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Proof that Prices which are Present-Discounted Certainty Equivalents Fluctuate Randomly

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DISCUSSION PAPER NO. 113

PROOF THAT PRICES WHICH ARE
PRESENT-DISCOUNTED CERTAINTY EQUIVALENTS
FLUCTUATE RANDOMLY*

by

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1. CONSISTENT BELIEFS

Consider a time sequence $\dots, \tilde{\phi}_t, \tilde{\phi}_{t+1}, \dots$ of random vectors. As in Samuelson 1965 [4], the price of spot # 2 wheat in Chicago at time t might be some component, say, \tilde{X}_t of the vector $\tilde{\phi}_t$; another component, say, \tilde{V}_t of the vector $\tilde{\phi}_t$ might denote, as in Samuelson 1973 [5], the price of common stock of General Motors at time t ; and so on. At any time t , the values $\dots, \tilde{\phi}_{t-1} = \varphi_{t-1}, \tilde{\phi}_t = \varphi_t$ are already history and, so, are fixed forever. But the same cannot be said about the values of $\tilde{\phi}_{t+1}, \tilde{\phi}_{t+2}, \dots$. These values are still in the future, and we may suppose that at time t they cannot be known with certainty.

Now, fix attention on \tilde{X}_T , where T is some arbitrarily chosen date; and suppose that at any time $t \leq T$ an individual assigns to \tilde{X}_T a probability distribution

$$\text{Prob}\{\tilde{X}_T \leq x_T \mid \varphi_t, \varphi_{t-1}, \dots\} = P_t(x_T \mid \varphi_t, \varphi_{t-1}, \dots). \quad (1)$$

The probability distribution that the individual assigns to \tilde{X}_T at time $t + 1$ would depend, in general, upon the value that $\tilde{\varphi}_{t+1}$ takes. Suppose that at time t the individual assigns to $\tilde{\varphi}_{t+1}$ the probability distribution

$$\text{Prob}\{\tilde{\varphi}_{t+1} \leq \varphi_{t+1} \mid \varphi_t, \varphi_{t-1}, \dots\} = P_t(\varphi_{t+1} \mid \varphi_t, \varphi_{t-1}, \dots); \quad (2)$$

and to \tilde{X}_T conditional upon the value that $\tilde{\varphi}_{t+1}$ takes, he assigns the (conditional) probability distribution

$$\text{Prob}\{\tilde{X}_T \leq x_T \mid \tilde{\varphi}_{t+1} = \varphi_{t+1}, \varphi_t, \dots\} = P_t(x_T \mid \varphi_{t+1}, \varphi_t, \dots). \quad (3)$$

ASSUMPTION 0: The individual's probability beliefs are consistent in the sense that they accord with the fundamental logic of probability calculus.

Then (1), (2) and (3) above necessarily obey the relation

$$P_t(x_T \mid \varphi_t, \varphi_{t-1}, \dots) = \int_{-\infty}^{\infty} P_t(x_T \mid \varphi_{t+1}, \varphi_t, \varphi_{t-1}, \dots) P_t(d\varphi_{t+1} \mid \varphi_t, \varphi_{t-1}, \dots). \quad (4)$$

Herein, $\int_{-\infty}^{\infty} f(x) g(dx)$ denotes a Stieltjes integral; and when x is a vector, it denotes a multiple Stieltjes integral.

2. ECONOMIC BEHAVIOR: A REVIEW

Suppose, now, that the individual owns a futures contract in # 2 wheat for delivery in Chicago at time T ; and, for this contract, let \tilde{Y}_t be his minimum asking price at any time $t \leq T$. Then at any time

$t < T$, the asking prices $\dots, \tilde{Y}_{t-1} = y_{t-1}, \tilde{Y}_t = y_t$ are already history and, hence, fixed. But the asking prices $\tilde{Y}_{t+1}, \tilde{Y}_{t+2}, \dots, \tilde{Y}_T$ are still in the future, and we may suppose that they cannot at time t be stated with certainty. We may, however, presume that when the due date T for the futures contract arrives arbitrage will ensure that

$$\tilde{Y}_T = x_T \text{ iff } \tilde{X}_T = x_T \text{ commissions aside.} \quad (5)$$

For all times $t < T$, the relation between y_t and \tilde{X}_T will depend upon what we posit about how the individual sets his asking price y_t .

For example, it might be posited that at any time $t < T$ the individual sets y_t equal to the now-expected level of the terminal spot price \tilde{X}_T . That is.

$$\begin{aligned} y_t &= \int_{-\infty}^{\infty} x_T P_*(dx_T | \varphi_t, \varphi_{t-1}, \dots) \\ &= E_t(\tilde{X}_T), \end{aligned} \quad (6)$$

where E_t denotes the "expectation operator" with respect to the probability distribution $P_*(x_T | \varphi_t, \varphi_{t-1}, \dots)$ which the individual assigns to \tilde{X}_T conditional upon $\tilde{\Phi}_t = \varphi_t, \tilde{\Phi}_{t-1} = \varphi_{t-1}, \dots$. This implicitly assumes that the individual has a linear utility for cash flow or income; further, it ignores the availability (to the individual) of risk-free investments yielding a positive interest.

Accordingly, it might more generally be posited that at any time $t < T$ the individual sets y_t equal to the present-discounted expected value of \tilde{X}_T , the discount rate $(r_{t+1}, r_{t+2}, \dots, r_T)$ being equal to the risk-free interest rate $(\rho_{t+1}, \rho_{t+2}, \dots, \rho_T)$ suitably

adjusted for the individual's risk attitude toward holding out the futures contract for the next period ((t to t+1), (t+1 to t+2), ..., (T-1 to T)). That is,

$$\begin{aligned} y_t &= \lambda_{t+1}^{-1} \cdot \lambda_{t+2}^{-1} \cdot \dots \cdot \lambda_T^{-1} \cdot E_t(\tilde{X}_T) \\ &= {}_t\Pi_T \circ E_t(\tilde{X}_T), \end{aligned} \tag{7}$$

where $\lambda_i = 1 + r_i$, with $\lambda_i > \rho_i$ [resp. $\lambda_i < \rho_i$] ($i = t+1, t+2, \dots, T$) according as the individual is risk averse [resp. risk loving]; and ${}_t\Pi_T$ denotes the "discounting operator" from time T to t at the individual's discount rate schedule.

A little reflection shows that both (6) and (7) above are subsumed under the more general behavioral assumption that at any time $t \leq T$ the individual sets y_t equal to the present-discounted certainty equivalent of the terminal spot price \tilde{X}_T , with the discount rate $(r_{t+1}, r_{t+2}, \dots, r_T)$ being equal to the risk-free interest rate $(\rho_{t+1}, \rho_{t+2}, \dots, \rho_T)$ for the period ((t to t+1), (t+1 to t+2), ..., (T-1 to T)).

The question then is what can we say about the sequence $\dots, y_{t-1}, y_t, \tilde{Y}_{t+1}, \dots, \tilde{Y}_T$ ($t < T$) ?

Samuelson 1965 [4] -- with a slight reinterpretation -- provides an answer to this question when y_t is related to \tilde{X}_T ($t < T$) by (6) and (7), respectively. The more general case, when y_t is equal to the present-discounted certainty equivalent of \tilde{X}_T , I shall now investigate.

3. CERTAINTY EQUIVALENT: CONSISTENT PREFERENCES WITH TIME DISCOUNTING

Make the following sufficient assumptions to guarantee the existence of cardinal utilities u_t for cash flow or income at time t ($t \in \mathbb{Z}$), where \mathbb{Z} is the set of natural numbers.

ASSUMPTION 1: For the purpose of the individual's preferences, a risky alternative is completely characterized by the probability distributions for cash flow or income at time t ($t \in \mathbb{Z}$).

ASSUMPTION 2: The individual has, over all risky alternatives, preferences which are consistent in the sense that he cannot, so to speak, make book against himself and end up winning -- or losing -- money! In other words, posit the Axiom of Complete Ordering of all risky alternatives, and the Axiom of "Strong Independence" (see, for example, Samuelson 1952 [3]).

This much assumption implies the existence of cardinal utility functions u_t for sure cash flow or income at time t ($t \in \mathbb{Z}$) and, hence, also the expected utility maximization rule for choice among risky alternatives. The following further assumption should be acceptable to all but the mystical few.

ASSUMPTION 3: All the utility functions u_t ($t \in \mathbb{Z}$) are strictly increasing monotonic in their argument, cash flow or income.

This now allows definition of the individual's "certainty equivalent operator" C_t :

$$C_t(\tilde{X}_\tau) = \zeta_\tau \quad \text{iff} \quad u_\tau(\zeta_\tau) = \int_{-\infty}^{\infty} u_\tau(x_\tau) P_*(dx_\tau | \varphi_t, \varphi_{t-1}, \dots), \quad (8)$$

where \tilde{X}_τ ($\tau \in Z$) is any random cash flow at time τ and $P_*(x_\tau | \varphi_t, \varphi_{t-1}, \dots)$ is the probability distribution which the individual assigns to \tilde{X}_τ conditional upon $\tilde{\varphi}_t = \varphi_t, \tilde{\varphi}_{t-1} = \varphi_{t-1}, \dots$ ($t \leq \tau$).

Thus, the behavioral axiom that at any time $t \leq T$ the individual sets y_t equal to the present-discounted certainty equivalent of the terminal spot price \tilde{X}_T may be formally stated as

$$\begin{aligned} y_t &= \lambda_{t+1}^{-1} \cdot \lambda_{t+2}^{-1} \cdot \dots \cdot \lambda_T^{-1} \cdot C_t(\tilde{X}_T) \\ &= {}_t\pi_T \circ C_t(\tilde{X}_T), \end{aligned} \quad (9)$$

where, in this case, $\lambda_i = 1 + \rho_i$ ($i = t+1, t+2, \dots, T$).

Lastly, make the following

ASSUMPTION 4: The individual's preferences among sure cash flows at different times accord with the usual present-discounted value calculus.

The following Fundamental Consistency Theorem may now be recaptured without proof from Prakash 1974 [1] and [2].

THEOREM (Prakash 1974): Grant Assumptions 1 through 4 above.

Then, the family $\{u_t | t \in Z\}$ of the individual's cardinal

utility functions is such that, for any $\tau \in \mathbb{Z}$, and any $t \leq \tau$,

$${}_t \pi_\tau \circ C_t \equiv C_t \circ {}_t \pi_\tau. \quad (10)$$

Not too roughly, this says that, if an individual has consistent preferences, then it must be that his present-discounted certainty equivalent of any random cash flow \tilde{X}_τ is the same as his present certainty equivalent of the random cash flow obtained by discounting \tilde{X}_τ to the present. Using (8) above, (10) translates into

$$u_t({}_t \pi_\tau(\zeta_\tau)) = \int_{-\infty}^{\infty} u_t({}_t \pi_\tau(x_\tau)) P_t(dx_\tau | \varphi_t, \varphi_{t-1}, \dots), \quad (11)$$

where $\zeta_\tau = C_t(\tilde{X}_\tau)$.

4. PRESENT-DISCOUNTED CERTAINTY EQUIVALENTS FLUCTUATE RANDOMLY

Toward enunciating the main theorems, let $\{\tilde{\Phi}_t\}$ be a "time" sequence of random vectors $\tilde{\Phi}_t$ of which some component \tilde{X}_t denotes random cash flow at time t . Fix a date T arbitrarily. For any $t \leq T$, let the probability laws (1) through (4) hold. Further, grant Assumptions 1 through 4, and let u_t be a cardinal utility function for random cash flows at time t ($t \leq T$).

THEOREM: For $t < T$, the sequence $y_t, \tilde{Y}_{t+1}, \dots, \tilde{Y}_T$ defined by (9) has the property

$$\begin{aligned} C_t(\tilde{Y}_{t+1} | \varphi_t) &= \lambda_{t+1} \cdot y_t, \\ C_t(\tilde{Y}_{t+k} | \varphi_t) &= \lambda_{t+1} \cdot \lambda_{t+2} \cdot \dots \cdot \lambda_{t+k} \cdot y_t, \end{aligned} \quad (12)$$

where λ_i is the discount rate for the period $(i-1)$ to i
 $(i = t+1, t+2, \dots, t+k)$ and $(t+k) \leq T$.

Proof: By definition (9), $(y_{t+1} | \varphi_{t+1}) = {}_{t+1}\pi_T \circ C_{t+1}(\tilde{X}_T | \varphi_{t+1})$. Then, using (3) and (11),

$$u_{t+1}(y_{t+1} | \varphi_{t+1}) = \int_{-\infty}^{\infty} u_{t+1}({}_{t+1}\pi_T(x_T)) P_t(dx_T | \varphi_{t+1}, \varphi_t, \varphi_{t-1}, \dots).$$

Now denote $C_t(\tilde{Y}_{t+1} | \varphi_t) = \zeta_{t+1}$ for short. Then, by (2) and (8),

$$\begin{aligned} u_{t+1}(\zeta_{t+1}) &= \int_{-\infty}^{\infty} u_{t+1}(y_{t+1} | \varphi_{t+1}) P_t(d\varphi_{t+1} | \varphi_t, \varphi_{t-1}, \dots) \\ &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} u_{t+1}({}_{t+1}\pi_T(x_T)) P_t(dx_T | \varphi_{t+1}, \varphi_t, \varphi_{t-1}, \dots) \\ &\quad P_t(d\varphi_{t+1} | \varphi_t, \varphi_{t-1}, \dots). \\ &= \int_{-\infty}^{\infty} u_{t+1}({}_{t+1}\pi_T(x_T)) P_t(dx_T | \varphi_t, \varphi_{t-1}, \dots), \text{ using (4)}. \end{aligned}$$

Denote $C_t(\tilde{X}_T) = \zeta_T$ for short. Then, using (11), the right side of the above equality is identified to be equal to $u_{t+1}({}_{t+1}\pi_T(\zeta_T))$. Hence,

$$u_{t+1}(\zeta_{t+1}) = u_{t+1}({}_{t+1}\pi_T(\zeta_T)), \text{ so that } \zeta_{t+1} = {}_{t+1}\pi_T(\zeta_T); \text{ and}$$

$${}_{t+1}\pi_T(\zeta_{t+1}) = {}_{t+1}\pi_T \circ {}_{t+1}\pi_T(\zeta_T) = {}_t\pi_T(\zeta_T) = y_t \text{ by definition. Recalling}$$

that ${}_{t+1}\pi_T = \lambda_{t+1}^{-1}$, we may rearrange the terms to yield the result

$$C_t(\tilde{Y}_{t+1} | \varphi_t) = \lambda_{t+1} \cdot y_t. \text{ The second part of (12) now follows by}$$

repeating the above argument k times. \diamond

COROLLARY (Samuelson's Theorem 2, 1965): For $t < T$, the sequence $y_t, \tilde{Y}_{t+1}, \dots, \tilde{Y}_T$ defined by (7) has the property

$$E_t(\tilde{Y}_{t+1} | \varphi_t) = \lambda_{t+1} \cdot y_t ;$$

$$E_t(Y_{t+k} | \varphi_t) = \lambda_{t+1} \cdot \lambda_{t+2} \cdot \dots \cdot \lambda_{t+k} \cdot y_t .$$

(13)