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Working Paper

Vector Measures are Open Maps

Discussion Paper, No. 510

Provided in Cooperation with:

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Suggested Citation: Samet, Dov (1981) : Vector Measures are Open Maps, Discussion Paper, No. 510, Northwestern University, Kellogg School of Management, Center for Mathematical Studies in Economics and Management Science, Evanston, IL

This Version is available at:

<https://hdl.handle.net/10419/220870>

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DISCUSSION PAPER NO. 510

VECTOR MEASURES ARE OPEN MAPS

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NOVEMBER, 1981

* The author acknowledges Zvi Artstein for a helpful discussion.

VECTOR MEASURES ARE OPEN MAPS

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ABSTRACT

Nonatomic vector measures are shown to be open maps from the σ -field on which they are defined to their range, where the σ -field is equipped with the pseudo-metric of the symmetric difference with respect to a given scalar measure.

We prove the following:

Main Theorem: Let $\lambda, \mu_1, \dots, \mu_n$ be nonatomic, σ -additive, finite measures on a measurable space (I, Σ) , and let λ be a nonnegative measure. Then the vector measure $\mu = (\mu_1, \dots, \mu_n)$ is an open map from Σ to the range of μ , where Σ is equipped with the topology induced by the pseudo-metric d_λ defined by

$d_\lambda(S, T) = \lambda[(S \setminus T) \cup (T \setminus S)]$, and the range of μ is equipped with its relative topology in R^n .

Let us introduce the following notations. For S in Σ we denote by \bar{S} the complementary set $I \setminus S$. The symmetric difference of S and T , $(S \setminus T) \cup (T \setminus S)$ is denoted by $S \Delta T$. The Euclidian norm in R^n is denoted by $\| \cdot \|$, and the scalar product of ξ and x in R^n is denoted by $\langle \xi, x \rangle$. By the relative boundary of a closed set K in R^n we mean the set of all points in K which are not in the relative interior of K . The face of a convex set K in the direction ξ , is the set $F(\xi) = \{x \mid x \in K, \langle \xi, x \rangle = \max_{y \in K} \langle \xi, y \rangle\}$. We say that a set K in R^n is strictly convex if all the points on the relative boundary of K are extreme, or alternatively if for each $\xi \in R^n$, $F(\xi)$ is either K or a singleton. For a scalar measure λ , we denote by $|\lambda|$ the sum of the positive and the negative parts of λ . For a vector measure $\mu = (\mu_1, \dots, \mu_n)$, $|\mu|$ is the sum $\sum_{i=1}^n |\mu_i|$. For each S we define $R(\mu, S) = \{\mu(T) \mid T \subseteq S\}$. Clearly $R(\mu, S) + R(\mu, \bar{S}) = R(\mu, I)$. By Lyapunov Theorem [1], $R(\mu, S)$ is a convex and compact set.

A convenient way to describe $R(\mu, I)$ is as follows. Let f_i be the Radon-Nikodym derivative of μ_i with respect to $|\mu|$. and let $f = (f_1, \dots, f_n)$. Then $\mu(S) = \int_S f d|\mu|$ and for $\xi \in R^n$, $\langle \xi, \mu(S) \rangle = \int_S \langle \xi, f \rangle d|\mu|$. Obviously $\mu(S) \in F(\xi)$ if and only if $\{t \mid \langle \xi, f(t) \rangle > 0\} \subseteq S \subseteq \{t \mid \langle \xi, f(t) \rangle \geq 0\}$ almost everywhere with respect to $|\mu|$. It follows then, that $R(\mu, I)$ is strictly convex if and only if, the set $\{t \mid \langle \xi, f(t) \rangle = 0\}$ is of $\bar{\mu}$ -measure zero for all

supporting hyperplanes ξ which do not contain $R(\mu, I)$, or alternatively if for each subspace V of R^n of dimension smaller than that of $R(\mu, I)$, the set $\{t, f(t) \in V\}$ is of $\bar{\mu}$ -measure zero.

We can prove now:

Lemma 1 There is a decomposition $R(\mu, I) = \sum_i R(\mu, S_i)$ such that $\bigcup_i S_i$ is a partition of I and $R(\mu, S_i)$ is strictly convex for each i .

Proof: The decomposition is built in n stages. In the stages $1, \dots, k-1$ a family of disjoint sets S_i^j , $1 \leq j \leq k-1$, $1 \leq i < i_j$ if defined (i_j is possibly ∞ or 0) such that $R(\mu, S_i^j)$ is strictly convex and of dimension j . Moreover, for each $k-1$ dimensional subspace of R^n , V , the set $\{t | t \in I \setminus \bigcup_i S_i^j, f(t) \in V\}$ is of $\bar{\mu}$ -measure zero. In the k -th stage we define the sets $S_i^{k,i,j}$, $1 \leq i < i_k$ which are all the subsets of $I \setminus \bigcup_{i,j} S_i^j$ of the form $\{t | f(t) \in V\}$ which have positive $\bar{\mu}$ measure, where V is a k -dimensional subspace of R^n . The disjointness of the sets S_i^k can be guaranteed since the intersection of such two sets is a set of t 's for which $f(t)$ belongs to a subspace of dimension less than k . The strict convexity of $R(\mu, S_i^k)$ follows similarly.

Q.E.D.

Let us call a vector measure $\mu = (\mu_1, \dots, \mu_n)$ monotonic if each μ_i ($1 \leq i \leq n$) is either nonnegative or nonpositive. We will show now that it suffices to prove the Main Theorem for monotonic μ with strictly convex range $R(\mu, I)$. Indeed, there is a partition $I = \bigcup_{i=1}^{2^n} I_i$ such that the restriction of μ to each I_i is monotonic. We can decompose, furthermore, each I_i

according to Lemma 1, to get eventually a partition $I = \bigcup_i S_i$ and a decomposition $R(\mu, I) = \sum_i R(\mu, S_i)$ such that for each i , μ is monotonic on S_i and $R(\mu, S_i)$ is strictly convex. For $\epsilon > 0$ and $S \in \Sigma$ denote

$$\Omega_i(S, \epsilon) = \{T | T \subseteq S_i, d_\lambda(T, S \cap S_i) < \epsilon\} \text{ and } \Omega(S, \epsilon) = \{\bigcup_i T_i | T_i \in \Omega_i(S, \epsilon)\}.$$

It is easy to verify that the family of sets $\Omega(S, \epsilon)$ where S ranges over Σ and ϵ ranges over the positive reals, is a basis to the topology induced by d_λ on Σ . Moreover $\mu(\Omega(S, \epsilon)) = \sum_i \mu(\Omega_i(S, \epsilon))$.

But $\mu(\Omega_i(S, \epsilon)) \subseteq R(\mu, S_i)$, $R(\mu, S_i)$ is strictly convex and the restriction of μ to S_i is monotonic. Therefore by proving the Main Theorem for monotonic μ with strictly convex range we prove that $\mu(\Omega_i(S, \epsilon))$ is relatively open in $R(\mu, S_i)$ which says that $\mu(\Omega(S, \epsilon))$ is relatively open in $R(\mu, I)$.

We assume now that μ is monotonic and that $R(\mu, I)$ is strictly convex. We start by proving the following lemma.

Lemma 2 If $x_0 = \mu(S_0)$ then for each $1 \leq i \leq n$ and $\epsilon > 0$ the set $\mu\{S | d_{|\mu_i|}(S, S_0) < \epsilon\}$ contains a set $\{x | x \in R(\mu, I), \|x - x_0\| < \delta\}$ for some $\delta > 0$.

We first prove the lemma in the case that x_0 is in the relative interior of $R(\mu, I)$, using lemma 3.

Lemma 3. If $x_0 = \mu(S_0)$ is in the relative interior of $R(\mu, I)$, then the intersection of the relative interiors of $R(\mu, S_0)$ and $R(\mu, \bar{S}_0)$ is not empty.

Proof of Lemma 3: Indeed, if this intersection is empty then there exists a hyperplane which separates the two sets and for at least one of them, say

$R(\mu, S_0)$, contains only points from its relative boundary. Since

$0 \in R(\mu, S_0) \cap R(\mu, \bar{S}_0)$ we conclude that there exists $\xi \in R^n$ such that

$\langle \xi, x \rangle > 0$ for $x \in R(\mu, S_0)$ and $\langle \xi, x \rangle \leq 0$ for $x \in R(\mu, \bar{S}_0)$ and moreover for some x in the relative interior of $R(\mu, S_0)$, $\langle \xi, x \rangle > 0$. Now let $S \in \Sigma$ and denote

$S_1 = S \cap S_0$, $S_2 = S \cap \bar{S}_0$. We have: $\langle \xi, \mu(S_2) \rangle \leq 0 \leq \langle \xi, \mu(S_0 \setminus \bar{S}_1) \rangle = \langle \xi, \mu(S_0) \rangle - \langle \xi, \mu(S_1) \rangle$ and therefore, $\langle \xi, \mu(S) \rangle = \langle \xi, \mu(S_1) + \mu(S_2) \rangle \leq \langle \xi, \mu(S_0) \rangle$. This inequality holds for each S in Σ and moreover, for some S the inequality is strict which shows that $\mu(S_0)$ is in the relative boundary of $R(\mu, I)$, contrary to our assumption.

Q.E.D.

Proof of Lemma 2: Assume first that x_0 is in the relative interior of

$R(\mu, I)$. Let E_0 , E_1 and E_2 be the linear spaces spanned by $R(\mu, I) \cap R(\mu, S_0)$ and $R(\mu, \bar{S}_0)$ respectively, and denote by B_0 , B_1 and B_2 the intersection of the unit ball in R^n with E_0 , E_1 and E_2 respectively. Since $0 \in R(\mu, S_0) \cap R(\mu, \bar{S}_0)$, we find, using Lemma 3, a point w which belongs to the relative interiors of both $R(\mu, \bar{S}_0)$ and $R(\mu, S_0)$ and for which $\|w\| < \frac{\epsilon}{4}$. Choose now $0 < \eta < \frac{\epsilon}{4}$ such that $w + \eta B_1 \subseteq R(\mu, S_0)$ and $w + \eta B_2 \subseteq R(\mu, \bar{S}_0)$. Clearly $E_0 = E_1 + E_2$ and therefore we can choose $0 < \delta < \frac{\epsilon}{4}$ such that

$\delta B_0 \subseteq \eta(B_2 + B_1) = \eta(B_2 - B_1)$. Now let $x \in R(\mu, I)$ with $\|x - x_0\| < \delta$ and denote $z = x - x_0$. Since $z \in \delta B_0$ there exist $z_1 \in \eta B_1$ and $z_2 \in \eta B_2$ such that $z = z_2 - z_1$. There exist also $S_1 \subseteq S_0$ and $S_2 \subseteq \bar{S}_0$ such that $\mu(S_1) = w + z_1$ and $\mu(S_2) = w + z_2$. Define $S = (S_0 \setminus S_1) \cup S_2$. We have

$$\mu(S) = \mu(S_0) - \mu(S_1) + \mu(S_2) = x_0 - z_1 + z_2 = x_0 + z = x,$$

and using the monotonicity of μ ,

$$d_{|\mu, I|}(S, S_0) \leq \|\mu(S \Delta S_0)\| = \|\mu(S_1) + \mu(S_2)\| = \|2w + z_1 + z_2\| < 2 \frac{\epsilon}{4} + 2\eta < \epsilon.$$

We continue now to prove Lemma 2 for x_0 on the relative boundary of $R(\mu, I)$.

Consider a sequence $x_n = \mu(S_n)$ such that $x_n \rightarrow x_0$. We will show that

$$\mu(S_n \Delta S_0) \rightarrow 0 \text{ which is more than we need to complete the proof of Lemma 3.}$$

let $T'_n = S_n \cap S_0$ and $T''_n = S_n \cap \bar{S}_0$. Since the sequences $\mu(T'_n)$ and $\mu(T''_n)$ belong to the compact sets $R(\mu, S_0)$ and $R(\mu, \bar{S}_0)$ we can assume without loss of generality that $\mu(T'_n) \rightarrow \mu(T')$ and $\mu(T''_n) \rightarrow \mu(T'')$ where $T' \subseteq S_0$ and $T'' \subseteq \bar{S}_0$. It follows that $\mu(T' \cup T'') = \mu(S_0)$ and since $R(\mu, I)$ is strictly convex $T' = \bar{S}_0$ and $T'' = \phi$ almost everywhere with respect to μ , which shows that

$$\mu(S_n \Delta S_0) = \mu(S_0) - \mu(T'_n) + \mu(T''_n) \rightarrow 0.$$

Q.E.D.

To complete the proof of the Main Theorem we have to show that d_λ can replace $d_{|\mu|}$ in Lemma 2. There is a partition $I = S_1 \cup S_2$ of I such that the restriction of λ to S_0 is continuous with respect to $|\mu|$ and $|\mu|(S_2) = 0$.

Define $\Omega_i(S, \varepsilon) = \{T \mid T \subseteq S_i, d_\lambda(T, S) < \varepsilon\}$ $i = 1, 2$, and

$\Omega(S, \varepsilon) = \{T_1 \cup T_2 \mid T_i \in \Omega_i(S, \varepsilon), i = 1, 2\}$. Clearly $\mu(\Omega_2(S, \varepsilon)) = 0$. But

$\Omega_1(S, \varepsilon)$ is open in the topology induced by $d_{|\mu|}$ on the σ -field

$\{T \mid T \in \Sigma, T \subseteq S_i\}$ and therefore by Lemma 2 $\mu(\Omega(S, \varepsilon)) = \mu(\Omega_1(S, \varepsilon))$ is

relatively open in $R(\mu, S_1) = R(\mu, I)$.

Corollary: The projection $\pi: R^{\mathbb{R}^{n+1}} \rightarrow R^n$ on the first n coordinates is an open map from $R((\mu_1, \dots, \mu_{n+1}), I)$ onto $R((\mu_1, \dots, \mu_n), I)$.

Proof: Denote $\mu = (\mu_1, \dots, \mu_{n+1})$. Then $\pi = (\pi\mu)\mu^{-1}$. The result follows from the continuity of $\pi\mu$ and the fact that μ^{-1} is an open map.

Finally let us remark that the Main Theorem is an extension of Lemma 2 in [2]. This lemma states for a nonnegative vector measure μ , that for each x in $R(\mu, I)$ there exists S with $\mu(S) = x$ such that any neighborhood of S (with respect to d_μ), is mapped by μ to a neighborhood of x . The Main Theorem is used in [3] where the weaker result of [2] is not enough.

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