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DISCUSSION PAPER NO. 272

Linear Functionals of Convex Sets with
Applications to Economics, Game-Theory and
Social Choice*

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1. Introduction

Let \mathcal{C} denote the set of all compact convex subsets of the n -dimensional Euclidean space, R^n . \mathcal{C} is a convex cone ($A, B \in \mathcal{C}$ imply $A+B = \{a+b : a \in A, b \in B\} \in \mathcal{C}$ and $rA = \{ra : a \in A\} \in \mathcal{C}$ for every $r \geq 0$). The main purpose of this paper is to characterize all the uniformly continuous (relative to the Hausdorff metric on \mathcal{C}) real valued linear functions ($F(A+B) = F(A) + F(B)$ and $F(rA) = rF(A)$ for $r \geq 0$) on \mathcal{C} . In section 2 we give examples to show that these functions are common in Economics and Game-Theory. Section 3 is devoted to the proof of an extended version of Theorem 1 below. Section 4 exhibits a method of recovering the measures associated with our characterization and presents an open problem.

For $C \in \mathcal{C}$ and $p \in R^n$ let $V_C(p)$ denote the support of C in the direction p , i.e. $V_C(p) = \sup_{c \in C} c \cdot p$. Let S denote the boundary of the unit ball in R^n .

Theorem 1: F is a uniformly continuous, real valued linear function on \mathcal{C} if and only if there exist non-negative real numbers a and b and probability measures μ and η on S such that for every $C \in \mathcal{C}$

$$F(C) = aE_{\mu} V_C - bE_{\eta} V_C$$

where E_{μ} denotes the expected value relative to the measure μ .

2. Examples

References to show the frequency of occurrence of linear functionals in Economics are too numerous to list (e.g. Debreu [1959] and Gale [1960]). We give the following examples in order to show four specific cases where the function F of Section 1 is natural.

Example 1: Decomposition of Production

We think of each of the coordinates of a point in R^n as representing output level of a certain good. Thus a point $x \in R^n$ represents output x^1 of good 1, x^2 of good 2 and so on. A set $C \subset R^n$ represents the feasible output levels for a producer. Given such a set C , let $F(C)$ be the profit (or cost) associated with the producer's choice in C . If a set A represents the feasible production levels in one location and B represents the feasible production levels in another location then $A+B$ represents the feasible production levels from the two locations. $F(A+B) = F(A) + F(B)$ means that as far as profit is concerned the problem is decomposable and the decision can be made jointly or separately. Similar considerations will justify the requirement that $F(kC) = kF(C)$ for every positive integer k , $F(rC) = rF(C)$ for every positive rational number r , and continuity argument will result in $F(tC) = tF(C)$ for every non-negative real number t . Of course the quantity $F(C)$ could represent many other concepts for which decomposition is desired.

Notice that $E_{\mu} V_C$ is the expected worth of the feasible production set C when prices have the distribution μ . Thus, by Theorem 1, it

follows that every uniformly continuous function which is decomposable can be viewed as the difference of the expected worth of the set C according to two different fixed probability distributions on potential prices.

Example 2: The Utility of Participating in a Cooperative Game

Linear, real valued functions arise in cooperative game theory when we consider the von-Neumann Morgenstern utility that an individual has for the options of participating in various cooperative games. For simplicity we illustrate this point on a class of very simple games of this type (see Aumann-Peleg [1960] for the general case and Nash [1950] for the special case that we consider here). In addition linear functions of convex sets arise naturally when we consider possible extensions of the Shapley Value (Shapley [1953]) to the family of cooperative games without sidepayments (Kalai-Myerson [1977]).

We consider two fixed individuals with fixed von-Neumann Morgenstern utility functions. A two-person game of these two individuals is represented by a pair $(a, C) \in \mathbb{R}^2 \times C$. $a = (a^1, a^2)$ represents the V-M utility levels resulting to the two of them when they do not cooperate. C represents the utility levels available to them when they do cooperate. Thus $c = (c^1, c^2) \in C$ if and only if there is a joint strategy yielding utility levels c^1 and c^2 to the two players respectively. We are interested in the V-M utility that player 1 may have for participating in the various games that he may encounter with 2. (See Roth [1976] for a treatment of the sidepayments case as a generalization

of the Shapley Value.) Consider three games, (a,A) (b,B) and the game resulting from the following lottery. With probability α they will play (a,A) and with probability $1-\alpha$ (b,B) . If we consider the ex-ante feasible V-M utility level associated with the third game (c,C) we obtain

$$(c,C) = (\alpha a + (1-\alpha)b, \alpha A + (1-\alpha)B).$$

Thus it follows from the V-M theory that player 1's utility for the games must satisfy

$$u(\alpha a + (1-\alpha)b, \alpha A + (1-\alpha)B) = \alpha u(a,A) + (1-\alpha)u(b,B).$$

By considering all the games with a fixed non-cooperation point, or by choosing a non-cooperation point that varies linearly with the choice of the feasible set we obtain that u must induce a linear, real valued function on \mathcal{C} .

Example 3: Decentralization of Social Choice Decisions

We let a set C denote the feasible production levels of n -public goods. $F(C)$ represents society's utility from the choice made out of C . The linearity of F means that if C is decomposed into sets that add up to C then making the choice on every one of the component sets will not effect society's utility of the final outcome.

Example 4: Utilitarian Social Welfare Functions

We consider a society of n individuals each having a von-Neumann Morgenstern utility function u_i ($1 \leq i \leq n$). A set $C \subset R^n$ represents (as in Example 2) feasible utility levels of the n individuals in a certain situation. Thus if society's options in a certain situation are given by a set A then $C = \{(u_1(a), u_2(a), \dots, u_n(a)) : a \in A\}$. C must be convex if we assume that the society has the option to randomize over alternatives in A . We let $F(C)$ denote the V-M utility of the society when it makes the choice from C . The argument given in Example 2 holds also in this case and implies that F must be linear on C .

Notice that if society's utility of a choice set C is given by the function $V_C(p)$ ($= \sup_{c \in C} c \cdot p$) for some $p \in R^n$ then society has a utilitarian social welfare function with the interpersonal weights given by $p = (p_1, p_2, \dots, p_n)$. (See Harsanyi [1955] for an axiomatization of this Social Welfare Function.) If society has a utility function of the form $u(C) = E_{\mu} V_C$ where μ is a probability distribution on the boundary of the unit ball in R^n then it is basically a utilitarian society which has a probability distribution μ over the interpersonal weights. Thus Theorem 1 tells us that every uniformly continuous V-M utility function for the society can be viewed as the difference of two utilitarian functions with two different probability distributions over the interpersonal weights.

3. Main Results

Let \mathcal{K} denote a convex cone of closed convex sets in \mathbb{R}^n with a common recession cone $T \neq \mathbb{R}^n$. Thus we assume that \mathcal{K} satisfies the following conditions.

1. Every element of \mathcal{K} is closed and convex.
2. $A, B \in \mathcal{K}$ imply that $A+B \in \mathcal{K}$ (+ denotes set addition)
3. $A \in \mathcal{K}$ and $t > 0$ imply that $tA \in \mathcal{K}$.
4. There exists a closed convex cone in \mathbb{R}^n , $T (\neq \mathbb{R}^n)$, such that for every $A \in \mathcal{K}$ the recession cone of A is T .

Recall that the recession cone is the set of directions to which A is unbounded, i.e.

$$T = \{v \in \mathbb{R}^n : A + \{v\} \subset A\} \text{ (see Rockafellar [1970])}.$$

Notice that when $T = \{0\}$ then $\mathcal{K} = \mathcal{C}$.

The cases where $T = -\mathbb{R}_+^n$ are very common in Game-Theory and Economics (free disposal). We let W denote the polar cone of T ($= \{w \in \mathbb{R}^n : w \cdot r \leq 0 \text{ for every } r \in T\}$) and for every $A \in \mathcal{K}$ we let V_A denote the support function of A then the effective domain of V_A is W . V_A is continuous, convex, and homogeneous of degree 1 on W . And there is a 1-1 correspondence between convex sets with T as a recession cone and functions on W which satisfy these conditions. We let \bar{W} denote the intersection of W with the boundary of the unit ball in \mathbb{R}^n , $\bar{W} = \{v \in W : \|v\| = 1\}$ where $\|\cdot\|$ denotes the Euclidean norm.

For $A, B \in \mathcal{K}$ the Hausdorff distance between A and B is defined as usual to be the supremum of radii of Euclidean balls which are centered around a point of one of the two sets A, B , without intersecting the other set. We discuss continuity of functions on \mathcal{K} relative to this Hausdorff metric.

A real valued function F on \mathcal{K} is linear if for every $A, B \in \mathcal{K}$ and every $r > 0$, $F(A+B) = F(A) + F(B)$ and $F(rA) = rF(A)$.

Theorem 2: F is a real valued uniformly continuous linear function on \mathcal{K} if and only if there is some signed regular Borel measure M on \bar{W} such that for every $C \in \mathcal{K}$

$$F(C) = \int_{\bar{W}} v_C(p) dM(p).$$

Let X be a normed linear space over the real numbers, let Y denote a convex cone in X , and let f be a real valued linear function on Y . We define $\|f\|_Y = \sup \left\{ \frac{|f(y)-f(z)|}{\|y-z\|} : y, z \in Y \right\}$ and we say that f is bounded on Y if $\|f\|_Y < \infty$.

Lemma 1: A real valued bounded linear function f defined on a convex cone Y can be extended to a bounded linear function \bar{f} on the entire space X with $\|\bar{f}\| = \|f\|_Y$.

Proof of Lemma 1: We let $Z = Y - Y$ then Z is a subspace of X .

We extend f to \bar{f} defined on Z by

$$\bar{f}(y_2 - y_z) = f(y_2) - f(y_z).$$

By the linearity of f it follows that \bar{f} is well defined, linear, and $\|\bar{f}\|_Z = \|f\|_Y$. The Hahn-Banach Theorem completes the proof of the lemma.

Proof of Theorem 2: Let \bar{W}^* denote the set of all real valued functions defined on \bar{W} which are the restrictions to \bar{W} of some support function V_A with effective domain W . Every element in \bar{W}^* can be associated uniquely with some V_A for some unique convex set $A \in \mathcal{K}$. We let \bar{V}_A denote the restriction of V_A to \bar{W} . Thus we can define F^* on \bar{W} by $F^*(\bar{V}_A) = F(A)$. So F^* is a well defined, linear, real valued function on \bar{W}^* . Also F^* is uniformly continuous (with the sup norm) because F is (see Artstein [1970] or Kalai [1975]). To show that F^* is bounded on \bar{W}^* we assume to the contrary that there exists a sequence of functions $\bar{V}_{A(i)}$ and $\bar{V}_{B(i)}$ on \bar{W}^* such that

$$R(A(i), B(i)) = \frac{|F^*(\bar{V}_{A(i)}) - F^*(\bar{V}_{B(i)})|}{\|\bar{V}_{A(i)} - \bar{V}_{B(i)}\|} \rightarrow \infty \text{ as } i \rightarrow \infty. \text{ By the uniform}$$

continuity of F^* there exists a $\delta > 0$ such that if $\|\bar{V}_{A(i)} - \bar{V}_{B(i)}\| \leq \delta$ then $\|F^*(\bar{V}_{A(i)}) - F^*(\bar{V}_{B(i)})\| \leq 1$. If we consider the sequence

$(A^1_{(i)}, B^1_{(i)}) \leq \frac{1}{\delta}$ for all i but also $R(A^1_{(i)}, B^1_{(i)}) \rightarrow \infty$ as $i \rightarrow \infty$, a contradiction. Thus F^* is a real valued, linear, bounded function on \bar{W}^* .

Since \bar{W}^* is a convex cone in the space $C_0(\bar{W})$ consisting of all continuous functions on \bar{W} we can extend, by Lemma 1, F^* to be a linear bounded real valued function on $C_0(\bar{W})$. Now Riesz Representation Theorem completes the proof of the theorem.

Corollary: Theorem 1 follows immediately from Theorem 2 by the Hahn Decomposition Theorem for signed measures. Notice also that if M is determined uniquely for a given F then Hahn's theorem implies that a, b, μ and η of Theorem 1 are determined uniquely after the obvious normalization.

4. Uniqueness, Recovering the Measure, and an Open Question

We give an example in \mathbb{R}^2 to show how to recover the measure M underlying a given uniformly continuous linear F defined on \mathcal{C} . We believe that this method can be applied to the general case. In particular this would show that the measure M described by Theorem 2 is unique.

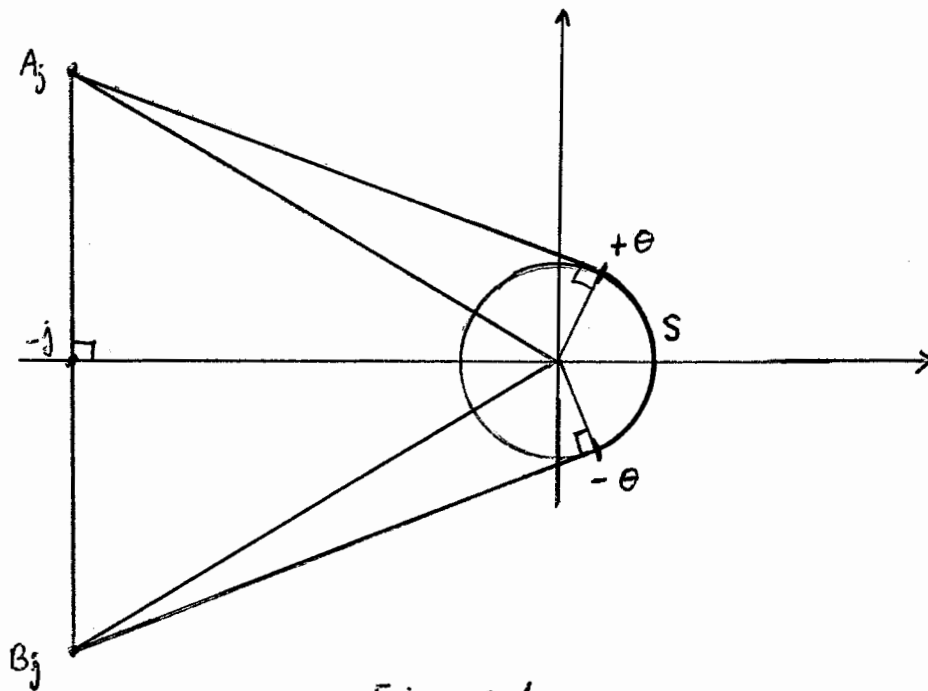


Figure 1

We let S be the interval described in Figure 1. S_j is the convex-hull of S and the points A_j and B_j . T_j is the convex-hull of A_j , B_j and $(0,0)$. We claim that $M(S) = \lim_{j \rightarrow \infty} [F(S_j) - F(T_j)]$. To justify this claim, consider the support functions of S_j and T_j

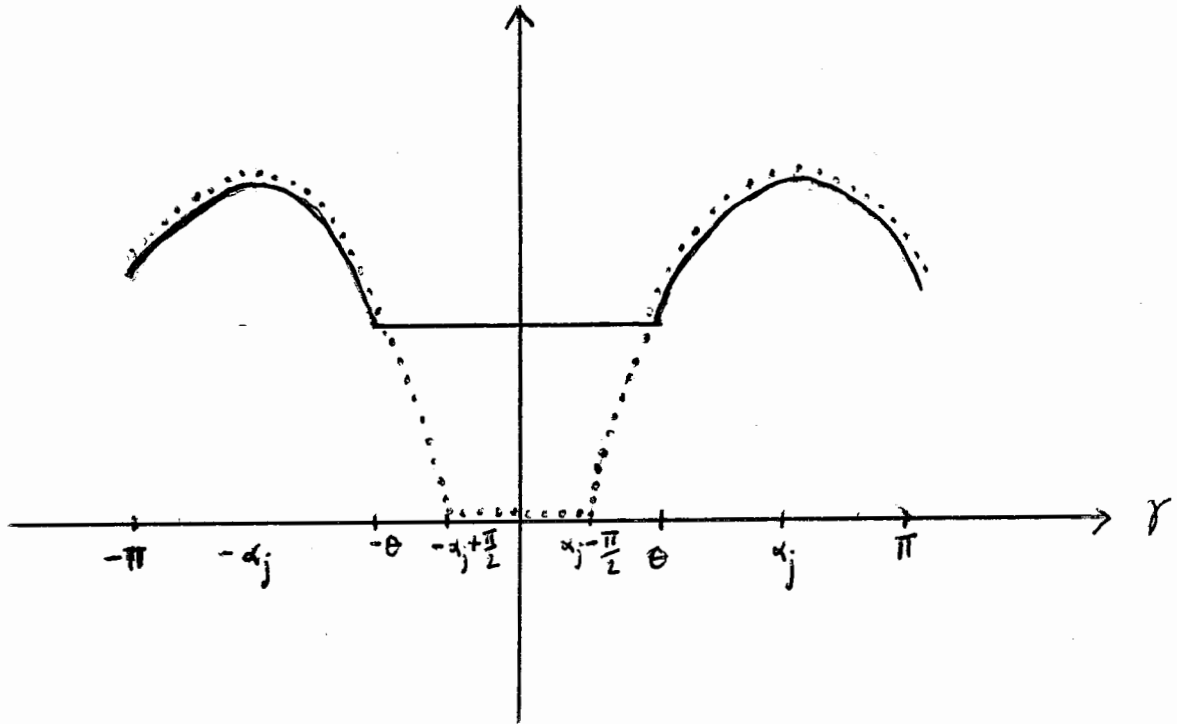


Figure 2.

$V_{T_j}(\gamma) = \max (0, |A_j| \cos(\gamma + \alpha_j), |A_j| \cos(\gamma - \alpha_j))$, the dotted line in Figure 2.

$V_{S_j}(\gamma) = \max (1, |A_j| \cos(\gamma + \alpha_j), |A_j| \cos(\gamma - \alpha_j))$, the solid line in Figure 2.

Since $-\alpha_j + \frac{\pi}{2} \rightarrow -\theta$ and $\alpha_j - \frac{\pi}{2} \rightarrow \theta$ as $j \rightarrow \infty$ it follows that

$$\int V_{S_j}(\gamma) - V_{T_j}(\gamma) dM(\gamma) \rightarrow \int_S 1 dM(\gamma) = M(S) \text{ as } j \rightarrow \infty.$$

Let $F = (F_1, F_2, \dots, F_n)$ consist of n uniformly continuous linear real valued functions on \mathcal{K} . Let M_1, M_2, \dots, M_n be the corresponding

measures. A question that arises naturally from the examples given in Section 2 is whether $F(C) \in C$ for every $C \in \mathcal{X}$. This would enable us to consider the actual choice that the decision maker makes out of the alternative in C . Necessary and sufficient conditions on the M_i 's that will induce this feasibility property would be of great interest. The condition that $F(C)$ be on the boundary of C for every $C \in \mathcal{X}$ (or Pareto Optimality) is also desirable but unfortunately it is inconsistent with continuity (see Kalai-Myerson [1977]).

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