

Prakash, Prem; Sertel, Murat R.

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On the Continuity of Cartesian Product and Factorisation

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ON THE CONTINUITY OF
CARTESIAN PRODUCT AND FACTORISATION

by

Prem Prakash and Murat R. Sertel

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Given a topological space X , we denote the set of non-empty subsets of X by $[X]$, the set of open nonempty subsets of X by $O[X]$, and the set of compact nonempty subsets of X by $K[X]$. Given a family $\{X_\alpha \mid \alpha \in A\} = \{X_\alpha\}_A$ of topological spaces, we sometimes abbreviate $\prod_B X_\alpha$ to X_B ($B \subset A$); we denote

$$\begin{aligned} B[\{X_\alpha\}_A] &= \{\prod_A P_\alpha \mid P_\alpha \in [X_\alpha] \text{ for each } \alpha \in A\}, \\ BO[\{X_\alpha\}_A] &= \{\prod_A P_\alpha \mid P_\alpha \in O[X_\alpha] \text{ for each } \alpha \in A\}, \\ BK[\{X_\alpha\}_A] &= \{\prod_A P_\alpha \mid P_\alpha \in K[X_\alpha] \text{ for each } \alpha \in A\}. \end{aligned}$$

The maps whose continuity we study are the Cartesian product map

$$\pi: \prod_A [X_\alpha] \rightarrow B[\{X_\alpha\}_A] \subset [X_A]$$

defined by $\pi(\{P_\alpha\}_A) = \prod_A P_\alpha = P_A$ ($\{P_\alpha\}_A \in \prod_A [X_\alpha]$) and factorisation, i.e., π^{-1} . (Clearly, π and π^{-1} are bijections.) In doing this, we always equip hyperspaces

(i.e., spaces of subsets) with the finite topology [1].

Given a topological space Y , by the finite topology on $[Y]$ is meant the topology generated by taking as a basis for open collections in $[Y]$ all collections of the form $\langle U^i \mid i \in M \rangle = \{P \in [Y] \mid P \subset \bigcup U^i \text{ and, for each } i \in M, P \cap U^i \neq \emptyset\}$ with M a finite set and $U^i \subset Y$ open for each $i \in M$. Given any hyperspace $H[Y] \subset [Y]$, the finite topology on $H[Y]$ is then the subspace topology on $H[Y]$ determined by the finite topology on $[Y]$.

Let $\{X_\alpha\}_A$ be a family of topological spaces.

1. PROPOSITION: Factorisation $\pi^{-1}: \mathcal{B}[\{X_\alpha\}_A]$

$\rightarrow \prod_A [X_\alpha]$ is continuous (i.e. π is an open map).

Proof: Let $\omega \subset \prod_A [X_\alpha]$ be a sub-basic open set, i.e., a set of the form $\omega = \omega_\beta \times_{A \setminus \{\beta\}} \prod [X_\alpha]$ with $\beta \in A$ and $\omega_\beta \subset [X_\beta]$ open, and assume furthermore that ω_β is basic, i.e., of the form $\omega_\beta = \langle W^i \mid i \in M \rangle$ with M finite and $W^i \subset X_\beta$ open for each $i \in M$. It suffices to show that $(\pi^{-1})^{-1}(\omega) = \pi(\omega) \subset \mathcal{B}[\{X_\alpha\}_A] \subset [X_A]$ is open. Defining $V^i = W^i \times_{A \setminus \{\beta\}} \prod X_\alpha \subset X_A$ for each $i \in M$, we see that the collection $V = \langle V^i \mid i \in M \rangle \subset [X_A]$ is open and that $V \cap \mathcal{B}[\{X_\alpha\}_A] = \pi(\omega)$. \diamond

2. LEMMA: Let $P_A \in BK[\{X_\alpha\}_A]$ and let $W \subset X_A$ be
any open set with $P_A \subset W$. Then there is an open "tube"
 $T_A = \prod_A T_\alpha \subset W$ (with $T_\alpha = X_\alpha$ for all but a finite set
 $N \subset A$ of indices) such that $P_A \subset T_A$.

Proof: For each $x \in P_A$ find an open tube nbd $T(x)$ of x
 with $T(x) \subset W$, so that, for each $x \in P_A$, $T_\alpha(x) = X_\alpha$

for all but a finite set $N(x) \subset A$ of indices.

$\{T(x) \mid x \in P_A\}$, being an open cover of the compact P_A ,

admits a finite subcover $\{T(x_i) \mid i \in M\}$ of P_A . Define

$N = \bigcup_M N(x_i)$ and $V = \bigcup_M T_N(x_i)$. Now $V \subset X_N$ is open with

$P_N \subset V$ and, since P_N is compact and N finite, there is

an open box $T_N \subset X_N$ with $P_N \subset T_N \subset V$. Writing

$T_A = T_N \times X_{A \setminus N}$, T_A is thus an open tube of the desired
 sort. \diamond

3. THEOREM: Cartesian product π is continuous on

$\prod_A K[X_\alpha]$ and so this space is homeomorphic to

$BK[\{X_\alpha\}_A]$.

Proof: π being a bijection, and noting that

$\pi(\prod_A K[X_\alpha]) = BK[\{X_\alpha\}_A]$, the above proposition leaves only the

continuity of π on $\prod_A K[X_\alpha]$ to show. Take any

$\{P_\alpha\}_A \in \prod_A K[X_\alpha]$ and let $W = \langle W^i \mid i \in M \rangle \cap BK[\{X_\alpha\}_A]$ be a

basic open nbd of $P_A = \pi(\{P_\alpha\}_A)$ with $M = \{1, \dots, m\}$.

By the lemma above, there is an open tube $T_A = T_N \times X_{A \setminus N}$ such that $P_A \subset T_A \subset W = \bigcup_M W^i$, where $N \subset A$ is finite and $T_N = \prod_N T_\alpha \subset X_N$ is an open box. For each $i \in M$, let $p^i \in W^i \cap P_A$, and find an open tube nbd T_A^i of p^i contained in $W^i \cap T_A$. Now defining $U_\alpha = \langle T_\alpha, T_\alpha^1, \dots, T_\alpha^m \rangle \cap K[X_\alpha]$ for each $\alpha \in N$, and writing $U = (\prod_N U_\alpha) \times \prod_{A \setminus N} K[X_\alpha]$, $U \subset \prod_A K[X_\alpha]$, is an open nbd of $\{P_\alpha\}_A$ and $\pi(U) \subset W$, so we conclude that π is continuous. \diamond

4. PROPOSITION: If A is a finite set, then Cartesian product π is continuous on $\prod_A O[X_\alpha]$ and so this space is homeomorphic to $BO[\{X_\alpha\}_A]$.

Proof: Imitate the last proof. \diamond

5. APPLICATIONS: Let X, Y be topological spaces and consider an application of X to Y , i.e., a continuous map $f: X \times Y \rightarrow Y$. Then the map $f^*: [X \times Y] \rightarrow [Y]$, defined by $f^*(S) = \{f(s) \mid s \in S\}$ ($S \in [X \times Y]$), is continuous (see Theorem 5.10.1, pp. 170, of [1]), so that the restriction of f^* to $B[X, Y] \subset [X \times Y]$ is also continuous. Let $H_X \subset [X]$ and $H_Y \subset [Y]$, and define $F: H_X \times H_Y \rightarrow [Y]$ through $F(P, Q) = f^*(P \times Q)$ ($P \in H_X, Q \in H_Y$). Now F is continuous if π is continuous on $H_X \times H_Y$, since F is the composition $f^* \circ \pi$; and F is actually an application if furthermore $F(H_X \times H_Y) \subset H_Y$.

Examples: (1) If X is a topological semigroup, then so is $K[X]$. (2) If X is a topological semigroup whose multiplication is an open map, then $O[X]$ is also a topological semigroup. (3) If X is a topological vector space, then the space of convex compact nonempty subsets of X forms a topological semivector space (see 2.1 of [2]), and this allows us to embed it in a topological vector space (see Theorem 3.1 of [2]).

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