhedral theory, that is they replace the constraint set of an integer programming problem by a convexification of the feasible zero-one points and extreme rays of the problem. For details on polyhedral structure of the SPP and its relatives see Padberg (1973, 1975 and 1979), Cornuejols and Sassano (1989) and Sassano (1989). Given that the problems are NP-hard a full polyhedral description of these problems is unlikely.

Branch and cut works likes branch and bound but tightens the bounds in every node of the tree by adding cuts. For a complete description of how

such cuts are embedded into a tree search structure along with other tricks of the trade, see Hoffman and Padberg (1993).

Because even small instances of the CAP1 may involve a huge number of columns (bids) the techniques described above need to be augmented with another method known as column generation. Introduced by Gilmore and Gomory (1961) it works by generating a column when needed rather than all at once. An overview of such methods can be found in Barnhart *et. al.* (1994). Later in this paper we illustrate how this idea could be implemented in an auction.

One sign of how successful exact approaches are can be found in Hoffman and Padberg (1993). They report being able to find an optimal solution to an instance of SPA with 1,053,137 variables and 145 constraints in under 25 minutes. In auction terms this corresponds to a problem with 145 items and 1,053,137 bids. A major impetus behind the desire to solve large instances of SPA (and SPC) quickly has been the airline industry. The problem of assigning crews to routes can be formulated as an SPA. The rows of the SPA correspond to flight legs and the columns to a sequence of flight legs that would be assigned to a crew. Like the CAP, in this problem the number of columns grows exponentially with the number of rows.[9] For the SPP, the large instances that have been studied have usually arisen from relaxations of SPA's. Given the above we believe that established integer programming methods will prove quite successful when applied to the solution of CAP.

Logistics.com's OptiBid^{TM} software has been used in situations where the number of bidders is between 12 to 350 with the average being around 120. The number of lanes (objects) has ranged between 500 and 10,000. Additionally, each lane bid can contain a capacity constraint as well as a budget capacity constraint covering multiple lanes. The typical number of lanes is 3000. OptiBid^{TM} does not limit the number of distinct subsets that bidders bid on or the number of items allowed within a package. OptiBid^{TM} is based on a integer program with a series of proprietary formulations and starting heuristic algorithms.[10]

SAITECH-INC's bidding software, SBID, is also based on integer programming. They report being able to handle problems of similar size as OptiBid^{TM}.[11]

Exact methods for CAP1 have been proposed by Fujishima *et. al.* (1999)

[9]However, these crew scheduling problems give rise to instances of SPA that have a large number of duplicate columns in the constraint matrix. In some cases as many as 60% of them. We thank Márta Eső for alerting us to this.

[10]We thank Dr. Christopher Caplice of Logistics.com for providing this information.

[11]We thank Dr. Yoshiro Ikura of SAITECH-INC for providing us with this information.

as well as Sandholm (1999) and Andersson *et. al.* (2000). The first two use variations of dynamic programming and the third uses integer programming. In the first, the method is tested on randomly generated instances the largest of which involved 500 objects (rows) and 20,000 bids (variables). The second also tests the method on randomly generated instances, the largest of which involved 400 objects (rows) and 2000 bids (variables). In these tests the number of bids examined is far less than the number of subsets of objects. The third uses integer programming methods on the test problems generated by the first two.

By comparison, a straightforward implementation on a commercially available code for solving linear integer programs (called CPLEX) runs into difficulties for instances of CAP involving more than 19 objects if one lists all the bids for the various subsets. There will be 2^{19} variables. This already requires one giga-byte of memory to store. CPLEX can handle in this straight forward approach on the order of 2^{19} variables and 19 constraints before running out of resident memory. Notice that this is large enough to handle the test problems considered in Sandholm (1999) and Fujishima *et. al.* (1999). We report on this in a later section where we also discuss schemes for generating test problems.[12]

The reader should note that size of an instance is not by itself an indicator of problem difficulty. Structure of the problem plays an important role. In a later section we will discuss some of the structure inherent in the generation schemes used by Sandholm and others.

2.5 Approximate Methods

One way of dealing with hard integer programs is to give up on finding the optimal solution. Rather one seeks a feasible solution fast and hopes that it is near optimal. This raises the obvious question of how close to optimal the solution is. There have traditionally been three ways to assess the accuracy of an approximate solution. The first is by worst-case analysis, the second by probabilistic analysis and the third empirically.

Before describing these approaches it is important to say that probably every heuristic approach for solving general integer programming problems has been applied to the SPP. The first one that almost everyone thinks off, called Greedy, is to iteratively select the column j that maximizes $c_j / \sum_i a_{ij}$, the weight to column sum ratio (see Fisher and Wolsey (1982) for example). Interchange/steepest ascent approaches have also been

[12]Since our findings duplicate those of Andersson *et. al.* (2000) we will comment on this paper in a later section.

used; a swap of one or more columns is executed whenever such a swap improves the objective function value (see Hurkens and Schrijver (1989) for the case of maximum cardinality packing and Arkin and Hassin (1998) for weighted packing problems). More fashionable approaches such as genetic algorithms Huang *et. al.* (1994), probabilistic search (Feo *et. al.*, 1989), simulated annealing (Johnson, *et. al.*, 1989), and neural networks (Aourid and Kaminska, 1994) have also been tried. Unfortunately, there has not been a comparative testing across such methods to determine under what circumstances a specific method might perform best. Beasley (1990) maintains (at `http://mscmga.ms.ic.ac.uk/info.html`) an extensive test set of covering and partitioning problem instances for those who would like to try their hand. We think it safe to say that anything one can think of for approximating the SPP has probably been thought of. In addition, one can embed approximation algorithms within exact algorithms so that one is attempting to get a sharp approximation to the lower bound for the problem at the same time that one iteratively tightens the upper bound.

2.5.1 Worst-Case Analysis

Let I denote an instance of the integer program one wishes to solve and $Z(I)$ its optimal objective function value. Assume the goal is to maximize the objective function as in the SPP. Let $Z_H(I)$ denote the value of the solution returned on the instance I by the polynomial time approximation algorithm (also called heuristic) H. The **worst-case ratio** for the performance of H is $\inf Z_H(I)/Z(I)$. Here the infimum is over all instances in the problem class. For a problem with a maximization objective, $Z_H(I) \leq Z(I)$ and so the worst-case ratio will always be less than 1. The objective is to bound it from below.

The SPP is difficult to approximate. It is proved by Håstad (1999) that unless P = NP, there is no polynomial time algorithm for the SPP that can deliver a worst case ratio larger than $n^{\epsilon-1}$ for any $\epsilon > 0$. On the positive side polynomial algorithms that have a worst case ratio of $O(n/(\log n)^2)$ are known. Restricting the instances of SPP does not help things very much. The reader interested in a full account of what is known about approximating the SPP should consult Crescenzi *et. al.* (1998) where an updated list of what is known about the worst-case approximation ratio of a whole range of optimization problems is given.

When interpreting these worst case results, two things should be kept in mind. The first is that they are *worst-case* results and so shed little light on the 'typical' accuracy of an approximation algorithm. The second is that

these results are lower bounds on $Z_H(I)$ as a *linear* function of $Z(I)$. It is very possible that bounds on $Z_H(I)$ that are non-linear functions of $Z(I)$ might not be so pessimistic.

2.5.2 Probabilistic Analysis

Probabilistic analysis is an attempt to characterize the typical behavior of an approximation algorithm. A probability distribution over problem instances is specified in which case $Z(I)$ and $Z_H(I)$ become random variables. The goal is to understand the behavior of the difference or ratio of the two variables as the size of the instances increase. For example, what is the mean difference, the variance etc. Since the results are asymptotic in nature, attention must be paid to the convergence results when interpreting the results. A problematic feature is that the distributions over instances that are chosen (because of ease of analysis) do not necessarily coincide with the distributions that actual instances will be drawn from. This issue arises also in the empirical testing of approximation algorithms.

2.5.3 Empirical Testing

Many approximation algorithms will be elaborate enough to defy theoretical analysis. For this reason it is common to resort to empirical testing. Further empirical testing allows one to consider issues not easily treated analytically.

A good guide to the consumption of an empirical study of approximation algorithms is given by Ball and Magazine (1981). They list the following evaluation criteria:

1. Proximity to the optimal solution.
2. Ease of implementation (coding and data requirements).
3. Flexibility; ability to handle changes in the model.
4. Robustness; ability to provide sensitivity analysis and bounds.

This is not the forum for an extensive discussion of the issues associated with the empirical testing of heuristics. However, some points are worth highlighting.

The most obvious is the choice of test problems. Are they realistic? Do they exhibit the features that one thinks one will find in the environment? Interestingly, probabilistic analysis has a role to play here in eliminating some schemes for randomly generating test problems. For example it is

known that certain generation schemes give rise to problems that are easy to solve; for example, a randomly generated solution is with high probability close to optimal. Success on a collection of problems generated in this way conveys no information. Is the accuracy due to the approximation algorithm or the structure of the test problems?

Some approximation algorithms involve a number of parameters that need to be fine tuned. Comparing their performance with heuristics whose parameters are not fine tuned becomes difficult because it is not clear whether one should include the overhead in the tuning stage in the comparison.

3 Decentralized Methods

One way of reducing some of the computational burden in solving the CAP is to set up a 'fictitious' market that will determine an allocation and prices in a decentralized way. The traditional auctioneer is replaced with a Walrasian one who sets prices for the objects. Agents announce which sets of objects they will purchase at the posted prices. If two or more agents compete for the same object, the Walrasian auctioneer adjusts the price vector. This saves bidders from specifying their bids for every possible combination and the auctioneer from having to process each bid function. Such methods also have the advantage that they can be adapted to dynamic environments where bidders and objects arrive and depart at different times. While such methods reduce the computational burden they cannot eliminate them altogether. A more compelling argument for decentralized methods is that the relevant information for choosing an allocation is itself decentralized.

Examples of decentralized approaches for solving the CAP can be found in Fujishima *et. al.* (1999) and Rassenti *et. al.* (1982). In the same spirit, Brewer (1999) and Wellman *et. al.* (1998) propose decentralized scheduling procedures in different contexts. In their set up the auctioneer chooses a feasible solution and 'bidders' are asked to submit improvements to the solution. In return for these improvements, the auctioneer agrees to share a portion of the revenue gain with the bidder. These methods can be viewed as instances of dual based procedures for solving an integer program. Auction or market interpretations of dual based procedures for optimization problems are not new. They appear, for example in Dantzig (1963). The updates on the dual variables that are executed in these algorithms can be interpreted as a form of myopic best response on the part of bidders. More recently, Bertsekas (1991) has proposed a collection of dual based algorithms for the class of linear network optimization problems. These algorithms he

dubs auctions algorithms. The incentive issues associated with such methods are discussed later.

3.1 Duality in Integer Programming

To describe the dual to SPP let $\mathbf{1}$ denote the m-vector of all 1's and a^j the j^{th} column of the constraint matrix A. The (superadditive) dual to SPP is the problem of finding a superadditive, non-decreasing function $F : \mathbb{R}^m \to \mathbb{R}^1$ that

$$\min F(\mathbf{1})$$
$$\text{s.t. } F(a^j) \geq c_j \ \forall j \in V$$
$$F(\mathbf{0}) = 0$$

We can think of F as being a non-linear price function that assigns a price to each bundle of goods (see Wolsey (1981)).

If the primal integer program has the integrality property, there is an optimal integer solution to its linear programming relaxation, the dual function F will be linear i.e. $F(u) = \sum_i y_i u_i$ for some y and all $u \in \mathbb{R}^m$. The dual becomes:

$$\min \sum_i y_i$$
$$\text{s.t. } \sum_i a_{ij} y_i \geq c_j \ \forall j \in V$$
$$y_i \geq 0 \ \forall i \in M$$

That is, the superadditive dual reduces to the dual of the linear programming relaxation of SPP. In this case we can interpret each y_i to be the price of object i. Thus an optimal allocation given by a solution to the CAP can be supported by prices on individual objects.

Optimal objective function values of SPP and its dual coincide (when both are well defined). There is also a complementary slackness condition:

THEOREM 3.1 *If x is an optimal solution to SPP and F an optimal solution to the superadditive dual then*

$$(F(a^j) - c_j)x_j = 0 \ \forall j.$$

Solving the superadditive dual problem is as hard as solving the original primal problem. It is possible to reformulate the superadditive dual problem as a linear program (the number of variables in the formulation is exponential in the size of the original problem). For small or specially structured

problems this can provide some insight. The interested reader is referred to Nemhauser and Wolsey (1988) for more details. In general one relies on the solution to the linear programming dual and uses its optimal value to guide the search for an optimal solution to the original primal integer program. The way this is done is through a technique known as Lagrangean Relaxation.

3.2 Lagrangean Relaxation

The basic idea is to 'relax' some of the constraints of the original problem by moving them into the objective function with a penalty term. That is infeasible solutions to the original problem are allowed, but they are penalized in the objective function in proportion to the amount of infeasibility. The constraints that are chosen to be relaxed, are selected so that the optimization problem over the remaining set of constraints is in some sense easy. We describe the bare bones of the method first and then give a 'market' interpretation of it.

Recall the SPP:

$$Z = \max \sum_{j \in V} c_j x_j$$

$$\text{s.t.} \quad \sum_{j \in V} a_{ij} x_j \leq 1 \ \forall i \in M$$

$$x_j = 0, 1 \ \forall j \in V$$

Let Z_{LP} denote the optimal objective function value to the linear programming relaxation of SPP. Note that $Z \leq Z_{LP}$. Consider now the following relaxed problem:

$$Z(\lambda) = \max \sum_{j \in V} c_j x_j + \sum_{i \in M} \lambda_i (1 - \sum_{j \in V} a_{ij} x_j)$$

$$\text{s.t. } 1 \geq x_j \geq 0 \ \forall j \in V$$

For a given λ, computing $Z(\lambda)$ is easy. To see why note that

$$\sum_{j \in V} c_j x_j + \sum_{i \in M} \lambda_i (1 - \sum_{j \in V} a_{ij} x_j) = \sum_{j \in V} (c_j - \sum_{i \in M} \lambda_i a_{ij}) x_j + \sum_{i \in M} \lambda_i.$$

Thus, to find $Z(\lambda)$, simply set $x_j = 1$ if $(c_j - \sum_{i \in M} \lambda_i a_{ij}) > 0$ and zero otherwise. It is also easy to see that $Z(\lambda)$ is piecewise linear and convex. A basic result that is easy to prove is:

THEOREM 3.2

$$Z_{LP} = \min_{\lambda \geq 0} Z(\lambda).$$

Why might this be useful? Since evaluating $Z(\lambda)$ for each λ is a snap, if we can find a fast way to determine the λ that solves $\min_{\lambda \geq 0} Z(\lambda)$ we would have fast procedure to find Z_{LP}. The resulting solution (values of the x variables) while integral need not be feasible. However it may not be 'too infeasible' and so could be fudged into a feasible solution without a great reduction in objective function value.

Finding the λ that solves $\min_{\lambda \geq 0} Z(\lambda)$ can be accomplished using the subgradient algorithm. Suppose the value of the lagrange multiplier λ at iteration t is λ^t. Choose any subgradient of $Z(\lambda^t)$ and call it s^t. Choose the lagrange multiplier for iteration $t+1$ to be $\lambda^t + \theta_t s^t$, where θ_t is a positive number called the step size. In fact if x^t is the optimal solution associated with $Z(\lambda^t)$,

$$\lambda^{t+1} = \lambda^t + \theta_t(Ax^t - \mathbf{1}).$$

Notice that $\lambda_i^{t+1} > \lambda_i^t$ for any i such that $\sum_j a_{ij} x_j^t > 1$. The penalty term is increased on any constraint currently being violated.

The algorithm is a natural adaptation of the steepest descent algorithm to non-differentiable functions. At each stage it adjusts the multiplier so as to produce a decrease in the function value. Since the function $Z(\lambda)$ is not differentiable, we must choose a subgradient rather than gradient. For an appropriate choice of step size at each iteration, this procedure can be shown to converge to the optimal solution. Specifically, $\theta_t \to 0$ as $t \to \infty$ but $\sum_t \theta_t$ diverges. The first condition ensures that as we get closer to the optimal solution, our step sizes go down, i.e. we won't leave the optimum once it is found. The second ensures that we don't stop prematurely, i.e., before finding the optimal solution. The subgradient algorithm is not guaranteed to converge to the optimal solution in a finite number of steps. Hence, when implemented, the user will incorporate a stopping rule, e.g., stop once an improvement in objective function value that does not exceed some threshold is observed and experiment with step sizes to ensure rapid convergence.

It should be emphasized that Lagrangean relaxation is not guaranteed to find the optimal solution to the underlying problem. Rather, it finds an optimal solution to a *relaxation* of it.

Here is the market interpretation. The Walrasian auctioneer chooses a price vector λ for the individual objects and bidders submit bids. If the highest bid, c_j, for the j^{th} bundle exceeds $\sum_{i \in M} a_{ij}\lambda_i$, this bundle is tentatively assigned to that bidder. Notice that the auctioneer need not know what

c_j is ahead of time. This is supplied by the bidders after λ is announced. In fact, the bidders need not announce bids, they could simply state which individual objects are acceptable to them at the announced prices. The auctioneer can randomly assign objects to bidders in case of ties. If there is a conflict in the assignments, the auctioneer uses the subgradient algorithm to adjust prices and repeats the process.

Now let us compare this market interpretation of Lagrangean relaxation with the simultaneous ascending auction (SAA) proposed by P. Milgrom, R. Wilson and P. McAfee (see Milgrom (1995)). In the SAA, bidders bid on individual items simultaneously in rounds. To stay in the auction for an item, bids must be increased by a specified minimum from one round to the next just like the step size. Winning bidders pay their bids. The only difference between this and Lagrangean relaxation, is that the bidders through their bids adjust their prices rather than the auctioneer. The adjustment is along a subgradient. Bids increase on those items for which there are two or more bidders competing.

One byproduct of the SAA is called the *exposure problem*. Bidders pay too much for individual items or bidders with preferences for certain bundles drop out early to limit losses. As an illustration consider an extreme example of a bidder who values the bundle of goods i and j at \$100 but each separately at \$0. In the SAA, this bidder may have to submit high bids on i and j to be able to secure them. Suppose that it loses the bidding on i. Then it is left standing with a high bid j which it values at zero. The presence of such a problem is easily seen within the Lagrangean relaxation framework. While Lagrangean relaxation will yield the optimal objective function value for the *linear relaxation* of the underlying integer program, it is not guaranteed to produce a feasible solution. Thus the solution generated may not satisfy the complementary slackness conditions. The violation of complementary slackness is the *exposure problem* associated with this auction scheme. Notice that any auction scheme that relies on prices for individual items will face this problem.

In contrast to the SAA outlined above is the Adaptive User Selection Mechanism (AUSM) proposed by Banks *et. al.* (1989). AUSM is asynchronous in that bids on subsets can be submitted at any time and so is difficult to connect to the Lagrangean ideas just described. An important feature of AUSM is an arena which allows bidders to aggregate bids to exploit synergies. DeMartini *et. al.* (1999) propose an iterative auction scheme that is a hybrid of the SAA and AUSM that is easier to connect to the Lagrangean framework. In this scheme, bidders submit bids on packages rather than on individual items. Like the SAA, bids on packages must be increased

by a specified amount from one round to the next. This minimum increment is a function of the bids submitted in the previous round. In addition, the number of items that a bidder may bid on in each round is limited by the number of items s/he bid on in previous rounds. The particular implementation of this scheme advanced by DeMartini *et. al.* (1999) can also be given a Lagrangean interpretation. They choose the multipliers (which can be interpreted as prices on individual items) so as to try and satisfy the complementary slackness conditions of linear programming. Given the bids in each round, they allocate the objects so as to maximize revenue. Then they solve a linear program (that is essentially the dual to CAP2) that finds a set of prices/multipliers that approximately satisfy the complementary slackness conditions associated with the allocation.

Kelly and Steinberg (1998) also propose an iterative auction scheme for combinatorial auctions.[13] It can also be described as a hybrid of the SAA and AUSM. The auction has two phases. The first phase is a sequential auction where bidders bid on individual items. In the second phase they use AUSM.

In Parkes (1999) an ascending auction, called *i*Bundle, that allows bidders to bid on combinations of items and uses non-linear prices is proposed. Bidders submit bids for subsets of items. At each iteration the auctioneer announces prices for those subsets of items that receive unsuccessful bids from agents. For a bid on a subset to be 'legal' it must exceed the price posted by the auctioneer. Given the bids, the auctioneer solves an instance of CAP2 and tentatively assigns the objects. For the next iteration, the prices on each subset are either kept the same or adjusted upwards. The upward adjustment is determined by the highest losing bid for the subset in question plus a user specified increment. The auction terminates when the bids from one round to the next do not show sufficient change. The scheme can be given a Lagrangian interpretation as well, however the underlying formulation is different from CAP1 or CAP2. We discuss it in Section 3.5.

3.3 Variations

By relaxing on a *subset* of the constraints as opposed to all of them we get different relaxations, some of which give upper bounds on Z that are smaller than Z_{LP}. Details can be found in Nemhauser and Wolsey (1988). Needless to say there have been many applications of Lagrangean relaxation to SPP,

[13] The description is tailored to the auction for assigning carrier of last resort rights in telecommunications.

SPA and SPC and hybrids with exact methods have also been investigated. See Balas and Carrera (1996) and Beasley (1990) for recent examples.

3.4 Column Generation

Column generation is a technique for solving linear programs with an exceedingly large number of variables. Each variable gives rise to a column in the constraint matrix, hence the name column generation. A naive implementation of a simplex type algorithm for linear programming would require recording and storing every column of the constraint matrix. However, only a small fraction of those columns would ever make it into an optimal basic feasible solution to the linear program. Further, of those columns not in the current basis, one only cares about the ones whose reduced cost will be of the appropriate sign. Column generation exploits this observation in the following way. First an optimal solution is found using a subset of the columns/variables. Next, given the dual variable implied by this preliminary solution, an optimization problem is solved to find a non-basic column/variable that has a reduced cost of appropriate sign. The trick is to design an optimization problem to find this non-basic column without listing all non-basic columns.

The column generation idea can be implemented in an auction setting as follows. In the first step the auctioneer chooses an extreme point solution to the CAP. It does not matter which one, any one will do. Note that this initial solution could involve fractional allocations of objects.

This extreme point solution is reported to all bidders. Each bidder, looking only at how they value the allocation proposes a column/variable/subset to enter the basis (along with its value to the bidder). The proposed column and its valuation must satisfy the appropriate reduced cost criterion for inclusion in the basis. In effect each bidder is being used as a subroutine to execute the column generation step.

The auctioneer now gathers up the proposed columns (along with their valuations) and using these columns and the columns from the initial basis only (and possibly previously generated nonbasic columns), solves a linear program to find a revenue maximizing (possibly fractional) allocation. The new extreme point solution generated is handed out to the bidders who are asked to each identify a new column (if any) to be added to the new basis that meets the reduced cost criterion for inclusion. The process is then repeated until an extreme point solution is identified that no bidder wishes to modify. To avoid cycling, the auctioneer can always implement one of the standard anti-cycling rules for linear programming.

This auction procedure eliminates the need to transmit and process long lists of subsets and their bids. Bids and subsets are generated only as needed. Second, the bidders are provided an opportunity to challenge an allocation provided they propose an alternative that increases the revenue to the seller. If the bids might lead to a nonintegral allocation, then this column generation has to be imbedded into a branch and cut/price scheme to produce an integer solution.[14]

Notice that the ellipsoid method provides a way to solve the fractional CAP to optimality while generating only a polynomially bounded number of columns.

3.5 Cuts, Extended Formulations and Non-linear Prices

The decentralized methods described above work by conveying 'price' information to the bidders. Given a set of bids and an allocation, prices for individual items that 'support' or are 'consistent' with the bids and allocations are derived and communicated to the bidders. Such prices, because they are linear cannot hope to fully capture the interactions between the parties. Here we show, with an example, how cutting plane methods can be used to generate prices that more closely reflect the interactions between bids on different sets of objects.

In the example we have 6 objects with highest bids on various subsets of objects shown below; subsets with bids of zero are not shown:

$$b(\{1,2\}) = b(\{2,3\}) = b(\{3,4\}) = b(\{4,5\}) = b(\{1,5,6\}) = 2,\ b(\{6\}) = 1.$$

Formulation CAP1 for this example (ignoring the integrality constraints) is:

$$\begin{array}{lccccccc}
\max & 2x_{12}+ & 2x_{23}+ & 2x_{34}+ & 2x_{45}+ & 2x_{156}+ & x_6 & \\
s.t. & x_{12}+ & & & & x_{156} & & \leq 1 \\
& x_{12}+ & x_{23} & & & & & \leq 1 \\
& & x_{23}+ & x_{34} & & & & \leq 1 \\
& & & x_{34}+ & x_{45} & & & \leq 1 \\
& & & & x_{45}+ & x_{156} & & \leq 1 \\
& & & & & x_{156}+ & x_6 & \leq 1 \\
& x_{12}, & x_{23}, & x_{34}, & x_{45}, & x_{156}, & x_6 & \geq 0
\end{array}$$

The optimal *fractional* solution is to set all variables equal to a half. The optimal dual variables are $y_i = 1/2$ for $i = 1, \ldots, 5$ and $y_6 = 1$. So, for example, the imputed price of the set $\{1,2\}$ is $y_1 + y_2 = 1$.

[14] We thank Márta Eső for suggesting this last refinement. See Eső (1999) for an example of such a branch and cut scheme.

Consider now the following inequality:

$$x_{12} + x_{23} + x_{34} + x_{45} + x_{156} \leq 2.$$

Every feasible *integer* solution to the formulation above satisfies this inequality but not all fractional solutions do. In particular the optimal fractional solution above does not satisfy this inequality. This inequality is an example of a cut. Classes of cuts for the SPP are known, the one above belongs to the class of odd-cycle cuts.

Now append this cut to our original formulation:

$$\begin{array}{lcccccccc}
\max & 2x_{12}+ & 2x_{23}+ & 2x_{34}+ & 2x_{45}+ & 2x_{156}+ & x_6 & & \\
s.t. & x_{12}+ & & & & x_{156} & & \leq & 1 \\
 & x_{12}+ & x_{23} & & & & & \leq & 1 \\
 & & x_{23}+ & x_{34} & & & & \leq & 1 \\
 & & & x_{34}+ & x_{45} & & & \leq & 1 \\
 & & & & x_{45}+ & x_{156} & & \leq & 1 \\
 & & & & & x_{156}+ & x_6 & \leq & 1 \\
 & x_{12}+ & x_{23}+ & x_{34}+ & x_{45}+ & x_{156} & & \leq & 2 \\
 & x_{12}, & x_{23}, & x_{34}, & x_{45}, & x_{156}, & x_6 & \geq & 0
\end{array}$$

The optimal solution to this linear program is *integral*. It is $x_{12} = 1, x_{34} = 1$ and $x_6 = 1$. There are now 7 dual variables. One for each of the six objects (y_i) and one more for the cut (μ). One optimal dual solution is $y_1 = y_5 = y_6 = 0$, $y_2 = y_3 = y_4 = 1$ and $\mu = 1$. The imputed price for the set $\{1, 2\}$ is now $y_1 + y_2 + \mu = 2$. In general the price of a set S will be the sum of the item prices, $\sum_{i \in S} y_i$ plus μ if the 'x' variable associated with the set S appears with coefficient 1 in the cut. Notice that pricing sets of objects in this way means that the price function will be superadditive.

It is instructive to compare the imputed price of the set $\{1, 2\}$ in the two formulations. The first formulation assigns a price of one to the set. The second a higher price. The first formulation ignores the fact that if the set $\{1, 2\}$ is assigned to a bidder, the sets $\{1, 5, 6\}$ and $\{2, 3\}$ cannot be assigned to anyone else. This fact is captured by the cut. The dual variable associated with the cut can be interpreted as the associated opportunity cost of assigning the set $\{1, 2\}$ to a bidder. Thus the actual price of the set $\{1, 2\}$ is the sum of the prices of the objects in it plus the opportunity cost associated with its sale.

Cuts can be derived in one of two ways. The first is by purely combinatorial reasoning and the other through an algebraic technique first proposed by Ralph Gomory (see Nemhauser and Wolsey (1988) for the details). For

CAP1, given a fractional extreme point, one can use the Gomory method to generate a cut involving only the variables that are basic in the current extreme point. This is useful for computational purposes as one does not have to lug all variables around to identify a cut. Second, the new inequality will be a non-negative linear combination of the current basic rows less equal than a non-negative number. Thus the dual variable associated with this new constraint will have an additive effect on the prices of various subsets as in the example.

The reader will notice that by picking an extreme point dual solution, the imputed prices for some sets are zero. Since there is some flexibility in the choice of dual variables, one can choose an interior (to the feasible region) dual solution.

Yet another way to get non-linear prices is by starting with a stronger formulation of the underlying optimization problem. One formulation is stronger than another if its set of feasible (fractional) solutions is strictly contained in the other. In the example above, the second formulation is stronger than the first. Both formulations share the same set of integer solutions, but not fractional solutions. The set of fractional solutions to the second formulation is a strict subset of the fractional solutions to the first one.

Stronger formulations can be obtained, as shown above, by the addition of inequalities. Yet another, standard way, of obtaining stronger formulations is through the use of additional or auxiliary variables, typically a large number of them. Geometrically, one is treating the problem formulated in the original set of variables as the projection of a higher dimensional but structurally simpler polyhedron. Formulations involving such additional variables are called extended formulations and developing these extended formulations is called lifting. Using lifting one can develop a hierarchy of successively stronger formulations of the underlying integer program.

There is a close connection between lifting and cutting plane approaches. When one projects out the auxiliary variables one obtains a formulation involving the original variables but with additional constraints which are cuts.

Extended formulations can be generated by the study of the problem at hand or algorithmically. Perhaps the most accessible introduction to these matters is Balas *et. al.* (1993) which also discusses the connection to cutting planes.

In the auction context, Bikchandani and Ostroy (1998), propose an extended formulation for the problem of selecting the winning set of bids. To describe this formulation let Π be the set of all possible partitions of the

objects in the set M. If π is an element of Π, we write $S \in \pi$ to mean that the set $S \subset M$ is a part of the partition π. Let $z_\pi = 1$ if the partition π is selected and zero otherwise. These are the auxiliary variables. Using them we can reformulate CAP2 as follows:

$$\max \sum_{j \in N} \sum_{S \subseteq M} b^j(S) y(S,j)$$

$$\text{s.t.} \sum_{S \subseteq M} y(S,j) \leq 1 \ \forall j \in N$$

$$\sum_{j \in N} y(S,j) \leq \sum_{\pi \ni S} z_\pi \ \forall S \subset M$$

$$\sum_{\pi \in \Pi} z_\pi \leq 1$$

$$y(S,j) = 0, 1 \ \forall S \subseteq M, \ j \in N$$

Call this formulation CAP3. In words, CAP3 chooses a partition of M and then assigns the sets of the partition to bidders in such a way as to maximize revenue. It is easy to see that this formulation is stronger than CAP1 or CAP2. Fix an $i \in M$ and add over all $S \ni i$ the inequalities

$$\sum_{j \in N} y(S,j) \leq \sum_{\pi \ni S} z_\pi \ \forall S \subset M$$

to obtain:

$$\sum_{S \ni i} \sum_{j \in N} y(S,j) \leq 1 \ \forall i \in M$$

which are the inequalities that appear in CAP2. While stronger than CAP2, formulation CAP3 still admits fractional extreme points.

The dual of the linear relaxation of CAP3 involves one variable for every constraint of the form:

$$\sum_{S \subseteq M} y(S,j) \leq 1 \ \forall j \in N,$$

call it s_j, which can be interpreted as the surplus that bidder j obtains. The dual involves one variable for every constraint of the form:

$$\sum_{j \in N} y(S,j) \leq \sum_{\pi \ni S} z_\pi \ \forall S \subset M$$

which we will denote p_S. It can be interpreted as the price of the subset S. In fact the dual will be:

$$\min \sum_{j \in N} s_j + \mu$$

$$\text{s.t. } s_j \geq b^j(S) - p_S \ \forall j \ S \subset M$$

$$\mu \geq \sum_{S \in \pi} p_S \ \forall \pi \in \Pi$$

$$s_j, p_S, \mu \geq 0$$

and has the obvious interpretation: minimizing the surplus that bidders obtain. Thus one can obtain non-linear prices from the extended formulation. These prices are not exact since CAP3 is not integral. Further they do not depend on the bidders, that is all bidders pay the same price for a given subset. The catch of course is this formulation involves many more variables than CAP1 or CAP2.

In Parkes and Ungar (2000) a condition on bidders preferences is identified that ensures that the linear relaxation of CAP3 has an integral solution. The condition, called bid safety, is difficult to interpret easily but has the effect of forcing complementary slackness to hold for an integer solution of CAP3. Under this condition any algorithm for solving CAP3's dual (or its lagrangean relaxation) will generate an optimal solution of CAP3 itself. Since many dual algorithms can be given an auction interpretation with the iterations being identified as adjustments in bids that a myopic best reply agent might execute, one can generate auction schemes that are arguably optimal. This is precisely the tack taken in Parkes and Ungar (2000) to support the adoption of the *i*Bundle auction scheme of Parkes (1999).

Bikchandani and Ostroy (1998) introduce yet another formulation stronger than CAP3 which is integral. The idea is to use a variable that represents both a partition of the objects *and* an allocation. The dual to this formulation gives rise to non-linear prices with the twist that they are bidder specific. Different bidders pay different prices for the same subset.

A warning about extended formulations is in order. One must be careful in invoking an extended formulation that simply formulates the problem away. As an example, consider the auxiliary variables introduced in CAP3, the z_π's. For each z^r, let y^r be an optimal extreme point solution to the following :

$$\max \sum_{j \in N} \sum_{S \subseteq M} b^j(S) y(S, j)$$

$$\text{s.t.} \sum_{S \subseteq M} y(S,j) \leq 1 \ \forall j \in N$$

$$\sum_{j \in N} y(S,j) \leq \sum_{\pi \ni S} z_\pi^r \ \forall S \subset M$$

$$y(S,j) \geq 0 \ \forall S \subseteq M, \ j \in N$$

Notice that y^r is integral because the constraint matrix is totally unimodular. Given this we can formulate the problem of finding the winning set of bids as:

$$\max \sum_{r \geq 1} [\sum_{j \in N} \sum_{S \subseteq M} b^j(S) y^r(S,j)] \nu_r$$

$$\sum_{r \geq 1} \nu_r = 1$$

$$\nu_r \geq 0$$

It is trivial to see that this linear program has the integrality property. It should also be clear that the formulation sheds no light on the original problem.

4 Incentive Issues

Thus far we have focused on the problem of choosing an allocation of the objects so as to maximize the sellers revenue. The revenue depends on the bids submitted but there is no guarantee that the submitted bids approximate the actual values that bidders assign to the various subsets. To illustrate how this can happen consider three bidders, 1, 2 and 3 and two objects $\{x, y\}$. Suppose:

$$v^1(x,y) = 100, \ v^1(x) = v^1(y) = 0, \ v^2(x) = v^2(y) = 75, \ v^2(x,y) = 0,$$

$$v^3(x) = v^3(y) = 40, \ v^3(x,y) = 0.$$

Here $v^i(\cdot)$ represents the value to bidder i of a particular subset. Notice that the bid that i submits on the set S, $b^i(S)$ need not equal $v^i(S)$.

If the bidders bid truthfully, the auctioneer should award x to 2 and y to 3, say, to maximize his revenue. Notice however that bidder 2 say, under the assumption that bidder 3 continues to bid truthfully, has an incentive to shade his bid down on x and y to, say, 65. Notice that bidders 2 and 3 still win but bidder 2 pays less. This argument applies to bidder 3 as well. However, if they both shade their bids downwards they can end up losing

the auction. This feature of combinatorial auctions is called the 'threshold problem' (see Bykowsky *et. al.* (1995)): a collection of bidders whose combined valuation for distinct portions of a subset of items exceeds the bid submitted on that subset by some other bidder. It may be difficult for them to coordinate their bids to outbid the large bidder on that subset.

In this section we describe what is known about auction mechanisms that give bidders the incentive to truthfully reveal their valuations.

To discuss incentive issues we need a model of bidders preferences. The simplest conceptual model endows bidder $j \in N$ with a list $\{v^j(S)\}_{S \subseteq M}$, abbreviated to v^j, that specifies how she values (monetarily) each subset of objects. Thus $v^j(S)$ represents how much bidder j values the subset S of objects.[15]

The auction scheme chosen and the bids submitted will be a function of the beliefs that seller and bidders have about each other. The simplest model of beliefs is the independent private values model. Each bidder's v^j is assumed by seller and all bidders to be an independent draw from a commonly known distribution over a compact, convex set.[16] Bidder j knows her v^j but not the valuations of the other bidders. Last, bidders and seller are assumed to be risk neutral.

The seller is interested in choosing an auction design that will do three things:

1. Induce bidders to reveal their actual valuations (**incentive compatibility**).

2. No bidder is made worse off (in expectation) by participating in the auction.

3. Subject to the two conditions above the seller maximizes expected revenue.

Call the auction design that satisfies these conditions optimal. Finding the optimal auction appears daunting given the cornucopia of elaborate schemes with multiple rounds that one can imagine. However the revelation principle (see Myerson (1981) for an explanation) allows one to concentrate on auctions with a single round where bidders submit bids and the seller decides, as a function of the reported bids, who gets what and for how much.

[15] In the language of mechanism design, this list of valuations becomes the bidders **type**. In this case, since the type is not a single number it is called multi-dimensional.

[16] A slightly different model replaces the convexity assumption with a differentiability condition on interim expected utilities. See Williams (1999) for details.

The optimal auction in this setting is still not known. However, if one is willing to accept an additional restriction, **efficiency**, then the optimal auction is known. An auction is efficient if the allocation of objects to bidders chosen by the seller solves the following:

$$\max \sum_{j \in N} \sum_{S \subseteq M} v^j(S) y(S,j)$$

$$\text{s.t.} \sum_{S \ni i} \sum_{j \in N} y(S,j) \leq 1 \ \forall i \in M$$

$$\sum_{S \subseteq M} y(S,j) \leq 1 \ \forall j \in N$$

$$y(S,j) = 0,1 \ \forall S \subseteq M, \ j \in N$$

Notice that this is just CAP2 with b^i replaced by v^i. The optimal objective function value of this integer program is an upper bound on the revenue that the seller can achieve. The fact the seller uses an auction that selects an allocation that solves this integer program does not imply that the seller achieves this revenue.[17]

It is proved in Krishna and Perry (1997) (see also Williams (1999) for the same result under slightly different assumptions) that (in the model described) the optimal auction that is also efficient is what is sometimes known as as Vickrey-Clarke-Groves (VCG) scheme. It works as follows:

1. Agent j reports v^j. There is nothing to prevent agent j from misrepresenting themselves. Since the VCG scheme is incentive compatible this does not happen.

2. The seller chooses the allocation that solves:

$$V = \max \sum_{j \in N} \sum_{S \subseteq M} v^j(S) y(S,j)$$

$$\text{s.t.} \sum_{S \ni i} \sum_{j \in N} y(S,j) \leq 1 \ \forall i \in M$$

$$\sum_{S \subseteq M} y(S,j) \leq 1 \ \forall j \in N$$

$$y(S,j) = 0,1 \ \forall S \subseteq M, \ j \in N$$

Call this optimal allocation y^*

[17] In Myerson (1981) it is shown that optimal auction for a single good is not guaranteed to be efficient.

3. To compute the payment that each bidder must make let, for each $k \in N$,

$$V^{-k} = \max \sum_{j \in N \setminus k} \sum_{S \subseteq M} v^j(S) y(S,j)$$

$$\text{s.t.} \sum_{S \ni i} \sum_{j \in N \setminus k} y(S,j) \leq 1 \ \forall i \in M$$

$$\sum_{S \subseteq M} y(S,j) \leq 1 \ \forall j \in N \setminus k$$

$$y(S,j) = 0, 1 \ \forall S \subseteq M, \ j \in N \setminus k$$

   Denote by y^k the optimal solution to this integer program. Thus y^k is the efficient allocation when bidder k is excluded.

4. The payment that bidder k makes is equal to

$$V^{-k} - [V - \sum_{S \subseteq M} v^k(S) y^*(S,k)].$$

   Thus bidder k's payment is the difference in 'welfare' of the other bidders without him and the welfare of others when he is included in the allocation. Notice that the payment made by each bidder to the auctioneer is non-negative. This is not true in all economic environments.

If a seller were to adopt the VCG scheme her total revenue would be

$$\sum_{k \in N} V^{-k} - \sum_{k \in N} [V - \sum_{S \subseteq M} v^k(S) y^*(S,k)]$$

$$= \sum_{k \in N} \sum_{S \subseteq N} v^k(S) y^*(S,k) + \sum_{k \in N} (V^{-k} - V)$$

$$= V + \sum_{k \in N} (V^{-k} - V).$$

If there were a large number of agents then no single agent can have a significant effect, i.e., one would expect that, on average, V is very close in value to V^{-k}. Thus the revenue to the seller would be close to V, the largest possible revenue that any auction could extract. To solidify this intuition we need that for all agents k that their valuation v^k is superadditive, i.e. $v^k(A) + v^k(B) \leq v^k(A \cup B)$ for all $k \in N$ and $A, B \subset M$ such that $A \cap B =$

$\emptyset$. With this assumption we can find the efficient allocation using CAP1 without the need to introduce dummy goods. Thus:

$$V = \max \sum_{S \subset M} \{\max_{j \in N} v^j(S)\} x_S$$

$$\text{s.t.} \quad \sum_{S \ni i} x_S \leq 1 \ \forall i \in M$$

$$x_S = 0, 1 \ \forall S \subset M$$

and

$$V^{-k} = \max \sum_{S \subset M} \{\max_{j \in N \setminus k} v^j(S)\} x_S$$

$$\text{s.t.} \quad \sum_{S \ni i} x_S \leq 1 \ \forall i \in M$$

$$x_S = 0, 1 \ \forall S \subset M$$

Notice now that if the number $|N|$ of bidders is large and given that the v^j's live in a compact set, the random variable $\max_{j \in N} v^j(S)$ is very close on average to $\max_{j \in N \setminus k} v^j(S)$.[18] Hence the objective function of the program that defines V is essentially the same as the objective function of the integer program that defines V^{-k}. This argument is made precise in Monderer and Tennenholtz (1999), where it is shown in the model used here that the VCG scheme generates a revenue for the seller that is asymptotically close to the revenue from the optimal auction.

The VCG scheme is clearly impractical to implement, if the number of bidders is very large. To overcome the computational difficulties one might replace y^* and y^k for all $k \in N$ with approximately optimal solutions. Such a modification in the scheme need not preserve incentive compatibility. In Lehmann *et. al.* (1999) such a direction is taken. They solve the embedded optimization problems using a greedy type algorithm and show that the resulting scheme is not incentive compatible. However if one is willing to restrict bidders valuations (drastically) it is possible to generate schemes based on the greedy algorithm that are incentive compatible. In Lehmann *et. al.* (1999) this restriction is called single mindedness. Each bidder values only one subset and no other.[19]

[18] In fact the difference of the two is essentially the difference between the first and second order statistic of a large collection of independent random numbers from a compact set.

[19] Another restriction that might be fruitful to explore is introduced in Eső and Maskin (1999) which they call partition preferences. The collection of subsets that a bidder assigns positive value to form a partition of the set of objects. Note that problem CAP is still NP-hard under this restriction.

Even if one is willing to relax incentive compatibility, an approximate solution to the underlying optimization problems in the VCG can lead to other problems. There can be many different solutions to an optimization problem whose objective function values are within a specified tolerance of the optimal objective function value. The payments specified by the VCG scheme are very sensitive to the choice of solution. Thus the choice of approximate solution can have a significant impact on the payments made by bidders. This issue is discussed in Johnson *et. al.* (1997) in the context of an electricity auction used to decide the scheduling of short term electricity needs. Through simulations they show that variations in near-optimal schedules that have negligible effect on total system cost can have significant consequences on the total payments by bidders.

Yet another way in which the computational difficulties can be side stepped is to ask for auction schemes that give **computationally bounded** bidders the incentive to truthfully reveal their valuations. Parkes and Ungar (2000b), for example attempts to do just this. They propose a method to adjust the final prices in *i*Bundle towards VCG prices. Under appropriate conditions on bidder behavior, the method allows *i*Bundle to inherit the strategy-proofness of the VCG scheme.

Experience with the VCG scheme in field settings is limited. Isaac and James (1998) report on an experiment using the VCG scheme for a combinatorial auction involving three bidders and two objects. On the basis of their results they argue that the VCG scheme can be operationalized and, in their words, 'achieve high allocative efficiency'. In Hobbs *et. al.* (2000) the possibility that the VCG scheme is vulnerable to collusion is explored. It is pointed out by these authors in environments with repeated interactions that not only are there many opportunities for collusion amongst bidders but incentive compatibility of the VCG scheme cannot be guaranteed.

5 Computational Experiments and Test Problems

To evaluate the strength of standard methods of combinatorial optimization we implemented a solver using the branch and bound solver [20] of the CPLEX 6.5 callable library (see ILOG (1997)). We wrote the (straight forward) program in C++. The running-times we report were obtained on an SGI O2 computer with one MIPS R10000 processor at 175MHz with 192MB of main memory running IRIX 6.5.5m. This machine is is slightly inferior to the one used in Sandholm (1999). As test cases we considered the same problems

[20] In fact at all nodes, clique-cuts might be added.

as studied in Sandholm (1999). They belong to four different classes of distributions:

Random: For each bid, pick the number of items randomly from $1, \ldots, m$. Randomly choose that many items without replacement. Pick the prices randomly from $[0, 1]$.

Weighted Random: As above, but pick the price as a real number between 0 and the number of items in the bid.

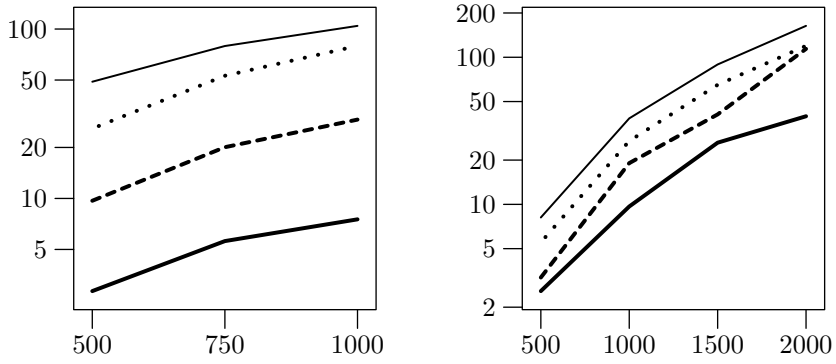
Uniform: Draw the same number of randomly chosen items for each bid. Pick the prices from $[0, 1]$.

Decay: Give the bid one random item. Then repeatedly add a new random item with probability α until an item is not added or the bid includes all m items. Pick the price as a real number between 0 and the number of items in the bid.

For these experiments we did not throw out duplicate sets. However for the given parameters duplicate sets occur with very small probability. No fine tuning of the parameters of `libcplex.a` was attempted. We used standard settings only. Our Figures 1, 2, and 3 show the runtimes for CAP1 instances from the same distribution as Sandholm (1999). In fact we use the same convention as Sandholm (1999) to summarize the data so that comparisons will be easier.

In Figure 1 we show the logarithm of the runtimes our program needed to solve instances from the random (left) and weighted random (right) distribution for different numbers of items and bids. In Figure 2 we show the logarithm of the runtimes our program needed to solve instances from the uniform distribution and Figure 3 shows the logarithms of runtimes for the decay distribution. A comparison with Sandholm's (1999) result graphs reveals that the runtimes with the CPLEX solver (on these instances) are at least 60 times faster for the random distribution (Figure 20), 10 times faster on the weighted random distribution (Figure 20). For the date in Figures 20 and 20 the CPLEX solver produces a speedup of at least two orders of magnitude. In Figure 3 we see at least an order of magnitude speedup.

While performing our experiments we became aware that Andersson *et. al.* (2000) had also used CPLEX on the Sandholm instances. They perform more extensive tests with similar results. Furthermore, they use CPLEX on some of Fujishima *et. al.*'s (1999) test problems. For the Binomial class of problems (described below) they report that for small problems



(a) Random distribution

(b) Weighted random distribution

Figure 1: The abscissa depicts the number of bids and the ordinate depicts the running time (on a logarithmic scale). The lines show the cases for 100 items (—), 200 items (– –), 300 items (· · ·), and 400 items (—).

Fujishma's *et. al.* times are better while CPLEX was able to solve substantially larger problems than Fujishima *et. al.*

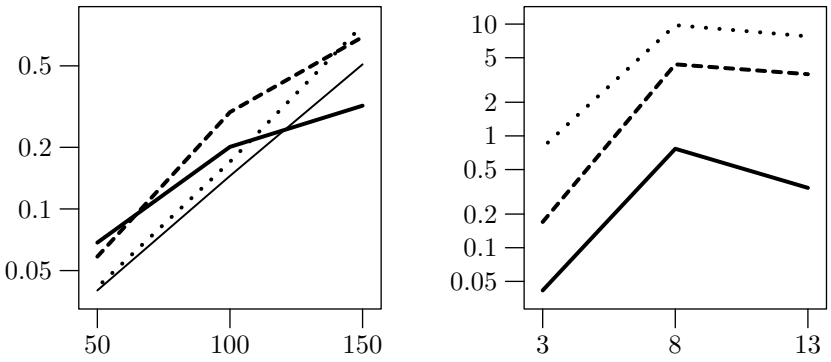
5.1 Test Problems

Performance of an algorithm for CAP1 will depend on the class of test problems chosen. Thus it is important to know whether good performance is a function of algorithm design or 'easy' problems.

As is pointed out in Andersson *et. al.* (2000), the random model in Sandholm (1999) is 'easy' when the problems become large. This is because the bid values assigned to each set are independent draws from the same distribution. Thus many small sets, in fact singletons, when combined are likely to generate a higher total revenue than a large set that contains the same elements.

The weighted random model suffers from the fact that on average the bid to column sum ratio is constant across columns. This implies that the model does not capture synergies among the items in a bidset. Further, for large enough problems, the optimal linear programming solution to the problem may be integral. In the uniform model bids have no relation to the size of subsets.

The binomial model of Fujishima *et. al.* (1999) can be described by



(a) Uniform distribution, 3 items per bid

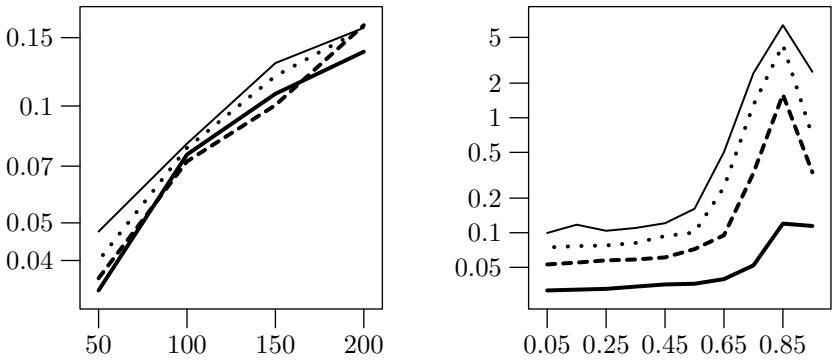
(b) Uniform distribution, 75 items

Figure 2: Left picture shows running times (on logarithmic scale) for 25 items (—), 50 items (- - -), 75 items (· · ·), and 100 items (—); the abscissa gives the number of bids. Right picture shows running times (on logarithmic scale) for 50 bids (—), 100 bids (- - -), and 150 bids (· · ·); the abscissa gives the number of items per bid.

saying that each entry of the constraint matrix of CAP1 is set to one with a given probability p independently of the others. The bid value of a subset/column is a uniform draw from a distribution whose mean depends on the size of the corresponding subset. Given the report of Andersson *et. al.* this class of problems may be of interest.

None of these schemes is based on a model of how bidders might value different subsets of objects. For example, each assumes that $b(S)$, the highest bid on the subset S, is a random draw from a uniform distribution. But, $b(S) = \max_{j \in N} b^j(S)$, i.e. it is the maximum of a collection of numbers. If we assume that each bidder's bid on S is an independent draw from some distribution then it is hard to imagine that $b(S)$ will be uniformly distributed.

Out of curiosity we generated some test problems involving 3 bidders. each bidders value for a single item was an independent uniform draw from $[0, 1]$. Let v_k^j be the value that bidder j assigns to object k. To capture complementarities, we assumed the value that bidder j assigns to a subset S is $\sum_{k \in S} v_k^j + \sum_{k,q} v_k^j v_q^k$. Hence, assuming truthful bidding, the highest



(a) Decay distribution, $\alpha = 0.55$

(b) Decay distribution, 100 items

Figure 3: Left picture shows running times (on logarithmic scale) for 50 items (—), 100 items (– –), 150 items (· · ·), and 200 items (—); the abscissa gives the number of bids. Right picture shows running times (on logarithmic scale) for 50 bids (—), 100 bids (– –), 150 bids (· · ·), and 200 bids (—); the abscissa gives the parameter α of the distribution.

bid on subset S is

$$\max_{1\leq j\leq 3} \sum_{k\in S} v_k^j + \frac{1}{2} \sum_{k\neq q} v_k^j v_q^j.$$

In this experiment we allowed only subsets of size 3 to be bid upon. Figure 4 reports on the solution times for various numbers of bids on instances involving 100 objects and uniform sets of size 3. There, the times for prices according to Sandholms uniform model and our just outlined model (for three bidders) are compared.

It is not clear from the data that the quadratic prices model is easier or harder to solve than the uniform model. To see how large a problem one could solve in the quadratic price model we solved an instance involving 100 items, 3000 bids (each on subsets of size 10) and 3 bidders. On an SGI Origin 200 computer[21] it took 410 minutes.

Motivated by the FCC spectrum auctions (see Ausubel *et. al.* (1997)) we outline a model, called the Quadratic Model (QM), of how bidders value different combinations of objects. In the spectrum auctions it was gener-

[21] With a four MIPS R10000 processor at 225MHz, 1GB of main memory that run using parallel CPLEX6.5.2.

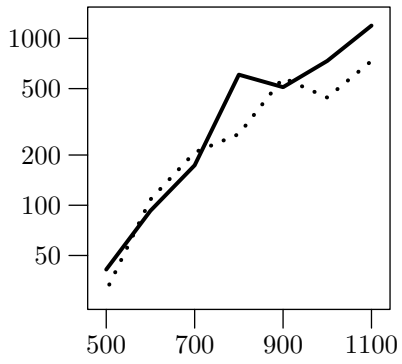


Figure 4: The abscissa depicts the number of bids and the ordinate depicts the mean running time (for 20 examples) (on a logarithmic scale) for 100 objects and uniform bidsets of three items. The lines show the cases for Sandholm's prices in the uniform case (/) and for our quadratic prices for three bidders ($\cdot\cdot$).

ally believed that the stand-alone value of a spectrum licence/object was proportional to the population covered by that spectrum licence. To model this we assign to each object k a value v_k which will be an iid draw from a distribution. This is the stand alone value of object k, which is common to all bidders.[22]

Second, as suggested in Ausubel *et. al.* (1997) if a bidder was interested in owning two spectrum licenses, the value of owning the two spectrum licences was proportional to the sum of the populations plus the products of the populations covered. The product term captures the synergies to be had from owning a pair of licences. However, these synergies were bidder specific. We model this feature in two steps.

First for each bidder j we choose a subset M^j of objects at random. The set M^j represents the set of objects that in bidder j's eyes complement each other. There are many ways to specify how the M^j can be selected. In some schemes the resulting sets may not overlap very much. In others they may be small, or mostly large or run the gamut from large to small. Second, the value to bidder j of the subset S of objects will be $\sum_{k\in S} v_k + \mu^j \sum_{k,q\in S\cap M^j} v_k v_q$. Here μ^j is a bidder specific parameter that allows one to vary the strength of the complementarities.[23] In this model, the largest

[22] In a more elaborate version we could allow each bidders value of object k to be drawn from a distribution centered at p_k.

[23] The parameters μ^k should not be set too large since this will bias a solution in favor

bid (assuming truthful bidding) on subset S would be

$$\max_{j \in N} \sum_{k \in S} v_k + \mu^j \sum_{k,q \in S \cap M^j} v_k v_q.$$

The QM model is by no means universally appropriate. In the transportation setting for example, it would make no sense whatsoever. The value of a set of lanes would depend on the transportation costs incurred to service those lanes. Complementarities come from combining different lanes together to reduce travel costs. These complementarities will be bidder specific in that they may already be servicing some collection of lanes. It is not hard to propose an alternative model for this set up as well. We do not do so here. Our goal is only to suggest that the specific application should be used to guide the model for generating test problems.

5.2 FCC Data

Perhaps the most well known combinatorial auction to be run to date is the FCC spectrum auction. Data from this auction is available from `ftp://ftp.fcc.gov/pub/Auctions/PCS/Broadband/MTA/Programs/`. Since the auction was run in rounds we have information on the bids submitted by each bidder on each licence. From this we can infer something about the subsets that each bidder had an interest in. However one can deduce very little about what bids they might have submitted on a package. In fact in some preliminary stabs we assigned bids to subsets in various ways (using the recorded bids as a a starting point) to generate instances of CAP1. All of these instances solved immediately using CPLEX but this is probably a function of how we assigned bid values.

Acknowledgements

We thank David Parkes, Sunil Chopra, Michael Rothkopf and John Ledyard for useful comments.

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