Prakash, Prem; Sertel, Murat R.

Working Paper

On the Continuity of Cartesian Product and Factorisation

Discussion Paper, No. 82

Provided in Cooperation with:
Kellogg School of Management - Center for Mathematical Studies in Economics and Management Science, Northwestern University

DISCUSSION PAPER NO. 82

ON THE CONTINUITY OF CARTESIAN PRODUCT AND FACTORISATION

by

Prem Prakash and Murat R. Sertel

April 10, 1974

Also issued as No. I/74-16 in the Preprint Series of the International Institute of Management, D-1000 Berlin 33, Griegstraße 5.

ON THE CONTINUITY OF CARTESIAN PRODUCT AND FACTORISATION

by

Prem Prakash and Murat R. Sertel

Given a topological space X, we denote the set of nonempty subsets of X by $[X]$, the set of open nonempty subsets of X by $\mathcal{O}[X]$, and the set of compact nonempty subsets of X by $\mathcal{K}[X]$. Given a family $\{X_\alpha \mid \alpha \varepsilon A\} = \{X_\alpha\}_A$ of topological spaces, we sometimes abbreviate $\prod_B X_\alpha$ to X_B $(B \subset A)$; we denote

$$\mathcal{B}[\{X_\alpha\}_A] = \{\prod_A P_\alpha \mid P_\alpha \varepsilon [X_\alpha] \text{ for each } \alpha \varepsilon A\},$$
$$\mathcal{B}\mathcal{O}[\{X_\alpha\}_A] = \{\prod_A P_\alpha \mid P_\alpha \varepsilon \mathcal{O}[X_\alpha] \text{ for each } \alpha \varepsilon A\},$$
$$\mathcal{B}\mathcal{K}[\{X_\alpha\}_A] = \{\prod_A P_\alpha \mid P_\alpha \varepsilon \mathcal{K}[X_\alpha] \text{ for each } \alpha \varepsilon A\}.$$

The maps whose continuity we study are the Cartesian <u>product</u> map

$$\pi\colon \prod_A [X_\alpha] \to \mathcal{B}[\{X_\alpha\}_A] \subset [X_A]$$

defined by $\pi(\{P_\alpha\}_A) = \prod_A P_\alpha = P_A$ $(\{P_\alpha\}_A \ \varepsilon \ \prod_A [X_\alpha])$ and <u>factorisation</u>, i.e., π^{-1}. (Clearly, π and π^{-1} are bijections.) In doing this, we always equip hyperspaces (i.e., spaces of subsets) with the <u>finite topology</u> [1]. Given a topological space Y, by the finite topology on $[Y]$ is meant the topology generated by taking as a basis for open collections in $[Y]$ all collections of the form $\langle U^i \mid i \in M\rangle = \{P \in [Y] \mid P \subset \bigcup_M U^i$ and, for each $i \in M$, $P \cap U^i \neq \emptyset\}$ with M a finite set and $U^i \subset Y$ open for each $i \in M$. Given any hyperspace $\mathcal{H}[Y] \subset [Y]$, the finite topology on $\mathcal{H}[Y]$ is then the subspace topology on $\mathcal{H}[Y]$ determined by the finite topology on $[Y]$.

Let $\{X_\alpha\}_A$ be a family of topological spaces.

1. <u>PROPOSITION</u>: <u>Factorisation</u> $\pi^{-1}: \mathcal{B}[\{X_\alpha\}_A] \to \prod_A [X_\alpha]$ <u>is continuous</u> (<u>i.e.</u> π <u>is an open map</u>).

<u>Proof</u>: Let $\mathcal{W} \subset \prod_A [X_\alpha]$ be a sub-basic open set, i.e., a set of the form $\mathcal{W} = \mathcal{W}_\beta \times \prod_{A\setminus\{\beta\}} [X_\alpha]$ with $\beta \in A$ and $\mathcal{W}_\beta \subset [X_\beta]$ open, and assume furthermore that $\mathcal{W}_\beta$ is basic, i.e., of the form $\mathcal{W}_\beta = \langle W^i \mid i \in M\rangle$ with M finite and $W^i \subset X_\beta$ open for each $i \in M$. It suffices to show that $(\pi^{-1})^{-1}(\mathcal{W}) = \pi(\mathcal{W}) \subset \mathcal{B}[\{X_\alpha\}_A] \subset [X_A]$ is open. Defining $V^i = W^i \times \prod_{A\setminus\{\beta\}} X_\alpha \subset X_A$ for each $i \in M$, we see that the collection $\mathcal{V} = \langle V^i \mid i \in M\rangle \subset [X_A]$ is open and that $\mathcal{V} \cap \mathcal{B}[\{X_\alpha\}_A] = \pi(\mathcal{W})$. $\diamond$

2. LEMMA: *Let $P_A \varepsilon \mathcal{BK}[\{X_\alpha\}_A]$ and let $W \subset X_A$ be any open set with $P_A \subset W$. Then there is an open "tube" $T_A = \prod_A T_\alpha \subset W$ (with $T_\alpha = X_\alpha$ for all but a finite set $N \subset A$ of indices) such that $P_A \subset T_A$.*

Proof: For each $x \varepsilon P_A$ find an open tube nbd $T(x)$ of x with $T(x) \subset W$, so that, for each $x \varepsilon P_A$, $T_\alpha(x) = X_\alpha$ for all but a finite set $N(x) \subset A$ of indices. $\{T(x) \mid x \varepsilon P_A\}$, being an open cover of the compact P_A, admits a finite subcover $\{T(x_i) \mid i \varepsilon M\}$ of P_A. Define $N = \bigcup_M N(x_i)$ and $V = \bigcup_M T_N(x_i)$. Now $V \subset X_N$ is open with $P_N \subset V$ and, since P_N is compact and N finite, there is an open box $T_N \subset X_N$ with $P_N \subset T_N \subset V$. Writing $T_A = T_N \times X_{A \setminus N}$, T_A is thus an open tube of the desired sort. ◊

3. THEOREM: *Cartesian product π is continuous on $\prod_A \mathcal{K}[X_\alpha]$ and so this space is homeomorphic to $\mathcal{BK}[\{X_\alpha\}_A]$.*

Proof: π being a bijection, and noting that $\pi(\prod_A \mathcal{K}[X_\alpha]) = \mathcal{BK}[\{X_\alpha\}_A]$, the above proposition leaves only the continuity of π on $\prod_A \mathcal{K}[X_\alpha]$ to show. Take any $\{P_\alpha\}_A \varepsilon \prod_A \mathcal{K}[X_\alpha]$ and let $\mathcal{W} = \langle W^i \mid i \varepsilon M\rangle \cap \mathcal{BK}[\{X_\alpha\}_A]$ be a basic open nbd of $P_A = \pi(\{P_\alpha\}_A)$ with $M = \{1, \ldots, m\}$.

By the lemma above, there is an open tube $T_A = T_N \times X_{A\setminus N}$ such that $P_A \subset T_A \subset W = \coprod_M W^i$, where $N \subset A$ is finite and $T_N = \prod_N T_\alpha \subset X_N$ is an open box. For each $i \in M$, let $p^i \in W^i \cap P_A$, and find an open tube nbd $T_A^{\,i}$ of p^i contained in $W^i \cap T_A$. Now defining $\mathcal{U}_\alpha = \langle T_\alpha, T_\alpha^1, \ldots, T_\alpha^m\rangle \cap K[X_\alpha]$ for each $\alpha \in N$, and writing $\mathcal{U} = (\prod_N \mathcal{U}_\alpha) \times \prod_{A\setminus N} K[X_\alpha]$, $\mathcal{U} \subset \prod_A K[X_\alpha]$, is an open nbd of $\{P_\alpha\}_A$ and $\pi(\mathcal{U}) \subset \mathcal{W}$, so we conclude that π is continuous. ◊

4. <u>PROPOSITION</u>: <u>If A is a finite set, then Cartesian product π is continuous on $\prod_A \mathcal{O}[X_\alpha]$ and so this space is homeomorphic to $\mathcal{BO}[\{X_\alpha\}_A]$.</u>

<u>Proof</u>: Imitate the last proof. ◊

5. <u>APPLICATIONS</u>: Let X, Y be topological spaces and consider an <u>application</u> of X to Y, i.e., a continuous map $\delta: X \times Y \to Y$. Then the map $\delta^*: [X \times Y] \to [Y]$, defined by $\delta^*(S) = \{\delta(s) \mid s \in S\}$ $(S \in [X \times Y])$, is continuous (see Theorem 5.10.1, pp. 170, of [1]), so that the restriction of δ^* to $\mathcal{B}[X, Y] \subset [X \times Y]$ is also continuous. Let $\mathcal{H}_X \subset [X]$ and $\mathcal{H}_Y \subset [Y]$, and define $F: \mathcal{H}_X \times \mathcal{H}_Y \to [Y]$ through $F(P, Q) = \delta^*(P \times Q)$ $(P \in \mathcal{H}_X, Q \in \mathcal{H}_Y)$. Now F is continuous if π is continuous on $\mathcal{H}_X \times \mathcal{H}_Y$, since F is the composition $\delta^* \circ \pi$; and F is actually an application if furthermore $F(\mathcal{H}_X \times \mathcal{H}_Y) \subset \mathcal{H}_Y$.

Examples: (1) If X is a topological semigroup, then so is $K[X]$. (2) If X is a topological semigroup whose multiplication is an open map, then $O[X]$ is also a topological semigroup. (3) If X is a topological vector space, then the space of convex compact nonempty subsets of X forms a topological semivector space (see 2.1 of [2]), and this allows us to embed it in a topological vector space (see Theorem 3.1 of [2]).

REFERENCES

[1] Michael, Ernest, "Topologies on Spaces of Subsets," Transactions of the American Mathematical Society, Vol. 71 (1951), pp. 152-182.

[2] Prakash, Prem and Murat R. Sertel, "Hyperspaces of Topological Vector Spaces: Their Embedding in Topological Vector Spaces," Preprint Series No. I/74-17, Easter 1974, International Institute of Management, West Berlin, or Discussion Paper No. 83 , Center for Mathematical Studies in Economics and Management Science, Northwestern University, Evanston, Illinois, (1974).