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Optimal Facility Location under Random Demand with General Cost Structure

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OPTIMAL FACILITY LOCATION UNDER RANDOM DEMAND WITH GENERAL COST STRUCTURE

by

V. Balachandran and Suresh Jain

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ABSTRACT

This paper investigates the problem of determining the optimal location of plants and their respective production and distribution levels, in order to meet demand at a finite number of centers. The possible location of the plants is restricted to a finite set of sites and the demand at the different centers is allowed to be a random variable. The cost structure of operating a plant is dependent on its location and is assumed to be a piecewise linear function of the production level, though not necessarily concave or convex. The paper is organized in three parts. In the first part a branch and bound procedure for the general piecewise linear cost problem is presented assuming that the demand is known. In the second part a solution procedure is presented for the case when the demand is random, assuming a linear cost of production. Finally, in the third part, a solution procedure is presented for the general problem utilizing the results of the earlier parts. Certain extensions, such as capacity expansion or reduction at existing plants, and geo-political configuration constraints can be easily incorporated within our framework.

1. Introduction

The facility location and related problems are of relevance in the long range planning of a firm's operations. These problems involve the determination of the location of the facilities, their associated capacity, and the distribution of the product from these facilities to the different demand centers. Different aspects of this problem have been investigated by a number of researchers under varied assumptions [2, 8, 10, 14, 15, 17]. In our paper we consider a generalized version of the above problem. Specifically, we consider the case where the location of facilities and their sizes are to be decided upon, in order to satisfy the demand from different centers. The demand at these different centers is assumed to be random and the cost associated with the production at any facility is assumed to be a piece-wise linear function though not necessarily convex or concave. In the next section we consider the problem formulation and its motivation.

2. Model Formulation

A firm manufactures a product which is required at n different demand centers. The demand b_j $(j = 1,2,..\ ,n)$, at each center, is assumed to be a random variable whose marginal density $f(b_j)$ is assumed to be known. The firm has the option of setting up facilities at m different sites, i, $(i = 1,2,..\ ,m)$. The possible capacities of the facility at site i could be any one element from the ordered set A_i, where $A_i = \{a_{ir} \mid r = 1,2,..\ ,m_i\}$. The first element of each of the sets $a_i \in A_i$, $(i = 1,2,..\ ,m)$ is assumed to be 0 and corresponds to the decision of not locating a facility at site i and the last element a_{im_i} corresponds to the maximum possible production at site i. The cost of producing

y_i units at site i is $f_i(y_i)$ where

$$(1)\quad f_i(y_i) = \begin{cases} 0 & \text{if } y_i = a_{i1} = 0 \\ K_{ir} + v_{ir}y_i & \text{if } a_{ir} < y_i \leq a_{i,r+1} \text{ for } r = 1,2,\ldots,m_{i-1} \\ \infty & \text{if } y_i > a_{im_i} \end{cases}$$

K_{ir} may be considered as the fixed component of the cost associated with setting up a plant of maximum capacity $a_{i,r+1}$, and v_{ir} is the per unit variable cost. Thus the cost structure at any particular site is a piece-wise linear function of the quantity produced. The cost of distributing x_{ij} units from a facility at site i to demand center j is $t_{ij}\, x_{ij}$ where t_{ij} is a constant. These costs may be considered as the discounted costs if a multi-period planning horizon is considered. With the above notation the problem can be formulated as follows:

$$(2)\quad \text{Minimize} \quad Z = \sum_{i=1}^{m} f_i(y_i) + \sum_{i=1}^{m}\sum_{j=1}^{n} t_{ij}x_{ij}$$

Subject to

$$(3)\quad \sum_{i=1}^{m} x_{ij} = B_j \quad \text{for} \quad j \in J\ ,$$

$$(4)\quad \sum_{j=1}^{n} x_{ij} = y_i \leq a_{im_i} \quad \text{for } i \in I\ ,$$

$$(5)\quad x_{ij} \geq 0,\ y_i \geq 0\ ,$$

where $J = \{1, 2,\ldots,n\}$, $I = \{1, 2,\ldots,m\}$ and B_j represents the realization of the random variable b_j. In order to ensure a feasible solution we assume

$$\sum_{i=1}^{n} a_{im_i} \geq \sum_{j=1}^{n} B_j\ .$$

The above formulation incorporates the problems posed by various authors. For instance, the fixed change transportation problem [8], the location-allocation

problem [14], and the warehouse location problem [2, 17], are all special cases of the problem posed above. A discussion of the recent research work done in this area has been presented by White and Francis [15], and also by Soland [14]. Our formulation is similar in some respects to that of Soland [14]. Soland considers the case where the demand is deterministic and the cost functions are concave. Since the demand is generally not known with certainty as discussed in [6, 16], the consideration of this problem with probabilistic demand appears to be more realistic. Francis and White [15] consider probabilistic demand for a different problem viz. that of determining optimum warehouse sizes only. The problem that they consider is different from ours, since they are not concerned with either the location, or the distribution aspects. Essentially their problem has no constraints of the type represented by equations (3) and (4). The consideration of random demands increases the complexity of the problem. To our knowledge, no computationally satisfactory solution of this entire problem exists in the published literature.

Further, the cost structure that is generally considered is either a fixed cost plus a variable cost such that the total cost of production is concave, or a general concave cost as in Soland [14]. In the real world, because of indivisibilities and economies/diseconomies of scale, the cost structure is often different. Hadley and Whitin [9, Chapter 2, p. 62] discuss quantity discounts where the cost function though piece-wise linear is neither concave nor convex. Rech and Barton [11] also consider nonconvex piece-wise linear cost functions for solving the transportation problem. As discussed by them this cost structure arises frequently in the real world. The location, capacity and distribution problem with such a cost structure has not been discussed

in the literature. In our framework such cost structures are treated.

For ease of exposition we present the solution procedure of this problem in three phases. Initially we consider the case where the demand is deterministic. We develop an algorithm treating B_j in (3) as known constants. This algorithm essentially solves a deterministic facility location, capacity, and distribution problem with a general piece-wise linear cost structure. A branch and bound procedure wherein sub-problems are solved using operator theory [12, 13] (an extension of parametric programming where simultaneous changes in the parameters of a transportation problem are investigated) is presented in Section 3 for the solution of this deterministic problem. In Section 4 we develop an algorithm to solve the probabilistic case when the cost is assumed to be a linear function. The algorithm for this probabilistic demand problem utilizes the operator theoretic approach and the Kuhn-Tucker conditions. Finally, in Section 5, we integrate the algorithms in Sections 3 and 4 to develop a solution procedure for the entire problem. We show that the branch and bound procedure of Section 3 for the entire problem, result in sub-problems of the type considered in Section 4. This three-phase approach provides flexibility for a user to solve either the general problem or the special cases considered in Sections 3 and 4.

3. The Deterministic Demand Model

In this section, we assume that the B_j's given in equation (3) are known constants. The cost $f_i(y_i)$ that is associated with the production of y_i units at site i is assumed to be piece-wise linear. Some of the different cost structures which arise in reality, and are permissible in our formulation are sketched below (Figures 1a - 1e).

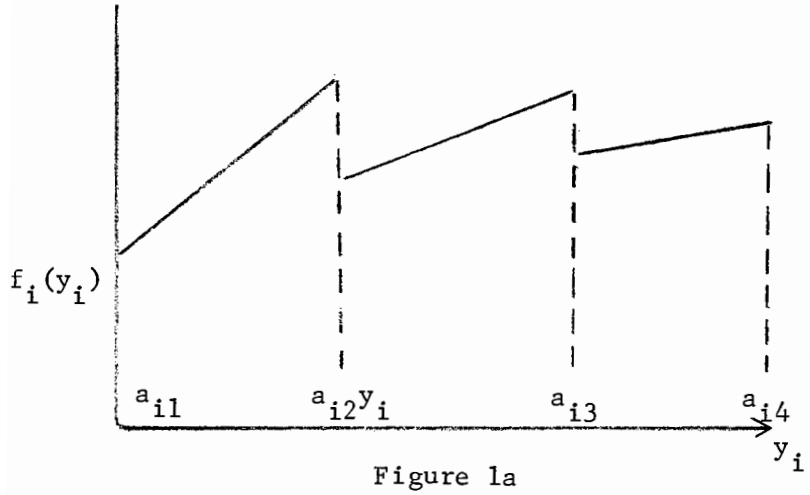


Figure 1a
Economies of scale both in fixed and variable cost

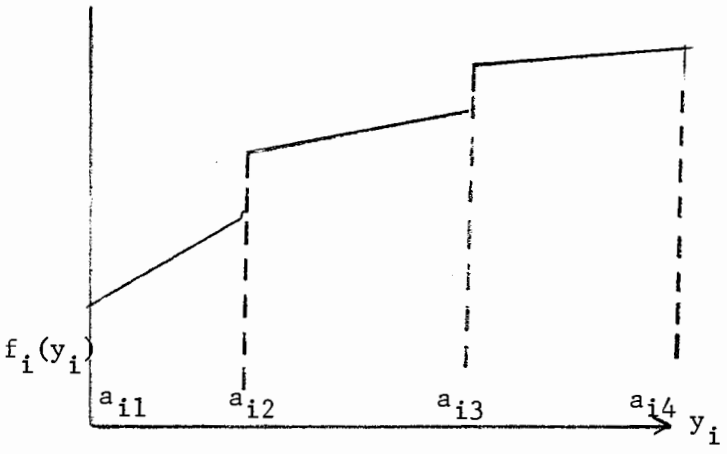


Figure 1b
Economies of scale in variable cost

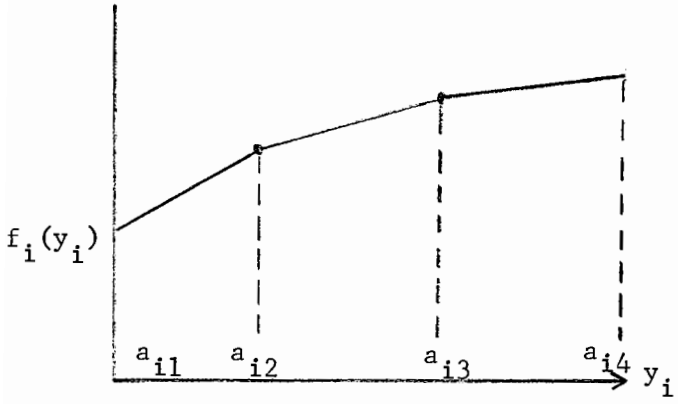


Figure 1c
Piece-wise Linear Concave Costs

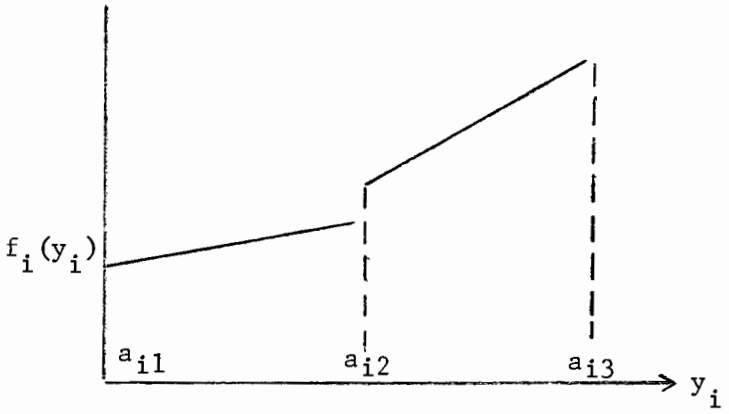


Figure 1d
Diseconomies of scale

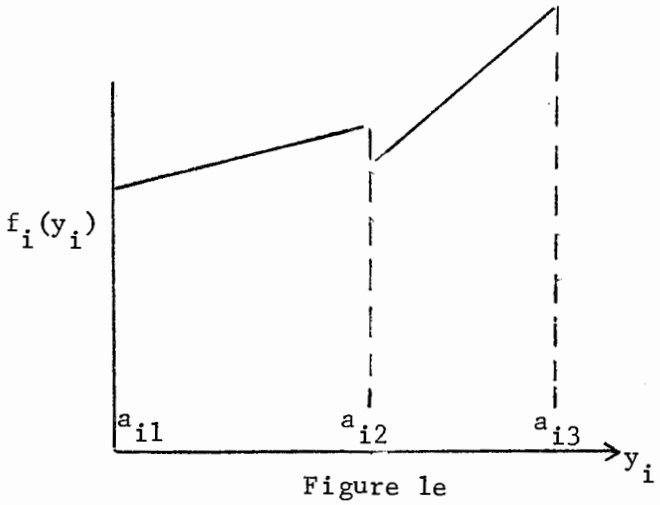


Figure 1e
Diseconomies of scale in variable cost

To solve this problem we apply the branch and bound procedure. We first approximate all the cost functions, $f_i(y_i)$, by their best linear underestimates.

Definition 1: A linear underestimate of the function $f_i(y_i)$ over the interval R_i is a linear function $\ell^0 + \ell_i y_i$ such that

$$\ell_0 + \ell_i y_i \leq f_i(y_i) \quad \text{for all } y_i \in R_i ,$$

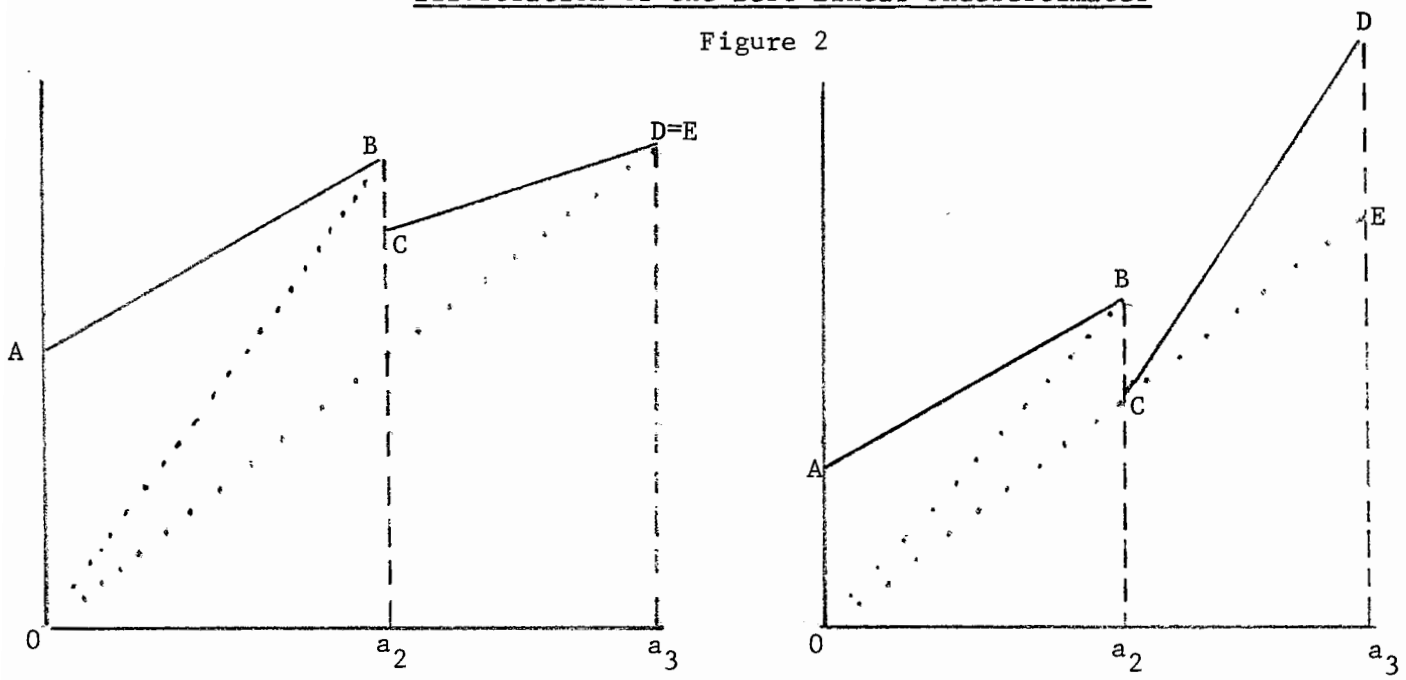
and

$$\ell_0 + \ell_i y_{io} = f_i(y_{io}) \text{ where } y_{io} \in R_i \text{ and } y_{io} \leq y_i \ \forall\ y_i \in R_i .$$

Definition 2: The best linear underestimate $\ell_0 + c_i y_i$ of the function $f_i(y_i)$ over the interval R_i is a linear underestimate such that if $\ell_0 + \ell_i y_i$ is any linear underestimate of the function $f_i(y_i)$ over R_i then $c_i \geq \ell_i$.

Illustration of the Best Linear Underestimates

Figure 2



Cost function is OABCD.

As illustrated in Figure 2, OE is best linear underestimate of the cost functions sketched in the interval $[0, a_3]$. The best linear underestimates of the cost function in the intervals $[0, a_2]$ and $[a_2, a_3]$ are OB and CD respectively.

Substituting the best initial linear underestimates $\ell^1_{i0}+c^1_i y_i$ for each of the function $f_i(y_i)$ over the intervals $[a_{io}, a_{im_i}]$ we obtain the following transportation problem after removing the constant term, if any, from the objective function (corresponding to node 1 in Figure 3).

$$\text{(6)} \qquad \text{Minimize} \quad Z_1 = \sum_{i=1}^{m} \sum_{j=1}^{n} (c^1_i + t_{ij})x_{ij}$$

subject to

$$\text{(7)} \qquad \sum_{i=1}^{m} x_{ij} = B_j \text{ for } j \in J \,,$$

$$\text{(8)} \qquad \sum_{j=1}^{n+1} x_{ij} = a_{im_i} \text{ for } i \in I \,,$$

$$\text{(9)} \qquad x_{ij} \geq 0 \text{ for } i \in I \text{ and } j \in J \cup \{n+1\} \,,$$

where, $x_{i,n+1}$ $(i = 1,2,\ldots,m)$ are the slack variables. Let us denote by J' the set $J \cup \{n+1\}$. Let the optimal solution vector to this approximate problem be X^1 and its optimal cost be $Z_1(X^1)$. If $Z(X^1)$ is the value of the original objective function (2) for the solution vector X^1 then we have the following result:

Lemma. If X^* is the optimal solution vector to the original problem represented by equations (2) - (5) then the optimal value $Z(X^*)$ is bounded above by $Z(X^1)$ and bounded below by $Z_1(X^1)$.

Proof. Since by definition $\ell^1_{i0}+c^1_i(y_i) \leq f_i(y_i)$ over $[a_{i0}, a_{im_i}]$, and by definition $y_i = \sum_{j \in J} x_{ij}$, for $i \in I$, we have $Z_1(X^*) \leq Z(X^*)$ for any feasible X^*.

Further, since X^1 is an optimal solution for (6) - (9), and X^* is a feasible solution to that problem, we have

$$Z_1(X^1) \leq Z_1(X^*) \ .$$

Hence we have

$$Z_1(X^1) \leq Z(X^*) \ .$$

Again X^1 is a feasible solution to the original problem and since X^* is an optimal solution to the original problem

$$Z(X^*) \leq Z(X^1) \ . \qquad \text{Q.E.D.}$$

If L and U denote the current lower and upper bounds, then after solving the first approximate problem corresponding to node 1 (Figure 3), we have

$$L = Z_1(X^1) \ ,$$

and

$$U = Z(X^1) \ .$$

After solving the first approximate problem we partition the domain of definition of one of the functions $f_i(y_i)$ so as to obtain better linear underestimates in each of the different segments of the partition. A number of different rules could be used to determine the index $i(i \in I)$ on which to partition. Further, there is also the option of determining the number of segments in the partition, which would equal the number of branches. In this paper we provide one rule for obtaining two branches. This rule is similar to the one proposed by Rech and Barton [11]. We determine the first index k, $k \in I$ such that

$$f_k(Y_k^1) - c_k^1 Y_k^1 - \ell_{k0}^1 \geq f_i(Y_i^1) - c_i^1 Y_i^1 - \ell_{i0}^1 \ , \text{ for } \forall\ i \in I, \tag{10}$$

and

$$f_k(Y_k^1) - c_k^1 Y_k^1 - \ell_{k0}^1 > 0 \ , \tag{11}$$

where

$$Y_i^1 = \sum_{j \in J} X_{ij}^1 \ . \tag{12}$$

The two branches that we obtain are

(13) $$y_k \leq a_{k,t} ,$$

and

(14) $$y_k > a_{k,t} ,$$

where

(15) $$a_{k,t} < Y_k^1 \leq a_{k,t+1} .$$

Thus at each stage of the branching process we generate two additional sub-problems which can be represented by nodes in the branching tree. These nodes are numbered sequentially as the sub-problems are generated. These sub-problems partition the domain of definition of $f_k(y_k)$. We substitute the best linear approximation of $f_k(y_k)$ in each of these partitions and add the relevant constraint from the set (12)-(13) to each of the sub-problems. The resultant problems are also transportation problems, since $y_k \leq a_{kt}$ simply means that one of the right-hand side constants in (8) is changed, whereas $y_k > a_{kt}$ implies that $x_{k,n+1} < a_{km_i} - a_{kt}$; that is, the variables $x_{k,n+1}$ is upper bounded (when we solve the resultant transportation problem $x_{k,n+1} \leq a_{km_i} - a_{kt} - 1$) Thus the following changes occur in each of the branches (represented by nodes 2 and 3 in Figure 3)

(i) cost coefficients of all x_{kj}, $j = 1,2,\ldots,n$ differ by some constant amount--k_1 in one of the branches and k_2 in the other branch.

(ii) capacity of plant k is changed in one of the branches and the upper bound of the slack variable ($x_{k,n+1}$) corresponding to plant k is changed in the second one.

We may remark here that the new solutions, taking into consideration the above changes, can be generated (without resolving) by the operator theoretic approach discussed in [12, 13]. If X^2 and X^3 are the optimal solutions to the

approximate problems generated by the branching process at nodes 2 and 3, then we can show that the current lower bound, L, and upper bound, U, satisfy the following: (by an argument similar to the one in Lemma 1).

$$L = \text{Minimum}[Z_1(X^2),\ Z_1(X^3) > Z_1(X^1)\ ,$$

$$U = \text{Minimum}[Z(X^1),\ Z(X^2),\ Z(X^3)]\ ,$$

and

$$L \leq Z(X^*) \leq U.$$

Remark: The strict inequality $L > Z_1(X^1)$ follows from the fact that the best linear underestimates in at least one of the partitions has to be strictly greater than the previous best linear underestimate since (11) holds.

If $L = Z_1(X^2)$ we branch on the node corresponding to the solution X^2 to obtain nodes 4 and 5 as shown in Figure 3. If X^4 and X^5 are the solutions at these nodes then the new current lower and upper bounds are given by

$$L = \text{Min}[Z_1(X^4),\ Z_1(X^5),\ Z_1(X^3)]\ ,$$

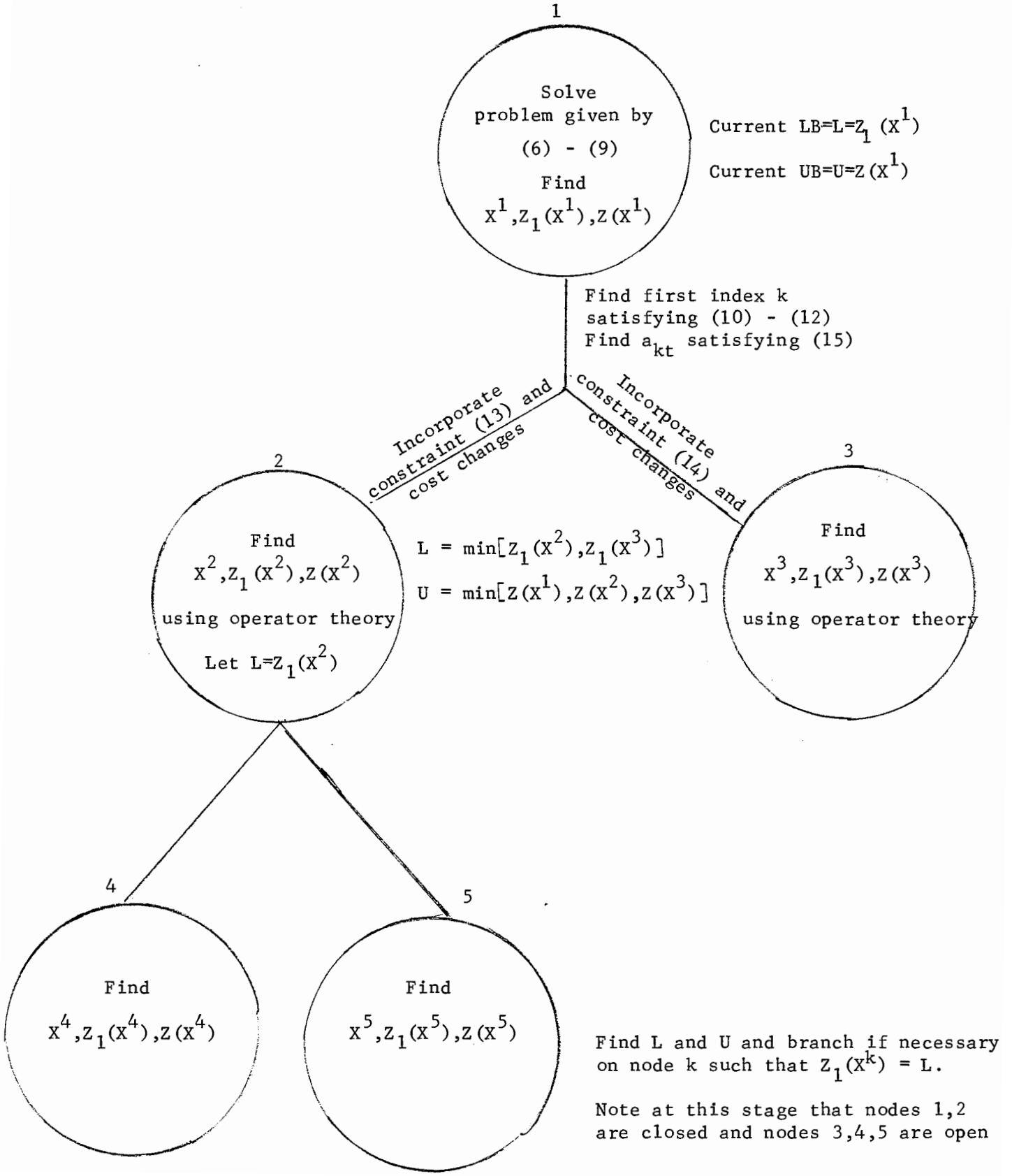
and

$$U = \text{Min}[Z(X^1),\ Z(X^2),\ Z(X^3),\ Z(X^4),\ Z(X^5)]$$

Thus the current lower bound, L, at any stage equals the minimum of the lower bounds at the open nodes (nodes from where there are no branches), whereas the current upper bound, U, is the minimum of the upper bound over all the nodes. The algorithm terminates when the current lower bound equals the current upper bound at the same node. This process terminates in a finite number of steps since the number of sites is finite and since at each branch we partition the domain of definition of the y_i's into disjoint intervals. Further, since each of the $f_i(y_i)$ is a piece-wise linear function we cannot have more than m_i

Figure 3

Branch and Bound Procedure



partition on the variable y_i before we have the best linear underestimate equalling the function itself, and thereby not allowing further branching on the variable y_i.

We therefore have the following algorithm.

Algorithm I:

Step 0: For each of the functions $f_i(y_i)$ substitute $\ell_{i0}^1 + c_i^1 y_i$, the best linear underestimate of $f_i(y_i)$ for $a_{i1} \leq y_i \leq a_{im_i}$. Solve the resultant transportation problem, and create an open node corresponding to the solution X^1. The current lower bound is $Z_1(X^1)$ and the current upper bound is $Z(X^1)$.

Step 1: If the current lower bound equals the current upper bound and occur at the same node then terminate. The optimal solution corresponds to the solution at the node where these bounds are equal. Otherwise go to step 2.

Step 2: Determine the open node with the current lower bound and partition on the index k that satisfies equations (11) - (13). Close this node and generate two additional open nodes corresponding to equations (13) and (14). Solve the two resultant transportation problems using operator theory from the solution of the old open node. If any of the resultant transportation problems has no feasible solution then close that node and drop it from consideration for further branching. Let the upper bound associated with such a node equal ∞. Determine the current lower bound over all the open nodes, and the current upper bound over all the nodes. Go to Step 1.

Remark: If at a branch (node) some $y_k \leq a_{k1} = 0$, (no plant at site k), then in the resultant transportation problem the variables x_{kj} for $j \in J'$ can be set to zero, or alternatively dropped.

4. The Probabilistic Model

In this section we investigate the problem in equations (2) - (5) with special emphasis to the case when the demands b_j, $j \in J$, are random and the functions $f_i(y_i)$ are linear. The demand is assumed to follow some known multivariate distribution, which allows interaction among the demands at the different centers. Let $f(b_j)$ represent the marginal density function of b_j. We therefore have the following problem:

$$\text{Min} \quad \sum_{i\in I} \sum_{j\in J} c_i x_{ij} + \sum_{i\in I} \sum_{j\in J} t_{ij} x_{ij} \tag{16}$$

Such that

$$\sum_{i\in I} x_{ij} = b_j \quad \forall\ j \in J\ , \tag{17}$$

$$\sum_{j\in J} x_{ij} + x_{i,n+1} = U_i \quad \forall\ i \in I\ , \tag{18}$$

$$x_{ij} \geq 0 \text{ for } \forall\ i \in I \text{ and } j \in J\ , \tag{19}$$

$$0 \leq x_{i,n+1} \leq u_i \text{ for } \forall\ i \in I, \tag{20}$$

where each b_j is assumed to be random with known marginal density (or mass) function $f(b_j)$.

The solution procedure presented in this section is based on the theory of "Two Stage Linear Programming Under Uncertainty" also called as "Stochastic Programming with Recourse", as proposed by Dantzig and Madansky [5], and others, [4, 6, 7, 16]. In order to solve this problem we make the following assumptions, which are similar to the ones used by all the above mentioned authors.

(A1) The marginal distribution $f(b_j)$ of each b_j is known.

(A2) This distribution $f(b_j)$ is independent of the choice of x_{ij}.

(A3) For every b_j and any set of $x_{ij} \geq 0$, satisfying the constraint set (18), (20), there exists x_{ij} satisfying the constraint set (17).

Charnes, Cooper and Thompson [4] have shown that the two stage linear program is equivalent to a constrained generalized median problem whose objective function has some absolute value terms. This objective function was shown to be equivalent to a mathematically tractable function by Garstka [7]. In this section we essentially follow the approach and notation given by Garstka [7].

Since each b_j is random, it is no longer true that all the constraints given by equation (17) are exactly met. We therefore assume that the firm experiences an opportunity cost of lost demand. This cost, $p_j \geq 0$, is assumed to be linear and treated as the per unit penalty cost of not satisfying demand at center j. Similarly, if we have more supply than the demand, then there may be a penalty due to holding, storing, and obsolescence. Let us assume that this per unit penalty, $d_j \geq 0$, due to overproduction is also linear at center j. Following Charnes, Cooper and Thompson, the total penalty costs associated with demand exceeding production is:

$$\frac{1}{2}p_j \left\{\left|b_j - \sum_{i\in I} x_{ij}\right| + \left(b_j - \sum_{i\in I} x_{ij}\right)\right\} . \tag{21}$$

It can be verified easily that if the production exceeds demand, viz., if $\sum_i x_{ij} > b_j$, then the above penalty is zero. Similarly, the total penalty cost associated with excess production is

$$\frac{1}{2}d_j \left\{\left|\sum_{i\in I} x_{ij} - b_j\right| + \left(\sum_{i\in I} x_{ij} - b_j\right)\right\} . \tag{22}$$

Hence, the objective function to be minimized for the problem given by (16) - (20), equals the expected value of the total production costs, distribution costs, and costs due to under and over production. The problem (16) - (20) is therefore equivalent to:

(23) Minimize $$E\Big[\sum_{i\in I}\sum_{j\in J} c_i x_{ij} + \sum_{i\in I}\sum_{j\in J} t_{ij}x_{ij} + \tfrac{1}{2}\sum_{j\in J} d_j\Big\{\Big|\sum_{i\in I} x_{ij} - b_j\Big| + \Big(\sum_{i\in I} x_{ij} - b_j\Big)\Big\} + \tfrac{1}{2}\sum_{j\in J} p_j\Big\{\Big|b_j - \sum_{i\in I} x_{ij}\Big| + \Big(b_j - \Sigma x_{ij}\Big)\Big\}\Big]\ ,$$

subject to

(24) $$\sum_{j\in J} x_{ij} + x_{i,n+1} = U_i \quad \text{for } \forall\ i \in I\ ,$$

(25) $$x_{ij} \geq 0 \quad \text{for } \forall\ i \in I \text{ and } j \in J\ ,$$

(26) $$0 \leq x_{i,n+1} \leq u_i \quad \text{for } \forall\ i \in I\ ,$$

Let us denote $\sum_{i\in I} x_{ij} = b_{jo}$ and $C_{ij} = c_i + t_{ij}$ for $i \in I$ and $j \in J$. It has been shown by Garstka [7] that the objective function given in (23) can be transformed into

(27) $$\text{Min}\ \ F_1 + F_2,$$

where

(28) $$F_1 = \sum_{i\in I}\sum_{j\in J} \tfrac{1}{2}(2C_{ij} - p_j + d_j)x_{ij},$$

and

(29) $$F_2 = \sum_{j\in J} (p_j + d_j) \int_{b_{jo}}^{b_{jm}} (b_j - \sum_{i\in I} x_{ij}) f(b_j)db_j$$

where b_{jm} is the median of the random variable b_j whose marginal density (mass) function is $f(b_j)$.

It is seen that F_1 is linear and it can be proved easily that F_2 is convex (see Garstka [7, page 11]). If λ_i and μ_i are the dual variables corresponding to constraints (17) and (20) respectively, then by the Kuhn-Tucker conditions the optimal solution $X^* = \{x_{ij}^*\}$, μ_i^*, and λ_i^* satisfying the following:

(30) λ_i^* are unconstrained for $i \in I$

(31) $\mu_i^* \leq 0$ for $i \in I$

$$(32)\qquad C_{ij} - \tfrac{1}{2}p_j + \tfrac{1}{2}d_j - \sum_{j\in J}(p_j + d_j)\int_{b_{jo}}^{b_{jm}} f(b_j)db_j - \lambda_i^* \geq 0 \text{ for } \forall\ i \in I \text{ and } j \in J,$$

$$(33)\qquad \sum_{j\in J'} x_{ij} = U_i \text{ for } \forall\ i \in I,$$

$$(34)\qquad x_{i,n+1} \leq u_i \text{ for } \forall\ i \in I,$$

$$(35)\qquad x_{ij} \geq 0 \text{ for } \forall\ i \in I \text{ and } j \in J',$$

$$(36)\qquad x_{ij} \cdot \{\text{left hand side of (32)}\} = 0 \quad \text{for } i \in I \text{ and } j \in J.$$

$$(37)\qquad x_{i,n+1}(-\mu_i - \lambda_i) = 0 \quad \text{for } i \in I,$$

$$(38)\qquad (u_i - x_{i,n+1})\mu_i = 0 \quad \text{for } i \in I.$$

These conditions given by (30) - (38) provide a basis for solving the stochastic capacitated transportation problem. Based on these conditions, the following propositions P1, P2, and P3 can be proved. The proofs are similar to those provided by Garstka [7].

P1. For any $i \in I$ and $j \in J$, if $C_{ij} > p_j$, then $x_{ij} \equiv 0$.

P2. For any $j \in J$ and all $i \in I$, if $C_{ij} \geq (p_j - d_j)/2$, then $\sum_{i\in I} x_{ij} \leq b_{jm}$ in the optimal solution. It may be noticed that if $p_j < d_j$, the above inequality trivially holds since $C_{ij} \geq 0$.

P3. For any $i \in I$ and all $j \in J$, if $C_{ij} + d_j < 0$ then $\sum_{j\in J'} x_{ij} = U_i$

In many situations the combined cost of production and transportation, C_{ij}, is greater than half of the difference $(p_j - d_j)$. Thus in the algorithm given below we assume that $C_{ij} \geq (p_j - d_j)/2$. Note, however, that if the per unit cost of underproduction, p_j, is less than that of overproduction, d_j, then the above assumption is unnecessary. In order to solve (30) - (38) let us pose a new deterministic transportation problem given below:

(39) Minimize $F_1 = \sum_{i\in I} \sum_{j\in J} \frac{1}{2}(2C_{ij} - p_j + d_j)x_{ij}$

such that

(40) $$\sum_{j\in J'} x_{ij} = U_i \text{ for } i \in I$$

(41) $$\sum_{i\in I} x_{ij} = b_{j*} \text{ for } j \in J$$

(42) $$x_{ij} \geq 0 \quad \text{for } i \in I \text{ and } j \in J \text{ and}$$

(43) $$0 \leq x_{i,n+1} \leq u_i \text{ for } i \in I$$

where b_{j*} are the realizations of b_j. If λ_i, μ_i, and ν_j are the dual variables corresponding to constraint sets (40), (43), and (41) respectively, then by the Kuhn-Tucker conditions the optimal primal and dual variables satisfy:

(44) λ_i^*, ν_j^* are unconstrained, and $\mu_i^* \leq 0$, for $i \in I$, $j \in J$,

(45) $C_{ij} - (p_j + d_j)/2 - \lambda_i^* - \nu_j^* \geq 0$, for $i \in I$, $j \in J$,

(46) $\sum_{j\in J'} x_{ij} = U_i$, for $i \in I$,

(47) $\sum_{i\in I} x_{ij} = b_{j*}$, for $j \in J$,

(48) $x_{i,n+1} \leq u_i$, for $i \in I$,

(49) $x_{ij} \geq 0$, for $i \in I$ and $j \in J'$,

(50) $x_{ij} \cdot (\text{left hand side of (45)}) = 0$ for $i \in I$ and $j \in J$,

(51) $x_{i,n+1}(-\mu_i - \lambda_i) = 0$ for $i \in I$,

(52) $(u_i - x_{i,n+1})\mu_i = 0$ for $i \in I$.

It is easy to observe the similarity of the Kuhn-Tucker conditions given in (44) - (52) to those of the original problem. Comparing equations (32) and (45), we see that a solution to (44) - (52) will satisfy the conditions of the original problem (30) - (38) if

(53) $$\nu_j^* = (p_j + d_j) \int_{b_{jo}}^{b_{jm}} f(b_j)db_j \quad .$$

Since, by assumption $C_{ij} \geq (p_j - d_j)/2$, by proposition P2 stated earlier

$$b_{jo} = \sum_{i \in I} x_{ij} \leq b_{jm} , \tag{54}$$

and therefore the problem reduces to finding a set of b_{j*}, $j \in J$, such that

$$\nu_j^* = (p_j + d_j) \int_{b_{j*}}^{b_{jm}} f(b_j) db_j \tag{55}$$

If such a b_{j*} exist, then the optimal solutions to (16) - (20) will be obtained by solving (39) - (43) with the b_{j*} replacing b_j in the constraint set (17). An optimal solution can now be determined by the following algorithm provided $C_{ij} \geq (p_j - d_j)/2$.

Algorithm II:

Step 0: Initialization: Let k the number of iterations be 1. Find b_{jm} the median of the random variable b_j for $\forall$ $j \in J$, and set the initial $b_{j*} = b_{jm}$ for $j \in J$ in equation (44). Find the optimal solution and optimal cost to the deterministic transportation problem (39) - (43) and find the dual variables λ_i, μ_i for $i \in I$ and ν_j for $j \in J$. (Though μ_i for $i \in I$ are obtained they are not directly needed.) The duals λ_i and ν_j are solved from the relation $\lambda_i + \nu_j = C'_{ij} = C_{ij} - p_{j/2} + d_{j/2}$ for (i,j) in the optimal basis. For $k = 1$ let the basis set be denoted by $B^1 = \{(i,j) | x_{ij} > 0\}$ and $\Lambda^1 = \{\lambda_i^1 |$ the optimal values of the dual variables for $i \in I\}$ and $V^1 = \{\nu_j^1 |$ the optimal value of the dual variables for $j \in J\}$. Let $b_{j*}^1 = b_{j*}$ for $j \in J$.

Step 1: Iteration Procedure: Find b_{j*}^{k+1} from the following (newsboy type) relationship:

$$(56) \qquad \nu_j^k = (p_j + d_j) \int_{b_{j*}^{k+1}}^{b_{jm}} f(b_j)db_j \ .$$

Step 2: If $b_{j*}^{k+1} = b_{j*}^{k}$ for every $j \in J$, then an optimal solution to the stochastic transportation problem (16) - (20) is found and hence STOP, otherwise go to Step 3.

Step 3: Area Rim Operator Application: Let $\beta_j^{k+1} = b_{j*}^{k+1} - b_{j*}^{k}$, for $j \in J$ and let $\alpha_i = 0$ for $i \in I$. Following the algorithm given by Srinivasan and Thompson [12, page 215], apply the area operator δR^A with $\delta = 1$ and the above computed α_i and β_j to generate the new optimal solution for the $(k+1)^{th}$ problem. Let the new duals set be Λ^{k+1}. Let $k = k+1$ and go to step (1). (Note that if V^{k+1} and V^k are the same for every j, then $b_{j*}^{k+1} = b_{j*}^{k}$ so that the iteration can be stopped.)

The convergence proof of this algorithm is based on the results derived by Charnes and Cooper [3] and Charnes, Cooper, Thompson [4]. Garstka [7] has given a proof based on [3, 4] and since the proof for our algorithm is similar to that given by Garstka [7] we are providing an outline of the proof. It can be shown that $F_2(b_j)$ is convex in b_j so that $F_1 + F_2$ in (27) is also convex. Further, it can be shown that $b_{j*}^{k+1} \leq b_{j*}^{k}$, and thus the b_{j*}^{k} are monotone and bounded by $b_{j*}^{1} = b_{jm}$. With the relationship between b_{j*}^{k+1} and ν_j^k which is one to one, and due to montonicity, and convexity, and convergence is established.

5. The General Model and Extensions

In this section we provide a solution procedure for the problem formulated in Section 2 given by equations (2) - (5). In Section 3 we provided Algorithm 1 based on a branch and bound procedure, to solve the above problem with b_j's being deterministic. At every branch in this algorithm, we are

faced with a deterministic transportation problem, where the right hand sides in the constraint set (3) and (4) change from branch to branch and the cost coefficients in (6) also change. However, due to the operator theory [12, 13], the optimal solutions at each branch are obtained in a computationally efficient manner. Let us represent for each of exposition, the costs and the right hand sides of a branch V as C_i^V, U_i^V, and u_i^V for $i \in I$. Now, to consider the problem (2) - (5) in its entirety it is enought, if we introduce the randomness in the b_j's, $j \in J$. This leads us directly into Section 4, where these appropriate costs and right hand sides mentioned above replaces the corresponding ones in equations (16) - (20). Thus, Algorithm II is directly applicable to this new stochastic transportation problem where the costs and right hand sides of the branch being considered replace the costs and right hand side of (16) - (20), provided Assumptions A1 - A3 hold. It is to be noticed that t_{ij}, p_j's and d_j's do not change. Assumptions A1 and A2 are unrestrictive and can be expected to be true in most practical situations. Since the plant capacities (u_i^V) differ among branches we need to check Assumption A3 before applying Algorithm II. If Assumption A3 does not hold at a branch, then there is no feasible solution at that branch, and we can therefore close that node and set the upper bound equal to ∞. This check of Assumption A3, therefore, reduces the number of potential branches. We now present the following unified Algorithm III to solve the original problem posed by equations (2) - (6) by utilizing Algorithm I first and applying Algorithm II to each branch of Algorithm I.

Algorithm III:

Step 0: For each of the functions $f_i(y_i)$ substitute $\ell_{i0}^1 + c_i^1 y_i$, the best linear underestimate of $f_i(y_i)$ for $a_{i1} \leq y_i \leq a_{im_i}$. Solve the resultant stochastic transportation problem (6) - (9) with the assumption that b_j^* for $j \in J$ in (7) as random, utilizing Algorithm II. Create an open node corresponding to this optimal solution X^1 to the stochastic transportation problem. Denote the current lower bound as $Z_1(X^1)$ and from the actual objective function Z given in (2), get the upper bound $Z(X^1)$.

Step 1: Same as Step 1 of Algorithm I.

Step 2: Determine that open node with the current lower bound and partition on the index k that satisfies equations (10) - (13). Close this open node and generate two new open nodes as branches, corresponding to (13) and (14). Note that the problem at each of these branches is a stochastic transportation problem which can be solved using Algorithm II provided assumption A3 holds. This assumption can be checked by determining that the resultant transportation problem has a feasible sllution. If assumption A3 does not hold, then close that node and set the upper bound equal to ∞. Determine the current lower bound over all the open nodes, and the current upper bound over all the nodes. Go to Step 1.

Remark: Since Algorithm III above is the unification of Algorithms I and II, the convergence follows due to the convergence of the earlier two algorithms.

Our algorithms facilitate easy consideration of certain extensions to the problem formulated in Section 2 such as

(1) Inclusion of constraints requiring mandatory operation of certain plants.

(2) Capacity expansion or reduction at existing plants.

(3) Geo-political consideration requiring the operation of plants at mutually exclusive or mutually dependent plant sites.

Consideration of extension (1) and (2) follows from the fact that a_{il}, and a_{im} for each $i \in I$ are arbitrary, while extensions 2 and 3 can be imposed when branching occurs.

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