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A LOWER BOUND FOR THE DIMENSION OF THE MESSAGE SPACE OF THE DECENTRALIZED MECHANISMS REALIZING A GIVEN GOAL*

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November 1989

Abstract

The dimension of the message space is an important indicator of the informational efficiency of the decentralized economic mechanism. A general method is developed which leads to obtaining a lower bound for the dimension of the message space of the decentralized mechanisms that realize a given allocation function. The lower bound is expressed in terms of certain differential property of the allocation function. In particular, for economies consisting of two agents the lower bound can be further expressed by the rank of the "bordered Hessian" of the allocation function. Our approach is analytic and requires differentiability assumption.

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*The paper is based on part 1 of my Ph.D. dissertation at Northwestern University. I would like to thank my adviser, Don Saari, for his encouragement and support during this work. I would also like to thank Ken Mount for several stimulating discussions.

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Abstract

The dimension of the message space is an important indicator of the informational efficiency of the decentralized economic mechanism. A general method is developed which leads to obtaining a lower bound for the dimension of the message space of the decentralized mechanisms that realize a given allocation function. The lower bound is expressed in terms of certain differential property of the allocation function. In particular, for economies consisting of two agents the lower bound can be further expressed by the rank of the "bordered Hessian"of the allocation function. Our approach is analytic and requires differentiability assumption.

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1. Introduction

Informational efficiency has been a subject of extensive study in the theory of economic mechanisms originated in Hurwicz(1960). In Hurwicz's formal model of decentralized mechanisms, the informational efficiency is characterized by the size of the message space as is further elaborated by Mount and Reiter(1974) et al. When euclidean spaces are taken as the message spaces, the notion of the size coincides with that of dimensionality. A mechanism that realizes a given allocation rule is informationally efficient if it has the smallest possible message space. In this framework, it has been proved that the price mechanism is informationally efficient for realizing the Walrasian allocation [see, for example, Mount and Reiter(1974), Hurwicz(1977), Osana(1978)].

For a more general study of allocation rules and realizing mechanisms, it is important to determine or at least to characterize the minimal dimension of the message space of the mechanisms that realize a given allocation rule. This problem has been studied extensively in the literature [see, for example, Mount and Reiter(1974), Hurwicz, Reiter and Saari(1978, 1980)]. Perhaps the only available general strategy for dealing with this problem is the so-called single-valuedness lemma, which is developed by Mount and Reiter(1974), Hurwicz(1977), Osana(1978), et al. The strategy is a state of art technique for obtaining a lower bound of the size of the message spaces that realize a given allocation rule. But it also leaves something to be desired. Besides that it is subject to several regularity conditions that are usually hard to verify, the main weakness is that its successful use depends on identification of a subset of environments that satisfies the *uniqueness property*; and this often requires considerable ingenuity and luck. Also since the dimension of the subset of environments is the lower bound according to the theory, it is usually far from being tight unless a subset of relatively large dimension can be found.

Our purpose in this paper is to provide another general method for obtaining a lower bound of the dimension of the message spaces. Our approach is analytic and requires differentiability assumption. In this setting , an explicit, straightforward lower bound is given. The lower bound is expressed in terms of certain differential property of the allocation function. In particular, for economies consisting of two agents the lower bound can be further expressed by the rank of the "bordered Hessian"of the allocation function.

The rest of the paper is organized as follows. In section 2, we briefly describe the model and specify the terminologies to be used. In section 3, some preliminary results are presented which will be used in section 4 and 5. We derive a lower bound for two-agent economies in section 4 and then generalize it to general economies in section 5.

2. The Model

Consider l-agent economies that are finitely characterized, i.e., each agent is characterized by finitely many (real) parameters. Let $\mathbf{x}_i = (x_1^{(i)}, \ldots, x_{k_i}^{(i)})$ characterize the i-agent, then $(\mathbf{x}_1, \ldots, \mathbf{x}_l)$ characterizes the economic environment. Let U be the set consisting of all admissible economic environments, then $U \subseteq \mathbf{R}^{k_1} \times \cdots \times \mathbf{R}^{k_l}$. We assume that U is an open subset in $\mathbf{R}^{k_1} \times \cdots \times \mathbf{R}^{k_l}$.

A goal function (or social goal) on the class of economies U is a function $\mathbf{P} : U \longrightarrow \mathbf{R}^m$, where $\mathbf{P}(\mathbf{x}_1, \ldots, \mathbf{x}_l) \in \mathbf{R}^m$ specifies certain social goal (e.g., desired resource allocation, social choice, etc) associated with the given economic environment $(\mathbf{x}_1, \ldots, \mathbf{x}_l)$. We assume that $\mathbf{P}$ is a C^2 function. Also we often write $\mathbf{P} : \mathbf{R}^{k_1} \times \cdots \times \mathbf{R}^{k_l} \longrightarrow \mathbf{R}^m$ with the understanding that $\mathbf{P}$ is defined only on U.

A mechanism Π is a triple $(G, \mathbf{R}^n, h)$, where $G = (\mathbf{g}_1, \ldots, \mathbf{g}_l)$ and $\mathbf{g}_i : \mathbf{R}^{k_i} \times \mathbf{R}^n \longrightarrow \mathbf{R}^{n_i}$, $i = 1, \ldots, l$, $h : \mathbf{R}^n \longrightarrow \mathbf{R}^m$ are C^2 functions. $\mathbf{R}^n$, $\mathbf{g}_i$, h are called the message space, the equilibrium function, the outcome function, respectively, of the mechanism Π.

Definition A mechanism $\Pi = (G, \mathbf{R}^n, h)$ is said to *realize* the goal function $\mathbf{P}$ if the following Mount-Reiter diagram [see Mount and Reiter(1974)] commutes

$$\begin{array}{ccc} \mathbf{R}^{k_1} \times \cdots \times \mathbf{R}^{k_l} & \xrightarrow{\mathrm{P}} & \mathbf{R}^m \\ \mathbf{g}_1 = 0 \searrow \quad \cdots \quad \searrow \mathbf{g}_l = 0 & & \nearrow h \\ & \mathbf{R}^n & \end{array}$$

The working of a mechanism is to be understood as follows. For each economic environment $(\mathbf{x}_1, \ldots, \mathbf{x}_l) \in U$, through certain dynamic process of information exchange among the agents or of information verification with a central agency (which we are not concerned with in this paper), an equilibrium message $\mathbf{m} \in \mathbf{R}^n$ is reached, which is characterized by the equilibrium equations

$$\begin{cases} \mathbf{g}_1(\mathbf{x}_1, \mathbf{m}) = 0 \\ \qquad \vdots \\ \mathbf{g}_l(\mathbf{x}_l, \mathbf{m}) = 0 \end{cases} \tag{1}$$

That each equation involves only one of $\mathbf{x}_1, \ldots, \mathbf{x}_l$ reflects the idea of informational decentralization, i.e., each agent acts, responds or communicates basing only on his own characteristics and public messages; he has no access to the others' characteristics. This is called *privacy-preserving* in the literature. Once an equilibrium message $\mathbf{m}$ is reached, the outcome function h determines the outcome $h(\mathbf{m})$ for the economy $(\mathbf{x}_1, \ldots, \mathbf{x}_l)$. If Π realizes $\mathbf{P}$ then $h(\mathbf{m}) = \mathbf{P}(\mathbf{x}_1, \ldots, \mathbf{x}_l)$, i.e., the outcome is precisely the social goal. This is a static picture of Hurwicz's model of decentralized economic mechanisms. For more detailed description of this model and its general theory, especially its informational properties, see Hurwicz(1986).

An important aspect of this theory concerns the evaluation of mechanisms with respect to informational efficiency, as captured by, besides other things, the dimension of the message space [see, for example, Mount and Reiter(1974), Hurwicz(1986)]. The dimension of the message space can be thought as the number of information channels that the mechanism needs to operate. Given a goal function $\mathbf{P} : \mathbf{R}^{k_1} \times \cdots \times \mathbf{R}^{k_l} \longrightarrow \mathbf{R}^m$, an important problem is to determine the minimal dimension of the message space that permits the existence of some mechanism to realize $\mathbf{P}$. The minimal dimensionality is clearly an attribute of the function $\mathbf{P}$. So reasonably it or its lower bound should be determined by certain characteristic of function $\mathbf{P}$.

3. Preliminaries

From now on we assume that the mechanisms $\Pi = (G, \mathbf{R}^n, h)$ satisfy following two conditions:

(a) the dimension of the message space $\mathbf{R}^n$ agrees with the number of component functions in G, i.e., $n = \sum_{i=1}^{l} n_i$;

(b) the Jacobian of G with respect to $\mathbf{m} \in \mathbf{R}^n$ is nonsingular.

Making these assumptions loses no generality. Since the (equilibrium) message $\mathbf{m}$ is determined in (1) for all given $(\mathbf{x}_1, \ldots, \mathbf{x}_l) \in U$, it is expected that the number of component functions in G is no greater than the number of variables $\mathbf{m}$ so that solutions in (1) always exist. On the other hand, the number of variables $\mathbf{m}$ should not exceed the number of component functions in G also, for otherwise the equilibrium messages $\mathbf{m}$ determined by (1) will usually form a manifold of positive dimension in $\mathbf{R}^n$. In that case, the original message space $\mathbf{R}^n$ can be replaced by one with smaller dimension. So the original dimension can not be the minimum.

Definition A mechanism $\Pi = (\{\mathbf{g}_1, \ldots, \mathbf{g}_l\}, \mathbf{R}^n, h)$ is said to be *efficient* if the Jacobians $\nabla_{\mathbf{x}_i} \mathbf{g}_i$, $i = 1, \ldots, l$, are all of maximal rank.

The following proposition shows that a mechanism that is not efficient does not achieve optimality in the dimension of the message space in realizing $\mathbf{P}$.

Proposition 1 If $\mathbf{P}$ can be realized by a mechanism which is not efficient with the message space of dimension n, then $\mathbf{P}$ can be realized by an efficient mechanism with the message space of dimension $< n$.

Proof Suppose $(\{\mathbf{g}_1, \ldots, \mathbf{g}_l\}, \mathbf{R}^n, h)$ is the mechanism that realizes $\mathbf{P}$ and is not efficient. Let $\mathbf{m}(\mathbf{x}_1, \ldots, \mathbf{x}_l)$ be the solution for $\mathbf{m}$ in (1). Differentiating (1) yields

$$\left(\frac{\partial \mathbf{m}(\mathbf{x}_1, \ldots, \mathbf{x}_l)}{\partial(\mathbf{x}_1, \ldots, \mathbf{x}_l)}\right)_{n \times \sum_{i=1}^{l} k_i} = -\left(\frac{\partial(\mathbf{g}_1, \ldots, \mathbf{g}_l)}{\partial \mathbf{m}}\right)^{-1}_{n \times n} \left(\frac{\partial(\mathbf{g}_1, \ldots, \mathbf{g}_l)}{\partial(\mathbf{x}_1, \ldots, \mathbf{x}_l)}\right)_{n \times \sum_{i=1}^{l} k_i}$$

So $\frac{\partial \mathbf{m}(\mathbf{x}_1,\ldots,\mathbf{x}_l)}{\partial(\mathbf{x}_1,\ldots,\mathbf{x}_l)}$ is not of maximal rank since $\frac{\partial(\mathbf{g}_1,\ldots,\mathbf{g}_l)}{\partial(\mathbf{x}_1,\ldots,\mathbf{x}_l)}$ is not. Hence the component functions $m_1(\mathbf{x}_1,\ldots,\mathbf{x}_l),\ldots,m_n(\mathbf{x}_1,\ldots,\mathbf{x}_l)$ of $\mathbf{m}(\mathbf{x}_1,\ldots,\mathbf{x}_l)$ are functionally dependent. Suppose that their rank is $r < n$. W.L.O.G., we can assume that

$$m_i(\mathbf{x}_1,\ldots,\mathbf{x}_l) = \gamma_i(m_1(\mathbf{x}_1,\ldots,\mathbf{x}_l),\ldots,m_r(\mathbf{x}_1,\ldots,\mathbf{x}_l)), \qquad i = r+1,\ldots,n$$

for some functions $\gamma_{r+1},\ldots,\gamma_n$. Let $\tilde{h} : \mathbf{R}^r \to \mathbf{R}^m$ and $\psi_1,\ldots,\psi_l$ be defined as

$$\tilde{h}(m_1,\ldots,m_r) = h(m_1,\ldots,m_r,\gamma_{r+1}(m_1,\ldots,m_r),\ldots,\gamma_n(m_1,\ldots,m_r))$$

$$\psi_i(\mathbf{x}_i; m_1,\ldots,m_r) = \mathbf{g}_i(\mathbf{x}_i; m_1,\ldots,m_r,\gamma_{r+1}(m_1,\ldots,m_r),\ldots,\gamma_n(m_1,\ldots,m_r)), \qquad i = 1,\ldots,l$$

then

$$\frac{\partial(\psi_1,\ldots,\psi_l)}{\partial(m_1,\ldots,m_r)} = \frac{\partial(\mathbf{g}_1,\ldots,\mathbf{g}_l)}{\partial(m_1,\ldots,m_r)} + \frac{\partial(\mathbf{g}_1,\ldots,\mathbf{g}_l)}{\partial(m_{r+1},\ldots,m_n)} \cdot \frac{\partial(\gamma_{r+1},\ldots,\gamma_n)}{\partial(m_1,\ldots,m_r)}$$

So $\frac{\partial(\psi_1,\ldots,\psi_l)}{\partial(m_1,\ldots,m_r)}$ is of maximal rank since $\frac{\partial(\mathbf{g}_1,\ldots,\mathbf{g}_l)}{\partial(m_1,\ldots,m_n)}$ is. Therefore we can delete some components in each of $\psi_1,\ldots,\psi_l$ to get $\tilde{\mathbf{g}}_1,\ldots,\tilde{\mathbf{g}}_l$ such that the $r \times r$ square matrix $\frac{\partial(\tilde{\mathbf{g}}_1,\ldots,\tilde{\mathbf{g}}_l)}{\partial(m_1,\ldots,m_r)}$ is of maximal rank. It is easily checked that the mechanism $(\{\tilde{\mathbf{g}}_1,\ldots,\tilde{\mathbf{g}}_l\},\mathbf{R}^r,\tilde{h})$ is efficient and realizes $\mathbf{P}$. Q.E.D.

For a local characterization of the efficient mechanisms and the dimension of their message spaces, we have an important theorem due to Hurwicz, Reiter and Saari(1978, 1980) and its generalized version by Saari(1984). The following is Saari's formulation of the theorem in terms of differential ideals.

Proposition 2 Let $\mathbf{P} : \mathbf{R}^{k_1} \times \cdots \times \mathbf{R}^{k_l} \to \mathbf{R}^m$ be a C^2 goal function. The following are sufficient and necessary conditions for the (local) existence of an efficient privacy-preserving mechanism that realizes $\mathbf{P}$ with a message space of dimension n

(a) there exist differential one-forms $w_i^1,\ldots,w_i^{n_i}$, $i = 1,\ldots,l$, with $n_1+\cdots+n_l = n$ such that $I_i = \langle w_i^1,\ldots,w_i^{n_i}; \{dx_k^{(j)} \mid j \neq i, k = 1,\ldots,k_j\}\rangle$ is a differential ideal of dimension $n_i + \sum_{j\neq i} k_j$, $i = 1,\ldots,l$;

(b) $I = \cap_{i=1}^l I_i$ is a differential ideal of dimension $n_1+\cdots+n_l$ and $dP \in I$.

4. Lower Bound for Two-Agent Economies

In this section we consider two-agent economies and derive a lower bound for the dimension of the message spaces when the goal function is a scaler function $P : \mathbf{R}^{k_1} \times \mathbf{R}^{k_2} \to \mathbf{R}$. To this end, the following lemmas will be needed.

Lemma 1 If $\langle w_1,\ldots,w_k\rangle$ is a differential ideal generated by k linearly independent one-forms $w_1,\ldots,w_k$, then $v_1 \wedge \cdots \wedge v_{k+1} = 0$, for all $v_1,\ldots,v_{k+1} \in \{w_1,\ldots,w_k,dw_1,\ldots,dw_k\}$.

Proof By definition of the differntial ideals, there exist one-forms α_l^i such that

$$dw_l = \sum_{i=1}^{k} \alpha_l^i \wedge w_i, \qquad l = 1, \ldots, k$$

It is clear that $v_1 \wedge \cdots \wedge v_{k+1}$ is a sum of terms like $f \wedge dw_l \wedge w_1 \wedge \cdots \wedge w_k$, which equals zero. Q.E.D.

Let $A = (a_{ij})_{p \times q}$ be a $p \times q$ matrix. Let $D(i_1 \cdots i_r; j_1 \cdots j_r)$ denote the determinant of its $r \times r$ submatrix formed by rows $i_1, \ldots, i_r$ and columns $j_1, \ldots, j_r$, where $i_1 < \cdots < i_r, j_1 < \cdots < j_r$, and $r \leq \min\{p, q\}$.

Lemma 2

$$(a)\ D(i_1 \cdots i_r; j_1 \cdots j_r) = \sum_{k=1}^{r} (-1)^{l+k} a_{i_l j_k} D(i_1 \cdots i_{l-1} i_{l+1} \cdots i_r; j_1 \cdots j_{k-1} j_{k+1} \cdots j_r)$$

$$(b)\ D(i_1 \cdots i_r; j_1 \cdots j_r) = \sum_{l=1}^{r} (-1)^{l+k} a_{i_l j_k} D(i_1 \cdots i_{l-1} i_{l+1} \cdots i_r; j_1 \cdots j_{k-1} j_{k+1} \cdots j_r)$$

Proof (a) This is a cofactor expansion along row i_l. (b) This is a cofactor expansion along column j_k. Q.E.D.

Let $w = \sum_{i=1}^{p} \sum_{j=1}^{q} a_{ij} dx_i dy_j$.

Lemma 3

$$\overbrace{w \wedge \cdots \wedge w}^{r \text{ times}} = (-1)^{\frac{r(r-1)}{2}} r! \sum_{\substack{i_1 < \cdots < i_r \\ j_1 < \cdots < j_r}} D(i_1 \cdots i_r; j_1 \cdots j_r) dx_{i_1} \cdots dx_{i_r} dy_{j_1} \cdots dy_{j_r}$$

Proof By induction on r. It is clear that the formula is valid for $r = 1$. Suppose that the formula is valid for $r - 1$, i.e.,

$$\overbrace{w \wedge \cdots \wedge w}^{r-1 \text{ times}} = (-1)^{\frac{(r-1)(r-2)}{2}} (r-1)! \sum_{\substack{i_1 < \cdots < i_{r-1} \\ j_1 < \cdots < j_{r-1}}} D(i_1 \cdots i_{r-1}; j_1 \cdots j_{r-1}) dx_{i_1} \cdots dx_{i_{r-1}} dy_{j_1} \cdots dy_{j_{r-1}}$$

then

$$\overbrace{w \wedge \cdots \wedge w}^{r} = w \wedge \left(\overbrace{w \wedge \cdots \wedge w}^{r-1} \right) = c(r) S$$

where $c(r) = (-1)^{\frac{(r-1)(r-2)}{2}} (r-1)!$ and

$$S = \sum_{i,j} \sum_{\substack{i_1<\cdots<i_{r-1} \\ j_1<\cdots<j_{r-1}}} a_{ij} D(i_1 \cdots i_{r-1}; j_1 \cdots j_{r-1}) dx_i dy_j dx_{i_1} \cdots dx_{i_{r-1}} dy_{j_1} \cdots dy_{j_{r-1}}$$

$$= \sum_{\substack{i_1<\cdots<i_r \\ j_1<\cdots<j_r}} \sum_{l,k=1}^{r} (-1)^{r-1+l+k} a_{i_l j_k} D(i_{-l}; j_{-k}) dx_{i_1} \cdots dx_{i_r} dy_{j_1} \cdots dy_{j_r}$$

$$= (-1)^{r-1} r \sum_{\substack{i_1<\cdots<i_r \\ j_1<\cdots<j_r}} D(i_1 \cdots i_r; j_1 \cdots j_r) dx_{i_1} \cdots dx_{i_r} dy_{j_1} \cdots dy_{j_r}$$

where $D(i_{-l}; j_{-k}) = D(i_1 \cdots i_{l-1} i_{l+1} \cdots i_r; j_1 \cdots j_{k-1} j_{k+1} \cdots j_r)$. The last equality follows from Lemma 2. Q.E.D.

Let $P : \mathbf{R}^{k_1} \times \mathbf{R}^{k_2} \longrightarrow \mathbf{R}$ be a C^2 function, and let

$$BH(P) = \begin{pmatrix} P_{x_1 y_1} & P_{x_1 y_2} & \cdots & P_{x_1 y_{k_2}} & P_{x_1} \\ P_{x_2 y_1} & P_{x_2 y_2} & \cdots & P_{x_2 y_{k_2}} & P_{x_2} \\ \vdots & \vdots & \ddots & \vdots & \\ P_{x_{k_1} y_1} & P_{x_{k_1} y_2} & \cdots & P_{x_{k_1} y_{k_2}} & P_{x_{k_1}} \\ P_{y_1} & P_{y_2} & \cdots & P_{y_{k_2}} & 0 \end{pmatrix}_{(k_1+1)\times(k_2+1)}$$

$$w_x = \sum_{i=1}^{k_1} P_{x_i} dx_i, \qquad w_y = \sum_{j=1}^{k_2} P_{y_j} dy_j$$

then $dP = w_x + w_y$, and

$$dw_y = -dw_x = \sum_{i=1}^{k_1} \sum_{j=1}^{k_2} P_{x_i y_j} dx_i dy_j$$

Lemma 4 Let $D(i_1 \cdots i_r; j_1 \cdots j_r)$ as above with $A = BH(P)$, then we have

(a) $\overbrace{dw_x \wedge \cdots \wedge dw_x}^{r-1} \wedge w_x \wedge w_y$

$$= c_1(r) \sum_{\substack{i_1<\cdots<i_r<(k_1+1) \\ j_1<\cdots<j_r<(k_2+1)}} D(i_1 \cdots i_r, (k_1+1); j_1 \cdots j_r, (k_2+1)) dx_{i_1} \cdots dx_{i_r} dy_{j_1} \cdots dy_{j_r}$$

where $c_1(r) = (-1)^{\frac{r(r-1)}{2}+1}(r-1)!$

(b) $\overbrace{dw_x \wedge \cdots \wedge dw_x}^{r} \wedge w_x$

$$= c_2(r) \sum_{\substack{i_1<\cdots<i_{r+1}<(k_1+1) \\ j_1<\cdots<j_r<(k_2+1)}} D(i_1 \cdots i_{r+1}; j_1 \cdots j_r, (k_2+1)) dx_{i_1} \cdots dx_{i_{r+1}} dy_{j_1} \cdots dy_{j_r}$$

where $c_2(r) = (-1)^{\frac{r(r+1)}{2}} r!$

(c) $\overbrace{dw_x \wedge \cdots \wedge dw_x}^{r} \wedge w_y$

$$= c_3(r) \sum_{\substack{i_1<\cdots<i_r<(k_1+1) \\ j_1<\cdots<j_{r+1}<(k_2+1)}} D(i_1 \cdots i_r, (k_1+1); j_1 \cdots j_{r+1}) dx_{i_1} \cdots dx_{i_r} dy_{j_1} \cdots dy_{j_{r+1}}$$

where $c_3(r) = (-1)^{\frac{r(r-1)}{2}} r!$

(d) $\overbrace{dw_x \wedge \cdots \wedge dw_x}^{r+1}$

$$= c_4(r) \sum_{\substack{i_1<\cdots<i_{r+1}<(k_1+1) \\ j_1<\cdots<j_{r+1}<(k_2+1)}} D(i_1 \cdots i_{r+1}; j_1 \cdots j_{r+1}) dx_{i_1} \cdots dx_{i_{r+1}} dy_{j_1} \cdots dy_{j_{r+1}}$$

where $c_4(r) = (-1)^{\frac{r(r+1)}{2}} (r+1)!$

Proof Part (d) follows from Lemma 3 with $w = dw_x$, and $a_{ij} = P_{x_i y_j}$. Now we prove (b); the proofs of (a) and (c) are similar. In (d) change $r+1$ to r then

$$\overbrace{dw_x \wedge \cdots \wedge dw_x}^{r} \wedge w_x$$

$$= c(r) \sum_{i=1}^{k_1} \sum_{\substack{i_1<\cdots<i_r<(k_1+1) \\ j_1<\cdots<j_r<(k_2+1)}} P_{x_i} D(i_1 \cdots i_r; j_1 \cdots j_r) dx_i dx_{i_1} \cdots dx_{i_r} dy_{j_1} \cdots dy_{j_r}$$

$$= c(r)(-1)^r \sum_{\substack{i_1<\cdots<i_{r+1}<(k_1+1) \\ j_1<\cdots<j_r<(k_2+1)}} D(i_1 \cdots i_{r+1}; j_1 \cdots j_r, (k_2+1)) dx_{i_1} \cdots dx_{i_{r+1}} dy_{j_1} \cdots dy_{j_r}$$

by Lemma 2(b). where $c(r) = (-1)^{\frac{r(r-1)}{2}} r!$. Q.E.D.

Lemma 5 If $\overbrace{dw_x \wedge \cdots \wedge dw_x}^{r-1} \wedge w_x \wedge w_y = 0$ in an open set $U \subseteq \mathbf{R}^{k_1} \times \mathbf{R}^{k_2}$, then $rank BH(P) \leq r$ in U.

Proof We show that

$$\overbrace{dw_x \wedge \cdots \wedge dw_x}^{r-1} \wedge w_x \wedge w_y = 0 \quad \text{in} \quad U \tag{2}$$

implies

$$\overbrace{dw_x \wedge \cdots \wedge dw_x}^{r} \wedge w_x = 0 \quad \text{in} \quad U, \tag{3}$$

$$\overbrace{dw_x \wedge \cdots \wedge dw_x}^{r} \wedge w_y = 0 \quad \text{in} \quad U, \tag{4}$$

$$\overbrace{dw_x \wedge \cdots \wedge dw_x}^{r+1} = 0 \quad \text{in} \quad U. \tag{5}$$

Then, by Lemma 4, we can conclude that all $(r+1)$-subdeterminants of $BH(P)$ are equal to zero. To establish (3), (4) and (5), differentiate (2) to yield

$$0 = d(\overbrace{dw_x \wedge \cdots \wedge dw_x}^{r-1} \wedge w_x \wedge w_y)$$

$$= \overbrace{dw_x \wedge \cdots \wedge dw_x}^{r-1} \wedge dw_x \wedge w_y - \overbrace{dw_x \wedge \cdots \wedge dw_x}^{r-1} \wedge w_x \wedge dw_y$$
$$= \overbrace{dw_x \wedge \cdots \wedge dw_x}^{r} \wedge w_x + \overbrace{dw_x \wedge \cdots \wedge dw_x}^{r} \wedge w_y$$

since $dw_x = -dw_y$. Note that bases for $\overbrace{dw_x \wedge \cdots \wedge dw_x}^{r} \wedge w_x$ consist of $r+1$ $dx_i's$ and r $dy_j's$ while bases for $\overbrace{dw_x \wedge \cdots \wedge dw_x}^{r} \wedge w_y$ consist of r $dx_i's$ and $r+1$ $dy_j's$, so

$$\overbrace{dw_x \wedge \cdots \wedge dw_x}^{r} \wedge w_x \neq -\overbrace{dw_x \wedge \cdots \wedge dw_x}^{r} \wedge w_y$$

unless

$$\overbrace{dw_x \wedge \cdots \wedge dw_x}^{r} \wedge w_x = 0 \quad \text{and} \quad \overbrace{dw_x \wedge \cdots \wedge dw_x}^{r} \wedge w_y = 0$$

This establishes (3) and (4). To get (5), just differentiate (3). Q.E.D.

Now we are ready to prove our main theorem.

Theorem 1 Let $P : \mathbf{R}^{k_1} \times \mathbf{R}^{k_2} \longrightarrow \mathbf{R}$ be a C^2 function. If P can be realized in an open set $U \subseteq \mathbf{R}^{k_1} \times \mathbf{R}^{k_2}$ by an efficient privacy-preserving mechanism with a message space of dimension n, then $rankBH(P) \leq n$ in U.

Proof Let $\Pi = (G, M, h)$ be the realizing mechanism with dim $M = n$. According to Proposition 2, there exist differential one-forms $w_x^1, \ldots, w_x^{n_1}, w_y^1, \ldots, w_y^{n_2}$ such that

$I_1 = \langle w_x^1, \ldots, w_x^{n_1}; dy_1, \ldots, dy_{k_2} \rangle$ is a differential ideal of dimension $n_1 + k_2$,

$I_2 = \langle w_y^1, \ldots, w_y^{n_2}; dx_1, \ldots, dx_{k_1} \rangle$ is a differential ideal of dimension $n_2 + k_1$,

$dP \in I = I_1 \cap I_2$, and I is a differential ideal of dimension $n_1 + n_2 = n$.

Without loss of generality, we can assume that w_x^i is a linear combination of $dx_1, \ldots, dx_{k_1}$ only, $i = 1, \ldots, n_1$; similarly w_y^j is a linear combination of $dy_1, \ldots, dy_{k_2}$ only, $j = 1, \ldots, n_2$. For otherwise, $w_x^i = \alpha_x^i + \alpha_y^i, w_y^j = \beta_x^j + \beta_y^j$, where α_x^i, β_x^j are linear combinations of $dx_1, \ldots, dx_{k_1}$ only and α_y^i, β_y^j are linear combinations of $dy_1, \ldots, dy_{k_2}$ only, $i = 1, \ldots, n_1; j = 1, \ldots, n_2$. Then just replace w_x^i by α_x^i and w_y^j by β_y^j. Recall that $dP = w_x + w_y$. Since $dP \in I_1 \cap I_2$, i.e., $w_x + w_y \in I_1$, and $w_x + w_y \in I_2$, there exist differentiable functions $a_i, b_j, i = 1, \ldots, n_1; j = 1, \ldots, n_2$ such that

$$w_x = \sum_{i=1}^{n_1} a_i w_x^i, \qquad w_y = \sum_{j=1}^{n_2} b_j w_y^j$$

then

$$dw_x = \sum_{i=1}^{n_1} da_i \wedge w_x^i + \sum_{i=1}^{n_1} a_i dw_x^i$$

Note that $I = I_1 \cap I_2 = \langle w_x^1, \ldots, w_x^{n_1}; w_y^1, \ldots, w_y^{n_2} \rangle$ is a differential ideal of dimension n, so

$$\overbrace{dw_x \wedge \cdots \wedge dw_x}^{n-1} \wedge w_x \wedge w_y$$

$$= \overbrace{\left(\sum_{i=1}^{n_1} da_i \wedge w_x^i + \sum_{i=1}^{n_1} a_i dw_x^i\right) \bigwedge \cdots \bigwedge \left(\sum_{i=1}^{n_1} da_i \wedge w_x^i + \sum_{i=1}^{n_1} a_i dw_x^i\right)}^{n-1} \bigwedge \left(\sum_{i=1}^{n_1} a_i w_x^i\right) \bigwedge \left(\sum_{j=1}^{n_2} b_j w_y^j\right)$$

$$= 0$$

by Lemma 1, since by cross wedging out, the above expression reduces to a sum of terms like $f \wedge v_1 \wedge \cdots \wedge v_{n+1}$ in the lemma, which are equal to zero by the lemma. Finally by Lemma 5, $rankBH(P) \leq n$ in U. Q.E.D.

Let $\min_{(P,U)} \dim M$ denote the minimal dimension of the message spaces in the privacy-preserving mechanisms realizing P on set U, and let $\max_U rankBH(P)$ denote the maximal rank of the "bordered Hessian" $BH(P)$ in set U. Then we have the following as a corollary of Proposition 1 and Theorem 1.

Corollary Let P and U be as in Theorem 1, then

$$\min_{(P,U)} \dim M \geq \max_U rankBH(P)$$

Remark The corollary is stronger than a result of Williams(1982) where he proved that P can be realized by a privacy-preserving mechanism with the dimension of the message space less than that of the parameter transfer process, i.e., $\min\{k_1, k_2\} + 1$, only if $BH(P)$ is not of full rank. It is clear that this conclusion follows from our theorem.

While it is still an open question whether $\min_{(P,U)} \dim M = \max_U rankBH(P)$ in general, we know that it is true if either one of them equals two, as the following theorem shows.

Theorem 2 Let $P : \mathbf{R}^{k_1} \times \mathbf{R}^{k_2} \longrightarrow \mathbf{R}$ be a C^2 function.

(a) $\min_{(P,U)} \dim M \leq 2$ iff $rankBH(P) \leq 2$ in U;

(b) If $\max_U rankBH(P) = 2$ then $\min_{(P,U)} \dim M = 2$;

(c) Suppose U is connected and $dP \neq 0$ in U. If $\min_{(P,U)} \dim M = 2$ then $\max_U rankBH(P) = 2$.

Proof (a) It follows from Theorem 1 that $\min_{(P,U)} \dim M \leq 2$ implies $rankBH(P) \leq 2$ in U. So suppose that $rankBH(P) \leq 2$ in U. Define ideals

$$I_1 = \langle w_x; dy_1, \ldots, dy_{k_2} \rangle, \qquad I_2 = \langle w_y; dx_1, \ldots, dx_{k_1} \rangle$$

Let $I = I_1 \cap I_2$, then $I = \langle w_x, w_y \rangle$. It is easy to see that I_1, I_2 are both differential ideals. By Lemma 4, $dw_x \wedge w_x \wedge w_y = 0$, and $dw_y \wedge w_x \wedge w_y = 0$, so I is also a differential ideal. It is clear

that $dP \in I$. Hence by Proposition 2, P can be realized in U by a 2-dimensional message space, hence $\min_{(P,U)} \dim M \leq 2$.

(b) By (a) we must have that $\min_{(P,U)} \dim M \leq 2$. But $\min_{(P,U)} \dim M = 1$ only when P is a function of $\mathbf{x}$ or $\mathbf{y}$ alone in U. In that case, $rankBH(P) \leq 1$ in U.

(c) By (a) we must have that $rankBH(P) \leq 2$ in U. Therefore to show that $\max_U rankBH(P) = 2$ it suffices to show that $rankBH(P) = 2$ at some $(\mathbf{x}_0, \mathbf{y}_0) \in U$. In turn it suffices to show that $w_x \wedge w_y \neq 0$ at some $(\mathbf{x}_0, \mathbf{y}_0) \in U$. But this must hold under the assumptions made in (c). For otherwise, $w_x \wedge w_y \equiv 0$ in U, or $P_{x_i} P_{y_j} \equiv 0$ in $U, i = 1, \ldots, k_1; j = 1, \ldots, k_2$. Hence

$$\left(\sum_{i=1}^{k_1} P_{x_i}^2 \right) \left(\sum_{j=1}^{k_2} P_{y_j}^2 \right) = \sum_{i=1}^{k_1} \sum_{j=1}^{k_2} \left(P_{x_i} P_{y_j} \right)^2 \equiv 0 \qquad \text{in} \quad U$$

i.e., $\sum_{i=1}^{k_1} P_{x_i}^2 = 0$ or $\sum_{j=1}^{k_2} P_{y_j}^2 = 0$ in U. Let $U_1 = \{(\mathbf{x}, \mathbf{y}) \in U : w_x = 0\}, U_2 = \{(\mathbf{x}, \mathbf{y}) \in U : w_y = 0\}$, then $U = U_1 \cup U_2$. Note that U_1, U_2 are relatively close in U. Since U is connected, it follows that either $U_1 \cap U_2 \neq 0$ or one of them is empty. The former case violates the assumption that $dP \neq 0$ in U; while in the latter case, P is a function of $\mathbf{x}$ or $\mathbf{y}$ alone, hence $\min_{(P,U)} \dim M = 1$, violating the assumption that $\min_{(P,U)} \dim M = 2$. Q.E.D.

Remark (1) The theorem above is an extension of a theorem of Hurwicz, Reiter and Saari(1978, 1980) [or see Hurwicz(1986)] which only concerns with the special case with $k_1 = k_2 = 2$.

(2) Theorem 2 essentially states that under reasonably mild regularity conditions $\min_{(P,U)} \dim M = 2$ iff $\max_U rankBH(P) = 2$.

5. Lower Bound for General Economies

In this section we extend Theorem 1 firstly to the vector-valued goal function $\mathbf{P}$ of two-agent economies and then further to the goal function $\mathbf{P}$ of the general l-agent economies. In both cases a necessary condition is obtained for $\mathbf{P}$ to be realized by an efficient privacy-preserving mechanism with a message space of dimension n. The necessary condition characterizes a lower bound for the dimension of the message spaces.

First of all, two-agent case with the goal function $\mathbf{P} : \mathbf{R}^{k_1} \times \mathbf{R}^{k_2} \to \mathbf{R}^m$. Let $P^1, \ldots, P^m$ denote the component functions of $\mathbf{P}$, and let

$$Q_x^k = \sum_{i=1}^{k_1} P_{x_i}^k dx_i, \qquad Q_y^k = \sum_{j=1}^{k_2} P_{y_j}^k dy_j, \qquad k = 1, \ldots, m$$

then $dP^k = Q_x^k + Q_y^k$ and

$$dQ_y^k = -dQ_x^k = \sum_{i=1}^{k_1} \sum_{j=1}^{k_2} P_{x_i y_j}^k dx_i dy_j$$

Theorem 3 Let $\mathbf{P} = (P^1, \ldots, P^m) : \mathbf{R}^{k_1} \times \mathbf{R}^{k_2} \to \mathbf{R}^m$ be a C^2 function, $m \geq 1$. If $\mathbf{P}$ can be realized in an open set $U \subseteq \mathbf{R}^{k_1} \times \mathbf{R}^{k_2}$ by an efficient privacy-preserving mechanism with a message space of dimension n, then the following condition must be satisfied in U

$$v_1 \wedge \ldots \wedge v_{n+1} = 0, \qquad \text{for all } v_1, \ldots, v_{n+1} \in \{Q_x^1, \ldots, Q_x^m, Q_y^1, \ldots, Q_y^m, dQ_x^1, \ldots, dQ_x^m\}$$

Proof As in the proof of Theorem 1, by using Proposition 2 we have

$$dP^k \in I = I_1 \cap I_2 = \langle w_x^1, \ldots, w_x^{n_1}; w_y^1, \ldots, w_y^{n_2} \rangle, \qquad k = 1, \ldots, m$$

So there exist differentiable functions a_i^k, b_j^k, $i = 1, \ldots, n_1; j = 1, \ldots, n_2; k = 1, \ldots, m$, such that

$$Q_x^k = \sum_{i=1}^{n_1} a_i^k w_x^i, \qquad Q_y^k = \sum_{j=1}^{n_2} b_j^k w_y^j, \qquad k = 1, \ldots, m$$

Then

$$dQ_x^k = \sum_{i=1}^{n_1} da_i^k \wedge w_x^i + \sum_{i=1}^{n_1} a_i^k dw_x^i, \qquad k = 1, \ldots, m$$

Now it is clear that the condition stated in the theorem follows from Lemma 1. Q.E.D.

Remark For $m = 1$, the condition in Theorem 3 reduces to the equivalent of the rank condition in Theorem 1. For $m > 1$, however, I know no neat representation of this condition. The condition is already complicated when $n = 2$, which, after omitting obvious redundancy, can be written explicitly as follows

$$\begin{aligned}
Q_x^{\nu_1} \wedge Q_x^{\nu_2} \wedge Q_y^{\nu_3} &= 0 \\
Q_x^{\nu_1} \wedge Q_y^{\nu_2} \wedge Q_y^{\nu_3} &= 0 \\
Q_x^{\nu_1} \wedge Q_y^{\nu_2} \wedge dQ_x^{\nu_3} &= 0 \\
Q_x^{\nu_1} \wedge dQ_x^{\nu_2} \wedge dQ_x^{\nu_3} &= 0 \\
Q_y^{\nu_1} \wedge dQ_x^{\nu_2} \wedge dQ_x^{\nu_3} &= 0 \\
dQ_x^{\nu_1} \wedge dQ_x^{\nu_2} \wedge dQ_x^{\nu_3} &= 0
\end{aligned}$$

For $\nu_1, \nu_2, \nu_3 = 1, \ldots, m$. Assuming $Q_x^i \neq 0$, $Q_y^i \neq 0$, $i = 1, \ldots, m$, and after further eliminating non-independent equations, the above can be written as the following in terms of partial derivatives of $\mathbf{P}$.

$$\begin{vmatrix} P_{x_i}^{\nu_1} & P_{x_j}^{\nu_1} \\ P_{x_i}^{\nu_2} & P_{x_j}^{\nu_2} \end{vmatrix} = 0, \qquad i < j,\ \nu_1, \nu_2 = 1, \ldots, m$$

$$\begin{vmatrix} P_{y_i}^{\nu_1} & P_{y_j}^{\nu_1} \\ P_{y_i}^{\nu_2} & P_{y_j}^{\nu_2} \end{vmatrix} = 0, \qquad i < j,\ \nu_1, \nu_2 = 1, \ldots, m$$

$$\begin{vmatrix} P^{\nu}_{x_{i_1}y_{j_1}} & P^{\nu}_{x_{i_1}y_{j_2}} & P^{1}_{x_{i_1}} \\ P^{\nu}_{x_{i_2}y_{j_1}} & P^{\nu}_{x_{i_2}y_{j_2}} & P^{1}_{x_{i_2}} \\ P^{1}_{y_{j_1}} & P^{1}_{y_{j_2}} & 0 \end{vmatrix} = 0, \qquad i_1 < i_2,\ j_1 < j_2,\ \nu = 1, \ldots, m$$

$$\begin{vmatrix} P^{\nu_2}_{x_{i_1}y_{j_1}} & P^{\nu_3}_{x_{i_1}y_{j_2}} & P^{1}_{x_{i_1}} \\ P^{\nu_2}_{x_{i_2}y_{j_1}} & P^{\nu_3}_{x_{i_2}y_{j_2}} & P^{1}_{x_{i_2}} \\ P^{\nu_2}_{x_{i_3}y_{j_1}} & P^{\nu_3}_{x_{i_3}y_{j_2}} & P^{1}_{x_{i_3}} \end{vmatrix} + \begin{vmatrix} P^{\nu_3}_{x_{i_1}y_{j_1}} & P^{\nu_2}_{x_{i_1}y_{j_2}} & P^{1}_{x_{i_1}} \\ P^{\nu_3}_{x_{i_2}y_{j_1}} & P^{\nu_2}_{x_{i_2}y_{j_2}} & P^{1}_{x_{i_2}} \\ P^{\nu_3}_{x_{i_3}y_{j_1}} & P^{\nu_2}_{x_{i_3}y_{j_2}} & P^{1}_{x_{i_3}} \end{vmatrix} = 0, \qquad \begin{array}{r} i_1 < i_2 < i_3,\ j_1 < j_2, \\ \nu_2, \nu_3 = 1, \ldots, m \end{array}$$

$$\begin{vmatrix} P^{\nu_2}_{x_{i_1}y_{j_1}} & P^{\nu_2}_{x_{i_1}y_{j_2}} & P^{\nu_2}_{x_{i_1}y_{j_3}} \\ P^{\nu_3}_{x_{i_2}y_{j_1}} & P^{\nu_3}_{x_{i_2}y_{j_2}} & P^{\nu_3}_{x_{i_2}y_{j_3}} \\ P^{1}_{y_{j_1}} & P^{1}_{y_{j_2}} & P^{1}_{y_{j_3}} \end{vmatrix} + \begin{vmatrix} P^{\nu_3}_{x_{i_1}y_{j_1}} & P^{\nu_3}_{x_{i_1}y_{j_2}} & P^{\nu_3}_{x_{i_1}y_{j_3}} \\ P^{\nu_2}_{x_{i_2}y_{j_1}} & P^{\nu_2}_{x_{i_2}y_{j_2}} & P^{\nu_2}_{x_{i_2}y_{j_3}} \\ P^{1}_{y_{j_1}} & P^{1}_{y_{j_2}} & P^{1}_{y_{j_3}} \end{vmatrix} = 0, \qquad \begin{array}{r} i_1 < i_2,\ j_1 < j_2 < j_3, \\ \nu_2, \nu_3 = 1, \ldots, m \end{array}$$

$$\sum_{(\sigma_1,\sigma_2,\sigma_3)} \begin{vmatrix} P^{\sigma_1}_{x_{i_1}y_{j_1}} & P^{\sigma_2}_{x_{i_1}y_{j_2}} & P^{\sigma_3}_{x_{i_1}y_{j_3}} \\ P^{\sigma_1}_{x_{i_2}y_{j_1}} & P^{\sigma_2}_{x_{i_2}y_{j_2}} & P^{\sigma_3}_{x_{i_2}y_{j_3}} \\ P^{\sigma_1}_{x_{i_3}y_{j_1}} & P^{\sigma_2}_{x_{i_3}y_{j_2}} & P^{\sigma_3}_{x_{i_3}y_{j_3}} \end{vmatrix} = 0, \quad i_1 < i_2 < i_3, j_1 < j_2 < j_3, \nu_1, \nu_2, \nu_3 = 1, \ldots, m$$

where $(\sigma_1, \sigma_2, \sigma_3)$ are permutations of (ν_1, ν_2, ν_3).

Now we deal with the general l-agent case.

Theorem 4 Let $\mathbf{P} : \mathbf{R}^{k_1} \times \cdots \times \mathbf{R}^{k_l} \to \mathbf{R}^m$ be a C^2 function. If $\mathbf{P}$ can be realized in an open set $U \subseteq \mathbf{R}^{k_1} \times \cdots \times \mathbf{R}^{k_l}$ by an efficient privacy-preserving mechanism with a message space of dimension n, then the following condition must be satisfied in U

$$v_1 \wedge \cdots \wedge v_{n+1} = 0, \qquad \text{for all } v_1, \ldots, v_{n+1} \in \{Q_i^k, dQ_i^k \mid i = 1, \ldots, l; k = 1, \ldots, m\}$$

where

$$Q_i^k = \sum_{j=1}^{k_i} P^k_{x_j^{(i)}} dx_j^{(i)}, \qquad i = 1, \ldots, l; k = 1, \ldots, m$$

Proof Let $\Pi = (G, \mathbf{R}^n, h)$ be the realizing mechanism, then Π satisfies the conditions stated in Proposition 2. As in the proof of Theorem 1 and 3, without loss of generality, we can assume that one-forms $w_i^1, \ldots, w_i^{n_i}$ are linear combinations of $dx_1^{(i)}, \ldots, dx_{k_i}^{(i)}$, $i = 1, \ldots, l$. Hence

$$I = \cap_{i=1}^{l} I_i = \langle w_i^j \mid j = 1, \ldots, n_i; i = 1, \ldots, l \rangle$$

Since $dP^k \in I$, and $dP^k = \sum_{i=1}^{l} Q_i^k$, $k = 1, \ldots, m$, there exist differentiable functions a_{ij}^k, $j = 1, \ldots, n_i; i = 1, \ldots, l; k = 1, \ldots, m$, such that

$$Q_i^k = \sum_{j=1}^{n_i} a_{ij}^k w_i^j, \qquad i = 1, \ldots, l; k = 1, \ldots, m$$

Then

$$dQ_i^k = \sum_{j=1}^{n_i} da_{ij}^k \wedge w_i^j + \sum_{j=1}^{n_i} a_{ij}^k dw_i^j, \quad i = 1, \ldots, l; k = 1, \ldots, m$$

Now it is clear that the condition stated in the theorem follows from Lemma 1. Q.E.D.

Corollary Let $\mathbf{P} : \mathbf{R}^{k_1} \times \cdots \times \mathbf{R}^{k_l} \to \mathbf{R}^m$ and U be as in Theorem 4, then

$$\min_{\mathbf{P},U} dim M \geq \min\{t \mid v_1 \wedge \cdots \wedge v_{t+1} = 0, \text{ for all } v_1, \ldots, v_{t+1} \in W\}$$

or equivalently

$$\min_{\mathbf{P},U} dim M \geq \max\{t \mid v_1 \wedge \cdots \wedge v_t \neq 0, \text{ for some } v_1, \ldots, v_t \in W\}$$

where W denotes $\{Q_i^k, dQ_i^k \mid i = 1, \ldots, l; k = 1, \ldots, m\}$.

Proof This follows from Proposition 1 and Theorem 4. Q.E.D.

References

Chen, Pengyuan, 1989, On the efficiency and complexity of computational and economic processes, Ph.D. dissertation (Northwestern University, Evanston, IL).

Hurwicz, Leonid, 1960, Optimality and informational efficiency in resource allocation processes, in: K. Arrow, S. Karlin and P. Suppes, eds., Mathematical methods in the social sciences (Stanford University Press, Stanford, CA) 27–46.

Hurwicz, Leonid, 1977, On the dimensional requirements of informationally decentralized Pareto satisfactory processes, in: K. Arrow and L. Hurwicz, eds., Studies in resource allocation processes (Cambridge University Press, Cambridge) 413–424.

Hurwicz, Leonid, 1986, On informational decentralization and efficiency in resource allocation mechanisms, in: S. Reiter, ed., Studies in mathematical economics (Mathematical Association of America, Washington, DC) 238–350.

Hurwicz, Leonid, Stanley Reiter and Donald Saari, 1978, On constructing mechanisms with message spaces of minimal dimension for smooth performance functions, Mimeo. (University of Minnesota, Minneapolis, MN).

Hurwicz, Leonid, Stanley Reiter and Donald Saari, 1980, On constructing an informationally decentralized process implementing a given performance function, Mimeo. (University of Minnesota, Minneapolis, MN) .

Mount, Kenneth and Stanley Reiter, 1974, The informational size of message spaces, Journal of Economic Theory 8, 161–192.

Osana, Hiroaki, 1978, On the informational size of message spaces for resource allocation processes, Journal of Economic Theory 17, 66–78.

Saari, Donald, 1984, A method for constructing message systems for smooth performance functions, Journal of Economic Theory 33, 249–274.

Williams, Steven, 1982, A geometric study of smooth decentralized economic mechanisms, Ph.D. dissertation (Northwestern University, Evanston, IL).

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A LOWER BOUND FOR THE DIMENSION OF THE MESSAGE SPACE OF THE DECENTRALIZED MECHANISMS REALIZING A GIVEN GOAL*

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Abstract

The dimension of the message space is an important indicator of the informational efficiency of the decentralized economic mechanism. A general method is developed which leads to obtaining a lower bound for the dimension of the message space of the decentralized mechanisms that realize a given allocation function. The lower bound is expressed in terms of certain differential property of the allocation function. In particular, for economies consisting of two agents the lower bound can be further expressed by the rank of the "bordered Hessian" of the allocation function. Our approach is analytic and requires differentiability assumption.

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