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DISCUSSION PAPER NO. 510

VECTOR MEASURES ARE OPEN MAPS

BY

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VECTOR MEASURES ARE OPEN MAPS

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ABSTRACT

Nonatomic vector measures are shown to be open maps from the σ -field on which they are defined to their range, where the σ -field is equipped with the pseudo-metric of the symmetric difference with respect to a given scalar measure.

We prove the following:

Main Theorem: Let $\lambda, \mu_1, \dots, \mu_n$ be nonatomic, σ -additive, finite measures on a measurable space (I, Σ) , and let λ be a nonnegative measure. Then the vector measure $\mu = (\mu_1, \dots, \mu_n)$ is an open map from Σ to the range of μ , where Σ is equipped with the topology induced by the pseudo-metric d_λ defined by

$d_\lambda(S, T) = \lambda[(S \setminus T) \cup (T \setminus S)]$, and the range of μ is equipped with its relative topology in \mathbb{R}^n .

Let us introduce the following notations. For S in Σ we denote by \bar{S} the complementary set $I \setminus S$. The symmetric difference of S and T , $(S \setminus T) \cup (T \setminus S)$ is denoted by $S \Delta T$. The Euclidian norm in \mathbb{R}^n is denoted by $\| \cdot \|$, and the scalar product of ξ and x in \mathbb{R}^n is denoted by $\langle \xi, x \rangle$. By the relative boundary of a closed set K in \mathbb{R}^n we mean the set of all points in K which are not in the relative interior of K . The face of a convex set K in the direction ξ , is the set $F(\xi) = \{x \mid x \in K, \langle \xi, x \rangle = \max_{y \in K} \langle \xi, y \rangle\}$. We say that a set K in \mathbb{R}^n is strictly convex if all the points on the relative boundary of K are extreme, or alternatively if for each $\xi \in \mathbb{R}^n$, $F(\xi)$ is either K or a singleton. For a scalar measure λ , we denote by $|\lambda|$ the sum of the positive and the negative parts of λ . For a vector measure $\mu = (\mu_1, \dots, \mu_n)$, $|\mu|$ is the sum $\sum_{i=1}^n |\mu_i|$. For each S we define $R(\mu, S) = \{\mu(T) \mid T \subseteq S\}$. Clearly $R(\mu, S) + R(\mu, \bar{S}) = R(\mu, I)$. By Lyapunov Theorem [1], $R(\mu, S)$ is a convex and compact set.

A convenient way to describe $R(\mu, I)$ is as follows. Let f_i be the Radon-Nikodym derivative of μ_i with respect to $|\mu|$. and let $f = (f_1, \dots, f_n)$. Then $\mu(S) = \int_S f d|\mu|$ and for $\xi \in \mathbb{R}^n$, $\langle \xi, \mu(S) \rangle = \int_S \langle \xi, f \rangle d|\mu|$. Obviously $\mu(S) \in F(\xi)$ if and only if $\{t \mid \langle \xi, f(t) \rangle > 0\} \subseteq S \subseteq \{t \mid \langle \xi, f(t) \rangle \geq 0\}$ almost everywhere with respect to $|\mu|$. It follows then, that $R(\mu, I)$ is strictly convex if and only if, the set $\{t \mid \langle \xi, f(t) \rangle = 0\}$ is of $\bar{\mu}$ -measure zero for all

supporting hyperplanes ξ which do not contain $R(\mu, I)$, or alternatively if for each subspace V of R^n of dimension smaller than that of $R(\mu, I)$, the set $\{t, f(t) \in V\}$ is of $\bar{\mu}$ -measure zero.

We can prove now:

Lemma 1 There is a decomposition $R(\mu, I) = \sum_i R(\mu, S_i)$ such that $\bigcup_i S_i$ is a partition of I and $R(\mu, S_i)$ is strictly convex for each i .

Proof: The decomposition is built in n stages. In the stages $1, \dots, k-1$ a family of disjoint sets S_i^j , $1 \leq j \leq k-1$, $1 \leq i < i_j$ if defined (i_j is possibly ∞ or 0) such that $R(\mu, S_i^j)$ is strictly convex and of dimension j . Moreover, for each $k-1$ dimensional subspace of R^n , V , the set $\{t | t \in I \setminus \bigcup_i S_i^j, f(t) \in V\}$ is of $\bar{\mu}$ -measure zero. In the k -th stage we define the sets $S_i^{k,i,j}$, $1 \leq i < i_k$ which are all the subsets of $I \setminus \bigcup_{i,j} S_i^j$ of the form $\{t | f(t) \in V\}$ which have positive $\bar{\mu}$ measure, where V is a k -dimensional subspace of R^n . The disjointness of the sets S_i^k can be guaranteed since the intersection of such two sets is a set of t 's for which $f(t)$ belongs to a subspace of dimension less than k . The strict convexity of $R(\mu, S_i^k)$ follows similarly.

Q.E.D.

Let us call a vector measure $\mu = (\mu_1, \dots, \mu_n)$ monotonic if each μ_i ($1 \leq i \leq n$) is either nonnegative or nonpositive. We will show now that it suffices to prove the Main Theorem for monotonic μ with strictly convex range $R(\mu, I)$. Indeed, there is a partition $I = \bigcup_{i=1}^{2^n} I_i$ such that the restriction of μ to each I_i is monotonic. We can decompose, furthermore, each I_i

according to Lemma 1, to get eventually a partition $I = \bigcup_i S_i$ and a decomposition $R(\mu, I) = \sum_i R(\mu, S_i)$ such that for each i , μ is monotonic on S_i and $R(\mu, S_i)$ is strictly convex. For $\epsilon > 0$ and $S \in \Sigma$ denote

$$\Omega_i(S, \epsilon) = \{T | T \subseteq S_i, d_\lambda(T, S \cap S_i) < \epsilon\} \text{ and } \Omega(S, \epsilon) = \{\bigcup_i T_i | T_i \in \Omega_i(S, \epsilon)\}.$$

It is easy to verify that the family of sets $\Omega(S, \epsilon)$ where S ranges over Σ and ϵ ranges over the positive reals, is a basis to the topology induced by d_λ on Σ . Moreover $\mu(\Omega(S, \epsilon)) = \sum_i \mu(\Omega_i(S, \epsilon))$.

But $\mu(\Omega_i(S, \epsilon)) \subseteq R(\mu, S_i)$, $R(\mu, S_i)$ is strictly convex and the restriction of μ to S_i is monotonic. Therefore by proving the Main Theorem for monotonic μ with strictly convex range we prove that $\mu(\Omega_i(S, \epsilon))$ is relatively open in $R(\mu, S_i)$ which says that $\mu(\Omega(S, \epsilon))$ is relatively open in $R(\mu, I)$.

We assume now that μ is monotonic and that $R(\mu, I)$ is strictly convex. We start by proving the following lemma.

Lemma 2 If $x_0 = \mu(S_0)$ then for each $1 \leq i \leq n$ and $\epsilon > 0$ the set $\mu\{S | d_{|\mu_i|}(S, S_0) < \epsilon\}$ contains a set $\{x | x \in R(\mu, I), \|x - x_0\| < \delta\}$ for some $\delta > 0$.

We first prove the lemma in the case that x_0 is in the relative interior of $R(\mu, I)$, using lemma 3.

Lemma 3. If $x_0 = \mu(S_0)$ is in the relative interior of $R(\mu, I)$, then the intersection of the relative interiors of $R(\mu, S_0)$ and $R(\mu, \bar{S}_0)$ is not empty.

Proof of Lemma 3: Indeed, if this intersection is empty then there exists a hyperplane which separates the two sets and for at least one of them, say

$R(\mu, S_0)$, contains only points from its relative boundary. Since

$0 \in R(\mu, S_0) \cap R(\mu, \bar{S}_0)$ we conclude that there exists $\xi \in R^n$ such that

$\langle \xi, x \rangle > 0$ for $x \in R(\mu, S_0)$ and $\langle \xi, x \rangle \leq 0$ for $x \in R(\mu, \bar{S}_0)$ and moreover for some x in the relative interior of $R(\mu, S_0)$, $\langle \xi, x \rangle > 0$. Now let $S \in \Sigma$ and denote $S_1 = S \cap S_0$, $S_2 = S \cap \bar{S}_0$. We have: $\langle \xi, \mu(S_2) \rangle \leq 0 \leq \langle \xi, \mu(S_0 \setminus \bar{S}_1) \rangle = \langle \xi, \mu(S_0) \rangle - \langle \xi, \mu(S_1) \rangle$ and therefore, $\langle \xi, \mu(S) \rangle = \langle \xi, \mu(S_1) + \mu(S_2) \rangle \leq \langle \xi, \mu(S_0) \rangle$. This inequality holds for each S in Σ and moreover, for some S the inequality is strict which shows that $\mu(S_0)$ is in the relative boundary of $R(\mu, I)$, contrary to our assumption.

Q.E.D.

Proof of Lemma 2: Assume first that x_0 is in the relative interior of $R(\mu, I)$. Let E_0 , E_1 and E_2 be the linear spaces spanned by $R(\mu, I) \cap R(\mu, S_0)$ and $R(\mu, \bar{S}_0)$ respectively, and denote by B_0 , B_1 and B_2 the intersection of the unit ball in R^n with E_0 , E_1 and E_2 respectively. Since $0 \in R(\mu, S_0) \cap R(\mu, \bar{S}_0)$, we find, using Lemma 3, a point w which belongs to the relative interiors of both $R(\mu, \bar{S}_0)$ and $R(\mu, S_0)$ and for which $\|w\| < \frac{\epsilon}{4}$. Choose now $0 < \eta < \frac{\epsilon}{4}$ such that $w + \eta B_1 \subseteq R(\mu, S_0)$ and $w + \eta B_2 \subseteq R(\mu, \bar{S}_0)$. Clearly $E_0 = E_1 + E_2$ and therefore we can choose $0 < \delta < \frac{\epsilon}{4}$ such that

$\delta B_0 \subseteq \eta(B_2 + B_1) = \eta(B_2 - B_1)$. Now let $x \in R(\mu, I)$ with $\|x - x_0\| < \delta$ and denote $z = x - x_0$. Since $z \in \delta B_0$ there exist $z_1 \in \eta B_1$ and $z_2 \in \eta B_2$ such that $z = z_2 - z_1$. There exist also $S_1 \subseteq S_0$ and $S_2 \subseteq \bar{S}_0$ such that $\mu(S_1) = w + z_1$ and $\mu(S_2) = w + z_2$. Define $S = (S_0 \setminus S_1) \cup S_2$. We have

$$\mu(S) = \mu(S_0) - \mu(S_1) + \mu(S_2) = x_0 - z_1 + z_2 = x_0 + z = x,$$

and using the monotonicity of μ ,

$$d_{|\mu, I|}(S, S_0) \leq \|\mu(S \Delta S_0)\| = \|\mu(S_1) + \mu(S_2)\| = \|2w + z_1 + z_2\| < 2 \frac{\epsilon}{4} + 2\eta < \epsilon.$$

We continue now to prove Lemma 2 for x_0 on the relative boundary of $R(\mu, I)$.

Consider a sequence $x_n = \mu(S_n)$ such that $x_n \rightarrow x_0$. We will show that

$$\mu(S_n \Delta S_0) \rightarrow 0 \text{ which is more than we need to complete the proof of Lemma 3.}$$

let $T'_n = S_n \cap S_0$ and $T''_n = S_n \cap \bar{S}_0$. Since the sequences $\mu(T'_n)$ and $\mu(T''_n)$ belong to the compact sets $R(\mu, S_0)$ and $R(\mu, \bar{S}_0)$ we can assume without loss of generality that $\mu(T'_n) \rightarrow \mu(T')$ and $\mu(T''_n) \rightarrow \mu(T'')$ where $T' \subseteq S_0$ and $T'' \subseteq \bar{S}_0$. It follows that $\mu(T' \cup T'') = \mu(S_0)$ and since $R(\mu, I)$ is strictly convex $T' = \bar{S}_0$ and $T'' = \phi$ almost everywhere with respect to μ , which shows that

$$\mu(S_n \Delta S_0) = \mu(S_0) - \mu(T'_n) + \mu(T''_n) \rightarrow 0.$$

Q.E.D.

To complete the proof of the Main Theorem we have to show that d_λ can replace $d_{|\mu|}$ in Lemma 2. There is a partition $I = S_1 \cup S_2$ of I such that the restriction of λ to S_0 is continuous with respect to $|\mu|$ and $|\mu|(S_2) = 0$.

Define $\Omega_i(S, \varepsilon) = \{T \mid T \subseteq S_i, d_\lambda(T, S) < \varepsilon\}$ $i = 1, 2$, and

$\Omega(S, \varepsilon) = \{T_1 \cup T_2 \mid T_i \in \Omega_i(S, \varepsilon), i = 1, 2\}$. Clearly $\mu(\Omega_2(S, \varepsilon)) = 0$. But

$\Omega_1(S, \varepsilon)$ is open in the topology induced by $d_{|\mu|}$ on the σ -field

$\{T \mid T \in \Sigma, T \subseteq S_i\}$ and therefore by Lemma 2 $\mu(\Omega(S, \varepsilon)) = \mu(\Omega_1(S, \varepsilon))$ is

relatively open in $R(\mu, S_1) = R(\mu, I)$.

Corollary: The projection $\pi: R^{n+1} \rightarrow R^n$ on the first n coordinates is an open map from $R((\mu_1, \dots, \mu_{n+1}), I)$ onto $R((\mu_1, \dots, \mu_n), I)$.

Proof: Denote $\mu = (\mu_1, \dots, \mu_{n+1})$. Then $\pi = (\pi\mu)\mu^{-1}$. The result follows from the continuity of $\pi\mu$ and the fact that μ^{-1} is an open map.

Finally let us remark that the Main Theorem is an extension of Lemma 2 in [2]. This lemma states for a nonnegative vector measure μ , that for each x in $R(\mu, I)$ there exists S with $\mu(S) = x$ such that any neighborhood of S (with respect to d_μ), is mapped by μ to a neighborhood of x . The Main Theorem is used in [3] where the weaker result of [2] is not enough.

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