Tamir, Arie

Working Paper

Pre-Leontief Functions and Least Elements

Discussion Paper, No. 137

Provided in Cooperation with:
Kellogg School of Management - Center for Mathematical Studies in Economics and Management Science, Northwestern University

Suggested Citation: Tamir, Arie (1975) : Pre-Leontief Functions and Least Elements, Discussion Paper, No. 137, Northwestern University, Kellogg School of Management, Center for Mathematical Studies in Economics and Management Science, Evanston, IL

This Version is available at:
https://hdl.handle.net/10419/220496

DISCUSSION PAPER NO. 137

PRE-LEONTIEF FUNCTIONS AND LEAST ELEMENTS

by

Arie Tamir*

May 1975

* Graduate School of Management
Northwestern University

Pre-Leontief functions and least elements

Introduction

In a previous paper [6] we introduced the Z-functions and showed that a nonempty feasible region defined by a continuous Z-function always contains a least element which is also a complementary solution. An iterative procedure to find the least element was also given in [6]. In this work we generalize the Z-property to introduce the pre-Leontief functions. It is then shown that these functions define feasible regions containing least elements.

Following the generalization of the linear complementarity problem presented in [2], we demonstrate a complementarity property associated with the least elements.

A modification of the algorithm presented in [6] for Z-functions, is then shown to be applicable for finding least elements of regions corresponding to pre-Leontief functions.

Definition: Let $f : R^n_+ \to R^m$ be a function from $R^n_+ = \{x \mid x \in R^n, x \geq 0\}$ to R^m, whose components are $f_1,\ldots,f_m$. f is a pre-Leontief function on R^n_+ if for each i, i=1,...,m, there exists an integer k(i), $1 \leq k(i) \leq n$, such that for all $x \in R^n_+$ and $t \geq 0$

$$f_i(x) \geq f_i(x + te_j) \quad \forall\ j \neq k(i),$$

where e_j is the j^{th} unit vector in R^n.

If m = n and k(i) = i f is said to be a Z-function.

Note that the pre-Leontief property ensures that each component of f is nonincreasing with respect to at least (n-1) of its arguments.

If f is linear and characterized by an $m \times n$ matrix then it is pre-Leontief if each of its m rows contains at most one positive element. If the matrix is square and all its off-diagonal elements are nonpositive f is a Z-function.

As indicated in [6], the Z-functions constitute a natural extension of the simple linear Leontief Interindustry Model, corresponding to n products and n industries, each with one type of output. Following the exposition of [6], we immediately observe that the pre-Leontief function can be used to describe the case where several industries may produce the same type of product.

In [6] we provided a constructive proof of the following theorem, dealing with the existence of a least element.

Theorem 1: Let $f : R_+^n \to R^n$ be a continuous Z-function and let q be in R^n. If $X_q^+ = \{x \mid f(x) + q \geq 0,\ x \geq 0\}$ is nonempty then it contains a least element $\bar{x}$ (i.e. $\bar{x} \in X_q^+$ and $\bar{x} \leq y$ for all $y \in X_q^+$), and $\bar{x}$ satisfies $\bar{x}'(f(\bar{x})+q) = 0$.

The proof given in [6] is based on a modification of the well known Gauss-Seidel and Jacobi iterative procedures.

Note that the main significance of the least element is that it (simultaneously) minimizes any real isotone objective function defined on X_q^+.

We now generalize Theorem 1 to pre-Leontief functions.

Existence of Least Elements

Theorem 2: Let $f : R_+^n \to R^m$ be a continuous pre-Leontief function. Given $a \geq 0$ in R_+^n and $q \in R^m$, if $X_{q,a}^+ = \{x \mid f(x)+q \geq 0,\ x \geq a\}$ is nonempty it contains a least element.

Proof:

If $a \in X^+_{q,a}$ it is clearly the least element. Hence, assume that a is not contained in $X^+_{q,a}$. The continuity of f implies that $X^+_{q,a}$ is closed. Let x be the element of $X^+_{q,a}$ which is closest to a, with respect to the Euclidean norm. We show that x is the least element. Suppose that $y \in X^+_{q,a}$ and $y_j < x_j$ for some j, $1 \leq j \leq n$. Define z in R^n_+ by $z_i = \min(x_i, y_i)$ for i, $i=1,\ldots,n$. To complete the proof it is then sufficient to show that $z \in X^+_{q,a}$.

We first note that $z \geq a$. Let i, $i=1,\ldots,m$, and consider f_i. Using the pre-Leontief property and supposing that $z_{k(i)} = x_{k(i)}$ we get $f_i(z) + q \geq f_i(x) + q \geq 0$. (If we had assumed that $z_{k(i)} = y_{k(i)}$ we would have obtained $f_i(z) + q \geq f_i(y) + q \geq 0$). Hence $f(z) + q \geq 0$ and $z \in X^+_{q,a}$.

We note that the preceding theorem generalizes a result due to Cottle and Veinott [3], who dealt with pre-Leontief matrices, i.e. matrices for which each row contains at most one positive element.

In fact, when $f(x)$ is linear the existence of a least element for $X^+_{q,a}$ for all $a \in R^n_+$ and $q \in R^m$, provided $X^+_{q,a}$ is not empty, implies that the matrix defining f is pre-Leontief. The following example illustrates that this is not always true when nonlinear functions are considered.

Example 1: Let $f : R^2_+ \to R^1$ be defined by

$$f(x_1,x_2) = \begin{cases} \dfrac{x_1+1}{x_2+1} & x_1 \leq x_2 \\ \dfrac{x_2+1}{x_1+1} & x_1 \geq x_2 \end{cases}$$

It is easily verified that for each $a \in R_+^2$ and scaler q $X_{q,a}^+$ contains a least element, provided it is not empty, but f is clearly not pre-Leontief.

Although the **pre-Leontief** property is not satisfied globally the following result can be interpreted as a local pre-Leontief property.

Theorem 3: Let $f : R_+^n \to R^m$ be such that for each $a \in R_+^n$ and $q \in R^n$, $X_{q,a}^+ \neq \emptyset$ implies the existence of a least element in $X_{q,a}^+$. Then for each x $x \in R_+^n$, and i, i=1,...,m there exists $k(i,x), 1 \leq k(i,x) \leq n$ such that $f_i(x+te_j) \leq f_i(x)$ for $0 \leq t$ and $j \neq k(i,x)$.

Proof:

Suppose on the contrary that for some $x \in R_+^n$ and i, i=1,...,m, there exist r and p $1 \leq r$, $p \leq n$ $r \neq p$ such that $f_i(x+s_o e_r) > f_i(x)$, and $f_i(x+s_1 e_p) > f_i(x)$ where $s_o > 0$, $s_1 > 0$. For any j, j=1,...,m, $j \neq i$ let $m_j = \min \{f_j(x), f_j(x+s_1 e_p), f_j(x+s_o e_r)\}$ and $m_i = \min \{f_i(x+s_o e_r), f_i(x+s_1 e_p)\}$. Consider $X_{q,x}^+$, where $q = -(m_1, m_2, \ldots, m_m)^1$. $x + s_o e_r$ and $x + s_1 e_p$ belong to $X_{q,x}^+$ which in turn implies that x is the least elemen of $X_{q,x}^+$ - a contradiction to the definition of m_i.

As shown by the next theorem, the result of Theorem 3 can be strengthened to achieve the (global) pre-Leontief property, if seperability is assumed.

Theorem 4: Let $f : R_+^n \to R^m$ be given by $f(x_1,\ldots,x_n) = f^1(x_1) + f^2(x_2) + \ldots + f^n(x_n)$ and suppose that for each $a \in R_+^n$ and $q \in R^m$, $X_{q,a}^+ \neq \emptyset$ implie the existence of a least element in $X_{q,a}^+$. Then f(x) is pre-Leontief.

Proof:

We show that for each i, i=1,...,m, there exists k(i), $1 \leq k(i) \leq n$, such that for any $j \neq k(i)$ j=1,...,n the scaler function $f_i^j(x_j)$ is nonincreasing.

Suppose on the contrary, that there exist r and p $1 \leq r, p \leq n$ $r \neq p$ such that $f_i^r(x_r)$ and $f_i^p(x_p)$ are not nonincreasing. Hence, there exist $\bar{x}_r \geq 0$, $\bar{x}_p \geq 0$, $s_o > 0$ and $s_1 > 0$ satisfying $f_i^p(\bar{x}_p+s_1) > f_i^p(\bar{x}_p)$ and $f_i^r(\bar{x}_r+s_o) > f_i^r(\bar{x}_r)$.

For any u, u=1,...,m, let $q_u = -\min[f_u((\bar{x}_p+s_1)e_p+\bar{x}_r e_r);$ $f_u(\bar{x}_p e_p + (\bar{x}_r+s_o)e_n)]$.

Then it is easily seen that $(\bar{x}_p+s_1)e_p + \bar{x}_r e_r$ as well as $\bar{x}_p e_p + (\bar{x}_r+s_o)e_r$ belong to $X^+_{q,a}$ where $q = (q_1,...,q_m)$ is defined above and $a = \bar{x}_p e_p + \bar{x}_r e_r$. This in turn implies that a is the least element of $X^+_{q,a}$ - a contradiction to the definition of q_i.

As shown in [3], if f is affine the (global) pre-Leontief property is equivalent to the following condition.

For each $q \in R^m$, $X^+_{q,o} \neq \emptyset$ implies the existence of a least element in $X^+_{q,o}$.

The next example shows that the latter condition, (i.e. when the existence of least elements for $X^+_{q,a}$ for all q and a is replaced by the existence of least elements for $X^+_{q,o}$ for all q), is not even sufficient to yield the (local) pre-Leontief property stated in Theorem 3.

Example 2: Let $f : R_+^2 \to R^1$ be defined by $f(x_1,x_2) = h(x_1) + g(x_2)$ where

$$h(x_1) = \begin{cases} x_1 & 0 \le x_1 \le 1 \\ 2-x_1 & 1 \le x_1 \le 2 \\ 0 & x_1 \ge 2 \end{cases} \quad \text{and} \quad g(x_2) = \begin{cases} 1 - x_2 & 0 \le x_2 \le 1 \\ x_2 - 1 & 1 \le x_2 \le 2 \\ 1 & x_2 \ge 2 \end{cases}$$

When some differentiability and regularity assumptions are imposed on the function f, one can derive additional necessary conditions for the existence of a least element. Specifically we will assume an arbitrary qualification for constrained optimization (see [4]).

Theorem 5: Let $f : R_+^n \to R^m$ be a continuously differentiable function and let $\bar{x}$ be a least element of $X_{q,a}^+$. Denote $I = \{i \mid f_i(\bar{x}) + q_i = 0\}$ and $J = \{i \mid \bar{x}_i = a_i\}$ and suppose that a constraint qualification is satisfied at $\bar{x}$.

Then $|I| + |J| \ge n$ and there exists an $n \times (|I| + |J|)$ nonnegative matrix A such that

$$A \cdot \begin{bmatrix} \nabla f_I(\bar{x}) \\ \tilde{I}_J \end{bmatrix} = I_{n \times n} \qquad (1)$$

where $I_{n \times n}$ is the identity matrix of order n, $\nabla f_I(x)$ is the Jacobian of the functions $f_i(x)$, $i \in I$, and $\tilde{I}_J$ is the $|J| \times n$ Jacobian matrix of the functions $g_i(x) = x_i$, $i \in J$.

Proof:

It is assumed that some constraint qualification which is required by the Kuhn-Tucker conditions is satisfied. (See [4]). $\bar{x}$ being a least element of $X^{+}_{q,a}$ implies that for each (row) unit vector e_j in R^n

$$e_j\bar{x} \leq e_j x \qquad \forall\ x \in X^{+}_{q,a} \ .$$

Hence for any j, j=1,...,n, there exists (row) vectors of multipliers $u^j \geq 0$, $u^j \in R^{|I|}$ and $v^j \geq 0$, $v^j \in R^{|J|}$ such that

$$e_j - u^j \nabla f_I(\bar{x}) - v^j = 0 \quad , \quad j=1,2,\ldots,n.$$

Let A be the $n \times (|I| + |J|)$ matrix having (u^j, v^j) as its j^{th} row, then

$$A \cdot \begin{bmatrix} \nabla f_I(\bar{x}) \\ \hline \tilde{I}_J \end{bmatrix} = I_{n \times n} \ .$$

In particular we obtain that $|I| + |J| \geq n$, and the proof is complete.

A similar version of the above theorem was also proved by Bod [1].

Following Mangasarian [5], we note that (1) is equivalent to the inverse isotonicity of $\begin{bmatrix} \nabla f_I(\bar{x}) \\ \hline \tilde{I}_J \end{bmatrix}$. (A matrix B is inverse isotone if for all x $Bx \geq 0$ implies $x \geq 0$.)

As a consequence of the theorem it follows that if $B = \left(\frac{\partial f_i(\bar{x})}{\partial x_j} \right)_{i \in I, j \notin J}$, then there exists a nonnegative matrix A of order $(n-|J|) \times |I|$ such that $AB = I_{(n-|J|)\times(n-|J|)}$. The latter implies that the partial Jacobian of the binding functions, f_i, $i \in I$, with respect to the nonbinding variables, x_j, $j \notin J$ is inverse isotone.

Finally we observe that the regularity assumptions in Theorem 3 cannot be omitted as illustrated by the function $f(x_1,x_2) = -(x_1-1)^2 - (x_2-1)^2$ and $X^+_{q,a}$ where $q = 0$ and $a = (0,0)$.

An Algorithm Finding the Least Element:

Focussing on a constructive approach to find a least element we next show that the algorithm suggested in [6] for Z-functions can be modified to be applied to pre-Leontief functions. (Note that by Theorem 2 the least element can be found by solving $\{\min \sum_{i=1}^{n} x_i \mid \text{s.t. } x \in X^+_{q,a}\}$.)

Let $f : R^n_+ \to R^m$ be a pre-Leontief function and $q \in R^m$. For any i, $i=1,\ldots,n$, let

$$I(i) = \{j \mid f_j \text{ is not monotonically nonincreasing in } x_i\} \quad (1)$$

Further, if $I(i) \neq \emptyset$ we define

$$g_i(x) = \min_{j \in I(i)} \{f_j(x) + q_j\} \quad (2)$$

We observe that the pre-Leontief property ensures that $I(i)$, $i=1,\ldots,n$ are well defined, i.e. $i \neq j \Rightarrow I(i) \cap I(j) = \emptyset$. We also verify that $g_i(x) : R_+^n \to R^1$ is pre-Leontief for each i such that $I(i) \neq \emptyset$. In fact, $g_i(x)$ is monotonically nonincreasing in x_j for all $j \neq i$. We define the following sets of indices

$$J = \{j \mid j \notin UI(i)\} \text{ and } I = \{i \mid I(i) \neq \emptyset\} \qquad (4)$$

We will further assume without loss of generality that

$$I = \{i(1), i(2), \ldots, i(t)\} \text{ where } i(k) < i(k+1),\ 1 \leq k < t \qquad (5)$$

Having associated the above notation with a pre-Leontief function f, we prove the following result, which will be found useful in the application of the algorithm of [6].

Lemma 6: Let $f : R_+^n \to R^m$ be pre-Leontief, and let $a \in R_+^n$ and $q \in R^m$. If $\bar{x}$ is a least element of $X_{q,a}^+$ then $\bar{x}_i = a_i$ for each $i \notin I$, where I is given by (4).

Proof:

$i \notin I$ implies that for each j, $j=1,\ldots,m$, $f_j(x)$ is monotonically nonincreasing in x_i, i.e. for any $x \in R_+^n$ and $t \geq 0$

$f_j(x) \leq f_j(x + te_i)$. If $\bar{x}$ is the least element in $X_{q,a}^+$ then

$0 \leq q_j + f_j(\bar{x}) \leq q_j + f_j(\bar{x} - (\bar{x}_i - a_i)e_i)$, for all j, $j=1,\ldots,n$,

which in turn proves that $\bar{x}_i = a_i$.

As a consequence of the above lemma we can set $x_i = a_i$ for all $i \notin I$, as a start in our effort to find the least element of $X^+_{q,a}$. Our next step is to introduce a Z-function which will be shown to be equivalent to f(x) in the sense of finding least elements.

Let I be given by (2)-(4), and $g_i(x)$, $i \in I$ be defined by (3). For each k=1,...,t define

$$h_k(y_1,\ldots,y_t) = g_{i(k)} \left(\sum_{k=1}^{t} y_k \, e_{i(k)} + \sum_{i \notin I} a_i e_i \right) \qquad (6)$$

Note that $h = (h_1,\ldots,h_t)$ is a Z-function mapping R^t_+ to R^t, where $t \leq \min(m,n)$. We next show that for our purposes it is sufficient to concentrate on the Z-function h.

Theorem 7: Let $f : R^n_+ \to R^m$ be pre-Leontief. Given $a \in R^n_+$ and $q \in R^m$, define $h : R^t_+ \to R^t$ by (6), and I,J by (4).

Let

$$\tilde{X}^+_{q,a} = \{x \mid f_j(x) + q_j \geq 0, \; j \notin J \text{ and } x \geq a\} \text{ and}$$

$$Y^+_{q,a} = \{y \mid h_k(y) \geq 0, \; y_k \geq a_{i(k)}, \; k=1,\ldots,t\}.$$

Then

(1) $Y^+_{q,a} = \emptyset$ if and only if $\tilde{X}^+_{q,a} = \emptyset$.

(2) If $\bar{y}$ is a least element in $Y^+_{q,a}$ then $\bar{x} = \sum_{k=1}^{t} \bar{y}_k \, e_{i(k)} + \sum_{i \notin I} a_i e_i$ is a least element in $\tilde{X}^+_{q,a}$. Further, if also $f_j(\bar{x}) + q_j \geq 0$ for all

$j \in J$ then $\bar{x}$ is also a least element of $X^+_{q,a}$;
otherwise $X^+_{q,a} = \emptyset$.

Proof:

(1) follows directly from the definition of h. If $y \in Y^+_{q,a}$ then it is easily verified that $x = \sum_{k=1}^{t} y_k e_{i(k)} + \sum_{i \notin I} a_i e_i \in \tilde{X}_{q,a}$. Conversely, if $x \in \tilde{X}_{q,a}$, observe that $f_j(x), j=1,\ldots,m$ is monotonically nonincreasing in x_i for all $i \notin I$. In particular for $j \notin J$

$$f_j(x) + q_j \geq 0 \text{ and } x \geq a \Rightarrow f_j\left(\sum_{k=1}^{t} x_{i(k)} e_{i(k)} + \sum_{i \notin I} a_i e_i\right) + q_j \geq 0.$$

The latter implication then yields

$$g_i\left(\sum_{k=1}^{t} x_{i(k)} e_{i(k)} + \sum_{i \notin I} a_i e_i\right) \geq 0, \quad \text{for all } i \in I.$$

Hence, $(x_{i(1)},\ldots,x_{i(t)}) \in Y^+_{q,a}$.

To prove (2), suppose that $\bar{y}$ is a least element in $Y^+_{q,a}$. Clearly, $x = \sum_{k=1}^{t} \bar{y}_k e_{i(k)} + \sum_{i \notin I} a_i e_i \in \tilde{X}^+_{q,a}$. If $\bar{x}$ was not the least element, there would exist $x^o \in \tilde{X}_{q,a}$ such that $x^o_{i(k)} < \bar{y}_k$ for some $1 \leq k \leq t$. As demonstrated above while proving (1), $(x^o_{i(1)},\ldots,x^o_{i(t)}) \in Y^+_{q,a}$, which in turn contradicts the fact that $\bar{y}$ is the least element in $Y^+_{q,a}$.

To prove the second part of (2), it will be shown that if $X^+_{q,a}$ is nonempty, then $\bar{x}$ is the least element of $X^+_{q,a}$. Suppose that x^1 is in $X^+_{q,a}$. In particular $x^1 \in \tilde{X}^+_{q,a}$ and $x^1 \geq \bar{x}$. The proof will be complete when we show that $f_j(\bar{x}) + q_j \geq 0$ for all $j \in J$. But the latter is implied by the monotonicity of f_j, $j \in J$ in all of its n arguments.

We are now ready to apply the algorithm of [6] to find the least element of $X^+_{q,a} = \{x \mid f(x) + q \geq 0,\ x \geq a\}$, provided $X^+_{q,a} \neq \emptyset$, when $f : R^n_+ \to R^m$ is a continuous pre-Leontief function.

Given a continuous Z-function $h : R^P_+ \to R^P$, and $q \in R^P$ the algorithm presented in [6] finds a least element of $X^+_{q,o} = \{x \mid h(x) + q \geq 0,\ x \geq 0\}$ or indicates that $X^+_{q,a}$ is empty.

To find the least element of $\{x \mid h(x) + q \geq 0,\ x \geq a\}$ when $a \in R^P_+$ one has to find the least element of $\{y \mid \tilde{h}(y) + q \geq 0,\ y \geq 0\}$, where $\tilde{h}(y) = h(y+a)$ and add the vector a to this least element.

Let $f : R^n_+ \to R^m$ be a continuous pre-Leontief function, $a \in R^n_+$ and $q \in R^m$. Using Theorem 7, one can apply the following procedure to verify the existence of a least element to $X^+_{q,a}$ and to find the element provided it exists.

Step 1: Define I, J and $h : R^t_+ \to R^t$ by (4) and (6) respectively.

Step 2: Apply the algorithm of [6] to find the least element of the set $\{y \mid h(y) \geq 0,\ y_k \geq a_{i(k)},\ k=1,\ldots,t\}$. If the set is empty $X^+_{q,a}$ is empty: terminate. Otherwise apply Step 3, where $\bar{y}$ is the least element.

Step 3: Define $\bar{x} = \sum_{k=1}^{t} \bar{y}_k e_{i(k)} + \sum_{i \notin I} a_i e_i$. If $f_j(\bar{x}) + q_j \geq 0$ for all $j \in J$, $\bar{x}$ is the least element of $X^+_{q,a}$. Otherwise, $X^+_{q,a}$ is empty.

Finally we demonstrate a complementarity property associated with pre-Leontief functions. Motivated by the generalization of the linear complementarity problem due to Cottle and Dantzig [2], we refer to the generalized nonlinear complementarity problem defined as follows.

Let $F : R^n_+ \to R^m$, have components $F_1,\ldots,F_m$, and suppose that these components are partitioned into n sets S_j, $j=1,\ldots,n$. The generalized complementarity problem is to find $x\in R^n$ such that

$$x \geq 0,\ F(x) \geq 0 \text{ and } x_j \cdot \prod_{r\in S_j} F_r(x) = 0,\ j=1,\ldots,n. \tag{7}$$

Cottle and Dantzig [2] treat the linear case i.e. when f is affine and provide conditions guaranteeing the existence of a complementary solution.

It is shown in [6] that if h is a continuous Z-function and y is a least element of $\{y|h(y) + q \geq 0,\ y \geq 0\}$, then $y'(h(y)+q)=0$. Consequently we can conclude that if f is a continuous pre-Leontief function and x is a least element of $X^+_{q,a}$, then for each $i\in I$, there exists $j = j(i) \in I(i)$ such that $x_i > a_i$ implies $f_j(x) + q_j = 0$. The latter observation can be interpreted in terms of the generalized complementarity property presented above. If $f : R^n_+ \to R^m$ is pre-Leontief and continuous define $S_i = I(i)$, $i=1,\ldots,n$ where $I(i)$ is given by (2). Assuming for simplicity that $a = 0$ we conclude that for any $q \in R^m$, $X^+_{q,o} \neq \emptyset$ implies the existence of a least element which is also a complementary solution to the generalized complementarity problem (7) defined by $F(x) = f(x) + q$.

References

[1] P. Bod, "On Closed Convex Sets Having a Least Element," Technical Report, 1974.

[2] R. W. Cottle and G. B. Dantzig, "A generalization of the linear complementarity problem," Journal of Combinatorial Theory, 8, 1, (1970)

[3] R. W. Cottle and A. F. Veinott, Jr., "Polyhedral sets having a least element," Mathematical Programming, 3 (1972) 238-249.

[4] F. J. Gould and J. W. Tolle, "A Necessary and Sufficient Qualification for Constrained Optimization," SIAM J. Appl. Math. 20 (1971)

[5] O. Mangasarian, "Characterization of real matrices of monotone kind," SIAM Review, 10 (1968) 439-441

[6] A. Tamir, "Minimality and Complementarity properties associated with Z-functions and M-functions," Mathematical Programming, 7 (1974).