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**Working Paper**

## Nash and Correlated Equilibria: Some Complexity Considerations

Discussion Paper, No. 777

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*Suggested Citation:* Gilboa, Itzhak; Zemel, Eitan (1988) : Nash and Correlated Equilibria: Some Complexity Considerations, Discussion Paper, No. 777, Northwestern University, Kellogg School of Management, Center for Mathematical Studies in Economics and Management Science, Evanston, IL

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Discussion Paper No. 777

NASH AND CORRELATED EQUILIBRIA:  
SOME COMPLEXITY CONSIDERATIONS\*

by

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and

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June 1988

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\*We wish to thank Ehud Kalai, Nimrod Megiddo, Roger Myerson, Dov Samet, and Israel Zang for helpful comments and stimulating discussions.

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## Abstract

This paper deals with the complexity of computing Nash and correlated equilibria for a finite game given in its normal form. In order to focus on the distinction between "easy" (i.e., polynomial) and "hard" (i.e., NP-hard) problems, we discuss YES/NO problems. However, the natural problems of existence are trivial from an algorithmic point of view: every game is known to have a Nash equilibrium (in mixed strategies)--hence, perforce, a correlated one. We therefore deal with the existence of equilibria satisfying a certain condition, such as: "Given a game  $G$  and a number  $r$ , is there a Nash (correlated) equilibrium of  $G$  in which all players obtain an expected payoff of at least  $r$ ?", or: "Given a game  $G$ , is there a unique Nash (correlated) equilibrium in  $G$ ?", and so forth.

We show that most of these problems turn out to be "hard" for Nash equilibria but "easy" for correlated equilibria. These results may be interpreted as implying that, at least for some purposes, the correlated equilibrium is a more reasonable solution concept than the Nash equilibrium.

## 1. Introduction

### 1.1 Motivation

Game-theoretic solution concepts may be theoretically interpreted and practically applied in numerous ways and in a variety of contexts. For some of these interpretations, the complexity of computing the equilibrium may be absolutely irrelevant. For instance, one may think of a Nash equilibrium as a condition which has to be satisfied by any steady state in a certain dynamic biological system. Such an application may be supported without assumptions on the players' rationality and, more specifically, without assuming that any of them "computed" the equilibrium.

However, there is a large class of applications--especially in economic theory--which do implicitly assume that a rational decision maker is faced with the technical problem of computing equilibria. For instance, whenever the Nash equilibrium concept is interpreted as a self-enforcing agreement among rational players, which is attained by negotiation or suggested to them by another party or even read by the players from a certain "game theory guide," it is implicitly assumed that someone computes Nash equilibria. This "someone" may be the players themselves, or the "other party", or the "Game Theory Guide" author. At any rate, this "someone" is not an omniscient super-being--it eventually turns out to be a person or a machine for which bounded rationality considerations and computational restrictions do apply.

We therefore believe that the complexity of computing a certain solution concept is one of the features determining its plausibility for a whole range of theoretically and practical applications.

This paper deals with two of the most widely-used solution concepts for

noncooperative games: the Nash equilibrium (introduced by Nash (1951)), and the correlated equilibrium (introduced by Aumann (1974)). The main results, given in the next sub-section, may be summarized, in very bold strokes, as saying that Nash equilibrium is a complicated solution concept, whereas correlated equilibrium is a simple one.

In Section 2 we give some preliminaries and provide the basic definitions. The proofs are to be found in Section 3. Section 4 is devoted to some technical remarks.

### 1.2 The Results

Assuming familiarity with the standard definitions quoted in Section 2, we may state our main results. We first define the problems.

In the following definitions, the word "game" should be interpreted as a finite game with rational payoffs given in its normal form. Each definition relates to two problems--one for Nash equilibrium (NE) and one for correlated equilibrium (CE):

- 1) NE (CE) Max Payoff: Given a game  $G$  and a number  $r$ , does there exist a NE (CE) in  $G$  in which each player obtains the expected payoff of at least  $r$ ?
- 2) NE (CE) Uniqueness: Given a game  $G$ , does there exist a unique NE (CE) in  $G$ ?
- 3) NE (CE) in a Subset: Given a game  $G$  and a subset of strategies  $T_i$  for each player  $i$ , does there exist a NE (CE) of  $G$  in which all strategies not included in  $T_i$  (for each  $i$ ) are played with probability zero?
- 4) NE (CE) Containing a Subset: Given a game  $G$  and a subset of

strategies  $T_i$  for each player  $i$ , does there exist a NE (CE) of  $G$ , in which every strategy in  $T_i$  (for every player  $i$ ) is played with positive probability?

- 5) NE (CE) Maximal Support: Given a game  $G$  and an integer  $k \geq 1$ , does there exist a NE (CE) of  $G$  in which each player uses at least  $k$  strategies with positive probability?
- 6) NE (CE) Minimal Support: Given a game  $G$  and an integer  $k \geq 1$ , does there exist a NE (CE) of  $G$  in which each player uses no more than  $k$  strategies with positive probability?

#### Theorem

(a) The following problems are NP-hard (NPH): NE max payoff; NE uniqueness; NE in a subset; NE containing a subset; NE maximal support; NE minimal support; CE minimal support.

(b) The following problems are of polynomial time complexity (P): CE max payoff; CE uniqueness; CE in a subset; CE containing a subset; CE maximal support.

These results may be summarized in the following table:

Table 1

	<u>NE</u>	<u>CE</u>
max payoff	$\text{NPH}^{(1)}$	P
uniqueness	$\text{NPH}^{(2)}$	P
in a subset	$\text{NPH}^{(1)}$	P
containing a subset	$\text{NPH}^{(1)}$	P
maximal support	$\text{NPH}^{(1)}$	P
minimal support	$\text{NPH}^{(1,4)}$	$\text{NPH}^{(3,4)}$

- (1) NPC for two player  
(2) CoNPC for two players  
(3) NPC for any number of players  
(4) NPH even for zero-sum games.

## 2. Preliminaries

### 2.1 Game Theory Definitions

A game (to be precise, a noncooperative game in normal form) is a triple  $(N, (S^i)_{i \in N}, (h^i)_{i \in N})$  where  $N$  is a nonempty set (of players),  $S^i$  is a nonempty set (of strategies of player  $i$ ) for every  $i \in N$  and  $h^i: S \rightarrow \mathbb{R}$  for every  $i$ , where  $S \equiv \prod_{i \in N} S^i$  ( $h^i$  is the payoff function of player  $i$ ). A game  $G = (N, (S^i)_{i \in N}, (h^i)_{i \in N})$  is called finite if the set  $N$  and all sets  $(S^i)_{i \in N}$  are finite. We will henceforth discuss only finite games. Since we are interested in computational issues, we will also assume that the game data is rational, i.e.,  $h^i: S \rightarrow \mathbb{Q}$  rather than  $S \rightarrow \mathbb{R}$ .

Given a finite game  $G = (N, (S^i)_{i \in N}, (h^i)_{i \in N})$  in which we assume, w.l.o.g. (without loss of generality), that  $N = \{1, \dots, n\}$ , we define the (mixture) extension of  $G$  to be the game  $\bar{G} = (N, (\Sigma^i)_{i \in N}, (H^i)_{i \in N})$  where:

- 1)  $\Sigma^i$  is the set of all probability vectors over  $S^i$  (the set of mixed

strategies of player  $i$ );

- 2) For every  $\sigma = (\sigma^1, \sigma^2, \dots, \sigma^n) \in \Sigma \equiv \prod_{i=1}^n \Sigma^i$  we define a measure  $P_\sigma$  on  $S$  by  $P_\sigma(s) = \prod_{i=1}^n \sigma^i(s^i)$  where  $s = (s^1, s^2, \dots, s^n)$ , and  $H^i(\sigma)$  is the expected payoff to player  $i$  according to  $P_\sigma$ , i.e.,

$$H^i(\sigma) = \sum_{s \in S} P_\sigma(s) h^i(s)$$

An  $n$ -tuple of mixed strategies  $\bar{\sigma} = (\bar{\sigma}^1, \bar{\sigma}^2, \dots, \bar{\sigma}^n) \in \Sigma$  is called a Nash equilibrium of  $G$  (in mixed strategies) if the following condition holds for every  $i \in N$  and  $\sigma^i \in \Sigma^i$ :

$$H^i(\bar{\sigma}) \geq H^i(\bar{\sigma}^1, \bar{\sigma}^2, \dots, \bar{\sigma}^{i-1}, \sigma^i, \bar{\sigma}^{i+1}, \dots, \bar{\sigma}^n).$$

That is to say,  $\bar{\sigma}$  is a Nash equilibrium if no player can increase his/her expected payoff by a unilateral deviation from the (equilibrium) strategy suggested for him/her by  $\bar{\sigma}$ .

Nash (1951) has shown that every finite game has a Nash equilibrium in mixed strategies as above. The proof uses topological arguments (Brouwer's fixed point theorem); to the best of our knowledge, there is no "elementary" proof of this fact. Hence it seems unlikely that the existing proofs of existence will be used to develop a polynomial algorithm for the computation of Nash equilibria, though they may give some insight for the development of iterative algorithms which, in turn, may prove useful for practical purposes. (See, for instance, Samuelson (1988).)

We now turn to correlated equilibria. In these, it is assumed that the players have some randomization device they may all observe simultaneously.



Hence any probability distribution on  $S$  may now be considered as a solution of the game, rather than the smaller set of distributions which are the product of independent marginal distributions. A correlated equilibrium is therefore defined to be a probability distribution  $p$  on  $S$  which satisfies the following condition:

For every  $s = (s_1, s_2, \dots, s_n) = (s_{j_1}^1, s_{j_2}^2, \dots, s_{j_n}^n) \in S$  such that  $p(s) > 0$ , for every player  $i \in N$  and for every strategy  $\bar{s}^i \in S^i$ ,

$$\sum_{\{s \in S | s_i = s_{j_i}^i\}} p(s) h^i(s) \geq \sum_{\{s \in S | s_i = s_{j_i}^i\}} p(s) h^i(s_{j_1}^1, \dots, \bar{s}^i, \dots, s_{j_n}^n).$$

The intuition which stands behind this definition is the following. Suppose an  $(n + 1)$ th party chooses each  $s \in S$  with probability  $p(s)$ , and reveals to each player only his/her component  $s_i$  of  $s$ . Given this information, and assuming that the other players will play the strategy "recommended" to them by the  $(n + 1)$ th party, player  $i$  has a conditional probability regarding the other players' choices. It is required that the strategy "recommended" to player  $i$ , that is,  $s_{j_i}^i$ , will be optimal for him/her given this conditional probability.

Aumann (1974), who introduced the concept of correlated equilibria, also noted that every Nash equilibrium in mixed strategies induces a correlated equilibrium defined by the product of the players' mixed strategies. This also implies that for every game there are correlated equilibria. However, Hart and Schmeidler (1986) noted that correlated equilibria are defined by a finite set of linear inequalities, and showed (without using Nash's result) that the feasible set induced by these inequalities is nonempty for all games. (Their result also deals with games

with an infinite number of players, for which there is no Nash equilibrium in general.) In fact, all the results presented in this paper, which prove that a certain CE problem is easy, use this observation and that linear programming problems can be solved in polynomial time.

## 2.2 Computer Science Definitions

Unfortunately, we cannot provide succinct and formal definitions for the terms we will use. For the sake of brevity, we will only provide short and intuitive explanations, and the interested reader is referred to Aho, Hopcroft, and Ullman (1974) for formal definitions.

By a problem we refer a YES/NO problem, i.e., a function  $A(\bullet)$  from the set of inputs to the set  $\{\text{YES}, \text{NO}\}$ . An instance of a problem is a given input. The size,  $|x|$ , of an instance  $x$  is the number of digits in the encoding of  $x$ .

An algorithm  $T$  is a well-defined set of instructions which may be identified with a Turing machine and thought of as a computer program with a specific output state denoted YES. Let  $T_1$  be the set of inputs such that  $T$ , when given  $x$ , reaches the output state YES within a finite number of steps. In that case, the number of steps is called the running time of  $T$  on  $x$ .

An algorithm  $T$  is said to solve the problem  $A$  if  $T_1 = \{x: A(x) = \text{YES}\}$ , i.e., it reaches the state YES precisely on the correct set of inputs. The computational complexity of  $T$ ,  $c(n)$ , is the maximum running time, over all inputs  $x \in T_1$  such that  $|x| \leq n$ . Note that this definition is not symmetric with respect to replacing YES by NO. We will focus on the order of magnitude of  $c(n)$ , rather than on the function itself. More specifically, we will be interested in the existence of "polynomial algorithms," that is,

algorithms for which the time complexity  $c(n)$  is bounded from above by some polynomial of  $n$ . The set of all problems for which there exists such an algorithm is denoted by  $P$ . Most of the well-known optimization problems, such as the traveling salesman problem, the set covering problem, the knapsack problem, etc., are generally believed to be outside  $P$ . Rather, they are known to be in a set containing  $P$ , which is called  $NP$ .

A problem is called NP (or belongs to the class  $NP$ ) if there is a nondeterministic Turing machine which solves it in polynomial time. One may think of a nondeterministic Turing machine as a computer with an unbounded number of processors working in parallel. Intuitively, a problem  $A$  is in  $NP$  if one can "prove" in polynomial time that  $A(x) = \text{YES}$ . For example, the problem of deciding whether a certain graph contains an Hamiltonian tour is not known to be in  $P$ , i.e., we do not know of a polynomial algorithm for it. However, this problem is in  $NP$  since we can prove in polynomial time that a given graph is in fact Hamiltonian by presenting the algorithm with a Hamiltonian tour. The only "polynomial" requirement is for the algorithm to check in polynomial time that the presented tour is in fact Hamiltonian.

As noted previously, the definition of running time is not invariant under complementation of YES and NO. The class of problems whose complements are in  $NP$  is called  $CoNP$ . In other words, a problem  $A$  is in  $CoNP$  if one can prove in polynomial time that  $A(x) = \text{NO}$ . Obviously,  $P \subseteq NP$  but the question of whether the containment is strict is still open. It is also not known whether or not  $NP = CoNP$ . The "evidence" so far suggests a negative answer to both these questions.

On the set of problems one may define the binary relation "is (polynomially) easier than" or "can be polynomially reduced to" as follows.

A problem A is easier than B if there is a polynomial algorithm which translates every possible input of A to an input of B, such that all A-inputs for which the A-answer is "YES," and only those, are mapped to B-inputs for which the B-answer is "YES." (In this case, we will also say that B is harder than A.) In particular, if A is easier than B according to this definition, then the following conditional statement is true: if there were a polynomial algorithm solving B, there would also be one for A.

A problem which is "harder," in the above sense, then all problems in NP is called NP hard (NPH). If such a problem is in NP it is called NP complete (NPC). Similarly, if it is in CoNP it is called CoNP complete (CoNPC). If a polynomial algorithm existed for any NP hard problem, then we would have a polynomial algorithm for all problems in NP and also in CoNP. In that case  $P = NP = CoNP$ . For this reason NP hard problems are considered hard: there are no known polynomial algorithms for them, and computer scientists tend to believe that such algorithms are unlikely to be found.

### 3. Proofs

In this section we provide the proof of our theorem. To each problem we devote a subsection showing whether it is NPH or P. Then, in Section 3.13, we show that for the case of two players, the NE problems are in NP except for NE uniqueness which is in CoNP. Finally, we show that minimal support CE is also in NP.

#### 3.1 NE Max Payoff

The proof is by reduction of the clique problem, defined as follows. Given an undirected graph  $G_r = (V, E)$  and an integer  $k$ , does there exist a

clique of size  $k$  in  $Gr$ ? That is, does there exist  $V' \subseteq V$ ,  $|V'| = k$  such that  $\{i,j\} \in E$  for all  $i,j \in V'$ ?

(The clique problem is known to be NPC.)

Given a graph  $Gr = (V,E)$  where, without loss of generality,  $V = \{1, \dots, n\}$  and a number  $k$ , construct a two-person game  $G$  as follows:

$$h^1((1,i),(1,j)) = h^2((1,i),(1,j)) = \begin{cases} 1 + \epsilon & \text{if } i = j \\ 1 & \text{if } i \neq j, \{i,j\} \in E \\ 0 & \text{otherwise} \end{cases}$$

where  $\epsilon = 1/nk$

$$h^1((2,i),(1,j)) = \begin{cases} k & i = j \\ 0 & i \neq j \end{cases}$$

$$h^2((2,i),(1,j)) = \begin{cases} -M & i = j \\ 0 & i \neq j \end{cases}$$

$$h^1((1,i),(2,j)) = \begin{cases} -M & i = j \\ 0 & i \neq j \end{cases}$$

$$h^2((1,i),(2,j)) = \begin{cases} k & i = j \\ 0 & i \neq j \end{cases}$$

$$h^1((2,i),(2,j)) = h^2((2,i),(2,j)) = 0.$$

where  $M = nk^2$ . The game matrix is given in Figure 1.

<Insert Figure 1 about here>

Claim:  $G$  has a NE with expected payoff of at least  $r \equiv 1 + \epsilon/k$  for both players iff  $G_r$  has a clique of size  $k$ .

Proof: First assume that  $G_r$  has such a clique, say  $\{i_1, i_2, \dots, i_k\}$ . Define mixed strategies  $p$  for player 1 and  $q$  for player 2 by

$$p_{(1,i_j)} = q_{(1,i_j)} = 1/k \text{ for } 1 \leq j \leq k.$$

It is easy to verify that  $p$  and  $q$  constitute a Nash equilibrium in which both players obtain the payoff  $r$ .

Conversely, assume that  $p = (p_{(1,1)}, \dots, p_{(2,n)})$  and  $q = (q_{(1,1)}, \dots, q_{(2,n)})$  are two strategies which form a NE in  $G$ , such that the expected payoff of each player is at least  $r$ . We want to show that  $G_r$  has a clique of size  $k$ .

Claim 1: For every  $1 \leq i \leq n$ ,  $p_{(2,i)}, q_{(2,i)} < 1/n(M + 1)$ .

Proof: For a given  $i \leq n$ , assume  $p_{(2,i)} > 0$ . This implies that  $E(n^1 | (2,i), q) \geq r$ . (Here and in the sequel, this expression means the expected payoff of player 1 given that he/she plays the pure strategy  $(2,i)$  and that player 2 plays the mixed strategy  $q$ . We will also use the obvious variations of this notation.) This is possible only if  $q_{(1,i)} > 0$ . But by the same argument, the latter implies  $E(n^2 | p, (1,i)) \geq r$ . A simple calculation shows that  $p_{(2,i)} < 1/n(M+1)$  follows. The proof for  $q$  is symmetric.  $[\ ]$

We now know that most of the probability mass of the mixed strategy of each player is concentrated on the first  $n$  strategies. Using this fact we will show that at least  $k$  of them are chosen with positive probability:

Claim 2:  $|\{i | p_{(1,i)} > 0\}|, |\{i | q_{(1,i)} > 0\}| \geq k$ .

Proof: Assume the contrary, say  $|\{i | q_{(1,i)} > 0\}| < k$ . Using Claim 1, this implies that for at least one index  $1 \leq i \leq n$ ,  $q_{(1,i)} > (1 - 1/nk^2)/(k-1)$ . Simple and not-too-tedious calculations then show that  $E(n^1 | (2,i), q) > 1 + \epsilon$ . But this is possible only if  $p_{(1,i)} = 0$  for all  $j \leq n$ , which is known to be false.  $[\ ]$

Claim 3: For all  $i \leq n$ , if  $p_{(1,i)} > 0$  then  $q_{(1,i)} \geq 1/k$ ;  
and if  $q_{(1,i)} > 0$  then  $p_{(1,i)} \geq 1/k$ .

Proof: Assume  $p_{(1,i)} > 0$ . The only payoff which exceeds 1 in the  $(1,i)$  row

is attained in column  $(1, i)$ . Hence  $E(h^1 | (1, i), q) \geq r = 1 + \epsilon/k$  only if  $q_{(1, i)} \geq 1/k$ . (And the other part is proved symmetrically.) []

Combining the conclusions of claims 2 and 3 we deduce that there are  $k$  indices  $1 \leq j_1 < j_2 < \dots < j_k \leq n$  such that  $p_{(1, j_\ell)} = q_{(1, j_\ell)} = 1/k$  for  $1 \leq \ell \leq k$ . It is now obvious that these indices correspond to a clique of size  $k$  in  $Gr$ . []

### 3.2 NE Uniqueness

Again we use the clique problem. Given a graph  $Gr$  we construct a game  $G$  as in subsection 3.1, only that now we add another strategy--say  $0$ --to each player. Each player may guarantee himself/herself the payoff  $r$  by choosing the strategy  $0$ , but if only one of them chooses  $0$ , the other one gets the payoff  $-M$ . Hence  $(0, 0)$  is certainly a NE. However, there are no Nash equilibria in which either one of the players obtains less than  $r$ . It is easy to see that  $(0, 0)$  is a unique NE iff the graph  $Gr$  does not have a clique of size  $k$ .

### 3.3 NE in a Subset

Use the construction of subsection 3.2 and define the subsets to be all strategies (of each player) but the one denoted  $0$ .

### 3.4 NE Containing a Subset

The proof uses the clique problem again. Given a graph  $Gr$  and a number  $k$ , construct a graph  $Gr'$  by adding one vertex which is connected to all the previous ones. Obviously  $Gr$  has a clique of size  $k$  iff  $Gr'$  has a clique of



size  $(k + 1)$  which includes the new node. Then construct a game  $G$  as described in subsection 3.2 for the graph  $Gr'$  and the integer  $(k + 1)$ . This game will have a NE in which the strategy, corresponding to the new node in  $Gr'$ , is played with positive probability iff  $Gr$  has a clique of size  $k$ .

### 3.5 NE Maximal Support

Again, use the construction of 3.2.

### 3.6 NE Minimal Support

In this subsection we will prove a stronger result than originally stated: we will prove that the NE minimal support problem is NPH even if the input is restricted to be a two-person zero-sum game. This result will also be used to show that CE-minimal support is also NPH. To this end we need a new construction, and this time we will use the set cover problem, which is also known to be NPC. The version we use is the following.

Set Cover: Given a number  $n \geq 1$  and  $r$  subsets  $T_1, T_2, \dots, T_r$  of  $N = \{1, \dots, n\}$  such that  $\bigcup_{1 \leq j \leq r} T_j = N$ , and a number  $k \leq r$ , are there  $k$  indices  $1 \leq j_1 < j_2 < \dots < j_k \leq r$  such that  $\bigcup_{1 \leq i \leq k} T_{j_i} = N$ ?

We will now show that Set Cover can be reduced to the following problem.

Zero-Sum NE Minimal Support: Given a two-person zero-sum game  $G$  and a number  $k \geq 1$ , is there a NE in  $G$  in which both players use no more than  $k$  strategies with positive probability?

Proof: Let there be given an integer  $n \geq 1$ ,  $r$  subsets  $T_1, \dots, T_r$  of  $N = \{1, \dots, n\}$  such that  $\bigcup_{1 \leq j \leq r} T_j = N$  and an integer  $k \leq r$ . Define the following game  $G$ :

$$S^1 = \{1, 2, \dots, r, r+1\}$$

$$S^2 = \{1, 2, \dots, n, n+1\}$$

$$h^1(j, i) = -h^2(j, i) = \begin{cases} 1 & j \leq r, i \leq n, i \in T_j \\ 0 & j \leq r, i \leq n, i \notin T_j \\ 1/k & j \leq r, i = n+1 \\ 1/2r & j = r+1 \end{cases}$$

Claim: The set  $N$  has a cover of size  $k$  out of  $\{T_1, \dots, T_r\}$  iff the game  $G$  has a NE in which both players do not use more than  $k$  strategies with positive probabilities.

Proof of Claim: First assume that  $G$  has a NE as required. Consider player 1's strategy given by  $p_j = 1/r$  for  $1 \leq j \leq r$  (and  $p_{r+1} = 0$ ). This strategy ensures player 1 the expected payoff  $1/r$ . Hence the pure strategy  $r+1$  is not an optimal (maxmin) strategy for player 1, and cannot be played with probability 1 at any equilibrium. Hence, if  $p$  and  $q$  are the equilibrium strategies of players 1 and 2, respectively, the set  $J = \{1 \leq j \leq r \mid p_j > 0\}$  is nonempty. We claim that  $\bigcup_{j \in J} T_j = N$ . Indeed, if the sets  $\{T_j\}_{j \in J}$  fail to cover the set  $N$ , there exists an  $i \in N$  for which  $E(h^1 | p, i) < 1/2r$ . In this case,  $p$  again is not a maxmin strategy for player 1. We then conclude

that  $\{T_j\}_{j \in J}$  is a cover of  $N$ . But our assumption on the Nash equilibrium under consideration implies that  $|J| \leq k$ . This completes the first half of the proof.

Conversely, assume that there are  $1 \leq j_1 < j_2 < \dots < j_k \leq r$  such that  $\bigcup_{1 \leq \ell \leq k} T_{j_\ell} = N$ . Let  $p$  be a mixed strategy of player 1 defined by  $P_{j_\ell} = 1/k$  for  $1 \leq \ell \leq k$ , and let  $q$  be player 2's strategy defined by  $q_{n+1} = 1$ . The minimal expected payoff for player 1 should he/she choose  $p$ , is  $1/k$ . This is also the maximal expected loss incurred on player 2 should he/she choose  $q$ . Hence these are optimal strategies and they constitute a NE of  $G$ . []

### 3.7 CE Minimal Support

In order to prove that this problem is NPH we will use the proof in subsection 3.6. The main point is that for zero-sum games the two concepts of equilibria coincide in terms of both the equilibrium payoffs and the strategies which may be used (at equilibrium) with positive probability.

We first note the following.

Claim 1: Let  $p$  be a correlated equilibrium in a two-person zero-sum game  $G$ .

Then  $E(h^1|p)$  equals the value of the game  $V(\equiv \max_p \min_q E(h^1|p,q) = \min_q \max_p E(h^1|p,q))$ .

Proof: Assume the contrary, e.g.,  $E(h^1|p) < V$ . (The other case is symmetric.) This implies that there are  $(i,j) \in S$  such that  $p_{ij} > 0$  and

$$\sum_j (p_{ij} / \sum_k p_{ik}) h^1(i,j) < V.$$

By definition, player 1 has an optimal strategy which assures him the payoff  $V$  against any strategy of player 2, in particular  $(p_{ij}/\sum_k p_{ik})_{j \in S^2}$ . This strategy may be a mixed one, but there must be at least one pure strategy  $q \in S^1$  such that  $E(h^1 | q, (p_{ij}/\sum_k p_{ik})_{j \in S^2}) \geq V$ . This implies that  $p$  is not a CE. []

Claim 2: Let  $p$  be a CE of a two-person zero-sum game  $G$ . Then for every  $(i, j) \in S$  such that  $p_{ij} > 0$ ,  $(p_{i\ell}/\sum_k p_{ik})_q$  and  $(p_{mj}/\sum_k p_{kj})_m$  are optimal strategies for players 2 and 1, respectively.

Proof: Let us consider player 2's strategy (the argument for player 1 is, of course, symmetric.) Consider

$$\text{Max}_{m \in S^1} \sum_{q \in S^2} (p_{i\ell}/\sum_k p_{ik}) h^1(m, q).$$

Since  $p$  is a CE, this maximum is obtained at  $m = i$ . But claim 1 shows that the maximal value is then the value of the game  $V$ . This is just the definition of a minimax strategy for player 2. []

The proof will now be completed by the following:

Proposition: Let there be given a two-person zero-sum  $G$  and an integer  $k$ .  $G$  has a NE in which each player uses no more than  $k$  strategies with positive probability iff it has such a CE.

Proof: The "only if" part is trivial since each NE also constitutes a CE

(in which exactly the same strategies are played with positive probability). For the "if" part, assume that there exists such a CE  $p$ , and pick a pair  $(i, j) \in S$  such that  $p_{ij} > 0$ . By Claim 2,  $(p_{i\ell} / \sum_k p_{ik})_{\ell}$  and  $(p_{mj} / \sum_k p_{kj})_m$  are optimal strategies, hence a NE. Obviously, in this NE the strategies which are played with positive probability are also played with positive probability in the CE  $p$ . []

### 3.8 CE Maximal Payoff

In view of the observations in Section 2.1, this problem is simply an LP (linear programming) one.

### 3.9 CE Uniqueness

Given the set of linear constraints on  $(p_s)_{s \in S}$  defining a CE in a given game, one may solve two LP problems for each  $s \in S$ . One of them will have the objective function  $\text{Max } p_s$ , and the other,  $\text{Min } p_s$ , while both share the same feasible set. Obviously, the constraints define a unique CE iff all these problems have the same solution.

### 3.10 CE in a Subset

This problem is again solved by linear programming where one constrains the appropriate variables to be zero.

### 3.11 CE Containing a Subset

Again, for each  $s \in S$ , one solves the LP problem defined by the feasible set of CE and the objective function  $\text{Max } p_s$ . Then one takes the arithmetic average of all solution vectors obtained. Of course, this is a

CE since the set of CE is convex. Furthermore, if there exists a CE at which  $p_s > 0$  for some  $s \in S$ , then  $p_s > 0$  also in this average solution. Hence, for given sets  $T_i$  (for every player  $i$ ) it only remains to check whether for every  $s_j^i \in T_i$  and every  $i \in N$  there is an  $s = (s^1, s^2, \dots, s_j^i, \dots, s^n)$  with  $p_s > 0$ . (Note that this may be carried out in time complexity which is polynomial in the size of the game.)

### 3.12 CE Maximal Support

Identical to 3.11.

### 3.13 Membership in NPC and CoNPC

We briefly show here that for two players the NE problems are in NP except for uniqueness which is in CoNP. Then we show that CE minimal support is also NPC (for any number of players).

We start by analyzing NE for two players. In this case, each NE is a solution to a polynomial set of equalities involving the (rational) matrices  $h^1$  and  $h^2$  as coefficients. It is well known that basic solutions to such systems are themselves rational, of size polynomial in the original data. Thus, all basic NE for a given game are of polynomial size. Also, given a polynomial set of alleged basic NE's, it is easy to verify in polynomial time that these are, in fact, NE's satisfying any of the additional properties such as Max Payoff, maximal support, etc. For the case of NE uniqueness, we can easily disprove this property in polynomial time by presenting a pair of distinct NE.

We now consider the case of CE minimal support for any number of players. As mentioned earlier, CE can be presented as a linear programming

problem so that its basic solution is of polynomial size. Furthermore, a CE satisfying the minimal support property can be chosen basic. Thus the problem is in NP.

This completes the proof of our main theorem.

#### 4. Some Remarks

4.1 Our results do not imply NP hardness for the problem of computing any NE for a given game. (The YES/NO problem which corresponds to this question is the trivial problem of existence of NE). In fact, Megiddo (1988) has shown that, for the case of two players, the problem is not NP hard unless  $NP = CoNP$ , an unlikely event. The problem for the general case is still open.

4.2 All the complexity analysis carried out here referred to the "worst case" complexity of exact algorithm. It is conceivable that problems which are hard with respect to this measure are in fact easy in the "average" case or if approximations, rather than exact algorithms, are concerned. This topic is currently under further study.

4.3 Our results about minimal and maximal support for CE can also be used to show that, for a given general set of linear constraints, it is "easy" to find a solution with the maximal number (or the maximal set) of positive variables, but it is "hard" to find the solution with the minimal number of positive variables.

	(1,1) ...	(1,n)	(2,1) ...	(2,n)
(1,1)	$(1+\epsilon, 1+\epsilon)$ $(1+\epsilon, 1+\epsilon)$ $\cdot$ $(e_{ij}, e_{ij})$ $\cdot$ $(e_{ij}, e_{ij})$		$(-M, k)$ $(-M, k)$ $\cdot$ $(0, 0)$ $\cdot$ $(0, 0)$	
(1,n)		$(1+\epsilon, 1+\epsilon)$		$(-M, k)$
(2,1)	$(k, -M)$ $(k, -M)$ $\cdot$ $(0, 0)$ $(0, 0)$ $\cdot$ $\cdot$			$(0, 0)$
(2,n)		$(k, -M)$		

$$e_{ij} = 1_{\{\{i,j\} \in E\}}$$

Figure 1



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