Kalai, Ehud; Muller, Eitan

Working Paper

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Discussion Paper, No. 234

Provided in Cooperation with:
Kellogg School of Management - Center for Mathematical Studies in Economics and Management Science, Northwestern University

Suggested Citation: Kalai, Ehud; Muller, Eitan (1977) : Characterization of Domains Admitting Nondictatorial Social Welfare Functions and Nonmanipulable Voting Procedures, Discussion Paper, No. 234, Northwestern University, Kellogg School of Management, Center for Mathematical Studies in Economics and Management Science, Evanston, IL

This Version is available at:
https://hdl.handle.net/10419/220594

Characterization of Domains Admitting Nondictatorial Social Welfare Functions and Nonmanipulable Voting Procedures* +

by
Ehud Kalai
Graduate School of Management
Northwestern University, Evanston, Illinois 60201
and
Eitan Muller
Graduate School of Management
Northwestern University
and
University of Pennsylvania, Philadelphia, PA. 19104

*The authors wish to thank Mark A. Satterthwaite for his very helpful contributions.

+This research was partly supported by a grant from the Graduate School of Management, Northwestern University.

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... Let A be a set of alternatives and Σ be a set of strong orderings of A. For an $\Omega \subset \Sigma$ we give a necessary and sufficient condition to determine the existance of an n-person nondictatorial social welfare function $\Omega^n \to \Sigma$ which satisfies unanimity and independence of irrelevant alternatives. The existence of such a function is independent of the number of individuals in the society (n). It is also equivalent to the existence of a nonmanipulable voting procedure on this domain. In this setup, Arrow Impossibility Theorem and Gibbard-Satterthwaite Theorem follow easily as special cases when $\Omega = \Sigma$. J. Econ. Theory,

(English) Kalai: Northwestern University, Evanston;
Muller: University of Pennsylvania, Philadelphia.

Journal of Economic Literature classification Number: 025

Characterization of Domains Admitting Nondictatorial Social Welfare Functions and Normanipulable Voting Procedures

by Ehud Kalai and Eitan Muller

1. Introduction

The purpose of this paper is to characterize the domains of individual preferences which admit n-person nondictatorial Arrow type social welfare functions, (see Arrow [1]), and the domains which admit nonmanipulable voting procedures (see Gibbard [6] and Satterthwaite [15]). We show that the existence of such a function or procedure for a given domain is independent of the number (n) of people for which they are desired, i.e. there exists an n-person social welfare function (voting procedure) for a given domain if and only if there exists a 2-person social welfare function (voting procedure) for the same domain. Thus a concept of a domain being nondictatorial or nonmanipulable (admitting a nondictatorial social welfare function or nonmanipulable voting procedure) can be defined independently of the number of individuals in the society. It turns out that these two concepts are completely equivalent and we give the characterization of those domains (our definition of a nonmanipulable voting procedure assumes a certain rationality condition).

Attempts to overcome Arrow's impossibility theorem by restricting the domains of individual preferences are numerous. The most celebrated example is the single peakedness condition originated by Black [2] and extensively discussed by Arrow [1]. Sen and Pattanaik [18] discussed the conditions under which majority rule which satisfies Arrow's conditions of unanimity,

independence of irrelevant alternatives and nondictatorship, would also satisfy transitivity. For an extensive discussion see Sen's book [17]. Recently additional negative results were demonstrated by Kalai-Muller-Satterthwaite [7]. In this paper we consider any social welfare function, not just those based on majority rule.

The equivalence of Arrow's axioms to axioms of nonmanipulability was treated by many authors. One direction of this equivalence was first proved by Gibbard [6] and very elegantly by Schmeidler-Sonnenschein [16]. Satterthwaite was the first to prove the full equivalence for the case of unrestricted preferences in his 1975 paper. Pattanaik [12] proved one direction of the equivalence for the cases in which individual preferences may be restricted, and a discussion of the possibility of full equivalence for these cases appears in Blin-Satterthwaite [3].

Maskin studied the question of social choice on restricted domains in great depth. In his two papers [8,9] and verbally, under the assumption of anonymity (symmetry of individuals), he characterized the domains which admit a 2-person social welfare function, gave the equivalence of an n-person function to two, three or five persons functions depending on n, and proved the equivalence of Arrow's axioms to axioms of nonmanipulability. Also, since then, independently of us, he studied similar questions to the ones we treat here: namely he replaced the restrictive anonymity assumptions with the well known assumption of nondictatorship, and obtained interesting results under a different approach (see Maskin [10,11]).

A by-product of our characterization is a generalization of Arrow's impossibility Theorem and Gibbard-Satterthwaite Impossibility

Theorem for all the dictatorial domains. (The unrestricted domain is easily shown to be dictatorial.) However we do not deal with the case where the individuals or the society are allowed to be indifferent over alternatives.

We let A denote a set of alternatives with at least two elements, Σ denote the set of all transitive antisymmetric total (i.e. if $p \in \Sigma$ then xpy or ypx or x=y) binary relations on A. An element of Σ is called a preference relation. We let Ω be a non-empty subset of Σ; the elements of Ω represent the admissible preference relations in the society. For an integer $n \geq 2$ Ω^n represents the set of all n-tuples of preferences from Ω and an element of Ω^n, $P = (p_1, p_2, \ldots, p_n) \in \Omega^n$ is called an n-person profile. An n-person social welfare function (SWF) on Ω is a function $f: \Omega^n \to \Sigma$ which satisfies the following two conditions.

1. Unanimity. For every $P \in \Omega^n$ if $P = (p_1, p_2, \ldots, p_n)$, $x, y \in A$ and for $i = 1, 2, \ldots, n$ $xp_i y$ then $xf(P)y$.
2. Independence of Irrelevant Alternatives (IIA). For $x, y \in A$ and $P, Q \in \Omega^n$ if [$xp_i y$ if and only if $xq_i y$ for $i = 1, 2, \ldots, n$] then [$xf(P)y$ if and only if $xf(Q)y$].

f is dictatorial if there exists an i, $1 \leq i \leq n$, for which $f(P) = p_i$ for every $P \in \Omega^n$. f is nondictatorial if it is not dictatorial.

2. Independence of n.

Theorem 1. For $n \geq 2$ there exists a nondictatorial n-person S.W.F. on Ω if and only if there exists a nondictatorial 2-person S.W.F. on Ω.

Before we proceed with the proof of Theorem 1 we need some additional definitions and lemmas. We say that the n-person S.W.F. f is dictatorial whenever two individuals agree if for every $1 \leq i,j \leq n$, there exists an integer $k(i,j)$ such that for every $P \in \Omega^n$, $f(P)=p_{k(i,j)}$ whenever $p_i=p_j$. That is, $k(i,j)$ is a dictator whenever i and j have the same preferences.

Lemma 1. If $n \geq 4$ and f is dictatorial whenever two individuals agree then f is dictatorial.

Proof. If $|\Omega| = 1$ then the proof is completed because then f is dictatorial. So we can assume that there are $p^1, p^2 \in \Omega$ and that $p^1 \neq p^2$. Next we observe that there must be a pair i,j such that $i \neq k(i,j) \neq j$. If not, consider $P = (p^1, p^1, p^2, \ldots, p^2)$. $f(P) = p^1$ and $f(P) = p^2$ a contradiction. So assume without loss of generality that $k(2,3) = 1$ and we will show that 1 is a dictator for f. We first show that $f(P) = p_1$ whenever $p_i = p_j$ for some $1 \leq i, j \leq n$ and $i \neq j$. If $i \neq 1 \neq j$ and $k(i,j) = s \neq 1$ let P be defined by $p_1 = p^1$ and $p_i = p^2$ for $i=2,3,\ldots,n$. Since 2 and 3 agree in P it follows that $f(p) = p^1$ and since i and j agree in P it follows that $f(p) = p_s = p^2 \neq p^1$ a contradiction. Therefore $k(i,j) = 1$. The other case is when one of the individuals i and j, say i, is 1. If $2 \neq j \neq 3$ then we let P be

defined by $p_1 = p_j = p^1$ and $p_s = p^2$ for $1 = s \neq j$. It follows (2 and 3 agree) that $f(P) = p_1$. Finally if $j = 2$ or $j = 3$, say $j = 2$, then since we know already that $k(3,4) = 1$ we can let 3 and 4 assume the rolls of 2 and 3 respectively and we are back in the case that j is different from the two distinct individuals.

So we showed that there must be an individual i such that whenever two individuals agree i is a dictator. Now since $n \geq 4$ for any pair of alternatives x,y at least two individuals must agree on this pair so (by IIA) i is a dictator for this pair. Q.E.D.

We call a pair of distinct alternatives x and y trivial if there are no p^1, $p^2 \in \Omega$ such that xp^1y and yp^2x. Thus the pair x, y is trivial if there is always unanimity on it.

Let $p \in \Omega$. Define $p^{-1} \in \Omega$ to be the preference relation which reverses the ordering of all the non trivial pairs and keeps the orderings of all trivial pairs. Notice that p^{-1} may not exist but when it does it is unique and $(p^{-1})^{-1} = p$.

Two preferences p^1, $p^2 \in \Omega$ are connected if there exists a non trivial pair $x,y \in A$ such that xp^1y and xp^2y, i.e. if they agree on a non trivial pair.

Notice that for every $p \in \Omega$ there is at most one (possibly none) $\bar{p} \in \Omega$ (namely p^{-1}) which is not connected to it. Two preference relations p^1, p^2 are indirectly connected if they are connected by a finite chain of connected preferences, i.e. there exists $q^1, q^2, \ldots, q^n$ such that $p^1 = q^1$ $p^2 = q^n$ and q^i is connected to q^{i+1} for $i=1,2,\ldots,n-1$.

Lemma 2. If any two elements of Ω are indirectly connected and f is a 3-person S.W.F. with the property that for every $p \in \Omega$ there is an $i(p)$, $1 \leq i(p) \leq 2$, such that $f(p_1,p_2,p) = p_{i(p)}$ then f is dictatorial.

Proof. If $|\Omega| = 1$ then the lemma is trivially true. If $|\Omega| \geq 2$ then clearly $i(p)$ is unique for every p (consider a profile with a conflict between individual 1 and individual 2). We will show that $i(p) = i(p')$ for every $p, p' \in \Omega$ and thus $i(p)$ is a dictator for f. It suffices to show that $i(p) = i(p')$ for every pair p,p' which are connected. Suppose p and p' are connected through the non-trivial pair x,y. Let p^1 and p^2 be preferences for which xp^1y and yp^2x. Consider the two profiles $P = (p^1,p^2,p)$ and $P' = (p^1,p^2,p')$. Since p and p' agree on x,y and since there is a conflict between 1 and 2 on x,y it follows by I.I.A. that $i(p) = i(p')$, Q.E.D.

We define the minority rule 3-person S.W.F. f as follows. For every $P = (p_1,p_2,p_3) \in \Omega^3$

> $xf(p)y$ if and only if either xp_iy for $i=1,2,3$ or two p_i's prefer y to x and the third p_i prefers x to y.

Lemma 3. If the 3-person minority rule is a well defined S.W.F. on Ω then there exists a 2-person nondictatorial S.W.F. on Ω.

Proof. Choose any $p \in \Omega$ and define $g_p(p_1,p_2)$ by

$xg_p(p_1,p_2)y$ if and only if either xp_1y and xp_2y or only one of p_1,p_2 prefers x to y and p prefers y to x.

It is clear that g_p is well defined and satisfies unanimity and I.I.A.. We have to show that $\succ = g_p(p_1,p_2)$ is transitive for every $p,p_1,p_2 \in \Omega$. Suppose this is not the case, i.e. there exist $x,y,z \in A$ such that

$$x \succ y \succ z \succ x.$$

Case 1. $x \succ y$ by unanimity of 1 and 2 and the same for $y \succ z$. But then $x \succ z$ by unanimity, a contradiction.

Case 2. $x \succ y$ by unanimity and $y \succ z$ not by unanimity. Assume without loss of generality yp_1z, zp_2y, and zpy. Since $z \succ x$ we must have zp_2x and xpz. But then we have for the minority rule function f

$$x\ f(p_1,p_2,p)\ y\ f(p_1,p_2,p)\ z\ f(p_1,p_2,p)x,$$

a contradiction.

Case 3. $y \succ z$ by unanimity and $x \succ y$ not by unanimity. Assume without loss of generality that xp_1y, yp_2x and ypx. Since $z \succ x$ we must have zp_2x and xpz. But then we have $x\ f(p_1,p_2,p)\ y\ f(p_1,p_2,p)z\ f(p_1,p_2,p)x$ a contradiction. Q. E. D.

Case 4. $x \succ y$ not by unanimity and $y \succ z$ not by unanimity. Assume without loss of generality that xp_1y, yp_2x and ypx. Also since $y \succ z$ not by unanimity we must have zpy. Thus by transitivity of p we have zpx. So in order to have $z \succ x$ we must have zp_1x and zp_2x. Therefore by transitivity we have zp_1y and since we have $y \succ z$ we must have yp_2z. But now we get $x\ f(p_1,p_2,p)\ y\ f(p_1,p_2,p)z\ f(p_1,p_2,p)x$ a contradiction. Q.E.D.

Proof of Theorem 1. Let f be a nondictatorial 2-person S.W.F. on Ω. Define $g:\Omega^n \to \Sigma$ by $g(p_1,p_2,\ldots,p_n) = f(p_1,p_2)$. It is easy to see that g is a nondictatorial n-person S.W.F.

To prove the other direction we show that for $n \geq 3$ if a nondictatorial n-person S.W.F. on Ω, f, exists then there exists a nondictatorial (n-1) person S.W.F. on Ω,g. We first show it for $n \geq 4$. For $1 \leq i < j \leq n$ we define $g_{i,j}$ by

$$g_{i,j}(p_1,p_2,\ldots,p_{n-1}) = f(p_1,p_2,\ldots,p_{j-1},p_i,p_j,p_{j+1},\ldots,p_{n-1}).$$

In other words $g_{i,j}$ replicates i's preferences in the j^{th} place, shifts $p_j,p_{j+1},\ldots,p_{n-1}$ up by one place and then uses f. It is easy to see that all the $g_{i,j}$'s are (n-1)-person S.W.F's and we claim that at least one of them is nondictatorial. Suppose that this is not the case, i.e. all the $g_{i,j}$'s are dictatorial. That implies, for f, that whenever two of its arguments are the same f is dictatorial. By lemma 1 it follows that f is dictatorial which is a contradiction.

Now we assume that $n = 3$. We consider first the case that Ω consists of only two elements of the type p and p^{-1}. In this case we define the 2-person nondictatorial function g by $g(p_1,p_2) = p$

if either $p_1 = p$ or $p_2 = p$ and $g(p_1,p_2) = p^{-1}$ if both $p_1 = p_2 = p^{-1}$. It is easy to check that g(P) is transitive, g satisfies the unanimity condition and since $\Omega = \{p,p^{-1}\}$ it follows that g satisfies I.I.A. . Thus in this case there exists a 2-person nondictatorial S.W.F. g. The second case is when Ω is not of the form described above, thus any two elements of Ω are indirectly connected. We assume first that there are two individuals, say 1 and 2, that are decisive for every pair of alternatives, i.e. for every $x,y \in A$ and every $p_1,p_2,p_3 \in \Omega$ if xp_iy for $i=1,2$ then $x\ f(p_1,p_2,p_3)\ y$. For every $p \in \Omega$ we define $g_p(p_1,p_2) = f(p_1,p_2,p)$. It is clear that all the g_p's are 2-person S.W.F.'s (unanimity follows from the decisiveness of $\{2,3\}$)and by Lemma 2 it follows that if they are all dictatorial then so is f. So there exists a 2-person nondictatorial S.W.F. g_p: Finally, if no pair of individuals is decisive for all the pairs we consider again two cases. Case one is when there is an individual, say 1, who is not weakly decisive for all the pairs, i.e. there exist $p^1,p^2 \in \Omega$ such that $f(p^1,p^2,p^2) \neq p^1$. We can define $g(p_1,p_2) = f(p_1,p_2,p_2)$ and g is a 2-person nondictatorial S.W.F. Case two is when every individual is weakly decisive for all the pairs. In this case f must be a minority rule S.W.F. on Ω and by Lemma 3 there exists a 2-person nondictatorial S.W.F. g on Ω.

Q.E.D.

3. Characterization of Nondictatorial Domains of Preferences

We say that the set of preferences $\Omega \subset E$ is nondictatorial if there exists a nondictatorial n-person S.W.F. on Ω. This definition is independent of n for $n \geq 2$ by Theorem 1 (for n=1 every S.W.F. is dictatorial by unanimity). Examples of dictatorial families are any Ω with $|\Omega| = 1$ (by unanimity) and the whole space Σ provided that there are at least three alternatives (by a well known theorem due to Arrow [1]). Single peaked preferences on a line (see Sen [17] and Black [2]) is an example of a non-dictatorial family. The purpose of this section is to characterize all the nondictatorial families of preferences.

We let $T = \{(x,y) \in A \times A : x \neq y\}$, $TR = \{(x,y) \in T$: there is no $p^1 \in \Omega$ and $p^2 \in \Omega$ such that xp^1y and $yp^2x\}$ and $NTR = T - TR$. Thus T consists of all distinct ordered pairs, NTR consists of the non-trivial ordered pairs (both xp^1y is feasible by some $p^1 \in \Omega$ and yp^2x is feasible by some $p^2 \in \Omega$), and TR consists of the trivial pairs (either xpy for all $p \in \Omega$ or ypx for all $p \in \Omega$).

We say that a set $R \subset T$ is closed under decisiveness implications (closed DI) if for every two pairs $(x,y),(x,z) \in NTR$ the following two conditions are true.

DI1. If there are $p^1, p^2 \in \Omega$ with xp^1y p^1z and yp^2z p^2x then

DI1a $(x,y) \in R$ implies that $(x,z) \in R$, and

DI1b $(z,x) \in R$ implies that $(y,x) \in R$.

DI2. If there is a $p \in \Omega$ with x p y p z then

DI2a $(x,y) \in R$ and $(y,z) \in R$ imply $(x,z) \in R$, and

DI2b $(z,x) \in R$ implies that either $(y,x) \in R$ or $(z,y) \in R$.

We say that Ω is decomposable if there exists a set R, with $TR \subsetneqq R \subsetneqq T$, which is closed under decisiveness implications.

Theorem 2. Ω is nondictatorial if and only if it is decomposable.

Lemma 4. Let $R_1 \subset T$, $TR \subsetneqq R_1 \subsetneqq T$, R_1 is closed under decisiveness implications, and $R_2 = TR \cup \{(x,y) \in T: (y,x) \notin R_1\}$, then $TR \subsetneqq R_2 \subsetneqq T$ and R_2 is closed under decisiveness implications.

Proof. It is clear that $TR \subsetneqq R_2 \subsetneqq T$. To show that R_2 is closed DI we assume that $(x,y),(x,z) \in NTR$.

To show DI1 for R_2 we assume that for some $p^1,p^2 \in \Omega$ xp^1y p^1z and yp^2z p^2x. We assume contrary to DI1a that $(x,y) \in R_2$ and $(x,z) \notin R_2$. Since $(x,y),(x,z) \in NTR$ it follows that $(y,x) \notin R_1$ and $(z,x) \in R_1$. By DI1b of R_1 we get a contradiction so that DI1a must hold for R_2. Assuming, contrary to DI1b for R_2, that $(z,x) \in R_2$ and $(y,x) \notin R_2$ we see that $(x,z) \notin R_1$ and $(x,y) \in R_1$. This contradicts DI1a for R_1, thus R_2 must satisfy DI1b.

To show that R_2 satisfies DI2 we assume that for some $p \in \Omega$ x p y p z. We assume, contrary to DI2a for R_2, that $(x,y) \in R_2$, $(y,z) \in R_2$ and $(x,z) \notin R_2$. This implies that $(y,x) \notin R_1$ and $(z,x) \in R_1$. By DI2b of R_1 it follows that $(z,y) \in R_1$. This implies that $(y,z) \in TR$ by the definition of R_2. Since $(x,z) \in NTR$ it follows that there must exist $p^2 \in \Omega$ for which yp^2z p^2x. Now DI1a for R_2, which was already proved, shows that $(x,z) \in R_2$, a contradiction. To show that R_2 satisfies DI2b we assume, per absurdum, that $(z,x) \in R_2$, $(y,x) \notin R_2$, and $(z,y) \notin R_2$. It follows that $(x,z) \notin R_1$ and $(x,y) \in R_1$.

If $(y,z) \in NTR$ then $(y,z) \in R_1$ which contradicts DI2a for R_1. So it must be that $(y,z) \in TR$. But then, by the fact that $(x,z) \in NT$, it follows that there is a $p^2 \in \Omega$ for which $yp^2\ zp^2x$. Now DI1a for R_1 is contradicted which completes the proof of the lemma.

Proof of Theorem 2. We first assume that Ω is nondictatorial. By Theorem 1 there exists a nondictatorial 2-person social welfare function f on Ω. We let R_1 be the set of pairs for which voter 1 is decisive, i.e.

$R_1 = \{(x,y) \in T$: for every $P \in \Omega^2$ if xp_1y then $xf(P)y\}$. It is clear that $R_1 \supset TR$. If $R_1 = TR$ then 2 is a dictator so $R_1 \supsetneq TR$. Also if $R_1 = T$ then 1 is a dictator so $TR \subsetneq R_1 \subsetneq T$.

Now we show that R_1 is closed DI, so we assume that (x,y), $(x,z) \in NTR$. To show DI1 we assume that for some $p^1,p^2 \in \Omega$ $xp^1y\ p^1z$ and $yp^2z\ p^2x$. Contrary to DI1a, we assume that $(x,y) \in R_1$ and $(x,z) \notin R_1$. Consider the profile $P = (p^1,p^2)$. $xf(P)y$ because $(x,y) \in R_1$ $yf(P)z$ by unanimity. So by transitivity $xf(P)z$. Thus, IIA implies $(x,z) \in R_1$, a contradiction. Contrary to DI1b we assume that $(z,x) \in R_1$ and $(y,x) \notin R_1$. Consider $P = (p^2,p^1)$. $yf(P)z$ by unanimity and $zf(P)x$ because $(z,x) \in R_1$. Thus $yf(P)x$. Therefore, by IIA, $(y,x) \in R_1$ which is a contradiction.

To show DI2 we assume that for some $p \in \Omega$ $x\ p\ y\ p\ z$. We assume, contrary to DI2a, that $(x,y) \in R_1$, $(y,z) \in R_1$, and $(x,z) \notin R_1$. Consider any P with $p_1 = p$. $xf(P)y$ because $(x,y) \in R_1$, and $yf(P)z$ because $(y,z) \in R_1$. By transitivity $xf(P)z$. IIA implies that $(x,z) \in R_1$, a contradiction. Finally we assume, contrary to DI2a, that $(z,x) \in R_1$,

$(y,x) \notin R_1$, and $(z,y) \notin R_1$. Since $(x,z) \in NTR$ there is a $p^1 \in \Omega$ with zp^1x. Consider $P = (p^1,p)$. $zf(P)x$ becuase $(z,x) \in R_1$, $xf(P)y$ because $(y,z) \notin R_1$. So $zf(P)y$ by transitivity. Thus IIA shows that $(z,y) \in R_1$, a contradition.

Notice that we could have defined R_2 to be the set of pairs for which 2 is decisive. This would demonstrate where the structure of Lemma 4 arises.

Now we assume that Ω is decomposable by a set R_1 which is closed DI and satisfies $TR \subsetneq R_1 \subsetneq T$. We define $R_2 = TR \cup \{(x,y) \in T: (y,x) \notin R_1\}$ then by Lemma 4 R_2 is closed DI and $TR \subsetneq R_2 \subsetneq T$. We define $f : \Omega^2 \to \Sigma$ as follows. $xf(P)y$ if and only if one of the following three situations occurs:

1. Unanimity: xp_iy for $i=1,2$.
2. Decisiveness of 1: xp_2y and $(x,y) \in R_1$.
3. Decisiveness of 2: xp_2y and $(x,y) \in R_2$.

We first show that for every $(x,y) \in T$ $xf(P)y$ or $yf(P)x$ but not both. If both, then neither of them could have occurred by unanimity, also they could not both occur by decisiveness of the same voter. So assume without loss of generality that xp_1y, $(x,y) \in R_1$, yp_2x, and $(y,x) \in R_2$. But this shows that $(x,y) \in NTR$ and contradicts the definition of R_2. Now assume that neither $xf(P)y$ nor $yf(P)x$. We can assume without loss of generality that xp_1y and yp_2x. So $(x,y) \in NTR$, $(x,y) \notin R_1$ and $(y,x) \notin R_2$, a contradiction.

Next we observe that f is nondictatorial because $R_1 \neq T$ and $R_2 \neq T$, f satisfies IIA since it is defined on pairs, and it satisfies unanimity by definition. Finally we show that for every P,

f(P) is transitive. We assume to the contrary that there is a P for which $\succ = f(P)$ is not transitive, i.e. for some $x,y,z, \in A$

$x \succ y \succ z \succ x$.

Case 1: $x \succ y$ by unanimity, and $y \succ z$ by unanimity. In this case $x \succ z$ by unanimity, a contradiction.

Case 2: $x \succ y$ by unanimity and $y \succ z$ not by unanmity. Since the properties of R_1 and R_2 are completely symmetric we can assume without loss of generality that $y \succ z$ by decisiveness of 1. Thus $xp_1y\ p_1z$, $(y,z) \in R_1$ and since $z \succ x$ we must have zp_2xp_2y and $(z,x) \in R_2$. DI1b of R_1 implies that $(x,z) \in R_1$ which contradicts the fact that $(z,x) \in R_2$.

Case 3: $x \succ y$ not by unanimity and $y \succ z$ by unanimity. We can assume without loss of generality that $x \succ y$ by decisiveness of 1, thus we must have xp_1yp_1z and $(x,y) \in R_1$. Since $z \succ x$ we must have yp_2zp_2x and $(z,x) \in R_2$. DI1a of R_1 implies that $(x,z) \in R_1$ which contradicts the fact that $(z,x) \in R_2$.

Case 4: $x \succ y$ not by unanimity and $y \succ z$ not by unanimity. If both of these preferences occure by the decisiveness of the same voter, say 1, then we must have xp_1yp_1z, zp_2yp_2x, $(x,y) \in R_1$ $(y,z) \in R_1$ and $(z,x) \in R_2$. But DI2a of R_1 implies that $(x,z) \in R_1$ which contradicts the fact that $(z,x) \in R_2$. So we assume without loss of generality that $x \succ y$ by the decisiveness of 1 and $y \succ z$ by the decisiveness of 2.

So we have xp_1y, zp_1y, yp_2z, yp_2x, $(x,y) \in R_1$, and $(y,z) \in R_2$. Since $z \succ x$ one of the following subcases must occur:

Subcase 4a: zp_ix for i=1,2. In this case DI1a of R_2 implies that $(y,x) \in R_2$ which contradicts the fact that $(x,y) \in R_1$.

Subcase 4b: zp_1x, xp_2z and $(z,x) \in R_1$. In this case (z,x), $(x,y) \in$NTR, thus DI2a of R_1 implies that $(z,y) \in R_1$. Since $(z,y) \in$NTR this contradicts the fact that $(y,z) \in R_2$.

Subcase 4c: xp_1z, zp_2x and $(z,x) \in R_2$. In this case DI2a of R_2 implies that $(y,x) \in R_2$ which contradicts the fact that $(x,y) \in R_1$.

This completes the proof of Theorem 2.

Remark:

From Lemma 4 and the proof of sufficiency it is clear that we could have defined decomposability somewhat differently: Ω is decomposable if there exist two sets R_1 and R_2 such that $TR \subsetneq R_i \subsetneq T$, closed under decisiveness implication and satisfying for all $(x,y) \in$NTR, $(x,y) \in R_1$ if and only if $(y,x) \notin R_2$. These two definitions are equivalent (see Lemma 4), and the difference is in appearance only. (It is easy to show that in this definition, condition DI2b is redundant. This adds somewhat to the external difference.) We let R_i be the set of pairs for which i is decisive, thus having the following intuitive meaning to the condition:

There exist at least two individuals with some power of decisiveness ($TR \subsetneqq R_i$). The condition that $(x,y) \in R_i$ iff $(y,x) \notin R_2$ guarantees the antisymmetry of the SWF. R_i being closed under decisiveness implication guarantees the transitivity of the SWF.

4. Applications

To show the usefulness of Theorems 1 and 2 we discuss the following examples.

Example 1: Arrow's Theorem. If $\Omega = \Sigma$ and $|A| \geq 3$ then all the relations between any three alternatives are possible. This shows that the only sets which are closed under decisiveness implications are $\emptyset$ and the set of all pairs, i.e. there is no nontrivial decomposition. Thus every SWF must be dictatorial.

Example 2: Single peak preferences (see Black [2] and the other standard texts). Let $q \in \Sigma$, and define the set of single peaked preferences relative to the linear order q by $\Omega_q = \{p \in Z$: for every three distinct alternative x,y,z if x q y q z then it is not the case that x p y and z p y$\}$. To show that Ω_q has nondictatorial n-person SWF's for every $n \geq 2$ we must show that Ω is decomposable. Let $R_1 = \{(x,y) \in T$: xqy$\}$. Clearly $\emptyset = TR \subsetneqq R_1 \subsetneqq T$. All that is left to show is that R_1 is closed under decisiveness implications.

DIIa. We suppose that $(x,y) \in R_1$ and for some $p^1, p^2 \in \Omega_q$ xp^1y p^1z and yp^2z p^2x. These relations imply that in q x cannot be between y and z and z cannot be between x and y. Thus y must be the middle one and since x q y we must have x q y q z . Thus $(x,z) \in R_1$.

DI1b. $(z,x) \in R_1$, $x p^1 y$ $p^1 z$ and $y p^2 z$ $p^2 x$. Again y must be the middle one so we must have $z\, q y\, q\, x$. Thus $(y,x) \in R_1$.

DI2a. $(x,y) \in R_1$, $(y,z) \in R_1$ and for some $p \in \Omega_q$, $x\, p\, y\, p\, z$. This shows that $x\, q\, y\, q\, z$. Thus $(x,z) \in R_1$.

DI2b: $(z,x) \in R_1$ and for some $p \in \Omega_q$, $x\, p\, y\, p\, z$. This shows that $z\, q y$ or equivalently $(z,y) \in R_1$.

Q.E.D.

It follows by Theorems 1 and 2 that a family of single peak preferences admits nondictatorial n-person SWF's for every $n \geq 2$.

5. A Characterization of Domains Admitting Nonmanipulable Voting Procedures

The existence of a nonmanipulable voting procedure on a given restricted domain is interesting on its own merits, and not just because of its equivalence to the existence of an Arrow social welfare function (which we will show). Imagine a society for which it is known a prori that all individuals have single peak preferences. This knowledge may come about by a prior analysis or by historical experience and it is shared by all individuals in this society. Restricting the individual to vote in a single peak fashion presents no restriction, and majority rule is a good nonmanipulable procedure for such a society. Are there other types of societies for which this situation is possible? The answer to this question is given by our characterization.

The question of strengthening the result by eliminating the requirement that the stated preferences (ballots), as well as true preferences, are restricted is still open. Blin and Satterthwaite

[4] dealt with the case of majority rule and single peakedness. They showed that the restriction of single peakedness on preferences alone without a restriction on admissible ballots is not sufficient to guarantee nonmanipulability of the (generalized) majority rule.

Our assumption here is that the voting procedure will count only those ballots which conform to the society's known restriction, since any other stated preference is insincere. The resulting voting procedure will be nonmanipulable if and only if Ω (the admissible true preferences) is decomposable. That is, the same restriction that guaranteed the existence of a SWF will guarantee the existence of a nonmanipulable voting procedure.

An n-person Voting procedure is a function $F: \Omega^n \times \mathcal{A} \to A$, where $\mathcal{A}$ is the set of all nonempty subsets of A. We will assume that all voting procedures satisfy the following three conditions.

1. Feasibility. For every $\alpha \in \mathcal{A}$ and every $P \in \Omega^n$ $F(P,\alpha) \in \alpha$.
2. Independence of Nonoptimal Alternatives (INOA). For every $P \in \Omega^n$ and every $\alpha \in \mathcal{A}$ if $\beta \subset \alpha$ and $F(P,\alpha) \in \beta$ then $F(P,\beta) = F(P,\alpha)$.
3. Unanimity. For every $P \in \Omega^n$ and every $\alpha \in \mathcal{A}$ if $x,y \in \alpha$ and xP_iy for $i=1,2,\ldots,n$ then $y \neq F(P,\alpha)$.

F is dictatorial if there exists an individual i such that for every $P \in \Omega^n$ and every $\alpha \in \mathcal{A}$ $F(P,\alpha)$ is i^{th} top choice among the alternatives of α, i.e. $F(P,\alpha)P_iy$ for every $y \in \alpha$ with $y \neq F(P,\alpha)$.

F is manipulable if there exists an $\alpha \in \mathcal{A}$, and $P, \overline{P} \in \Omega^n$ such that for some i $p_i \neq \overline{p}_i$, for every $s \neq i$ $p_s = \overline{p}_s$ and $F(\overline{P}, \alpha) p_i F(P, \alpha)$. See Blin-Satterthwaite [3] for a discussion of the definitions above.

Theorem 3. Let n be any integer, $n \geq 2$. The following three statements are equivalent for every $\Omega \subset \Sigma$.

1. Ω admits an n-person nondictatorial nonmanipulable voting procedure.
2. Ω admits an n-person nondictatorial social welfare function.
3. Ω is decomposable (recall that being decomposable is a property which is independent of n).

Thus Ω admitting an n-person nondictatorial nonmanipulable voting procedure is a property which is independent of n, is equivalent to admitting a nondictatorial social welfare function, and can be checked through the decomposability property given in Theorem 2. The equivalence was discussed by Blin-Satterthwaite [3] after Pattanaik [12] had proved it in one direction. Our proof is similar to a proof by Maskin but it relies heavily on our Theorem 1 which enables us to discard his assumption of Positive Association.

Clearly our applications in examples 1 and 2 are still valid. So that in our setup we obtain the Gibbard-Satterthwaite result as a corollary to Theorem 3. Also Theorem 3 assures us the existence of a nonmanipulable voting procedures for any number of people in the cases where the preferences are restricted to be single peaked.

Proof of Theorem 3. Clearly if Ω admits a 2-person nondictatorial nonmanipulable voting procedure then it does the same for n people (take the extra players as dummies as in the proof of Theorem 1). Therefore, by Theorem 2, it suffices to show the following two facts. If Ω admits an n-person nondictatorial nonmanipulable voting procedure then Ω is decomposable. And, if Ω is decomposable then it admits a 2-person nondictatorial nonmanipulable voting procedure.

To establish the first fact we assume that F is a nondictatorial nonmanipulable n-person voting procedure on Ω and we define the n-person S.W.F. $\tilde{F}$ as follows. For $P \in \Omega^n$ and $x,y \in A$ $x\tilde{F}(P)y$ if and only if $F(P,\{x,y\}) = x$. INOA guarantees that $\tilde{F}(P)$ is well defined. Unanimity of $\tilde{F}$ follows by unanimity of F. Also the nondictatorship of F implies that $\tilde{F}$ is nondictatorial. To show that $\tilde{F}$ satisfies IIA we use the Schmeidler-Sonnenschein method [16]. If $\tilde{F}$ does not satisfy IIA then there are two profile P and Q and a voter j such that $p_i = q_i$ for $i \neq j$, p_j agrees with q_j on the pair $\{x,y\}$, and $\tilde{F}(P)$ disagrees with $\tilde{F}(Q)$ on the pair $\{x,y\}$. It is clear then that voter j can manipulate either $F(P,\{x,y\}$ or $F(Q,\{x,y\})$ in this case. Thus $\tilde{F}$ is a well defined nondictatorial n-person SWF.

To establish the second fact we assume that Ω is decomposable. By Theorem 2 there exists a 2-person nondictatorial S.W.F. f on Ω. We define the voting procedure F by taking $F(P,\alpha)$ to be the most preferred alternative in α according to f(P). It is easy to observe that F is a nondictatorial voting procedure and it is left to show that it is nonmanipulable.

We suppose that F is manipulable. We can assume without loss of generality that there exists an $\alpha \in \mathcal{A}$, $P, \overline{P} \in \Omega$ such that $P_2 = \overline{P}_2$ and a pair of distinct alternatives $x, y \in \alpha$ such that $y = F(\overline{P}, \alpha) \, P_1 \, F(P, \alpha) = x$. Therefore, $yf(\overline{P})x$ and $xf(P)y$. It follows that xP_2y (otherwise unanimity would imply $yf(P)x$) hence $x\overline{P}_2y$. Now since $\overline{P}_1 \neq P_1$ and since $f(P)$ and $f(\overline{P})$ differ when restricted to $\{x,y\}$ if follows by IIA of f that $x\overline{P}_1y$. Therefore, by unanimity, $xf(\overline{P})y$ which is a contradiction. The nondictatorship of f implies that F is nondictatorial, Q.E.D.

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