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Factorization and Decomposition of Relations

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Discussion Paper No. 1211

Factorization and Decomposition of Relations

by

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March 1998

Factorization and Decomposition of Relations

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Abstract

The paper extends the factor and decomposition theorems for functions to the category *Rel* of sets and relations. The converse theorems are also stated and proved. We show that the (retraction, isomorphism, section)-decomposable relations form a closure system on *Mor(Rel)*, and characterize it in terms of a closure operator.

1 Introduction

Let (A, \mathcal{E}) be a relational structure consisting of a set A and an equivalence \mathcal{E} on A . It is well known that a function $f : A \rightarrow B$ decomposes into a canonical surjection $\nu_A : A \rightarrow A/(f^{-1} \circ f)$, bijection $\psi_f : A/(f^{-1} \circ f) \rightarrow \text{im } f$, and an injection (inclusion) $\iota : \text{im } f \rightarrow B$. The celebrated Homomorphism Theorems of E. Nöther (van der Waerden [3], Hungerford [2]) provide the corresponding decompositions for sets with operators (*e.g.*, Cohn [1]).

Recall that in the category **Set** of sets and functions, injections are sections and surjections are retractions. If R, S , and I are, respectively, the classes of retractions, sections, and isomorphisms, then $\nu_A \circ \psi_f \circ \iota$ is the essentially unique (R, I, S) -decomposition of an arbitrary morphism f of **Set**. In the present paper we investigate the (R, I, S) -decomposability in a larger category **Rel** of sets and relations. Although **Rel** is not (R, I, S) -decomposable as a category, we identify the class of those morphisms which do have such a decomposition. We find, furthermore, that this class is a closure system, and characterize it by the associated closure operator on $\text{Mor}(\mathbf{Rel})$. This development allows us to formulate and prove the respective converse statements for the factorization and decomposition theorems.

2 The Closure of a Relation

Recall that in the category **Rel**, the objects are sets and morphisms the triples of sets (A, \mathcal{R}, B) such that $\mathcal{R} \subseteq A \times B$. Composability of morphisms is defined as in **Set**, and the domain and codomain functions $\text{dom}, \text{codom} : \text{Mor}(\mathbf{Rel}) \rightarrow \text{Ob}(\mathbf{Rel})$ are defined by $\text{dom}(A, \mathcal{R}, B) = A$ and $\text{codom}(A, \mathcal{R}, B) = B$. For each $S \in \text{Ob}(\mathbf{Rel})$, the morphism (S, id_S, S) with the diagonal id_S of $S \times S$ is the identity on S . The composition of morphisms is defined in a natural way in terms of the composition of the corresponding relations. Since, furthermore, $\text{hom}(A, B) = \mathcal{P}(A \times B)$ is a set, the smallness of the morphism class condition is satisfied, and **Rel** is indeed a category.

Some special morphisms in **Rel** are natural generalizations from their counterparts in **Set**. Below, we refer to the image of the inverse of a relation as the coimage of that relation:

Definition 2.1 *We shall say that a relation \mathcal{R} is:*

- (i) *A correspondence if $\text{coim } \mathcal{R} = \text{dom } \mathcal{R}$.*
- (ii) *A partial function if for all $x \in \text{dom } \mathcal{R}$, $\text{card } \mathcal{R}[x] \leq 1$.*

And dually, \mathcal{R} is:

- (i) *A surjection if $\text{im } \mathcal{R} = \text{codom } \mathcal{R}$.*
- (ii) *An injection if for all $x \in \text{codom } \mathcal{R}$, $\text{card } \mathcal{R}^{-1}[x] \leq 1$.*

We shall further say that \mathcal{R} is a bijection if it is both surjective and injective. And dually, \mathcal{R} is a function if it is both a partial function and a correspondence.

Single-valuedness of a relation is, thus, in duality with bijectivity. The bijective functions are the only isomorphisms of \mathbf{Rel} , as follows from the next easily verifiable observation.

Proposition 2.2 *A morphism (A, \mathcal{R}, B) is a section in \mathbf{Rel} if and only if it is an injective correspondence. In this case \mathcal{R}^{-1} is a left inverse of \mathcal{R} :*

$$\mathcal{R}^{-1} \circ \mathcal{R} = \text{id}_{\text{dom } \mathcal{R}}. \quad (2.1)$$

Dually, a morphism (A, \mathcal{R}, B) is a retraction in \mathbf{Rel} if and only if it is a surjective partial function. In this case \mathcal{R}^{-1} is a right inverse of \mathcal{R} :

$$\mathcal{R} \circ \mathcal{R}^{-1} = \text{id}_{\text{codom } \mathcal{R}}. \quad (2.2)$$

Each relation \mathcal{R} gives rise to an equivalence generated on its coimage by $\mathcal{R}^{-1} \circ \mathcal{R}$; that is, there exists a mapping

$$E = \begin{cases} \text{Mor}(\mathbf{Rel}) & \rightarrow \text{Mor}(\mathbf{Rel}) \\ \mathcal{R} & \mapsto \bigcup_{n \in \mathbf{N}} (\mathcal{R}^{-1} \circ \mathcal{R})^n. \end{cases} \quad (2.3)$$

Generalizing from the terminology for functions, we shall refer to $\ker \mathcal{R} = \mathcal{R}^{-1} \circ \mathcal{R}$ as the *kernel* of the relation \mathcal{R} . If \mathcal{R} is a partial function, $E_{\mathcal{R}} = \ker \mathcal{R}$, since by (2.1) the kernel is an equivalence.

The mapping (2.3) allows us to define the closure operator $\langle \cdot \rangle$ on the class of relations (Definition 2.4 below). The proof of the closure property is based on two lemmata.

Lemma 2.3 *For any relation \mathcal{R} ,*

$$\mathcal{R} \circ E_{\mathcal{R}} \circ \mathcal{R}^{-1} = \text{Hom}_{\text{Rel}}(\mathcal{R}^{-1}, \mathcal{R})(E_{\mathcal{R}}) = E_{\mathcal{R}^{-1}}. \quad (2.4)$$

Also, if $f_A : A \rightarrow \text{coim } \mathcal{R}$ and $f_B : B \rightarrow \text{im } \mathcal{R}$ are retractions, then

$$\text{Hom}_{\text{Rel}}(f_A, f_A^{-1})(E_{\mathcal{R}}) = E\text{Hom}_{\text{Rel}}(f_A, f_B^{-1})(\mathcal{R}). \quad (2.5)$$

Proof. The identity (2.4) follows directly from definition (2.3) and the associativity of composition: $\mathcal{R} \circ E_{\mathcal{R}} \circ \mathcal{R}^{-1} = \mathcal{R} \circ \bigcup_{n \in \mathbf{N}} (\mathcal{R}^{-1} \circ \mathcal{R})^n \circ \mathcal{R}^{-1} = \bigcup_{n \in \mathbf{N}} (\mathcal{R} \circ \mathcal{R}^{-1})^{n+1} = E_{\mathcal{R}^{-1}}$, since the sequence $\{(\ker \mathcal{R})^n\}_{n \in \mathbf{N}}$ is monotone increasing in view of the reflexivity of $\ker \mathcal{R}$.

In order to prove (2.5), we may utilize (2.2) since f_A and f_B are retractions:

$$E_{\text{Hom}_{\text{Rel}}(f_A, f_B^{-1})(\mathcal{R})} = \bigcup_{n \in \mathbf{N}} (f_A^{-1} \circ \mathcal{R}^{-1} \circ f_B \circ f_B^{-1} \circ \mathcal{R} \circ f_A)^n = f_A^{-1} \circ \left(\bigcup_{n \in \mathbf{N}} (\ker \mathcal{R} \Big|_{\text{im } f_A}^{\text{im } f_B})^n \right) \circ f_A,$$

and (2.5) follows from the surjectivity of f_A and f_B guaranteed by Proposition 2.2.

Q.E.D.

A weaker than in Lemma 2.3 requirement, namely, that only f_B be a partial surjective function, permits us to conclude that

$$\text{Hom}_{\text{Rel}}(f_A, f_A^{-1})(\ker \mathcal{R}) = \ker \text{Hom}_{\text{Rel}}(f_A, f_B^{-1})(\mathcal{R}). \quad (2.6)$$

By composing the identity (2.4) with \mathcal{R} on the right, we arrive at the following **Proposition and Definition 2.4** For any \mathcal{R} , the diagram

$$\begin{array}{ccc} \text{dom } \mathcal{R} & \xrightarrow{\quad \mathcal{R} \quad} & \text{codom } \mathcal{R} \\ \downarrow E_{\mathcal{R}} & & \downarrow E_{\mathcal{R}^{-1}} \\ \text{dom } \mathcal{R} & \xrightarrow{\quad \mathcal{R} \quad} & \text{codom } \mathcal{R} \end{array}$$

is commutative. We shall say that the diagonal of the diagram,

$$\langle \mathcal{R} \rangle = \mathcal{R} \circ E_{\mathcal{R}} = E_{\mathcal{R}^{-1}} \circ \mathcal{R}, \quad (2.7)$$

is the closure of \mathcal{R} .

The name of $\langle \cdot \rangle$ is justified by Proposition 2.6 below, in the proof of which we utilize **Lemma 2.5** For any relation \mathcal{R} :

$$E_{\langle \mathcal{R} \rangle} = E_{\mathcal{R}} \quad (2.8)$$

$$\langle \mathcal{R} \rangle^{-1} = \langle \mathcal{R}^{-1} \rangle. \quad (2.9)$$

Also, if $f_A : A \rightarrow \text{coim } \mathcal{R}$ and $f_B : B \rightarrow \text{im } \mathcal{R}$ are retractions, then

$$\langle \text{Hom}_{\text{Rel}}(f_A, f_B^{-1})(\mathcal{R}) \rangle = \text{Hom}_{\text{Rel}}(f_A, f_B^{-1})(\langle \mathcal{R} \rangle). \quad (2.10)$$

Proof. From definitions (2.3) and (2.7), we obtain $E_{\langle \mathcal{R} \rangle} = \bigcup_{n \in \mathbf{N}} (E_{\mathcal{R}} \circ (\ker \mathcal{R}) \circ E_{\mathcal{R}})^n = E_{\mathcal{R}}$, since $E_{\mathcal{R}} \circ (\ker \mathcal{R}) = E_{\mathcal{R}}$, and $E_{\mathcal{R}}$ is idempotent; hence (2.8). Further, the unary operations $\langle \cdot \rangle$ and $(\cdot)^{-1}$ commute:

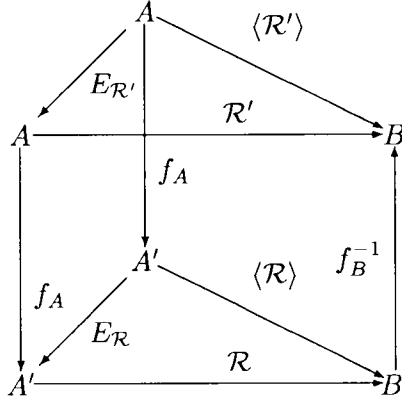
$$\langle \mathcal{R} \rangle^{-1} = (\mathcal{R} \circ E_{\mathcal{R}})^{-1} = E_{\mathcal{R}} \circ \mathcal{R}^{-1} = \mathcal{R}^{-1} \circ E_{\mathcal{R}^{-1}} = \langle \mathcal{R}^{-1} \rangle.$$

Finally,

$$\langle \text{Hom}_{\text{Rel}}(f_A, f_B^{-1})(\mathcal{R}) \rangle = \text{Hom}_{\text{Rel}}(f_A, f_B^{-1})(\mathcal{R}) \circ E_{\text{Hom}_{\text{Rel}}(f_A, f_B^{-1})(\mathcal{R})} = (f_B^{-1} \circ \mathcal{R} \circ f_A) \circ (f_A^{-1} \circ E_{\mathcal{R}} \circ f_A)$$

by (2.5) of Lemma 2.3. The last composition is equal to $f_B^{-1} \circ \langle \mathcal{R} \rangle \circ f_A$ since f_A is, by assumption, a retraction, and (2.10) follows at once. Q.E.D.

Lemmata 2.3 and 2.5 may be summarized as the commutativity of the diagram



in which $\mathcal{R}' = \text{Hom}_{\text{Rel}}(f_A, f_B^{-1})(\mathcal{R})$. We are in a position to state and prove

Proposition 2.6 *Let $\langle \cdot \rangle: \text{Mor}(\mathbf{Rel}) \rightarrow \text{Mor}(\mathbf{Rel})$ be the map, the action of which is defined by (2.7). Then $\langle \cdot \rangle$ is a closure operator on the Boolean $(\text{Mor}(\mathbf{Rel}), \subseteq)$; that is, for all $\mathcal{R}, \mathcal{S} \in \text{Mor}(\mathbf{Rel})$,*

$$\mathcal{R} \subseteq \mathcal{S} \implies \langle \mathcal{R} \rangle \subseteq \langle \mathcal{S} \rangle \quad (2.11)$$

$$\mathcal{R} \subseteq \langle \mathcal{R} \rangle \quad (2.12)$$

$$\langle \langle \mathcal{R} \rangle \rangle = \langle \mathcal{R} \rangle. \quad (2.13)$$

Proof. The monotonicity of the map E implies that, whenever relations \mathcal{R} and \mathcal{S} are such that $\mathcal{R} \subseteq \mathcal{S}$, we have $\langle \mathcal{R} \rangle = \mathcal{R} \circ E_{\mathcal{R}} \subseteq \mathcal{S} \circ E_{\mathcal{S}} = \langle \mathcal{S} \rangle$, hence the monotonicity (2.11) of $\langle \cdot \rangle$. The reflexivity of the equivalence $E_{\mathcal{R}}$ on $\text{coim } \mathcal{R}$ implies (2.12): $\langle \mathcal{R} \rangle = \mathcal{R} \circ E_{\mathcal{R}} \supseteq \mathcal{R} \circ \text{id}_{\text{coim } \mathcal{R}}$. Finally, (2.13) is a consequence of (2.8) and idempotency of equivalences: $\langle \langle \mathcal{R} \rangle \rangle = (\mathcal{R} \circ E_{\mathcal{R}}) \circ E_{\langle \mathcal{R} \rangle} = (\mathcal{R} \circ E_{\mathcal{R}}) \circ E_{\mathcal{R}} = \langle \mathcal{R} \rangle$. Q.E.D.

The commutativity of $\langle \cdot \rangle$ and $(\cdot)^{-1}$ (Lemma 2.5) implies that a relation is $\langle \cdot \rangle$ -closed if and only if its inverse is. Furthermore, whenever $\ker \mathcal{R}$ is an equivalence, $\langle \mathcal{R} \rangle = \mathcal{R} \circ \ker \mathcal{R} = \mathcal{R}$. This observation and its dual prove the following

Corollary 2.7 *For a relation \mathcal{R} to be closed it is sufficient that either (or both) $\ker \mathcal{R}$ or $\ker \mathcal{R}^{-1}$ be an equivalence.*

Thus, all equivalences, partial functions and injections are closed. In addition, identity (2.10) implies that $\text{Hom}_{\text{Rel}}(f_A, f_B^{-1})(\mathcal{R})$ is closed if and only if the relation \mathcal{R} is.

3 Factorizations

In the category \mathbf{Set} , the requirement that the kernel of a function f include as its subset an equivalence \mathcal{E} on $\text{dom } f$ is sufficient for f to be uniquely factorizable through the canonical

surjection $\nu_{\mathcal{E}} : \text{dom } f \rightarrow \text{dom } f/\mathcal{E}$. The following characterization theorem extends this result to the category **Rel** and, in addition, provides its converse.

Theorem 3.1 [Factor Theorem for Relations] *Let \mathcal{R} be a relation and \mathcal{E} an equivalence on $\text{coim } \mathcal{R}$ with the canonical surjection $\nu_{\mathcal{E}} : \text{coim } \mathcal{R} \rightarrow \text{coim } \mathcal{R}/\mathcal{E}$. Then a decomposition $\mathcal{R} = \hat{\mathcal{R}} \circ \nu_{\mathcal{E}}$ exists if and only if*

$$\mathcal{R} \circ \mathcal{E} \subseteq \mathcal{R}, \quad (3.1)$$

in which case $\hat{\mathcal{R}}$ is unique and equal to $\mathcal{R} \circ \nu_{\mathcal{E}}^{-1}$.

Proof. Suppose first that $\hat{\mathcal{R}}$ satisfying $\hat{\mathcal{R}} \circ \nu_{\mathcal{E}} = \mathcal{R}$ exists. Since the canonical surjection $\nu_{\mathcal{E}}$ is a retraction (Proposition 2.2), this yields a unique $\hat{\mathcal{R}} = \mathcal{R} \circ \nu_{\mathcal{E}}^{-1}$. A composition of $\hat{\mathcal{R}}$ with $\nu_{\mathcal{E}}$ on the right results in $\mathcal{R} = \hat{\mathcal{R}} \circ \nu_{\mathcal{E}} = \mathcal{R} \circ \nu_{\mathcal{E}}^{-1} \circ \nu_{\mathcal{E}} = \mathcal{R} \circ \mathcal{E}$.

Conversely, if $\mathcal{R} \subseteq \mathcal{R} \circ \mathcal{E}$, then $\mathcal{R} = \mathcal{R} \circ \mathcal{E}$ by the reflexivity of \mathcal{E} , and $\hat{\mathcal{R}} = \mathcal{R} \circ \nu_{\mathcal{E}}^{-1}$ provides the stated decomposition of \mathcal{R} . Q.E.D.

The next proposition presents upper bounds on $\ker \hat{\mathcal{R}}$ and $\ker \hat{\mathcal{R}}^{-1}$ for the factor $\hat{\mathcal{R}}$ in the decomposition $\mathcal{R} = \hat{\mathcal{R}} \circ \nu_{\mathcal{E}}$ of a relation \mathcal{R} . It utilizes the canonical ‘‘insertion’’ $\rho_{\mathcal{E}} : \text{coim } \mathcal{R}/\mathcal{E} \rightarrow \text{coim } \mathcal{R}/(\mathcal{E} \vee E_{\mathcal{R}})$ defined, more specifically, as follows. Let \mathcal{E}_1 and \mathcal{E}_2 be two equivalences on a set S with canonical identification maps $\nu_{\mathcal{E}_1}$ and $\nu_{\mathcal{E}_2}$, respectively. If $\mathcal{E}_1 \subseteq \mathcal{E}_2$, then $\rho_{12} = \nu_{\mathcal{E}_2} \circ \nu_{\mathcal{E}_1}^{-1}$ is a unique function that assigns to each equivalence class α of \mathcal{E}_1 the unique equivalence class of \mathcal{E}_2 that includes α as a subset (the existence and uniqueness of ρ_{12} follow immediately from Theorem 3.1 specialized to $\mathcal{R} = \nu_{\mathcal{E}}$, the conditions of which hold since functions, by Corollary 2.7, are closed; the fact that $\nu_{\mathcal{E}_i}$ is canonical for \mathcal{E}_i , $i = 1, 2$, ensures that ρ_{12} is a function). The kernel of the map $\rho_{\mathcal{E}} = \nu_{\mathcal{E} \vee E_{\mathcal{R}}} \circ \nu_{\mathcal{E}}^{-1}$ that inserts the equivalence classes of $\text{coim } \mathcal{R}/\mathcal{E}$ into the classes of $\text{coim } \mathcal{R}/(\mathcal{E} \vee E_{\mathcal{R}})$ provides an upper bound on the $\ker \hat{\mathcal{R}}$:

Proposition 3.2 *Let \mathcal{R} be a relation and \mathcal{E} an equivalence on $\text{coim } \mathcal{R}$ with the canonical surjection $\nu_{\mathcal{E}} : \text{coim } \mathcal{R} \rightarrow \text{coim } \mathcal{R}/\mathcal{E}$. If $\mathcal{R} = \hat{\mathcal{R}} \circ \nu_{\mathcal{E}}$ for some $\hat{\mathcal{R}}$, then*

$$\ker \hat{\mathcal{R}} \subseteq \ker \rho_{\mathcal{E}} \quad (3.2)$$

$$\ker \hat{\mathcal{R}}^{-1} \subseteq \ker \mathcal{R}^{-1}. \quad (3.3)$$

In particular, $\hat{\mathcal{R}}$ is: (i) injective if and only if $\mathcal{E} \supseteq E_{\mathcal{R}}$; and, (ii) a partial function if and only if \mathcal{R} is a partial function.

Proof. Let $(\alpha', \alpha'') \in \ker \hat{\mathcal{R}} = \nu_{\mathcal{E}} \circ \ker \mathcal{R} \circ \nu_{\mathcal{E}}^{-1}$ (if $\mathcal{R} = \emptyset$, what follows is vacuously true), which implies that there exist $a' \in \alpha'$ and $a'' \in \alpha''$ such that $(a', a'') \in \ker \mathcal{R} \subseteq E_{\mathcal{R}}$, or $a' \sim_{E_{\mathcal{R}}} a''$. Thus, both α' and α'' intersect the same equivalence class of $E_{\mathcal{R}}$. They both and the intersected class of $E_{\mathcal{R}}$ are subsets, therefore, of the same equivalence class of $\mathcal{E} \vee E_{\mathcal{R}}$: $\rho_{\mathcal{E}}(\alpha') = \rho_{\mathcal{E}}(\alpha'')$, or $(\alpha', \alpha'') \in \ker \rho_{\mathcal{E}}$. Since the pair $(\alpha', \alpha'') \in \ker \hat{\mathcal{R}}$ is arbitrary, we conclude that (3.2) holds.

Suppose further that $E_{\mathcal{R}} \subseteq \mathcal{E}$. Then $\mathcal{E} \vee E_{\mathcal{R}} = \mathcal{E}$, and $\rho_{\mathcal{E}}$ is the identity on $\text{coim } \mathcal{R}/\mathcal{E}$. In this case, the property (3.2) reads $\ker \hat{\mathcal{R}} \subseteq \text{id}_{\text{coim } \mathcal{R}/\mathcal{E}}$ and implies that $\hat{\mathcal{R}}$ is injective, as stated.

Conversely, if $\hat{\mathcal{R}}$ is injective, then $id_{coim \mathcal{R}/\mathcal{E}} = \ker \hat{\mathcal{R}} = \nu_{\mathcal{E}} \circ \ker \mathcal{R} \circ \nu_{\mathcal{E}}^{-1}$. That is, whenever $\alpha', \alpha'' \in coim \mathcal{R}/\mathcal{E}$ are such that there exist $a' \in \alpha'$ and $a'' \in \alpha''$ and $a'(\ker \mathcal{R})a''$, then $\alpha' = \alpha''$. This may take place if and only if, for each $a \in coim \mathcal{E}$, $\ker \mathcal{R}[a] \subseteq \mathcal{E}[a]$ and, consequently, $E_{\mathcal{R}} \subseteq \mathcal{E}$.

The kernel $\ker \hat{\mathcal{R}}^{-1} = \mathcal{R} \circ \nu_{\mathcal{E}}^{-1} \circ \nu_{\mathcal{E}} \circ \mathcal{R}^{-1} = \ker \mathcal{R}^{-1}$ since $\ker \nu_{\mathcal{E}} = \mathcal{E}$ and, by Theorem 3.1, the condition (3.1) holds. In particular, the relations \mathcal{R} and $\hat{\mathcal{R}}$ are single- or multi-valued simultaneously. Q.E.D.

The condition (3.1), which in view of the reflexivity of \mathcal{E} is equivalent to $\mathcal{R} \circ \mathcal{E} = \mathcal{R}$, may be restated to say that the inverse image sets of \mathcal{R} are unions of equivalence classes of \mathcal{E} . This, of course, is also true for all positive powers of $\ker \mathcal{R}$. As a consequence, $E_{\mathcal{R}} \supseteq \mathcal{E}$ is a necessary condition for \mathcal{R} to factor through a surjective function. The case when \mathcal{R} is a function is special in that this condition is also sufficient:

Corollary 3.3 [Factor Theorem for Functions] *Let $f : A \rightarrow B$ be a function and \mathcal{E} an equivalence on A with the canonical surjection $\nu_{\mathcal{E}} : A \rightarrow A/\mathcal{E}$. Then a decomposition $f = \hat{f} \circ \nu_{\mathcal{E}}$ exists if and only if $\ker f \supseteq \mathcal{E}$, in which case \hat{f} is unique and is a function.*

Proof. Suppose that the factorization exists. Then, by Theorem 3.1, the condition (3.1) holds, and consequently, $E_f = \ker f \supseteq \mathcal{E}$. By Proposition 3.2, \hat{f} is a function. Conversely, suppose that $E_f = \ker f \supseteq \mathcal{E}$. The composition with f^{-1} on the right and inversion then yield $f \supseteq f \circ \mathcal{E}$ — the condition (3.1) of Theorem 3.1. Hence, f may be factored as $\hat{f} \circ \nu_{\mathcal{E}}$. Q.E.D.

The sufficiency part of the preceding corollary is well known. The corollary indicates, in addition, that the condition (3.1) is also necessary: even if one is willing to forego the requirement that \hat{f} be single-valued, the factorization $f = \hat{f} \circ \nu_{\mathcal{E}}$ is not possible unless (3.1) is satisfied.

4 (R, I, S) -Decompositions

Turning to (R, I, S) -decompositions in **Rel**, we observe that Theorem 3.1 and Proposition 3.2 allow us to concentrate only on relations that are closed:

Theorem 4.1 *Let \mathcal{R} be a relation and \mathcal{E} an equivalence on $coim \mathcal{R}$ with the canonical surjection $\nu_{\mathcal{E}} : coim \mathcal{R} \rightarrow coim \mathcal{R}/\mathcal{E}$. Then a decomposition $\mathcal{R} = \hat{\mathcal{R}} \circ \nu_{\mathcal{E}}$ in which $\hat{\mathcal{R}}$ is injective exists if and only if \mathcal{R} is closed, i.e., $\mathcal{R} \circ E_{\mathcal{R}} = \mathcal{R}$.*

Under the conditions of the preceding characterization theorem, $\hat{\mathcal{R}}$ is closed and, in view of Lemma 2.5, its inverse is closed as well. The duality considerations suggest, then, the (injective) inverse of $\mu_{\mathcal{R}^{-1}} : im \mathcal{R} \rightarrow im \mathcal{R}/E_{\mathcal{R}^{-1}}$ as a natural candidate for $\hat{\mathcal{R}}$ in the decomposition of \mathcal{R} .

Theorem 4.2 [Decomposition of Relations] *A relation \mathcal{R} is (R, I, S) -decomposable if and only if it is closed. In this case $\mathcal{R} = \mu_{\mathcal{R}^{-1}}^{-1} \circ \psi_{\mathcal{R}} \circ \mu_{\mathcal{R}}$, where $\mu_{\mathcal{R}} : coim \mathcal{R} \rightarrow coim \mathcal{R}/E_{\mathcal{R}}$ and $\mu_{\mathcal{R}^{-1}} : im \mathcal{R} \rightarrow im \mathcal{R}/E_{\mathcal{R}^{-1}}$ are canonical surjections, and $\psi_{\mathcal{R}} : coim \mathcal{R}/E_{\mathcal{R}} \rightarrow im \mathcal{R}/E_{\mathcal{R}^{-1}}$ is a unique isomorphism.*

Proof. Suppose that \mathcal{R} is (R, I, S) -decomposable: $\mathcal{R} = \nu_{\mathcal{F}}^{-1} \circ \psi_{\mathcal{R}} \circ \nu_{\mathcal{E}}$ where $\nu_{\mathcal{E}}, \nu_{\mathcal{F}}$ are surjective functions, and f a bijection. Then $\hat{\mathcal{R}}$ is injective and, by Theorem 4.1, \mathcal{R} is closed.

Conversely, let \mathcal{R} be closed. Then, also by Theorem 4.1, \mathcal{R} is factorizable: $\mathcal{R} = \hat{\mathcal{R}} \circ \mu_{\mathcal{R}}$ for some $\hat{\mathcal{R}}$. Since $\hat{\mathcal{R}}$ is injective, both \mathcal{R} and its inverse are closed (Corollary 2.7). Hence, the conditions of the preceding theorem hold also for $\hat{\mathcal{R}}^{-1}$. Applying the dual restatement of Theorem 3.1 to $\hat{\mathcal{R}}^{-1}$, we conclude that there exists a relation $\psi_{\hat{\mathcal{R}}}^{-1}$ such that $\hat{\mathcal{R}}^{-1} = \psi_{\hat{\mathcal{R}}}^{-1} \circ \mu_{\hat{\mathcal{R}}^{-1}}$ and, consequently, $\mathcal{R} = \mu_{\hat{\mathcal{R}}^{-1}}^{-1} \circ \psi_{\hat{\mathcal{R}}} \circ \mu_{\mathcal{R}}$.

Further, $\psi_{\mathcal{R}}$ is unique, as may be deduced by composing $\mathcal{R} = \mu_{\hat{\mathcal{R}}^{-1}}^{-1} \circ \psi_{\hat{\mathcal{R}}} \circ \mu_{\mathcal{R}}$ with $\mu_{\mathcal{R}^{-1}}$ on the left and $\mu_{\mathcal{R}}^{-1}$ on the right to yield

$$\psi_{\mathcal{R}} = \mu_{\mathcal{R}^{-1}} \circ \mathcal{R} \circ \mu_{\mathcal{R}}^{-1}. \quad (4.1)$$

Finally, the above factorizations guarantee that both $\psi_{\mathcal{R}}$ and $\psi_{\mathcal{R}}^{-1}$ are injective. Alternatively, this follows from the following computation: from (4.1),

$$\ker \psi_{\mathcal{R}} = \mu_{\mathcal{R}} \circ \mathcal{R}^{-1} \circ E_{\mathcal{R}^{-1}} \circ \mathcal{R} \circ \mu_{\mathcal{R}}^{-1} = \mu_{\mathcal{R}} \circ (\ker \mathcal{R}) \circ E_{\mathcal{R}} \circ \mathcal{R} \circ \mu_{\mathcal{R}}^{-1}$$

in view of (2.7). Since $(\ker \mathcal{R}) \circ E_{\mathcal{R}} \subseteq E_{\mathcal{R}}$, we have $\ker \psi_{\mathcal{R}} \subseteq \mu_{\mathcal{R}} \circ E_{\mathcal{R}} \circ \mu_{\mathcal{R}}^{-1} = id_{coim \mathcal{R}/E_{\mathcal{R}}}$, so $\psi_{\mathcal{R}}$ is injective. Similarly, $\ker \psi_{\mathcal{R}}^{-1} \subseteq \mu_{\mathcal{R}^{-1}} \circ E_{\mathcal{R}^{-1}} \circ \mu_{\mathcal{R}^{-1}}^{-1} = id_{im \mathcal{R}/E_{\mathcal{R}^{-1}}}$, so $\psi_{\mathcal{R}}$ is a partial function. Q.E.D.

The fact that the sets $coim \mathcal{R}/E_{\mathcal{R}}$ and $im \mathcal{R}/E_{\mathcal{R}^{-1}}$ are isomorphic is well known (e.g., Cohn [1]). To establish this fact, it suffices to merely define $\psi_{\mathcal{R}}$ as in (4.1), which leads, as we have seen, to $\ker \psi_{\mathcal{R}} = id_{coim \mathcal{R}/E_{\mathcal{R}}}$ and $\ker \psi_{\mathcal{R}}^{-1} = id_{im \mathcal{R}/E_{\mathcal{R}^{-1}}}$. This choice of $\psi_{\mathcal{R}}$ is not, however, sufficient for \mathcal{R} to be decomposable: \mathcal{R} must be closed. Thus, besides yielding the form (4.1) of $\psi_{\mathcal{R}}$, Theorem 4.2 characterizes the (R, I, M) -decomposable relations in terms of the closure operator $\langle \cdot \rangle$, and establishes the uniqueness of such decomposition.

References

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