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# INFINITE HISTORIES AND STEADY ORBITS IN REPEATED GAMES*

by

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# INFINITE HISTORIES AND STEADY ORBITS IN REPEATED GAMES


## Abstract

We study a model of repeated games with the following features:

(a) Infinite Histories: The game has been played since days of yore, or is so perceived by the players;

(b) Turing Machines with Memory: Since regular Turing machines coincide with bounded recall strategies (in the presence of infinite histories), we endow them with "external" memory;

(c) Nonstrategic Players: The players ignore complicated strategic considerations and speculations about them. Instead, each player uses his/her machine to update some statistics regarding the others' behavior, and chooses a best response to observed behavior.

Relying on these assumptions, we define a solution concept for the one shot game, called steady orbit. The (closure of the) set of steady orbits payoffs strictly includes the convex hull of the Nash equilibria payoffs and is strictly included in the correlated equilibria payoffs.

Assumptions (a)-(c) above are independent to a large extent. In particular, one may define steady orbits without explicitly dealing with histories or machines.

# INFINITE HISTORIES AND STEADY ORBITS IN REPEATED GAMES

"The thing that hath been, it is that which
shall be; and that which is done is that which
shall be done: and there is no new thing
under the sun. . . ."

Ecclesiastes 1:9

## 1. Introduction

Multi-period decision problems in which the decision maker is faced with uncertainty are prevalent both in the scientific literature and in what is sometimes referred to as the "real world." In fact, it is difficult to give examples of "real world" problems faced by individuals or organizations which do not involve the time dimension or some uncertainty. Many models dealing with such problems may be found in the classical literature on statistical inference, dynamic programming, and repeated games. The strategies devised in these contexts tend to be very complex. Indeed, in many cases it became unlikely to assume that decision makers do use such strategies. Following Simon (1972, 1978), who introduced the idea that decision makers are only boundedly rational, game theorists have recently suggested models in which computational models such as finite automata and Turing machines are used to capture the intuitive notion that the decision maker can implement only strategies with a bounded complexity (defined in the appropriate sense). (See Aumann (1981), Rubinstein (1986), Neyman (1984), Ben Porath (1985), Kalai and Stanford (1985, 1986), Megiddo and Wigderson (1985), Binmore (1986), Gilboa (1986), Gilboa and Samet (1987), Abreu and Rubinstein (1987), Stanford (1987), and others.)

The paper studies three new assumptions.

a. Infinite History. In many cases of interest, there is no "stage 0." Organizations and individuals alike have to solve problems for

which a certain history is already given. Even if an initial stage did occur somewhere in the past, the decision makers tend to perceive long histories as infinite ones. This is especially true for organizations, where a certain decision maker has typically assumed his/her position long after the organization was founded. (See also Schwartz (1974) for a discussion of and results on repeated games with infinite histories.)

b. Turing Machines with Memory. A very simple model presented in the sequel shows that in the context of infinite histories, a decision maker's strategy which is implementable by a Turing machine which always halts is no more than a finite recall strategy, i.e., each choice is determined by the last k periods for some $k \geq 0$. We therefore suggest strengthening the computational model by allowing some memory to be carried over from one stage to the next.

c. Non-Strategic Players. We introduce a behavioral assumption that captures a different dimension of bounded rationality: each player considers the behavior of all other players as a Nature phenomenon, rather than strategic players. Observing a certain history, each player assumes that the others are about to play in the same way they have played in the past after similar histories. This assumption, which may also be made in a repeated game with "stage 0," is supposed to express a somewhat simplistic approach of an individual player: not knowing the other players' repeated games strategies, rather than getting involved in sophisticated reasoning about them and probability distributions on these huge spaces, the player simply assumes that they are as close to constant as possible. Of course, it would be silly on the part of the player to assume they are actually constant when he/she has contradictory evidence. However, the player

chooses to believe in the simplest theory that explains his/her observations, namely, that the other players' behavior after each sub-history this player can remember is governed by a fixed distribution function. Based on past experience, the player would compute an estimate of this distribution and choose a best response action (or mixed action) with respect to it.

Applying these three assumptions, we consider "steady orbits" that are defined, roughly, as follows: given a one-shot game and an integer for each player, indicating the memory length of this player, each player is allowed to choose a (possibly different) mixed strategy (in the one-shot game) for every possible sub-history he/she may observe. Given these choices, we can compute, for every history, the probability of each play of the game in the next stage. Thus, we have a Markov chain, the states of which are (finite) sub-histories.

Assuming that the game is played long enough with these strategies, each player would have the chance to estimate the distribution over the other players' moves (given his memory) quite precisely, and should his/her strategies fail to be best response, he/she would change it. A steady orbit would therefore be defined, loosely, as a selection of strategies that are a fixed point of this process.

We continue to study the set of payoffs that correspond to steady orbits. We prove that the closure of this set strictly includes the convex hull of Nash equilibria payoffs and is strictly included in the correlated equilibria payoffs.

We note that the defintion of steady orbits in repeated games is

independent of the preceding analysis. Thus, one may study steady orbits by themselves, where a conceptual basis may be given by assumption (c) above and the assumption of bounded recall. (In this context, see also Lehrer (1988a,b) and Aumann-Sorin (1989).)

In Section 2 we present the basic model. This includes only one decision maker who confronts uncertainty (modeled as "nature's choice"). We also suppose that the same decision maker has lived since days of yore and will live to eternity, and that both he/she and nature choose their actions without randomization. This very primitive model captures some of the important features of a model with infinite history and these are discussed in this section.

Section 3 introduces bounded rationality into the model by defining, discussing and studying the implications of recursive strategies. Its main point is that the standard model of Turing machine is not powerful enough to implement some intuitively simple strategies in the presence of infinite histories.

Section 4 briefly comments on the extension of the basic model to mixed moves. The definition of "Turing machine with memory" is given in Section 5. We also prove there that there are $\epsilon$-optimal Turing machines with memory for a decision maker facing a Nature phenomenon that does not "remember" more than he/she does.

In Section 6 we apply these results as a conceptual basis for the analysis of games. We define steady orbits in repeated games and prove the results mentioned above. This section may be read separately.

Finally, Section 7 contains some concluding remarks.

## 2. The Basic Model for One Decision Maker

Let A be a finite and nonempty set of actions available to the decision maker (henceforth, DM) at each period. Let S denote a finite and nonempty set of possible environments or states of nature which may occur at each period. Define $C = A \times S$ to be the set of possible circumstances. A history (of circumstances) is a function $\underline{c}: \{-i \mid i \geq 0\} \to C$. A circumstance $\underline{c}(-i)$ will also be denoted by $\underline{c}_{-i}$ and, when no confusion is likely to arise, by $c_{-i}$. Thus $\underline{c}$ may also be written as $(\ldots,c_{-i},\ldots,c_{-2},c_{-1},c_0)$. The set of all histories will be denoted by $C^{-\infty}$. A future (of circumstances) is simply an element $\bar{c}$ of $C^{\infty}$. We will write $\bar{c} = (\bar{c}_1,\bar{c}_2,\ldots)$ or, when possible, $(c_1,c_2,\ldots)$.

It will prove useful to define the natural projections of the set of circumstances C on A and S: let $a: C \to A$ and $s: C \to S$ be the unique functions satisfying $c = (a(c),s(c))$ for all $c \in C$. The projection functions a and s are extended to $C^{-\infty}$ and $C^{\infty}$ by the natural pointwise definition. I.e., $a(\underline{c}) = (\ldots,a(c_{-1}),a(c_0))$ and $s(\underline{c}) = (\ldots,s(c_{-1}),s(c_0))$ for all $\underline{c} \in C^{-\infty}$, and $a(\bar{c}) = (a(c_1),a(c_2),\ldots)$ and $s(\bar{c}) = (s(c_1),s(c_2),\ldots)$ for $\bar{c} \in C^{\infty}$.

We now define some operations on histories and futures:

(1) For a history $\underline{c} \in C^{-\infty}$ and $n \geq 0$, define the n-truncation of $\underline{c}$, denoted by $\underline{c}^{-n*}$, to be the history $(\ldots,c_{-(n+1)},c_{-n}) \in C^{-\infty}$.

(2) For a future $\bar{c} \in C^{\infty}$ and $n \geq 0$, the n-truncation of $\bar{c}$, denoted $\bar{c}^{n*}$, is the future $(c_{n+1},c_{n+2},\ldots) \in C^{\infty}$.

(3) For a history $\underline{c} \in C^{-\infty}$ and $n \geq 0$, let the n-suffix of $\underline{c}$, denoted $\underline{c}^{-n}$, be the finite sequence $(c_{-n+1},c_{-n+2},\ldots,c_0) \in C^n$. (For $n = 0$, $\underline{c}^{-n}$ is always the empty string, henceforth denoted by $c^0$.)

(4) For a future $\bar{c} \in C^{\infty}$ and $n \geq 0$, define the n-prefix of $\bar{c}$, denoted by $\bar{c}^n$, to be the finite sequence $(c_1, c_2, \ldots, c_n) \in C^n$. (For $n = 0$, $\bar{c}^n$ is always the empty string $c^0$.)

(5) For a history $\underline{c} \in C^{-\infty}$ and a finite sequence $\bar{c}^n = (c_1, c_2, \ldots, c_n)$, define the concatenation of $\underline{c}$ and $\bar{c}^n$, denoted $\underline{c} \bullet \bar{c}^n$, as the history $(\ldots, c_{-1}, c_0, c_1, c_2, \ldots, c_n) \in C^{-\infty}$.

Next we turn to define strategies. A DM strategy is a function $\sigma: C^{-\infty} \rightarrow A$. A set of all DM strategies will be denoted by $\Sigma$. Nature's strategy is a function $\theta: s(C^{-\infty}) \rightarrow S$. Let $\Theta$ denote the set of all nature strategies. The future function f maps $C^{-\infty} \times \Sigma \times \Theta$ into $C^{\infty}$: for $(\underline{c}, \sigma, \theta) \in C^{-\infty} \times \Sigma \times \Theta$, the future determined by $(\underline{c}, \sigma, \theta)$, i.e., $f(\underline{c}, \sigma, \theta)$, is the element $\bar{c} \in C^{\infty}$ such that for all $n \geq 1$, $c_n = (\sigma(\underline{c} \bullet \bar{c}^{n-1}), \theta(s(\underline{c} \bullet \bar{c}^{n-1})))$.

Given a triple $(\underline{c}, \sigma, \theta) \in C^{-\infty} \times \Sigma \times \Theta$, we say that $\underline{c}$ is consistent with $\sigma$ and $\theta$ if for every $n \geq 1$, $c_{-(n-1)} = (\sigma(\underline{c}^{-n}), \theta(s(\underline{c}^{-n})))$. A history $\underline{c}$ is possible if there exist $(\sigma, \theta) \in \Sigma \times \Theta$ such that $\underline{c}$ is consistent with $\sigma$ and $\theta$.

A history $\underline{c} \in C^{-\infty}$ is said to be cyclical if, for some $k \geq 1$, $\underline{c}^{-k*} = \underline{c}$. In this case $\underline{c}$ will also be said to be cyclical of order k and $\underline{c}^{-k}$ will be called its cycle. Likewise, a future $\bar{c} \in C^{\infty}$ is cyclical if for some $k \geq 1$ $\bar{c}^{k*} = \bar{c}$, $\bar{c}$ is cyclical of order k and $\bar{c}^k$ is its cycle.

We can now state two observations (the proofs of which are immediate).

Observation 2.1: Suppose that $\underline{c} \in C^{-\infty}$ is consistent with $\sigma \in \Sigma$ and $\theta \in \Theta$, and that $\underline{c}$ is cyclical of order k for some $k \geq 1$. Then $f(\underline{c}, \sigma, \theta)$ is also cyclical of order k and its cycle equals that of $\underline{c}$.

Observation 2.2: A history $\underline{c} \in C^{-\infty}$ is possible iff one of the following two conditions holds:

(i) $\underline{c}$ is cyclical;

(ii) For all $k \geq 0$, $\underline{c}^{-k}$ is not cyclical.

These observations show that our very definitions of strategies already entail some assumptions. These assumptions deserve comment. Let us first consider the definition of a DM strategy. In the bulk of literature in dynamic programming and repeated games, the decision maker's strategy is also defined as a function from histories to actions. However, in the case of finite histories, this definition constitutes no loss of generality: finite histories differ in their length. Hence, a strategy which depends only on the history also implicitly depends on "time," i.e., the stage at which the process is in. In our case, on the other hand, the same definition of a strategy does not allow the DM's choices to depend on "time," or on some "extraneous clock." The "clock," that is, the specific enumeration of stages from $-\infty$ to $\infty$ is only known to the outside observer and, indeed, this enumeration may be shifted by any integer without changing the model.

This implicit "no clock" assumption may also be represented as follows: we assume that all those circumstances relevant to the DM's choice are described in A and S. That it is say, the decision maker in this model is not allowed to deviate from a certain behavior pattern "just because" time has passed. In fact, given any model with a "clock," one may construct an equivalent model without a "clock" by incorporating the state of the clock

into the state of nature. If the original model's clock had infinitely many states, e.g., all integers, then S would not be finite. However, no problem would arise for finite-state clocks. (Consider, for instance, a good old-fashioned clock showing only the time of day but not the date, and so forth.)

Since our main motivation is to study some notions of "stage $\infty$," the no-clock assumption can also be explained by stating that at "stage infinity" there is no "sense of time"; put differently, $\infty + 1 = \infty$.

Let us now turn to nature's strategy. The only mathematical difference between nature and the DM is assumed to be that nature's choices do not depend on those of the DM (while the converse is, of course, false). Though philosophically questionable, this assumption seems to be a reasonable one for practical purposes. As regards the descriptive aspects of the model, it should also be noted that in many cases of interest even if this assumption fails to hold, it does describe the way the decision maker perceives the problem. For instance, weather conditions are known to be affected by people's actions; nevertheless, people tend not to take these effects into consideration while facing a decision problem. Moreover, neglecting these effects appears to give a rather good approximation to the real problem, especially if computability and complexity constraints are considered.

The fact that nature's strategy is also defined as a function of history alone (without dependency on a "clock") has a similar meaning to the corresponding definition of the DM's strategy. The assumption may seem too restrictive, since it concerns nature for which "bounded rationality" arguments do not seem to apply (as opposed to a human being). However, this assumption is equivalent to the following: if the states of nature were

alternating in a given cycle from days of yore until today, we assume that the same pattern will persist. If no cycle has been followed throughout history, this assumption is trivially satisfied.

## 3. Recursive Strategies

We now turn to impose bounded rationality assumptions on the strategies. The main assumption we will use is that these strategies are recursive, i.e., that there exist algorithms which can compute the next action of the DM and the next state of nature given the infinite history. Before we turn to discuss these assumptions, let us first specify the computational model we use.

Our model is basically a standard Turing machine with an assignment of an action in A (or state in S) to each final state (an internal state at which the computation may terminate). We assume that the input tape always contains an infinitely-long input string. It seems more convenient to think of two tapes--the (read only) input tape and the working tape (which is always empty at the beginning of the computation). For the sake of brevity, we omit the formal definition of such a machine. However, it is a straightforward adaptation of the standard one. (See, e.g., Hopcroft and Ullman (1979).)

We also assume that the Turing machines describing the relevant strategies are such which always halt for every conceivable input. By "conceivable" we mean that <u>any</u> history $\underline{c} \in C^{-\infty}$ and its projection $s(\underline{c})$ should be considered as potential inputs to the DM's and nature's machines, respectively. It should be emphasized that it does not suffice to assume that the machines halt for every history consistent with them. For some

cases, especially where the DM considers a change of strategy, we would like the future to be well-defined for any history and any pair of strategies.

Let us then denote by $\Sigma^0$ the subset of the set of all DM's strategies $\Sigma$ which are implementable by a Turing machine which always halts. Let $\Theta^0$ denote the corresponding subset of $\Theta$.

As for the DM's strategies, the restriction of allowable classes to $\Sigma^0$ seems almost innocuous, or, at least, a very weak assumption of bounded rationality. It only states that the DM's strategy can be unambiguously defined by a finite number of instructions. However, the corresponding assumption imposed on nature may seem unjustified. In fact, it is equivalent to the hypothesis that there exists a (finite) "scientific" theory which is "true" in the sense of perfectly predicting nature's choice. The philosophical grounds of such an hypothesis are beyond the scope of this paper. Nonetheless, we will not adhere to the deterministic model for very long. Once we allow for randomized choices this assumption would only mean that the <u>distribution over</u> states of nature, rather than the specific choice of one of them, is describable by a finite algorithm. It seems to us that such an assumption is not too restrictive for many cases of interest. (Consider, for instance, the concept of a Markov chain: there are several states, to each of which there corresponds a distribution over the set of states, and that distribution may be computed by an algorithm given the history.)

In the literature of repeated games there are several notions of bounded rationality. The most restrictive assumption seems to be that of <u>finite recall</u>. In our terms, $\sigma \in \Sigma$ ($\theta \in \Theta$) is said to be a <u>finite recall strategy</u> if there exists a number $n \geq 0$ and a function $f: C^n \to A$

($f: s(C^n) \rightarrow S$) such that $\sigma(\underline{c}) = f(\underline{c}^{-n})$ ($\theta(s(\underline{c})) = f(s(\underline{c}^{-n}))$) for all $\underline{c} \in C^{-\infty}$.

A simple but somewhat surprising result is the following:

Proposition 3.1: The set of finite recall strategies in $\Sigma$ ($\Theta$) is exactly $\Sigma^0$ ($\Theta^0$).

Proof: The fact that every finite recall strategy is recursive is trivial. The converse is an application of Konig's lemma and may be proved as follows: let $\sigma \in \Sigma^0$. (The proof for $\Theta^0$ is symmetric.) Consider the infinite tree, (V,E), which describes the conceivable histories:

$$V = \{\underline{c}^{-n} | \underline{c} \in C^{-\infty},\ n \geq 0\}$$
$$E = \{(\underline{c}^{-n}, \underline{c}^{-n-1}) | \underline{c} \in C^{-\infty},\ n \geq 0\}$$

That is, the vertices are all finite sequences of circumstances $(c_{-n+1}, \ldots, c_{-0})$ includes the empty sequence $c^0$ which is the root of the tree, and two such histories are connected by an edge if and only if the first one may be obtained from the second by deleting the first component of the latter.

For a given $\sigma \in \Sigma^0$ and a given history $\underline{c} \in C^{-\infty}$, one may consider the path in this tree along which the computation of $\sigma$ will proceed. Since $\sigma$ is required to halt, this is simply a finite path generated by $\underline{c}^{-i}$ for $0 \leq i \leq k$ for some k. To prove that $\sigma$ is a finite recall strategy, we only need to show that the length of this path k is bounded from above for all conceivable histories $\underline{c}$. Assume the contrary, i.e., that such a bound does

not exist. Consider the root of the tree $c^0$, and the finite number of branches emanating from it. If in each branch the computation paths of $\sigma$ were bounded, the maximal bound plus 1 would have been a bound for all conceivable histories. Hence there exists as least one branch (sub-tree) for which there is no such bound. Continuing in this fashion one obtains a conceivable history $\underline{c}$ for which the computation of $\sigma$ does not halt. []

Observation 3.2: Suppose $\sigma \in \Sigma^0$ and $\theta \in \Theta^0$ are consistent with the history $\underline{c} \in C^{-\infty}$. Then $\underline{c}$ is cyclical.

Proof: In view of Result 3.1, both $\sigma$ and $\theta$ are finite-recall strategies. Assume $\sigma(\underline{c})$ depends only on $\underline{c}^{-k_1}$ and $\theta(s(\underline{c}))$ on $s(\underline{c}^{-k_2})$, and let $k = \max\{k_1,k_2\}$. It is trivial to see that $\underline{c}$ is cyclical of order $m \leq k$. []

The last two results show that our model is too restrictive, since it can only describe cyclical phenomena. We are now going to generalize it in two ways by introducing (i) machines with memory and (ii) randomized actions. However, the main two assumptions, that of time stationarity and that of bounded rationality, will essentially be retained.

## 4. Randomized Actions

As mentioned in Section 3, the assumption that nature has a recursive strategy (with the "no clock" assumption) seem far too restrictive to describe uncertainty. On the other hand, we would not like to drop these assumptions altogether since we are interested in modeling situations in which there is something the decision maker may infer from the past

regarding the future.

The most natural thing to do at this point seems to be to allow randomized actions (at least for nature), and to require that the assumptions discussed above be satisfied with respect to these, rather than the actual actions (or states of the world) chosen. Thus no conceivable history will be ruled out and, in particular, the model will not be restricted to cyclical phenomena--but there will still be enough regularity for the DM to apply statistical inference techniques.

Let $\Delta(A)$ and $\Delta(S)$ denote the set of distributions over A and S, respectively, and define $\Sigma^* = \{\sigma: C^{-\infty} \to \Delta(A)\}$; $\Theta^* = \{\theta: s(C^{-\infty}) \to \Delta(S)\}$. As in the deterministic model, these definitions already impose the "time stationarity" or "no clock" assumptions. In order to formulate the bounded rationality assumption we will define Turing machines as before, save that now a distribution over A (or S) will be attached to each final internal state of the machine rather than a single element of it.

## 5. Machines with Memory

Result 3.1 may suggest that recursive strategies are too restrictive. In fact, they cannot implement reasonably simple strategies. Consider the following example: nature chooses whether it will rain or not. The DM chooses to take an umbrella or not. A possible DM strategy is the following "2-trigger strategy": if it never rained before, or if it rained exactly once in the entire history, do not take an umbrella. Otherwise, take an umbrella. This strategy is certainly not a finite recall strategy. Hence, it is not recursive. However, such a strategy is implementable by a finite automaton playing a repeated game as in Aumann (1981), Rubinstein (1986),

Neyman (1984), Kalai and Stanford (1985), and others. (These models assume that histories are finite, but the extension of the automaton notion is straightforward.)

This apparently paradoxical result, namely that a Turing machine is weaker than a finite automaton, is due to an abuse of terms: when a finite automaton is said to "implement" a strategy in a repeated game, it is meant that each stage of the game corresponds to one application of the automaton's transition function. When a Turing machine is said to "compute" a strategy, the role of the machine's internal states is quite different: they are used for a "background" computation, and only when the whole computation is completed is an action chosen and one single stage of the game is over. In other words, the automaton may use its states to "remember" information from one stage to the next. The Turing machine uses its states for the computation alone, and at each stage it is assumed to begin at the same state. Thus it cannot carry information bits computed in previous stages to the next one. In the context of finite histories, this lack of memory on the Turing machine's part constitutes no loss of generality: the machine can always simulate its computations in the previous stages. (This is, of course, true if only computability rather than complexity aspects are taken into account.) However, in the case of infinite histories this is no longer the case. It therefore seems natural to consider a larger set of machines which have another "memory" tape which may written to and read from during a computation. This tape is kept unaltered between the end of one computation and the beginning of the next.

We would therefore like to endow a DM's machine with memory, and it would seem suitable to allow it to remember real numbers such as relative

frequencies. However, if we do not impose any additional restrictions and we consider a Turing machine with real-valued registers on which arithmetic operations and comparisons can be performed, we make it significantly stronger than we originally suggested: such a machine can use a real number as a code and analyze it to determine its behavior. Thus, a very simple machine may actually implement a non-recursive strategy. This is obviously not what we had in mind. However, if we limit the memory to a finite number of cells (each of which may contain one of a finite set of symbols) we may be unduly restricting the machine's ability to compute numbers in a naive way.

We have to distinguish between the memory the machine has for numbers per se and numbers as encoding of information. We were unable to find an elegant computational model which will draw this distinction by its computational abilities. Rather, we suggest adopting a Turing machine with several real-valued registers, but restricting its complexity. That is, each computation performed by the machine may use the registers up to T times for some fixed T (which will be a part of the machine's specifications).

We therefore define an M-T-Turing machine (for $M,T \geq 0$) as a machine with M real-valued registers such that each computation may involve no more than T arithmetical operations ($+$, $-$ , $\times$, $/$) and comparisons ($=$, $\neq$, $>$, $\geq$, $<$, $\leq$). The formal definition of such a machine, a computation of it, and so forth, are omitted. They are straightforward adaptations of the classical definitions.

Observation 5.1: Suppose M is a M-T-Turing machine which computes an action

$a \in A$ (or $a \in \Delta(A)$) for every possible history. Then there exists a constant $k(M)$ such that for every history $\underline{c} \in C^{-\infty}$, M does not consult $\underline{c}^{-k(M)*}$.

Proof: Similar to that of Proposition 3.1. //

Therefore, we can conclude again that, apart from the memory the machine carries, it is still a bounded recall machine. However, such machines can already learn a stochastic process. That is, suppose that Nature's strategy is a finite recall mixed strategy $\theta \in \Theta^*$. Assume that $\theta$ depends only on the last k stages, and that $h: A \times S \to \mathbb{R}$ is the DM's payoff function. For every history $\underline{c}$ we consider

$$\sup_{\sigma \in \Sigma^*} \lim \inf_{T \to \infty} 1/T \sum_{t=1}^{T} E(h(f_t(\underline{c}, \sigma, \theta))).$$

A strategy $\sigma$ is ($\varepsilon$-) optimal against $\theta$ if it ($\varepsilon$-) obtains this payoff for all $\underline{c}$.

Theorem 5.2: Given nonempty and finite sets A, S, a payoff function $h: A \times S \to \mathbb{R}$, an integer $k \geq 0$ and $\varepsilon > 0$, there exists an M-T-Turing machine M with $k(M) = k$ which is $\varepsilon$-optimal against every $\theta$ with recall k.

Proof: Obviously, all M needs to do is to compute the relative frequency of each state of Nature s after each history $\underline{s} \in (S)^k$. Thus, $|S|^{k+1}$ registers is all that is needed. The problem is how to compute the relative frequencies given infinite history and one additional observation. The

solution is the following: for $\varepsilon_1 > 0$, let $N \geq 1/\varepsilon_1^3$. It follows from Markov inequality that if $X_1, X_2, \ldots, X_n$ are i.i.d. random variables, each of which is distributed over S, and $\mu \in \Delta(S)$ is their expectation, then

$$P(\|(1/n)\sum_{i=1}^{n} x_i - \mu\| \geq \varepsilon_1) \leq \varepsilon_1, \text{ for all } n \geq N.$$

Hence, we will let the machine M have $2*(|S|^{k+1} + |S|^k)$ registers. For each history $\underline{s}$ of length k the machine would have <u>two</u> relative frequencies: one obtained over the last $N_0$ occurrences of $\underline{s}$ and the other over $N_0 + N$ last occurrences, where $0 \leq N_0 \leq N - 1$. (Thus, it has $|S|$ registers for the distribution over S and one for the counter.) At each stage the machine uses the statistics obtained over the longer history for choosing a best response action, and updates both relative frequencies of the corresponding $\underline{s} \in (S)^k$. Correspondingly, it advances the counters $N_0$ and $N_0 + N$ by 1. When $N_0 + N = 2N$, the machine sets $N_0$ to zero and ignores the relative frequencies obtained over the longer history.

Thus, at each stage the machine has a relative frequency which is $\varepsilon_1$-close to the true Nature strategy, and it chooses a best response act versus the approximated distribution.

Obviously, for small enough $\varepsilon_1$, the machine obtains an $\varepsilon$-optimal payoff. //

## 6. <u>Steady Orbits in Games</u>

Consider a one-shot finite normal form game $G = (N, (S^i)_{i\in N}, (h^i)_{i\in N})$ where $N = \{1,2,\ldots,n\}$ $(n \geq 1)$ is the set of <u>players</u>, $S^i$ is a (finite and nonempty) set of <u>moves</u> of player i, and $h^i: S \to \mathbb{R}$ is player i's <u>payoff</u>

function where $S = \Pi_{i \in N} S^i$.

We would like to define a solution concept for one-shot games which would rely on the assumptions that G is an infinitely repeated game with infinite history and that each player considers all other players (in conjunction) as Nature. For brevity's sake we will not provide formal definitions of the repeated game and its strategies. These definitions are straightforward adaptations of the ones given above. However, bearing this interpretation in mind and applying the previous sections' results we know that a recursive strategy with a bounded number of memory registers is, in fact, a bounded recall strategy (with the same number of memory registers). Let $k_i \geq 0$ denote player i's recall, i.e., the number of periods player i remembers. Assuming that each player uses an $\varepsilon$-optimal machine, we are interested in the limit frequency of each $s \in S$. Furthermore, we are interested in those distributions over S which are sustainable by $\varepsilon$-optimal machines for all $\varepsilon > 0$.

We first define the set of player i's (mixed) strategies to be

$$\Sigma^i = \{\sigma^i : (S)^{k_i} \rightarrow \Delta(S^i)\}.$$

Given the recall bounds $\{k_i\}_{i \in N}$ we define $k = \max_i k_i$. We will be interested in states which are the $|S|^k$ k-tuples of one-shot plays which summarize the relevant information of the history.

Some notational conventions which will prove useful are the following. For a set X (such as $S^i$, S, etc.) and $\underline{x} \in X^m$, say $\underline{x} = (x_1, x_2, \ldots, x_m)$, we define $\text{suf}(\underline{x}, \ell)$ $(1 \leq \ell \leq m)$ as the element of $X^\ell$ defined by $(x_{m-\ell+1}, x_{m-\ell+2}, \ldots, x_m)$. Likewise, $\text{pref}(\underline{x}, \ell)$ will denote $(x_1, x_2, \ldots, x_\ell) \in$

$X^{\ell}$. For $\underline{x} = (x_1,\ldots,x_m) \in X^m$ and $\underline{y} = (y_1,\ldots,y_\ell) \in X^\ell$ we define $\underline{x}\bullet\underline{y}$ to be the element $(x_1,\ldots,x_m,y_1,\ldots,y_\ell) \in X^{m+\ell}$.

We also define $\Sigma = \Pi_{i\in N}\ \Sigma^i$ and a typical element $\sigma \in \Sigma$ will be understood to define $\sigma^i \in \Sigma^i$ (for all i) such that $\sigma = (\sigma^1,\sigma^2,\ldots,\sigma^n)$. For $i \in N$ we define $s^{-i}$ ($\sigma^{-i}$) to be an element of $S^{-i} \equiv \Pi_{j\neq i}\ S^j$ ($\Sigma^{-i} \equiv \Pi_{j\neq i}\ \Sigma^j$). the symbol $(s^{-i})^j$ ($(\sigma^{-i})^j$) will stand for player j's component in $s^{-i}$ ($\sigma^{-i}$). The symbols $(s^{-i},t^i)$ and $(\sigma^{-i},\tau^i)$ would be elements of S and $\Sigma$, respectively, with the obvious meaning.

Given $\sigma \in \Sigma$ we define a Markov chain whose set of states is $(S)^k$ with the transition probability matrix $A(\sigma)$ defined by the elements:

$$a(\sigma)_{\underline{s},\underline{s}'} = \begin{cases} \Pi_{i\in N}\ \sigma^i(\mathrm{suf}(\underline{s},k_i))(s'_k), & \text{if } \mathrm{suf}(\underline{s},\ k-1) = \mathrm{pref}\ (\underline{s}',\ k-1) \\ 0, & \text{otherwise.} \end{cases}$$

In other words, $a(\sigma)_{\underline{s},\underline{s}'}$ is the conditional probability that the play of the game will have a history (of length k) $\underline{s}'$ at time t, given that at time (t - 1) it has a history $\underline{s}$ and the players are playing according to $\sigma$.

The probabilities $A(\sigma)$ describe all the relevant information about the play of the game given a certain history. However, we also have to introduce the stationary distribution as a part of the definition of a steady orbit.

Let there by given $p \in \Delta((S)^k)$ and $\sigma^{-i} \in \Sigma^{-i}$. We define a function $\tau^i(p,\sigma)\colon (S)^{k_i} \to \Delta(S^{-i})$, which should be interpreted as the list of values of the statistics player i retains in his/her machine registers, by

$$\tau^i(p,\sigma)(\underline{s})(s_0^{-i}) = \sum_{\underline{s}'\in(S)^{k-k_i}} \bar{p}(\underline{s}') \prod_{j\neq i} \sigma^j(\mathrm{suf}(\underline{s}'\bullet\underline{s},k_j))((s_o^{-i})^j)$$

where

$$\bar{p}(\underline{s}') = p(\underline{s}'\bullet s)/\sum_{\underline{t}'\in(S)^{k-k_i}} p(\underline{t}'\bullet s)$$

if the denominator does not vanish. In case it does, i.e., $p(\underline{t}'\bullet s) = 0$ for all $\underline{t}' \in (S)^{k-k_i}$, the definition of $\tau^i$ is immaterial.

Note that in case $k_i = k$, $\tau^i(p,\sigma)(\underline{s})$ is the actual distribution on the next $(n-1)$-tuple of the other players' moves. In particular, it is independent of the stationary distribution p. However, if $k_i < k$ the probability vector $\tau^i(p,\sigma)(\underline{s})$ kept in player i's machine registers is a convex combination of several such distributions (for different histories).

Given a distribution on the other players' moves $q^{-i} \in \Delta(s^{-i})$, and a mixed move for player i, $q^i \in \Delta(S^i)$, we say that $q^i$ is a <u>best response</u> to $q^{-i}$ if it maximizes $E(h^i)$ given $q^{-i}$.

At last we can define a <u>steady orbit</u> of the game G with recall profile $\{k_i\}_i$ to be a pair $(p,\sigma)$ such that:

(i) $[A(\sigma)]^t p = p$ (where "t" indicates transpose);

(ii) $\forall\, i \in N$, $\forall\, \underline{s} \in (S)^{k_i}$, $\sigma^i(\underline{s})$ is a best response to $\tau^i(p,\sigma)(\underline{s})$.

The first condition states that p is indeed a stationary distribution for the Markov chain defined by $A(\sigma)$. The second condition simply requires that all players will choose a best response (mixed) move, given their information, while their information is consistent with the stationary distribution p and all players' strategies.

It is important to note that the interpretation we suggest for this

concept does not assume that the players are aware of the stationary distribution p, or even of the other players' recall bounds $\{k_i\}_i$, let alone other players' strategies. Each player gathers information and computes relative frequencies to the best of his/her ability, without knowing whether his/her bounded recall is large enough or not. If the players choose a certain strategy $\sigma$, an outside observer could define the resulting Markov chain and compute the stationary distribution. Should such an observer compute $\tau^i(p,\sigma)$ as defined above (s)he would find that it coincides with the statistics computed by player i, but this computation of $\{\tau^i\}$ via p and $\sigma$ cannot be done by the players themselves.

The first question which arises at this point seems to be existence. And indeed, a rather standard fixed-point argument ensures that the following holds:

Theorem 6.1: Given a one-shot game $G = (N,(S^i)_{i\in N},(h^i)_{i\in N})$ and $k_i \geq 0$ for $i \in N$, G has a steady orbit $(p,\sigma)$ with recall profile $\{k_i\}_{i\in N}$.

Proof: Let $B = \Delta((S)^k) \times \Pi_i \Delta(S^i)^{(|S|^{k_i})}$. Note that each element $b \in B$ can be interpreted as a pair $(p,\sigma)$ where $p \in \Delta((S)^k)$ and $\sigma$ is an n-tuple of strategies. Furthermore, B is a convex and compact subset of $\mathbb{R}^m$ for some m $(= |S|^k + \Sigma_{i=1}^n |S|^{k_i}|S^i|)$. Let $f: B \to 2^B$ be the following correspondence: for $(p,\sigma) \in B$, $f((p,\sigma))$ is the set of all pairs $(p',\sigma')$ such that: (i) $p' = [A(\sigma)]^t p$, and (ii) $(\sigma')^i$ is a best response strategy to $\tau^i(p,\sigma)$. (That is, the first component of all points in $f(p,\sigma)$ is the same p'.) Note that f is convex valued and upper semi-continuous. Hence, it has a fixed point by Kakutani's theorem. //

Given a stationary distribution $p \in \Delta((S)^k)$ we define $\hat{p}$ to be the induced distribution on S, i.e., $\hat{p}(s) = \sum_{\underline{s}' \in (S)^{k-1}} p(\underline{s}' \cdot s)$. We will also define for $i \in N$ $h^i(\hat{p})$ to be $\sum_{s \in S} p(s)h^i(s)$ and $h(\hat{p})$ will denote the vector of expected payoffs $(h^i(\hat{p}))_{i \in N}$. With a convenient abuse of notation, we will also use $h^i(p)$ and $h(p)$ for $p \in \Delta((S)^k)$ referring to $h^i(\hat{p})$ and $h(\hat{p})$, respectively.

We will be interested in:

$$SO(k_1,\ldots,k_n) = \{h(\hat{p}) \mid \quad (p,\sigma) \text{ is a steady orbit of G with recall profile } \{k_i\}_{i \in N} \text{ for some } \sigma \in \Sigma\}$$

and

$$SO = \bigcup_{(k_1,\ldots,k_m) \in (Z_+)^n} SO(k_1,\ldots,k_n).$$

with $Z_+ = \{0,1,2,\ldots\}$.

Let us denote by NE (CE) the set of Nash (correlated) equilibria payoffs (see Nash (1951) and Aumann (1974)). We will now state and prove some results regarding the relationships between the set of steady orbit payoffs and Nash/correlated equilibria payoffs. All these results hold for any given one-shot game $G = (N,(S^i)_{i \in N},(h^i)_{i \in N})$.

Proposition 6.2: For all $k_1,\ldots,k_n$,

$$NE \subseteq SO(k_1,\ldots,k_n).$$

Furthermore, $NE = SO(0,\ldots,0)$, hence $NE = \bigcap_{(k_1,\ldots,k_m) \in (Z_+)^n} SO(k_1,\ldots,k_n)$.

Proof: Given $k_1,\ldots,k_n$ and a Nash equilibrium of G, define $\sigma$ to be the n-tuple of strategies which play the given equilibrium regardless of the history. Let p be some stationary distribution of $A(\sigma)$ and note that $(p,\sigma)$ is a SO of G with $\{k_i\}_{i\in N}$.

For the "furthermore" part, let $k_i = 0$ for $i \in N$, and assume $(p,\sigma)$ is a steady orbit of G with $(k_1,\ldots,k_n)$. Note that $\sigma^i$ is no more than a mixed move in G (since there is only one zero-length history) and correspondingly, the Markov chain has only one state. The best response condition implies that $\sigma$ induces a NE in G. //

Proposition 6.3: Co (NE) $\subseteq \bar{S}\bar{O}$. (Co stands for convex hull, and the bar for closure in the standard topology.)

Proof: Let there be given m Nash equilibria of G denoted $NE_j = (r_j^i)_{i=1}^N$ $(i \le j \le m)$ where $r_j^i \in \Delta(S^i)$. We will also denote by $NE_j$ the payoff vector $h(\Pi_i\, r_j^i)$ (where $\Pi_i r_j^i$ is the product distribution on S defined by the distributions $r_j^i$ on $S^i$). It suffices to show that all rational convex combinations of $\{NE_j\}_{j=1}^m$ are in $\bar{S}\bar{O}$. Let there be given, then, positive integers $\{t_j\}_{j=1}^m$ with $T = \Sigma_{j=1}^m\, t_j$. We would like to show that $\Sigma_{j=1}^m\, (t_j/T)NE_j \in \bar{S}\bar{O}$.

Let us first explain the main idea of the proof. The obvious way to obtain the desired average payoff is to let all players have identical recall k and play one of the Nash equilibria each stage. Since all players would observe the same history, each one will know the exact mixed move of

every other player and the strategies will be best-response ones.

Let us first consider the case in which all the Nash equilibria are pure. In this case it suffices to set k = T and let the players play $NE_j$ $t_j$ times (j = 1,...,m) in a cycle. However, the same technique cannot be directly applied to mixed equilibria since the history (i.e., the actual realizations) does not identify the Nash equilibria that were played, and one cannot uniquely define the next equilibrium which should now be played.

We note here that the difficulty could be avoided if one assumes that each player remembers his/her own mixed actions. In this case the proof is identical to the pure Nash equilibria case. However, one need not deviate from our framework in order to obtain the result, and we therefore stick to it.

The main idea, which is not very surprising, will be the following: instead of a single play of an equilibrium (which does not identify it) we should have a long sequence of plays, which will identify it with high enough probability.

Thus, a history in which each equilibrium was played long enough according to a certain cycle is likely to regenerate a similar history. We should also verify that histories that do not correspond to the desired cycle will not have too high a (stationary distribution) probability, and we can guarantee that by deciding that the players would play an arbitrarily chosen NE, say $NE_1$, at those histories.

The formal proof is, naturally, slightly more delicate: let there be given $\{NE_j\}_{j=1}^m$ (without loss of generality, $NE_i \neq NE_j$ for $i \neq j$), $\{t_j\}_{j=1}^m$ and $1 > \varepsilon_0 > 0$, with $T = \sum_{j=1}^m t_j$.

Let us define

$$d = \min_{1 \le i \ne j \le m} \|NE_i - NE_j\| > 0$$

Without loss of generality we will assume that for every $s,s' \in S$ $\|h(s) - h(s')\| \le 1$. For $x \in \Delta(S)$ and $\varepsilon > 0$ denote $N_\varepsilon(x) = \{y \in \Delta(S) \mid \|x - y\| < \varepsilon\}$.

Let M be an integer satisfying $M > 3/d$. For such an M, if $x_1, x_2, \ldots, x_M \in \Delta(S)$ and $h(x_1), h(x_2), \ldots, h(x_{M-1}) \in N_{d/3}(NE_j)$, then

$$1/M \sum_{i=1}^{M} h(x_i) \notin N_{d/3}(NE_i) \text{ for } i \ne j.$$

Next, let $\varepsilon_1 = \varepsilon_0/3m(m-1)$ and choose L to be an integer such that $L > 1/\varepsilon_1$.

Let $K_0$ be a large enough integer such that for all $k \ge K_0$ and every $1 \le j \le m$, if $NE_j$ is played k times, forming a sequence $X_1, X_2, \ldots, X_k$ of i.i.d. random variables on $h(\Delta(S))$, then

$$\text{Prob}(\|(1/k) \sum_{i=1}^{k} X_i - NE_j\| < d/3) > (1 - \varepsilon_1)^{1/ML(T+1)}.$$

A "long sequence" of a certain equilibrium will be of length $K_0ML$, and thus we define the recall length to be $k = K_0ML(T+1)$. For each player $i \in N$, then, $k_i = k$.

We need some additional definitions: a sequence $\underline{s} \in (S)^{MK_0}$ is an <u>M-j-sequence</u> if

$$(1/MK_0) \sum_{r=1}^{MK_0} h(s_r) \in N_{d/3}(NE_j).$$

(Here and in the sequel we do not distinguish between an element of S and its corresponding element of $\Delta(S)$.)

A sequence $\underline{s} \in (S)^{MLK_0}$ is an L-M-j-sequence if it is the concatenation of L M-j-sequences. We can finally turn to define what is a "nice" history, in which the desired Nash equilibria seem to have been played according to the correct order. Basically, it is a sequence of T L-M-j-sequences (with the j's taking the value i exactly $t_i$ times), but we have to take into account the following points: (i) the sequences may have some overlap (our choice of M would guarantee that this overlap is no longer than $MK_0$, i.e., 1/L of an L-M-j-sequence); (ii) the end (most recent part) of the sequence does not have to belong to any L-M-j-sequence (but rather to a new one in the process of formation); (iii) the order of the Nash equilibria in the sequence is

$$\underbrace{\text{L-M-1-sequence},\ldots,\text{L-M-1-sequence}}_{t_1 \text{ times}},\ldots,\underbrace{\text{L-M-m-sequence},\ldots,\text{L-M-m-sequence}}_{t_m \text{ times}}$$

in some cyclical permutation.

We therefore define, for $0 \leq c \leq T - 1$, a history, $\underline{s} \in (S)^k$ to be a k-c-sequence if these are integers $0 < r_1 < r_2 < r_3 < \ldots < r_T$ such that for all $1 \leq v \leq T$ the subsequence $(s_{k-r_v},\ldots,s_{k-r_v+MLK_0-1})$ is an L-M-j-sequence where

$$\Sigma_{\ell=1}^{j-1} t_\ell < v + c \pmod{T} \leq \Sigma_{\ell=1}^{j} t_\ell .$$

A k-c-sequence for some c is also called a k-sequence.

We can finally define the steady orbit by assigning a Nash equilibrium to each history $\underline{s} \in (S)^k$. Given such an $\underline{s}$, apply the following algorithm: start at the end, $s_k$ (the most recent stage) and, going backward, look for an L-M-j-sequence for some j. If no such sequence is found, attach $NE_1$ to $\underline{s}$. Otherwise, assume L-M-$j_0$-sequence $\underline{s}^1$ is found. Continue the search from $s^1_M$ (the M-th component of $\underline{s}^1$) backward (in $\underline{s}$), this time looking only for an L-M-$j_0$-sequence. If there is none which beings at most $MLK_0$ stages before $s^1_M$, play $NE_{j_0}$ if $t_{j_0} > 1$ and $NE_{j_0+1(\text{mod } m)}$ if $t_{j_0} = 1$.

If, on the other hand, such an L-M-$j_0$-sequence $\underline{s}^2$ was found, continue with it in the same fashion. For $t < t_{j_0}$, if exactly t such sequences are found, play $NE_{j_0}$. If $t_{j_0}$ are found, play $NE_{j_0+1(\text{mod } m)}$.

We now contend that any stationary distribution p of this process induces a steady orbit as required. First, it is easy to see that for every $\underline{s} \in (S)^k$, if the process is in state s at time t, the probability of being in a k-sequence at time t + k is at least $(1 - \varepsilon_1)$, which means that the stationary distribution probability of these states is at least $(1 - \varepsilon_1)$. Furthermore, given any $\underline{s} \in (S)^k$ and every $i \leq j \leq m$, the probability that $NE_j$ will be played at least $(t_jL - 1)MK_0$ times during the next $TMLK_0$ stages is at least $(1 - \varepsilon_1)$.

We now wish to show that the stationary distribution probability of all states at which $NE_j$ is played, denoted $p(NE_j)$, is at least $(t_j/T - \varepsilon_0/m(m - 1))$. This would also mean that it is at most $(t_j/T - \varepsilon_0/m)$ and then

$$\| (1/T) \sum_{j=1}^{m} t_j \, NE_j - h(\hat{p}) \| = \| 1/T \sum_{j=1}^{m} t_j \, NE_j - \sum_{j=1}^{m} p(NE_j) NE_j \| \leq$$

$$\leq \sum_{j=1}^{m} |t_j/T - p(NE_j)| \; \|NE_j\| < \varepsilon_0.$$

Let $A_j$ be the set of histories $\underline{s} \in (S)^k$ at which $NE_j$ is played. Let $X_t$ be the random variable associated with the Markov chain. Then

$$p(NE_j) = \text{Prob}(X_t \in A_j) \qquad \text{(for all t)}$$

$$= 1/TMLK_0 \sum_{i=1}^{TMLK_0} \text{Prob}(X_{t+i} \in A_j) =$$

$$= 1/TMLK_0 \sum_{i=1}^{TMLK_0} \sum_{\underline{s} \in (S)^k} \text{Prob}(X_{t+i} \in A_j | X_t = \underline{s}) p(\underline{s}) =$$

$$= \sum_{\underline{s} \in (S)^k} p(\underline{s}) \; (1/TMLK_0) \sum_{i=1}^{TMLK_0} \text{Prob}(X_{t+i} \in A_j | X_t = \underline{s}).$$

By the previous argument, for each $\underline{s}$ one can find a set of indices $I \subseteq \{1, \ldots, TMLK_0\}$ such that $|I| = (t_j L - 1)MK_0$ and that

$$\text{Prob}(X_{t+i} \in A_j, \; \forall \; i \in I | X_t = \underline{s}) \geq 1 - \varepsilon_1.$$

Hence, for $i \in I$

$$P(X_{t+i} \in A_j | X_t = \underline{s}) \geq 1 - \varepsilon_1$$

and

$$p(NE_j) \geq \sum_{\underline{s} \in (S)^k} p(\underline{s}) \; (1/TMLK_0) \sum_{i \in I} \text{Prob}(X_{t+1} \in A_j | X_t = \underline{s})$$

$$\geq \sum_{\underline{s}\in(S)^k} p(\underline{s})\ (1/TMLK_0)(t_jL - 1)MK_0(1 - \varepsilon_1)$$

$$\geq (t_j/T - 1/LT)(1 - \varepsilon_1) =$$

$$= t_j/T - 1/LT - \varepsilon_1 t_j/T + \varepsilon_1/LT >$$

$$> t_j/T - 3\varepsilon_1 = t_j/T - \varepsilon_0/m(m - 1),$$

which completes the proof. //

Remark 6.4: It is obvious that for all $k \geq 0$ $SO(k,k,\ldots,k) \subseteq Co(NE)$.

Hence, we have proved that

$$\overline{\bigcup_{k\geq 0} SO(k,k,\ldots,k)} = Co(NE).$$

Proposition 6.5: Assume that $k_1 \geq k_2 \geq k_3 \geq \ldots \geq k_n$. Then $SO(k_1,k_2,\ldots,k_n) = SO(k_2,k_2,k_3,\ldots,k_n)$. (I.e., the player with the longest recall may restrict himself (herself) to strategies which only depend on histories of length equal to the second-longest recall.)

Proof: Assume $(p,\sigma)$ is a steady orbit with recall profile $(k_1,k_2,\ldots,k_n)$. Define a steady orbit $(p',\sigma')$ for the recall profile $(k_2,k_2,\ldots,k_n)$ as follows: for $i > 1$, $\sigma'^i = \sigma^i$. For $i = 1$, $\sigma'^1(\underline{s})$ $(\underline{s} \in (S)^{k_1})$ is the p-mixture of $\sigma^1(\underline{s}'\bullet\underline{s})$ (overall $\underline{s}' \in (S)^{k_1-k_2}$). Since player I's best response strategies constitute a convex set, $\sigma'^1$ is also a best response to

all other players' strategies given $\underline{s}$. On the other hand, $\tau^i$ for $i \neq 1$ has not changed, so that $\sigma'^i$ is also a best response strategy. Finally, define $p'(\underline{s})$ as $\sum_{\underline{s} \in (S)^{k_1 - k_2}} p(\underline{s}' \bullet \underline{s})$. //

Corollary 6.6: For $n = 2$, $\overline{SO} = Co(NE)$. //

Proposition 6.7: $SO \subseteq CE$.

Proof: Given a certain recall profile $\{k_i\}_i$ and a steady orbit $(p,\sigma)$, we will show that $\hat{p}$ is a correlated equilibrium. Hence, perforce, $h(\hat{p}) \in CE$. For some $s \in S$ with $\hat{p}(s) > 0$ and $i \in N$, and we have to show that $s^i$ is a best response move for player $i$ while the other players are playing according to

$$\frac{\hat{p}(\bullet, s^i)}{\sum_{t^{-i} \in S^{-i}} \hat{p}(t^{-i}, s^i)} \in \Delta(S^{-i}).$$

Note that

$$\hat{p}(t^{-i}, s^i) = \sum_{\underline{s} \in (S)^{k-1}} p(\underline{s} \bullet (t^{-i}, s^i)) =$$

$$= \sum_{\underline{s}' \in (S)^k} p(\underline{s}) \, a(\sigma)_{(\underline{s}', \underline{s} \bullet (t^{-i}, s^i))} =$$

$$= \sum_{\underline{r}_1 \in (S)^{k_i}} \sum_{\underline{r}_2 \in (S)^{k - k_i}} p(\underline{r}_2, \underline{r}_1) a(\sigma)_{(\underline{r}_2 \bullet \underline{r}_2, \underline{s} \bullet (t^{-i}, s^i))}$$

$$= \sum_{\underline{r}_1 \in (S)^{k_i}} \tilde{p}(\underline{r}_1) \sum_{\underline{r}_2 \in (S)^{k - k_i}} \bar{p}(r_2) a(\sigma)_{(\underline{r}_2 \bullet \underline{r}_1, \underline{s} \bullet (t^{-i}, s^i))}$$

where

$$\tilde{p}(r_1) = \sum_{\underline{r}_2 \in (S)^{k-k_i}} p(\underline{r}_2 \bullet \underline{r}_1)$$

and

$$\bar{p}(r_2) = p(\underline{r}_2 \bullet \underline{r}_1)/\tilde{p}(\underline{r}_1) \text{ or zero if } \tilde{p}(\underline{r}_1) = 0.$$

By definition of $\tau^i$,

$$\sum_{\underline{r}_2 \in (S)^{k-k_i}} \bar{p}(r_2) a(\sigma)_{(\underline{r}_2 \bullet \underline{r}_1, \underline{s} \bullet (t^{-i}, s^i))} =$$

$$= \sigma^i(\underline{r}_1)(s^i) \bullet \tau^i(p,\sigma)(\underline{r}_1)(t^{-i})$$

whence

$$\hat{p}(t^{-i}, s^i) = \sum_{\underline{r}_1 \in (S)^{k_i}} \tilde{p}(\underline{r}_1)\sigma^i(\underline{r}_1)(s^i)\tau^i(p,\sigma)(\underline{r}_1)(t^{-i})$$

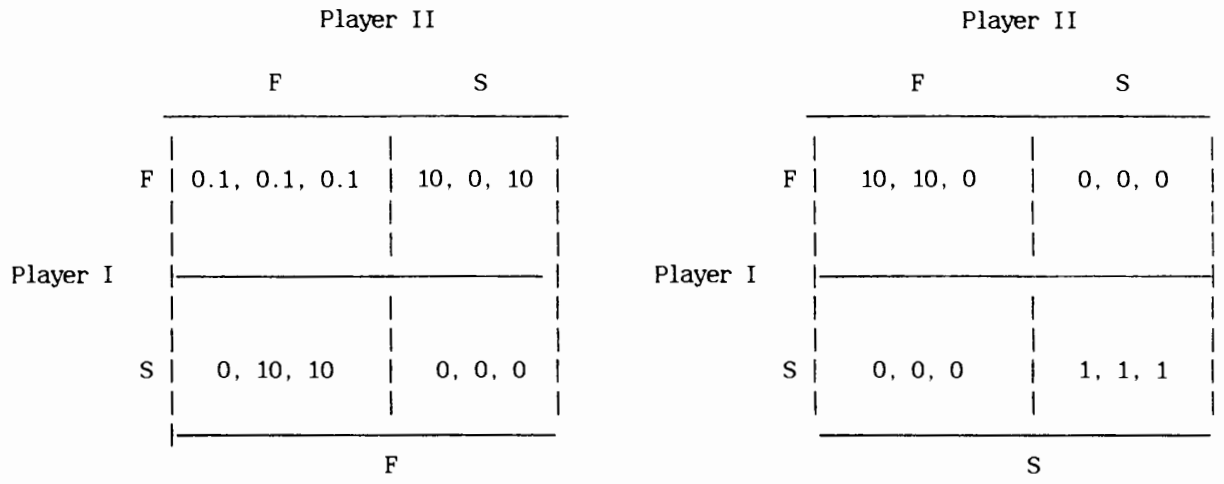
and

$$\sum_{t^{-i} \in S^{-i}} \hat{p}(t^{-i}, s^i) = \sum_{\underline{r}_1 \in (S)^{k_i}} \tilde{p}(\underline{r}_1)\sigma^i(\underline{r}_1)(s^i).$$

Combining these equalities, the conditional distribution of player i on the other players' moves (the ratio of the last two expressions) is a convex combination of $\{\tau^i(p,\sigma)(\underline{r}_1)\}_{\underline{r}_1 \in (S)^{k_i}}$. However, only $\underline{r}_1 \in (S)^{k_i}$ for which $\sigma^i(\underline{r}_1)(s^i) > 0$ have a positive coefficient. Note that for such $\underline{r}_1$ the move $s^i$ has to be a best response to $\tau^i(p,\sigma)(\underline{r}_1)$. Hence $s^i$ is also a best response to the convex combination of these distribution. //

Remark 6.8: SO is not necessarily convex.

Proof: Consider the following three-person game:




(Player III chooses the matrix.) For a recall profile (9,9,0) (that is, $k_1 = k_2 = 9$, $k_3 = 0$), the following is a (pure) steady orbit: Players I and II play S if they have observed (F,F,S) in the last nine periods, and F otherwise. Player III always plays S.

Thus, the steady orbit play consists of nine (F,F,S) and one (S,S,S) repeated cyclically. For players I and II the strategy is obviously a best response one. Player III observes that the other players play (F,F) with 90 percent frequency and (S,S) otherwise. Hence his/her strategy is also a best response one. The average payoff vector is (9.1, 9.1, 0.1).

Similarly, (9.1, 0.1, 9.1) and (0.1, 9.1, 9.1) are also in SO (for different recall profiles). If SO were convex, we would have to conclude that (6.1, 6.1, 6.1) is also in SO. Let us prove that this is impossible. Suppose, then, that $(k_1,k_2,k_3)$ is a recall profile for which there is a steady orbit $(p,\sigma)$ such that $h(\hat{p}) = (6.1, 6.1, 6.1)$ and assume, without loss of generality, that $k_1 \geq k_2 \geq k_3$. Consider $H(s) = h^1(s) + h^2(s) + h^3(s)$ and extend H linearly to $\Delta(S)$. Note that $H(\hat{p}) = 18.3$ and that the only three

points $s \in S$ for which $H(s) \geq 18.3$ are (F,F,S), (F,S,F), and (S,F,F). For those $H(s) = 20$. The maximal value of $H(s)$ for other points $s$ is 3, hence $\hat{p}((S,F,F)) + \hat{p}((F,S,F)) + \hat{p}((F,F,S)) \geq 0.9$. If $\epsilon$ satisfies $\hat{p}((S,F,F)) < \epsilon$, then $h^1(\hat{p}) \geq 9 - 10\epsilon$ whence $\hat{p}((S,F,F)) \geq 0.29$. Hence there must be $\underline{s} \in S^k$ such that the probability of (S,F,F) in the next move is at least 0.29. Suppose that at this node ($\underline{s}$) player II plays S with probability p and player III--with probability q. Since $k_1 = k$, player I knows these probabilities. For him/her to play S with positive probability the following inequality should hold:

$$(1 - p)(1 - q) \geq 0.1\, pq + 10(1 - p)q + 10(1 - q)p.$$

Some algebra shows that this is impossible if $pq \geq 0.29$.

(The above inequality means that

$$20.9\, pq + 1 \geq 11(p + q).$$

Since $pq \geq 0.29$ this means that $(p + q) \leq 7.061/11 \leq 0.65$. for such p,q max $pq \leq 0.11$.) //

Remark 6.9: The above example can also be used to show that $\bar{S}\bar{O}$ is not necessarily convex. Hence $\bar{S}\bar{O}$ is strictly included in CE.

## 7. Concluding Remarks

7.1. It has been argued that steady orbits, with the unavoidable interpretation of repeated games, are not robust with respect to partitions

of the time periods: the infinitely repeated game may be thought of as an infinite repetition of k-repetitions of the one-shot game for some $k > 1$. In this case the set of actions will be larger, and so will the set of equilibria payoffs; in fact, one may get the Folk Theorem.

There is, however, a crucial difference between $k = 1$ and $k > 1$: for $k = 1$ it is reasonable to assume that each player's action is observed by the others. The strategy in a repeated game cannot be observed in the same way.

Admittedly, by the same logic one concludes that steady orbits make more sense for a one-stage simultaneous move game than for general extensive form games. At least for this subclass of games, which may successfully model a wide range of interaction situations, we find steady orbits to be a viable solution concept.

7.2. The study presented above may be viewed as an attempt to formulate the "repeated game" interpretation of Nash equilibrium in the one shot game: a possible motivation for this concept (which is quite often used) is that if the game is repeated, and should a certain play of it be constantly chosen, this play must be a Nash equilibrium. Indeed, this intuition is reinforced by our results if all the players have zero memory length. However, the bounded rationality arguments only imply bounded (and not necessarily equal) memory lengths. Thus, we have found that a larger set of payoffs--namely, SO--may be justified on the same grounds.

7.3. In order to distinguish between real numbers as people seem to perceive them and real numbers as encoding of extremely complicated

strategies we used M-T-Turing machines, having finitely many real-valued registers, and restricted to use them only a bounded number of times in each computation. This sufficed for the bounded recall result to hold (Observation 5.1), and then the behavioral assumption of non-strategic players specified what the players actually will do with their memory registers.

However, we are not quite happy with the computational model presented here: while it restricts the complexity of each single computation, the sequence of moves a player chooses in the repeated game may still be quite complicated. For instance, a player may decide to encode his/her opponent's moves during N periods and then play according to some function of them (say, "tit-for-tat") for the next N periods. It is easy to see that this strategy does not require any recall at all, and that only one register, which is used once in each computation, suffices to implement it.

A natural way to solve this problem is to allow only finite memory instead of real-valued registers. Thus, the memory may contain approximations of real numbers, but cannot be infinitely complex. We rejected this solution because it does not draw the intended distinction: the precision of the approximation will also determine the complexity of the strategy.

Another solution may be to simply define a computational model that can do exactly what we want it to do, namely, update the appropriate statistics. Of course, this is a very restrictive model.

It is therefore left as an open problem to find a general computational model that distinguishes between the way man and machine think of numbers.

7.4. Finally, we note that one may obtain similar results with various versions of the assumptions: one may use a game with "stage 0" and explicitly assume (rather than deduce) bounded recall strategies; one may use Turing machines with finite memory, and so forth. However, we find the version presented here the most satisfactory from a conceptual viewpoint.

## References

Abreu, D. and A. Rubinstein (1986), "The Structure of Nash Equilibrium in Repeated Games with Finite Automata," manuscript.

Aumann, R. J. (1974), "Subjectivity and Correlation in Randomized Strategies," Journal of Math. Economics, 1, 67-95.

Aumann, R. J. (1981), "Survey of Repeated Games," in Essays in Game Theory and Mathematical Economics in Honor of Oskar Morgenstern. Bibliographisches Institut Manheim/Weir/Zurich.

Aumann, R. J. and S. Sorin (1989), "Cooperation and Bounded Recall," Games and Economic Behavior, 1, 5-39.

Ben-Porath, E. (1985), "Repeated Games with Bounded Complexity," manuscript.

Binmore, K. (1986), "Modeling Rational Players" (I,II), to appear in J. Econ. and Phil.

Gilboa, I. (1986), "The Complexity of Computing Best-Response Automata in Repeated Games," to appear in J. of Econ. Theory.

Gilboa, I. and D. Samet (1987), "Bounded Versus Unbounded Rationality: The Strength of Weakness," manuscript.

Hopcroft, J. E. and J. D. Ullman (1979), Introduction to Automata Theory, Languages and Computation, Reading, Mass.

Kalai, E. and W. Stanford (1985), "Equally Sophisticated Players: On the Complexity, Memory and Automation of Repeated Game Strategies," manuscript.

Kalai, E. and W. Stanford (1986), "Finite Rationality and Interpersonal Complexity in Repeated Games," manuscript.

Lehrer, E. (1988a), "Repeated Games with Stationary Bounded Recall

Strategies," J. of Econ. Theory, 46, 130-144.

Lehrer, E. (1988b), "n Players with Bounded Recall in Infinitely Repeated Games," manuscript.

Megiddo, N. and A. Wigderson (1986), On Play by Means of Computing Machines," manuscript, Northwestern University.

Nash, J. (1951), "Non-Cooperative Games," Annals of Mathematics, 54, 286-295.

Neyman, A. (1984), "Bounded Complexity Justifies Cooperation in the Finitely Repeated Prisoner's Dilemma," manuscript.

Rubinstein, A. (1986), "Finite Automata Play the Prisoner's Dilemma," J. of Econ. Theory, 39 (1), 83-96.

Schwartz, G. (1974), "Randomizing When Time is not Well-Ordered," Israel J. of Math., 19, 241-245.

Simon, H. A. (1972), "Theories of Bounded Rationality," in Decision and Organization, C. B. McGuire and R. Radner, eds., North-Holland, Amsterdam.

Simon, H. A. (1978), "On How to Decide What to Do," Bell J. Econ., 9, 494-507.

Stanford, W. (1987), "Some Simple Equilibria in Repeated Games," manuscript.