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Working Paper

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Discussion Paper, No. 468

Provided in Cooperation with:
Kellogg School of Management - Center for Mathematical Studies in Economics and Management Science, Northwestern University

Suggested Citation: Mirman, Leonard J. (1981) : An Axiomatic Approach to the Allocation of a Fixed Cost Through Prices, Discussion Paper, No. 468, Northwestern University, Kellogg School of Management, Center for Mathematical Studies in Economics and Management Science, Evanston, IL

This Version is available at:
https://hdl.handle.net/10419/220828

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AN AXIOMATIC APPROACH TO THE ALLOCATION OF A FIXED COST THROUGH PRICES

by

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April 1981

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Partial Support from the NSF is gratefully acknowledged.

The allocation of joint costs of production through prices remains to this day an important practical question ranging in applications from accounting procedures to the regulation of monopolies. There are many aspects as well as applications of this problem. For firms or organizations, the problem of allocating joint costs has been studied from an accounting point of view by Roth and Verrechia [RV] and Billera, Heath and Verrechia [BHV] as well as from the point of view of the allocation of resources within the organization, Billera, Heath and Raanan [BHR] and Zajac [Z]. From the economy wide point of view the problem has been studied as a regulatory problem in economies with increasing returns to scale (Zajac [Z], Ramsey [R], Boiteux [B], and Baumol-Bradford [BB]). Questions involving efficiency, equity and fairness have all been studied and have given rise to many and varied solutions to the problem of allocating shared fixed costs and joint costs (arising generally from increasing returns to scale production functions).

The marginal cost pricing system is widely accepted as the efficient price system under conditions of non-increasing returns to scale. However, as a practical matter, it is necessary to specify prices when the underlying production process does not meet the conditions of non-increasing returns or when revenues must equal costs (which, in general, is not true of marginal cost prices). For that purpose one can use the Aumann-Shapley price system, as axiomatically defined (independently) by Billera Heath [BH] and Mirman-Tauman [MT]. These axioms are reasonable properties of price systems and yield a unique price system which can be used easily in practice. In fact, Samet, Tauman and Zang [STZ] provide an algorithm which computes these prices for a large class of cost functions. Moreover, it turns out that marginal cost prices and Aumann-Shapley prices are strongley connected. Marginal cost prices obey all the axioms characterizing Aumann-Shapley prices but the cost

sharing axiom. Moreover by dropping this axiom and strengthening the positivity axiom (axiom 3 below) Samet and Tauman [ST] are able to characterize marginal cost prices. It is worth mentioning that marginal cost prices coincide with Aumann-Shapely prices on the class of cost fuctions stemming from constant returns to scale economies because it is precisely in this case that marginal cost prices are cost sharing prices. In this paper an axiomatic approach to the allocation of a fixed cost through prices is proposed. The methodology is similar to that used in the axiomatic approach to both the Aumann-Shapley and marginal cost price system. Before discussing our proposal for the allocation of fixed costs we shall discuss other pricing schemes, then discuss the axiomatic approach to marginal cost and Aumann-Shapley prices.

The two most important suggestions for pricing in these cases have been Fully Distributed Costs (FDC), see Braeutigam [B1], and Ramsey prices. FDC, which allocate shared costs based on directly attributable costs, has been used widely in practical situations, but rejected on the grounds that they are unjustified from an economic point of view. Ramsey prices, which depend on demand as well as on the cost structure, has the theoretical advantage of leading to efficient allocations, although in the second best sense (no first best solution is guaranteed in non-convex economies). From a practical point of view they are difficult (or even impossible) to calculate. Also informational requirements needed to calculate these prices are huge. Finally they depend upon the weights assigned to individual utilities, i.e., interpersonal comparisons of utility are involved.

In comparison, Aumann-Shapley prices are simple to apply since they depend only on the cost structure and on the quantities consumed, not on the utilities of consumers. However, it is shown in [MT] that standard

assumptions on preferences and weak assumptions on the cost function guarantee the existence of a supply decision such that the corresponding A-S prices lead to demands that match supply. Hence although the A-S price system depends only on costs, as with perfect competition, the equilibrium depends on both supply (or costs) and demand. This equilibrium result is independent of the returns to scale properties of production. This is true in both the partial equilibrium and the general equilibrium setting. In the latter case there is a sector of the economy with increasing returns and a sector with standard assumptions on production. This economy can be viewed as a mixed economy having a public sector and a competitive or private sector. It can be shown that these two sectors are compatible with A-S prices in the public sector and the usual marginal cost prices in the competitive sector. Although the above discussion holds for cost functions which do not contain a fixed cost component, theorem 2.2 of [MT] enables us to extend the equilibrium result to cases in which prices are continuous at each point except prehaps zero. Since the proposed prices have this property (as will be shown below) the equilibrium result holds also for cost functions with a fixed cost component.

To discuss the axiom proposed by [MT] let E^m be the m dimensional euclidean space and E^m_+ be the non negative orthant of E^m.

Let $\mathcal{F}$ be the family of functions[1] F defined, for a given m, on a full dimensional comprehensive subset C^F of E^m_+ (by comprehensive, we mean that for each $\alpha \in C^F$, $C_\alpha \subseteq C^F$, where $C_\alpha = \{x \in E^m_+ | x \leq \alpha\}$).

By a price mechanism, (p.m.) we mean a function $P(\cdot,\cdot)$ that assigns to each F in $\mathcal{F}$ and to each vector α in C^F, a vector of prices,

$$P(F,\alpha) = (P_1(F,\alpha), \ldots, P_m(F,\alpha)).$$

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[1] The results are also valid for the set consisting of only non decreasing cost functions, when Axioms 3 is changed as in footnote 2.

Here m is the number of components in α.

We will characterize those cost sharing price mechanisms which satisfy the following axioms.

Axiom 1 Cost-Sharing. For every $F \in \mathcal{F}$ and every $\alpha \in C^F$,

$$\alpha \cdot P(F,\alpha) = F(\alpha),$$

i.e., total cost equals total revenue.

Axiom 2 Additivity. If F, G and H are three functions in $\mathcal{F}$, such that $C^F = C^G = C^H$ and $F = G + H$, then,

$$P(F,\alpha) = P(G,\alpha) + P(H,\alpha),$$

for each $\alpha \in C^F$. I.e., if the cost function F can be broken into two components G and H (e.g., management and production), then calculating the prices determined by the cost function F can be accomplished by adding the prices determined by G and by H, separately.

The next axiom requires non negative prices if costs increase with increasing outputs.[2]

Axiom 3 Positivity. If $F \in \mathcal{F}$, $\alpha \in C^F$ and if F is nondecreasing on C_α, then,

$$P(F,\alpha) \geq 0.$$

The next axiom requires that each unit of the "same good" have the same price. The question is what is the criterion for being the "same good"?

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2 If $\mathcal{F}$ is the class of monotonic functions, then to preserve the uniqueness of the p.m. on this smaller class, Axiom 3 must state that if F, $G \in \mathcal{F}$ with $C^F = C^G$ and if F - G is monotonic then $P(F,\alpha) \geq P(G,\alpha)$ for each $\alpha \in C^F$.

Since the price mechanism yields prices which depend upon the cost function and not the demand functions it is clear that being the "same good" means playing the same role in the cost function. As an illustration, suppose that red and blue cars are produced. The cost function is a two-variable function $F(x_1,x_2)$ where x_1 and x_2 are the quantities of red and blue cars, respectively. But, in fact, the cost of producing a red car is the same as the cost of producing a blue car. This can be formulated as follows: There is a one-variable function G for which G(x) is the cost of producing a total of x cars (red ones, blue ones or both) and

$$F(x_1,x_2) = G(x_1 + x_2).$$

In this case the axiom asserts that the price of a blue car is the same as the price of a red car, i.e.,

$$P_1(F,(\alpha_1,\alpha_2)) = P_2(F,(\alpha_1,\alpha_2)).$$

Axiom 4 Consistency. Let F be in $\mathcal{F}$ and assume that $C^F \subseteq E_+^m$. Let $C = \{y \in E_+^1 \mid y = \sum_{i=1}^{m} x_i,\ x \in C^F\}$. If there is a function G defined on C such that,

$$F(x_1, x_2,\ldots,x_m) = G(\sum_{i=1}^{m} x_i),$$

then, for each i, $1 \leq i \leq m$, and for each $\alpha \in C^F$,

$$P_i(F,\alpha) = P(G, \sum_{i=1}^{m} \alpha_i).$$

Axiom 5 Rescaling. Let F be in $\mathcal{F}$ with $C^F \subseteq E_+^m$. Let $\lambda_1,\ldots,\lambda_m$ be m positive real numbers. Define $C = \{(x_1,\ldots,x_m) | (\lambda_1 x_1,\ldots,\lambda_m X_m) \in C^F\}$ and let G be a function on C defined by:

$$G(x_1,\ldots,x_m) = F(\lambda_1 x_1,\ldots,\lambda_m x_m).$$

Then, for each $\alpha \in C$ and each i, $1 \leq i \leq m$,

$$P_i(G,\alpha) = \lambda_i P_i(F, (\lambda_1\alpha_1,\ldots,\lambda_m\alpha_m)).$$

Thus, changing the scale of a commodity yields an equivalent change in prices.

Remark In [ST] it is shown that the additivity axiom can be replaced by other natural axioms. Basically, it can be replaced by a separability axiom, where the separability is on the set of commodities produced.

Definition. Let $\mathcal{F}_0$ be the subfamily of $\mathcal{F}$ consisting of all functions satisfying,

(1) $F(0) = 0$, i.e., F does not contain a fixed cost component.

(2) F is cotinuously differentiable (c.d.) on C_α for each $\alpha \in C^F$.

Theorem 1 [MT]. There exists one and only one price mechanism $P(\cdot,\cdot)$ on $\mathcal{F}_0$ which obeys the above five axioms. This is the A-S price mechanism, i.e.,

$$P_i(F,\alpha) = \int_0^1 \frac{\partial F}{\partial x_i}(t\alpha)\, dt,$$

for each $F \in \mathcal{F}$ and $\alpha \in C^F$ with $\alpha \neq 0$.

A similar result is due to Billera and Heath [BH]. Also see Billera Heath and Verrechia [BHV] for a discussion of these results in the framework of accounting.

It is worth mentioning two related results obtain in [ST]. The first result provides a full characterization of price mechanisms obeying all the axioms but cost-sharing and is stated in Theorem 2 below. The second emphasizes the relation between A-S prices and MC prices. It characterizes MC prices by a similar set of axioms when cost-sharing is excluded.

Theorem 2 ([ST]) $P(\cdot,\cdot)$ is a price mechanism on $\mathcal{F}_0$ obeying Axioms 2-5 if and only if there is a nonegative measure μ on $([0,1],\beta)$ (β is the family of all Borel subsets of $[0,1]$ such that for each $F \in \mathcal{F}_0$ and for each $\alpha \in C^F(\subseteq E^m_+)$,

$$(*) \quad P_i(F,\alpha) = \int_0^1 \frac{\partial F}{\partial x_i}(t\alpha)d\mu(t) \quad i = 1,\ldots, m.$$

Moreover, for a given price mechanism $P(\ ,\ )$ which obeys axioms 2-5, there is a unique measure μ which satisfies (*).

In other words, (*) defines a one-to-one mapping from the set of all nonnegative measurs on $([0,1]),\beta)$ onto the set of all price mechanisms obeying axioms 2-5.

Formula (*) asserts that the prices associated with each $F \in \mathcal{F}_0$ and $\alpha \in C^F$ are the "weighted" average of the marginal costs of F along the line segment $[0,\alpha]$. The weights are given by the measure μ which characterizes the given price mechanism. If this measure happens to be the Lebesgue measure on $[0,1]$, then the associated price mechanism is the A-S price mechanism. Hence, the A-S price is just the uniform average of all

marginal costs along the line segment $[0,\alpha]$. If μ happens to be the (atomic probability) measure whose whole mass is concentrated at the point $t = 1$, i.e., $\mu(\{1\}) = 1$, then the associated price mechanism $P(\cdot,\cdot)$ is the marginal cost price mechanism, i.e., for any $F \in \mathcal{F}_0$ and for any $\alpha \in C^F(\subseteq E^m_+)$,

$$P_i(F,\alpha) = \frac{\partial F}{\partial x_i}(\alpha) \qquad i = 1,\ldots,m.$$

Now, let us change slightly the Positivity Axiom (Axiom 3).

Axiom 3* Let $F \in \mathcal{F}$ and let $\alpha \in C^F$ $(\subseteq E^m_+)$. If F is non-decreasing in a neighborhood of α, then $P(F,\alpha) \geq 0$.

Note that Axiom 3 is implied by Axiom 3* and therefore, by Theorem 2, a price mechanism $P(\cdot,\cdot)$ which satisfies Axioms 2, 3*, 4 and 5, is of the form

$$P_i(F,\alpha) = \int_0^1 \frac{\partial F}{\partial x_i}(t\alpha)\,d\mu(t).$$

However, an even stronger statement is true.

Theorem 3 ([ST]) *$P(\cdot,\cdot)$ is a price mechanism on $\mathcal{F}_0$ which satisfies axioms 2, 3*, 4 and 5 if and only if there is a constant $c \geq 0$ such that for each $F \in \mathcal{F}_0$ and each $\alpha \in C^F(\subseteq E^m_+)$,*

$$P_i(F,\alpha) = c \frac{\partial F}{\partial x_i}(\alpha) \qquad i = 1,\ldots,m.$$

Moreover, if we require, in addition, that for the identity function $G(x) = x$, $P(G,1) = 1$ then, for each $F \in \mathcal{F}_0$ and $\alpha \in C^F$,

$$P_i(F,\alpha) = \frac{\partial F}{\partial x_i}(\alpha).$$

Hence, $P_i(F,\alpha)$ is exactly the marginal cost price.

The following four properties of A-S prices are worth mentioning.

Property 1. For each $F \in F_0$ and each $\alpha \in C^F$, if $\lambda \in E^1$, then

$$P(\lambda F,\alpha) = \lambda\ P(F,\alpha)$$

(this follows immediately from the formula of A-S prices).

Property 2. If $F \in F_0$ is a homogeneous function of degree $r > 0$, i.e., if for each $x \in C^F$ and each $0 < \lambda < 1$,

$$F(\lambda x) = \lambda^r\ F(x),$$

then for each $\alpha \in C^F$,

$$P(F,\alpha) = \frac{1}{r}\ MC(F,\alpha),$$

where MC is the marginal cost associated with F at the point α. In particular if $r = 1$ (constant return to scale) the A-S and the MC prices coincide. This property follows from the fact that

$$\frac{\partial F}{\partial x_i}\ (t\alpha) = t^{r-1}\ \frac{\partial F}{\partial x}\ (\alpha)$$

whenever F is homoeneous of degree r. Therefore,

$$P(F,\alpha) = \frac{\partial F}{\partial x}\ (\alpha) \int_0^1 t^{r-1}\ dt = \frac{1}{r}\ MC(F,\alpha).$$

For the next property, we need the following notation: For each function G in F_0 and for each $\alpha \in C^G$, $\alpha \neq 0$, define G^α to be the restriction of G to C_α.

Property 3. The A-S price mechanism $P(\cdot,\cdot)$ on $\mathcal{F}_0$ is continuous in both variables, i.e., for a given $F \in \mathcal{F}_0$ and $\alpha \in C^F$, if $\alpha_n \in C^F$ and $\alpha_n \to \alpha$, as $n \to \infty$, then

$$P(F,\alpha_n) \to P(F,\alpha), \text{ as } n \to \infty.$$

Moreover, if $(F_n)_{n=1}^{\infty}$ and F are in $\mathcal{F}_0$ and defined on the same domain C and if F_n^{α} and $\frac{\partial F_n^{\alpha}}{\partial x_i}$ converge uniformly on C_α to F^{α} and $\frac{\partial F^{\alpha}}{\partial x_i}$ respectively, then

$$P(F_n,\alpha) \to P(F,\alpha), \text{ as } n \to \infty,$$

for each $\alpha \in C$.

Property 3 is a simple consequence of the formula defining the A-S prices. The last property requires some further notation. For each $x \in E^m$, $x = (x_1,\ldots,x_m)$, denote by $x^{(i)}$ the vector,

$$x^{(i)} = (x_1,\ldots,x_{i-1}, x_{i+1},\ldots,x_m).$$

If $F \in \mathcal{F}$ and if $C^F \subseteq E_+^m$ define the function $F^{(i)}$ on $\{x^{(i)} \mid x \in C^F\}$ by

$$F^{(i)}(x^{(i)}) = F(x_1,\ldots,x_{i-1}, 0, x_{i+1},\ldots,x_m).$$

Property 4 Dummy Commodity. Let $F \in \mathcal{F}$. If F is independent of its i-th coordinate, i.e., if

$$F(x) = F^{(i)}(x^{(i)}),$$

for each $x \in C^F$, then

$$P^{(i)}(F,\alpha) = P(F^{(i)},\alpha^{(i)}),$$

and

$$P_i(F,\alpha) = 0$$

i.e., if commodity i has no effect on costs then its price is zero. This commodity may be dropped without affecting the other prices.

Remark: It is worth mentioning that marginal cost prices also obey the above four properties.

As mentioned above various types of fully distribution costs (FDC) were suggested to allocate costs but were rejected on the grounds that there is no economic justification for their use (for a discussion of these issues see Breautigam [B1]). Since reasonable properties of prices have been presented to allocate joint costs it seems natural to extned these properties to the case of fixed costs. Unfortunately this is not stratightforward. We shall describe proposed attempts at solving this problem. On the basis of the properties suggested it becomes clear why several of the FDC methods should be rejected.

Let us consider the set of all continuously differentiable cost functions with a fixed cost component. Each such function $\hat{F}$ can be written in the form $\hat{F}(x_1,\ldots,x_n) = F(x_1,\ldots,x_n) + C$ when F is c.d. and $F(0) = 0$. Unfortunately the result obtained in [MT] and [ST] cannot be immediately extended to this class since there is no price mechanism for this class that obeys the five axioms. To see this consider a p.m., Q which operates on this class. Now, define $\hat{F}(x_1,x_2) = x_1 + x_2 + C$. If Q were to obey the axioms then whenever $C_1 + C_2 = C$,

(1) $$Q(\hat{F},(\alpha_1,\alpha_2)) = Q(x_1 + C_1,(\alpha_1,\alpha_2)) + Q(x_2 + C_2,(\alpha_1,\alpha_2)),$$

for each α_1 and α_2. Since $\hat{F}$ is a function of $x_1 + x_2$ the consistency axiom implies that the two commodities have the same price, moreover,

(2) $$Q_i(\hat{F},(\alpha_1,\alpha_2)) = Q(x + C, \alpha_1 + \alpha_2), \quad i = 1,2.$$

Since Q is a cost sharing mechanism

(3) $$Q(x + C,(\alpha_1 + \alpha_2)) = \frac{\alpha_1 + \alpha_2 + C}{\alpha_1 + \alpha_2} = 1 + \frac{C}{\alpha_1 + \alpha_2}.$$

From (1), (2) and (3) it follows that,

(4) $$Q(\hat{F},(\alpha_1, \alpha_2)) = (1 + \frac{C}{\alpha_1 + \alpha_2}, 1 + \frac{C}{\alpha_1 + \alpha_2}).$$

On the other hand it can be shown (as in [MT] and [ST]) that Q has the dummy property, i.e., if a cost function is independent of one of its coordinates then the corresponding price is zero. Using this property and the cost sharing axiom we conclude

(5)
$$Q(x_1 + C_1,(\alpha_1,\alpha_2)) = (\frac{\alpha_1 + C_1}{\alpha_1}, 0) = (1 + \frac{C_1}{\alpha_1}, 0)$$
$$Q(x_2 + C_2,(\alpha_1,\alpha_2)) = (0,1 + \frac{C_2}{\alpha_2})$$

Using (1), (4) and (5) we can transform (1) into

$$1 + \frac{C}{\alpha_1 + \alpha_2} = 1 + \frac{C_1}{\alpha_1}$$

$$1 + \frac{C}{\alpha_1 + \alpha_2} = 1 + \frac{C_2}{\alpha_2}$$

These two equalities hold if and only if $C_1 = \frac{\alpha_1}{\alpha_1 + \alpha_2} \cdot C$ and $C_2 = \frac{\alpha_2}{\alpha_1 + \alpha_2} \cdot C$. Since the two equalities must hold for any α_1 and α_2 we have a contradiction. Hence there is no p.m. on this class that obeys all five axioms. In particular the following two price mechanisms that allocate the fixed cost independently from its variable part fail to have the desired properties,

I. $\quad Q(C,\alpha) = (\frac{C}{\Sigma\alpha_i}, \ldots, \frac{C}{\Sigma\alpha_i}), \qquad \alpha = (\alpha_1, \ldots, \alpha_m).$

II. $\quad Q(C,\alpha) = (\frac{C}{m\alpha_1}, \ldots, \frac{C}{m\alpha_m}).$

Let us assume, for example, that the variable part is allocated in both cases through the A-S prices. For the first p.m. prices are equal for all commodities, e.g., the i-th commodity is charged in total $\frac{C}{m}$ and thus its price is $\frac{C}{m\alpha_i}$.

The p.m. depends strongly on the definition of a unit of the commodity; changing the scale will not yield an appropriate change in price, i.e., it violates the rescaling axiom.

For example, consider again the production of red cars and blue cars in amounts α_1 and α_2 where $\alpha_1 \neq \alpha_2$. By the consistency axiom the price of red cars and blue cars must be the same. But the second mechanism treates these two type of cars unequally by imposing a higher proportion of the fixed costs on the cars which are produced in the smaller amount.

In fact it can easily be shown that <u>each</u> p.m. that alloates the fixed

cost independently of its variable cost violates either the rescaling axiom or the consistency axiom.

As we already concluded above there is no p.m. on the class of c.d. cost functions allowing a fixed cost component. Hence the existence of a p.m. on this class requires a relaxation of some of the axioms. It seems to us that the most natural candidate is the additivity axiom. We shall show that a weaker version of the additivity axiom insures the existence and the uniqueness of a p.m. on this class. This is stated as Axiom 2*.

Axiom 2* Let F and G be two c.d. functions on E_+^m such that, $F(0) = G(0)$.

Let $\alpha \in E_+^m$, with $F(\alpha) + G(\alpha) \neq 0$. Then

$$Q(F + G + C,\alpha) = Q(F + C_F, \alpha) + Q(G + C_G, \alpha)$$

where

$$C_F = \frac{F(\alpha)}{F(\alpha) + G(\alpha)} \text{ and } C_G = \frac{G(\alpha)}{F(\alpha) + G(\alpha)}.$$

The axiom states that if the variable cost is broken into two components then the part of the fixed cost associated with each of them is proportional to the total variable cost.

Consider the class $\hat{F}$ of all cost functions $\hat{F}$ of the form $F + C$ where F is a continuously differentiable function on E_+^m, $F(0) = 0$ and C is a real number (the fixed cost).

The following result follows (a sketch of the proof appears in the appendix).

Theorem There exists a unique price mechanism Q on $\hat{F}$ which obeys Axioms 1, 2,

3*, 4 and 5. This mechanism is defined for each F and for each α with $F(\alpha) \neq 0$ by

$$Q(F + C,\alpha) = P(F,\alpha) + \frac{C}{F(\alpha)} P(F,\alpha) = (1 + \frac{C}{F(\alpha)}) P(F,\alpha)$$

where $P(F,\alpha)$ is the A-S price vector associated with the variable part F for quantities α. I.e.,

$$P_i(F,\alpha) = \int_0^1 \frac{\partial F}{\partial x_i} (t\alpha)\, dt.$$

In other words the price mechanism Q associates, with each $\hat{F}$ and α, a price vector which is a scalar multiple of the A-S prices associated with F and α. The proof of the theorem appears in the appendix.

Remark: Notice that the above prices are continuous at each point α, except $\alpha = 0$.

Appendix

We provide a sketch of the proof of the main result. The proof here is similar to the proof of Theorem 1.2 of [MT1].

Proof: First, it is easy to verify that the price mechanism $Q(\cdot,\cdot)$ defined on F by

$$Q_i(F + C, \alpha) = (1 + \frac{C}{F(\alpha)}) \int_0^1 \frac{\partial F}{\partial x_i} (t\alpha)dt,$$

for each i, $i < i \leq m$, obeys the five axioms. Therefore we have to prove the uniqueness part only. Let us assume that $Q^1(\cdot,\cdot)$ and $Q^2(\cdot,\cdot)$ are two price mechanism on F that obey the five axioms. As in [MT1] (or in [ST]) it is enough to prove that the two coincide on polynomials i.e. that $Q^1(F,\alpha) = Q^2(F,\alpha)$, for each α in E_+^m and for each F which is a polynomial on E_+^m. The general case is then obtained by using a continuity arguments. Continuity is implied by the positivity axiom (For details see Proposition 4 of [ST]).

Any polynomial F in F is a linear combinations of monomials $F_1,\ldots,F_r$. i.e. F can be written as

$$F = a_1 F_1 + a_2 F_2 + \ldots + a_r F_r + a,$$

where F_i is of the form

$$F_i(x_1 \ldots x_m) = x_1^{k_1^i} \cdot x_2^{k_2^i} \cdot \ldots \cdot x_m^{k_m^i},$$

and where $(a_1,\ldots a_r, a)$ are real numbers and k^i_j is a non-negative integer for each j, $1 \leq j \leq m$. By formula 7.3 of [AS, p.41] any monomial F_i is a linear combination of polynomials P of the form

$$P(x_1\ldots x_m) = (n_1x_1 +\ldots+ n_mx_m)^{\ell}$$

where n_j $(1 \leq j \leq m)$ and ℓ are non-negative integers. Therefore any polynomial F can be written as

$$(1)\quad F(x_1\ldots x_m) = c_1 P_1(x_1\ldots x_m) +\ldots+ c_t P_t (x_1\ldots x_m) + C$$

where for each i, $1 \leq i \leq t$,

$$(2)\quad P_i(x_1\ldots x_m) = (n^i_1x_1 +\ldots+ n^i_m x_m)^{\ell_i},$$

and where n^i_j($1 \leq j \leq m$,) and ℓ_i are non-negative integers. Moreover we can assume that for each i, $P_i(x_1,\ldots,x_m)$ is not identically zero.

Let $\alpha \in E^m_{++}$ such that $F(\alpha) \neq 0$. By (2) for each i, $1 \leq i \leq t$, $P_i(\alpha) \neq 0$. Therefore by (1) and the additivity axiom 2^*

$$Q^k(F,\alpha) = \sum_{i=1}^{t} Q^k(c_iP_i + b_i, \alpha), \qquad k = 1,2$$

where $$b_i = \frac{C \cdot c_i \cdot P_i(\alpha)}{F(\alpha)}.$$

Thus it is enough to prove that Q^1 and Q^2 coincide on polynomials L of the form

the form

$$L(x_1 \ldots x_m) = c \cdot (n_1 x_1 + \ldots + n_m x_m)^{\ell} + b \quad) \quad b, c \in E^1.$$

Using the same arguments as of [ST] it is sufficient to prove the above statement in the case where $n_j > 0$, for each j, $1 \leq j \leq m$. Define a function $H: E_+^m \to E^1$ by

$$H(x_1 \ldots x_m) = c(x_1 + \ldots + x_m)^{\ell} + b,$$

and define a function $G: E_+^1 \to E^1$ by,

$$G(x) = cx^{\ell} + b.$$

Since

$$L(x_1 \ldots x_m) = H(n_1 x_1, \ldots, n_m x_m)$$

we have, by the rescaling axiom,

$$(3) \quad Q_j^k(L,\alpha) = n_i \, Q_j^k \cdot (H, (n_1\alpha_1, \ldots, n_m\alpha_m)), \quad k = 1,2.$$

By the consistency axiom, for each j and j', $1 \leq j, j' \leq m$,

$$Q_j^k(H, (n_1\alpha_1, \ldots, n_m\alpha_m)) = Q_{j'}^k \cdot (H, (n_1\alpha_1, \ldots, n_m\alpha_m)), \quad k = 1,2.$$

i.e., all the components of the vector $Q^k(H,(n_1\alpha_1, \ldots, n_m\alpha_m))$ are equal. Denote their size by d^k. Using the cost sharing axiom for the function H and the vector $(n_1\alpha_1, \ldots, n_m\alpha_m)$ implies that,,

$$d^k = \frac{H(n_1\alpha_1, \ldots, n_m\alpha_m)}{\sum_{j=1}^{m} n_j \alpha_j}, \quad k = 1,2.$$

This implies that $d^1 = d^2$. Hence by (3)

$$Q_j^1(L,\alpha) = Q_j^2(L,\alpha),$$

for each j, $1 \leq j \leq m$, and the proof is complete.

References

[AS] Aumann, R. J. and L. S. Shapley, Values of Non-Atomic Games, Princeton University Press, (1974).

[BB] Baumol, W.J. and D. Bradford, "Optimal Departures From Marginal Cost Pricing", A.E.R.60 (1970) 265-283.

[BH] Billera, L.J. and D.C. Heath, "Allocation of Shared Costs: A Set of Axioms Yielding a Unique Procedure," Cornell University (1979).

[BHV] Billera, L.J., D.C. Heath and R.E. Verrecchia, "A Unique Procedure for Allocating Common Costs From a Production Process," April 1980.

[B] Boiteux, M., "Le ' revenue distubuable' et les pertes 'economiques," Econometrica 19 (1951) 112-33.

[Bl] Braeutigam, R.R., "An Analysis of Fully Distributed Cost Pricing in Regulated Industries," Bell Journal of Economics Spring, 1980.

[MT1] Mirman, L. J., and Y. Tauman, "Demand Compatible, Equitable, Cost Sharing Prices," CORE Discussion Paper 7930 (1979).

[MT] Mirman, L. J., and Y. Tauman "Demand Compatible, Equitable, Cost Sharing Prices," CORE Discussion Paper 7930 Revised Version (1981).

[R] Ramsey, F., "A Contribution to the Theory of Taxation," Economic Journal 37 (1927) 47-61.

[RV] Roth, A. E. and R. E. Verrecchia, "The Shapley Value as Applied to Cost Allocation: A Re-Interpretation," Journal of Accounting Research 17 (1979) 295-303.

[ST] Samet D. and Y. Tauman, "A Characterization of Price Mechanisms and the Determination of Marginal Cost Prices Under a set of Axioms," (1979).

[STZ] Samet, D., Y. Tauman and I. Zang, "An Application of the Aumann-Shapley Prices for Cost Allocations in Transportation Problems," (1980).

[Z] Zajac, E. E., Fairness or Efficiency: An Introduction to Public Utility Pricing, Ballinger Publishing Co. (1978).