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Article

Subjective Homophily and the Fixtures Problem

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Abstract: The Stable Fixtures problem (Irving and Scott (2007)) is a generalized matching model that nests the well-known Stable Roommates, Stable Marriage, and College Admissions problems as special cases. This paper extends a result of the Stable Roommates problem to demonstrate that a class of homophilic preferences with an appealing psychological interpretation is sufficient to ensure that starting from an arbitrary matching, a decentralized process of allowing the sequential matching of randomly chosen blocking pairs will converge to a pairwise-stable matching with probability one. Strategic implications of this class of preferences are examined and further possible generalizations and directions for future research are discussed.

Keywords: many-to-many matching; stability; stable fixtures problem; homophilic preferences

JEL Classification: C78; D47

1. Introduction

The Stable Fixtures problem first studied by Irving and Scott [1] is a generalization of the Stable Roommates problem. The classical Stable Roommates problem entails matching each of $2n$ individuals so that no two people prefer each other over their assigned partners. The Stable Fixtures problem generalizes the Stable Roommates problem from a one-sided one-to-one matching model to a one-sided many-to-many matching model by allowing each individual to have a unique capacity representing his maximum possible number of matches. As in the Stable Roommates problem, there are instances of the Stable Fixtures problem for which there are no stable solutions (Irving and Scott [1]). Irving and Scott develop an algorithm that determines, for any given instance of the Stable Fixtures problem, if a stable solution exists. The Stable Fixtures problem is of theoretical interest because it nests several different matching models: the Stable Roommates problem, the Stable Marriage problem, and the College Admissions problem. The relationship between the Stable Roommates and Stable Marriage problems has been extensively studied,¹ as has the relationship between the Stable Marriage and College Admissions problems.² As noted by Chung [4], the Stable Marriage problem is the only point of contact between the Stable Roommates problem and the College Admissions problem. Understanding stability in the more general Stable Fixtures problem has the potential to yield insights into our understanding of how preferences, stability, and market structure interact in analyzing different matching economies.

We present two main results in this paper. We consider a class of preferences referred to as “subjective homophily”, which means that agents prefer to form matches with individuals they view as more similar to themselves according to some subjective heuristic. However, we do not require that

¹ See Gusfield and Irving [2] for an in depth review of the Stable Roommates Problem and its relationship with the Stable Marriage problem.

² See Roth and Sotomayor [3] for an extensive overview of two-sided matching.

the agents agree perfectly on the degree of their similarity or difference. Instead, we place a restriction on the degree of allowable divergence in their perceived difference we refer to as “approximate symmetry”. This assumption enforces a certain degree of consistency on the subjective similarity or dissimilarity of the agents, ensuring that their subjective views of one another are not radically inconsistent. We demonstrate via an algorithm that the subjective homophily preference restriction is sufficient to ensure the existence of a pairwise-stable matching in the Stable Fixtures problem, leading to the first main result: starting from an arbitrary matching, a pairwise-stable matching can be achieved via a decentralized process of randomly satisfying a finite sequence of blocking pairs. We then prove that a direct mechanism fails to be strategy-proof when preferences are weak, but a direct mechanism is strategy-proof in the case of strict preferences.

The paper is organized as follows. Section 2 provides a survey of relevant matching literature. Section 3 presents the Stable Fixtures Problem. Section 4 introduces subjective homophily and its relationship to preference. Section 5 discusses a strategic consideration while Section 6 presents potential applications of the results. Section 7 concludes.

2. Related Literature

Gale and Shapley [5] proved that the two-sided marriage and college admissions markets always admit stable matches. However, they also demonstrated that the one-sided generalization of the Stable Marriage problem, the Stable Roommates problem, does not always admit stable matches. Stable assignments for the roommates problem were studied further by Irving [6] and Gusfield [7]. Tan [8] developed necessary and sufficient conditions for the existence of stable matches in the Stable Roommates problem. These results were generalized to the weak preferences case by Chung [4] to obtain a sufficient condition for the existence of stable matches in the roommates problem.

Okumura [9] considers a one-sided many-to-many matching model that is similar, but not identical, to the Stable Fixtures problem. In Okumura’s framework, agents are teams that are looking to schedule games with one another. Unlike the Stable Fixtures problem, each team may play multiple games against the same opponent. Okumura’s model is a generalization of matching models under dichotomous preferences; that is, teams have an ideal number of games that they are willing to play against each acceptable opponent, but are indifferent between acceptable teams. Okumura examines stability and efficiency of matchings and the strategy-proofness of a direct mechanism in this context.

Another strand of the matching literature concerns itself with decentralized matching markets. In the absence of a centralized algorithmic mechanism, it is common for many markets to allow agents to freely form matches among themselves at random. Roth and Vande Vate [10] proved a random paths result for the Stable Marriage problem, generalized by Chung [4] to the Stable Roommates problem. This setting was further explored and generalized by Diamantoudi et al. [11]. Kojima and Ünver [12] demonstrated a random paths to pairwise-stability result for two-sided many-to-many matching markets. Ackerman et al. [13] study a decentralized matching process for two-sided markets with an emphasis on the question of convergence time. They also study a particular class of preferences called correlated markets, which is a similar notion to that of subjective homophily explored in this paper. Cseh and Skutella [14] study better- and best-response dynamics in a two-sided matching framework from an algorithmic perspective, and examine a case of correlated markets. Both Ackerman et al. [13] and Cseh and Skutella [14] analyze two-sided matching markets, rather than the one-sided many-to-many matching market that is the focus of this study.

Yet another aspect of the matching literature relevant to the research pursued herein consists of coarse matching (McAfee [15]). Coarse matching is a type of matching framework where the agents are broken into broad classes, and then matched on the basis of class membership. McAfee [15] illustrated that efficiency gains could be achieved by using a coarse matching scheme to ration electricity, rather than relying on the traditional system of rolling blackouts (analogous to a random matching scheme). Hospitals, for example, are categorized in the high priority class under this system, and, as such, are guaranteed access to power in a way that agents classed as lower priority are not.

This type of rank based matching system has similarities with what are called assortative matching markets (see Hoppe et al. [16]). The preference class examined herein has much in common with this notion of assortative matching, as the agents are all able to be ranked in a manner conducive to deriving pairwise-stable matchings.

Bartholdi and Trick [17] demonstrated that when preferences are derived from a simple psychological model, there always exists a stable matching for the Roommates Problem. The idea behind their preference restriction is intuitive: agents have preferences that are derived from a common framework that allow the agents to be ordered sequentially. For example, they consider roommates who wish to live with people who have similar preferences for setting the thermostat. Another example is an individual who prefers to live with someone who comes from a town closer to his own hometown over someone who comes from farther away. This class of preferences has an intuitive psychological appeal because agents prefer other agents who are closer or “more like them” in the sense of the metric. Implicit in this framework is a type of symmetry; agents agree on their differences. We show how this can be relaxed by introducing the notion of “subjective homophily” between and among agents. This reflects the possibility that agents may not agree completely on their differences or similarities, but so long as those disagreements are not too large the existence of a stable matching is guaranteed. We refer to this notion as approximate symmetry. We demonstrate that this class of preferences confers a nice structure to the Stable Fixtures problem: when agents can be sequentially ordered in this way, preferring those who are closer to those who are farther away, a stable solution will always exist. In this vein, Abraham et al. [18] consider the stable roommates problem with globally-ranked pairs. This preference structure is very similar to the notion of subjective homophily discussed herein, however, they examine the Stable Roommates problem whereas this work concerns the more general Stable Fixtures problem. Indeed, the globally-ranked pairs preference structure is akin to the correlated markets structure of Ackerman et al. [13] and Cseh and Skutella [14]. Subjective homophily with approximate symmetry is a slightly more general notion than correlated markets while representing a particular case of the globally-acyclic preferences discussed by Abraham et al. [18].

3. The Stable Fixtures Problem

To define the Stable Fixtures problem, let $\mathcal{X} = \{x_1, x_2, \dots, x_n\}$ denote the set of agents. For all $x_i \in \mathcal{X}$, there exists an integer c_i which we call x_i 's *capacity*, representing the maximum number of possible matches for x_i . Every agent $x_i \in \mathcal{X}$ has a preference ordering over $\mathcal{X} \cup \emptyset$ and his preference relation is denoted by \succeq_i . For each $x_i \in \mathcal{X}$, let \succ_i denote the strict preference relation derived from \succeq_i . We assume that the preference ordering is a weak order, that is, complete, and transitive; thus preferences are assumed to be weak.³ If x_i weakly prefers x_k to x_l , then we write $x_k \succeq_i x_l$. An instance of the Stable Fixtures problem is completely defined by the collection of agents, \mathcal{X} , their capacities, $\mathbf{c} = (c_1, c_2, \dots, c_n)$, and the preference profiles of the agents, $\succeq = (\succeq_1, \succeq_2, \dots, \succeq_n)$, that is, $(\mathcal{X}, \mathbf{c}, \succeq)$.

When $c_i = 1$ for every $x_i \in \mathcal{X}$, this is the Stable Roommates problem. If $c_i = 1$ for every $x_i \in \mathcal{X}$, and the agents can be partitioned into two sets, $M \subset \mathcal{X}$ and $W \subset \mathcal{X}$ such that $M \cap W = \emptyset$, $M \cup W = \mathcal{X}$, and agents in M only have preferences over agents in W and vice versa, this is the Stable Marriage problem. Allowing agents on one side of the aforementioned partition to have capacities greater than one is the College Admissions problem.

Definition 1 (Acceptable). If $x_j \succeq_i \emptyset$, x_j is acceptable to x_i .

³ The assumption of weak preferences is important when examining the strategy-proofness of a direct mechanism. In this case, a direct mechanism is not strategy-proof.

In words, x_j is acceptable to x_i if and only if x_i prefers being matched with x_j over remaining unmatched or leaving excess capacity open. In the event that x_i would rather remain unmatched or leave excess capacity open to matching with x_j , we state that x_j is *unacceptable* to x_i , which we denote by $\emptyset \succ_i x_j$.

Definition 2 (Acceptable Pair). *A pair $\{x_i, x_j\}$ is an acceptable pair if x_i is acceptable to x_j and x_j is acceptable to x_i .*

For the sake of consistency with Irving and Scott's [1] original formulation of the Stable Fixtures problem, we retain their notational conventions to define a matching as follows:

Definition 3 (Matching). *A matching, μ , is a set of pairs of agents $\{x_i, x_j\} \subset \mathcal{X}$ such that, for all $x_i \in \mathcal{X}$,*

$$|\{x_j : \{x_i, x_j\} \in \mu\}| \leq c_i.$$

The size of μ is the number of pairs in μ . The members of the set $\mu(x_i) = \{x_j : \{x_i, x_j\} \in \mu\}$ are referred to as the *matches* of x_i in μ . We denote the set of all possible matchings by \mathcal{M} .

Definition 4 (Individually Rational). *A matching μ is said to be individually rational if no agent is matched to an agent he considers unacceptable.*

Definition 5 (Blocking Pair). *An acceptable pair $\{x_i, x_j\} \notin \mu$ is a blocking pair for matching μ , or blocks μ if*

1. *Either x_i has fewer than c_i matches or strictly prefers x_j to at least one of his matches in μ ; and*
2. *Either x_j has fewer than c_j matches or strictly prefers x_i to at least one of his matches in μ .*

In words, this says that for $\{x_i, x_j\}$ to be a blocking pair, either x_i must have excess capacity or x_i must strictly prefer x_j to one of his current matches, and either x_j has excess capacity or strictly prefers x_i to one of his current matches.

Definition 6 (Pairwise-Stable). ⁴ *A matching for which there is no blocking pair is said to be pairwise-stable. Otherwise, the matching is said to be pairwise-unstable.*

4. Preferences and Subjective Homophily

In this section, we build a framework for examining a certain class of preferences. Suppose that a given agent, $x_i \in \mathcal{X}$, is looking to form relationships with other agents in \mathcal{X} . In choosing his matches, x_i desires to be matched with agents he deems to be “closer” to him according to some measure of likeness. This can be thought of as representing some notion of similarity, compatibility, or any other trait that x_i desires. For example, this could encompass where other agents hail from geographically or their political views. To capture this idea, we introduce the notion of *subjective homophily*.

Definition 7 (Subjective Homophily Function). *A subjective homophily function is a mapping, $\sigma_i : \mathcal{X} \rightarrow \mathbb{R}_+$ such that $\sigma_i(j) > 0$ for all $x_i, x_j \in \mathcal{X}$ such that $i \neq j$.*

The subjective homophily function represents the “difference” an agent perceives between himself and a fellow agent. The homophily function is called “subjective” because we do not require that $\sigma_i(j) = \sigma_j(i)$, that is, we do not require that x_i 's view of the difference between himself and x_j be the same as x_j 's view of the difference between himself and x_i . This relaxes the symmetry assumption of

⁴ There are other notions of stability in many-to-many matching markets. See Echenique and Oviedo [19].

Bartholdi and Trick [17]. We are interested in the case where agents would like to match with those who are perceived to be closest in terms of this notion of subjective homophily.

Given the subjective homophily function, we write that each agent in some sense represents his own *ideal point*. The greater the similarity between two agents (the smaller their differences according to the subjective homophily function), the more preferred they are.

When agent preferences are determined by subjective homophily functions, we have that:

$$x_j \succeq_i x_k \Leftrightarrow \sigma_i(j) \leq \sigma_i(k) \text{ for all } x_i, x_j, x_k \in \mathcal{X}.$$

Remark 1. The above formulation implicitly defines every agent as acceptable to every other agent. We can accommodate unacceptability into the framework in the following way: if agent x_i deems x_j unacceptable, we write $\sigma_i(j) = \infty$.

We now define a particular restriction on the subjective homophily function that will prove useful in demonstrating our main results.

Definition 8 (Approximate Symmetry). A subjective homophily function satisfies approximate symmetry if, for all $x_i, x_j, x_m, x_n \in \mathcal{X}$

$$\sigma_i(j) \leq \sigma_m(n) \Rightarrow \max\{\sigma_i(j), \sigma_j(i)\} \leq \min\{\sigma_n(m), \sigma_m(n)\}.$$

This condition imposes some degree of structure on preferences. We move away from the requirement of complete symmetry of Bartholdi and Trick [17], but rather only require that preferences be close enough to capture the effects of pure symmetry. The idea is that even if agents are not in perfect agreement regarding their differences, there exists some common framework that allows them to evaluate one another in such a way that their perceptions of one another are not radically different. The notion of approximate symmetry essentially captures the key features of acyclical preferences, as discussed by Chung [4] and Abraham et al. [18].

Definition 9 (Subjective Homophily Vector). Let $\sigma = (\sigma_i(j))_{x_i, x_j \in \mathcal{X}}$ be the vector of subjective homophily function outputs between all agents. We call σ the Subjective Homophily Vector.

Definition 10 (Ordered Subjective Homophily Vector). Consider only entries $\sigma_i(j)$ in σ such that $i \neq j$, ordered from smallest to largest. The resulting vector is the Ordered Subjective Homophily Vector, denoted by $\bar{\sigma}$.

Remark 2. When the subjective homophily function satisfies approximate symmetry, then $\bar{\sigma}$ can be written so that, for all $x_i, x_j \in \mathcal{X}$, either $\sigma_i(j)$ immediately follows or immediately precedes $\sigma_j(i)$. This is true even in the event of ties.

4.1. Existence of Pairwise-Stable Fixture Matchings

The following lemma will be useful in constructing our algorithm to prove the existence of pairwise-stable matchings for the Stable Fixtures Problem when agent preferences are consistent with subjective homophily and satisfy approximate symmetry.

Lemma 1 (Bartholdi and Trick). If among all available choices, agent x_i most prefers agent x_j , and agent x_j most prefers agent x_i , then in any pairwise-stable matching x_i and x_j must be matched.

Proof. If x_i and x_j are not matched, they form a blocking pair. \square

We now demonstrate that the existence of a pairwise-stable fixtures matching is guaranteed when preferences are consistent with subjective homophily satisfying approximate symmetry. The following

lemma can be viewed as a fairly straightforward extension of Abraham et al.'s [18] results for the Stable Roommates problem to the case of the more general Stable Fixtures problem.

Lemma 2. *If agent preferences are consistent with subjective homophily and satisfy approximate symmetry then there exist pairwise-stable fixture matchings.*

Proof. We demonstrate a constructive algorithm for obtaining a pairwise-stable fixture matching:

Step 1: Begin with the first entry in $\bar{\sigma}$. Let $\sigma_i(j)$ be this entry. Since neither x_i nor x_j currently has any matches, we match them. Remove $\sigma_i(j)$ from $\bar{\sigma}$. By approximate symmetry, the next entry in $\bar{\sigma}$ is $\sigma_j(i)$. Because x_i and x_j are matched, this entry can be removed as well. Reduce both c_i and c_j by 1. If either x_i or x_j has filled his capacity, remove any remaining subjective homophily function output corresponding to that agent from $\bar{\sigma}$. If neither x_i nor x_j has filled his capacity, no entries are removed from $\bar{\sigma}$. Define $\bar{\sigma}_1$ as the vector of subjective homophily function outputs remaining following Step 1.

Step k : If $\bar{\sigma}_{k-1} = \emptyset$, the algorithm terminates. If $\bar{\sigma}_{k-1} \neq \emptyset$, we match the agents corresponding to the first entry in $\bar{\sigma}_{k-1}$. Remove the subjective homophily function outputs corresponding to a match between these two agents from $\bar{\sigma}_{k-1}$, and reduce each agent's capacity by 1. If either agent has filled his capacity, we remove all subjective homophily function outputs corresponding to that agent from $\bar{\sigma}_{k-1}$ and rename the resulting vector $\bar{\sigma}_k$.

This algorithm will terminate after a finite number of steps (there are a finite number of agents, and therefore a finite number of subjective homophily function outputs to consider), when either all subjective homophily function outputs have been removed or there is a single agent remaining with excess capacity. We now demonstrate that the resulting matching is pairwise-stable. Assume for contradiction that the matching resulting at the termination of the above algorithm is not pairwise-stable. Then there exists a blocking pair, $\{x_i, x_j\}$, such that

1. either $x_j \succ_i x_k$ for some x_k that x_i is matched with, or x_i has not filled his quota and has excess capacity remaining, and
2. either $x_i \succ_j x_l$ for some x_l that x_j is matched with, or x_j has not filled his quota and has excess capacity remaining.

Assume that $x_j \succ_i x_k$ for some x_k that x_i is matched with. This implies that $\sigma_i(j) < \sigma_i(k)$. But since x_i and x_j are not matched, it must be the case that when the algorithm reached $\sigma_i(k)$, x_j 's entries must have already been removed, meaning that his quota was filled. Therefore, there is no agent, x_l , matched with x_j such that $x_i \succ_j x_l$. We now assume that x_i has not filled his quota and therefore has excess capacity available. This implies that there is no one remaining to whom he can be matched as all other agents must have filled their quotas with agents closer to them than x_i . Thus, $\{x_i, x_j\}$ can not be a blocking pair, and we have obtained the contradiction.

The matching is pairwise-stable. \square

It is important to here note the relevance of the weak preference ordering. Given that the subjective homophily function may admit ties between and among agents, it is possible for there to be multiple pairwise-stable matchings. When preferences are strict, the pairwise-stable matching achieved at the termination of the above algorithm will be unique. However, in the case of weak preferences where there are ties among agents in terms of subjective homophily, the outcome of the above algorithm will depend on how the subjective homophily outputs are ordered. However, the assumption of approximate symmetry means that such an ordering is possible, but may depend on the order in which agents with equivalent subjective homophily function outputs are ordered. This generates strategic implications that will be explored in more detail in Section 5.

4.2. Random Paths to Pairwise-Stable Fixture Matchings

We have demonstrated that when preferences are consistent with subjective homophily and satisfy approximate symmetry, there exists a pairwise-stable solution to the Stable Fixtures problem.

A natural question is whether stable matchings can be obtained through a decentralized matching process as opposed to a centralized algorithmic mechanism.

Our main result shows that when preferences are consistent with subjective homophily and satisfy approximate symmetry, a pairwise-stable matching can always be attained from a pairwise-unstable matching by satisfying a finite sequence of blocking pairs. Starting from an arbitrary matching μ , if $\{x_i, x_j\}$ form a blocking pair and it is true that both $|\mu(x_i)| < c_i$ and $|\mu(x_j)| < c_j$, then we can simply match both agents to generate a new matching, μ' . In this case, no other agents are affected by the match. However, it is possible that an agent x_i may have his entire capacity filled under μ , that is, $|\mu(x_i)| = c_i$. If $\{x_i, x_j\}$ form a blocking pair for matching μ in this case, this means that x_i must “dump” one of his current matches in order to match with x_j . In this case, x_i will dump his least preferred match among his current matches, that is, x_i will dump $x_k \in \mu(x_i)$ such that $\sigma_i(k) \geq \sigma_i(l)$ for all $x_l \in \mu(x_i)$. In words, x_i will dump the current match of his who is farthest away from him, according to the perceived subjective homophily, of all his current matches under μ in favor of matching with x_j . The dumped agent will then gain one unit of excess capacity.

Definition 11 (Satisfying the Blocking Pair). *Let \mathcal{X} be a set of agents with preferences consistent with subjective homophily that satisfy approximate symmetry. Let $\mu \in \mathcal{M}$ be a matching. Let $\{x_i, x_j\} \subset \mathcal{X}$ be a blocking pair for μ . A new matching, μ' , is obtained from μ by satisfying the blocking pair $\{x_i, x_j\}$ if:*

1. $x_j \in \mu'(x_i)$ and $x_i \in \mu'(x_j)$.
2. If $|\mu(x_i)| = c_i$, then $\exists x_k \in \mu(x_i)$ s.t. $\sigma_i(k) \geq \sigma_i(l) \forall x_l \in \mu(x_i)$ and x_i dumps x_k in favor of matching with x_j .
3. If $|\mu(x_j)| = c_j$, then $\exists x_m \in \mu(x_j)$ s.t. $\sigma_j(m) \geq \sigma_j(h) \forall x_h \in \mu(x_j)$ and x_j dumps x_m in favor of matching with x_i .
4. If $x_k = x_m = x_d$, then $\mu'(x_d) = \mu(x_d) \setminus \{x_i, x_j\}$ and if $x_k \neq x_m$, then $\mu'(x_k) = \mu(x_k) \setminus \{x_i\}$ and $\mu'(x_m) = \mu(x_m) \setminus \{x_j\}$.
5. $\forall x_r \in \mathcal{X} \setminus \{x_i, x_j, x_k, x_m\}, \mu'(x_r) = \mu(x_r)$.

Condition (1) states that after satisfying the blocking pair, x_i and x_j must now be matched with each other. Conditions (2) and (3) state that if x_i or x_j is currently matched at full capacity under the original matching μ , they must dump their least preferred current match to satisfy the blocking pair. Condition (4) states that under the new matching μ' , dumped agents remain matched to the same set of agents that they were matched with under μ minus the members of the satisfied blocking pair. Condition (5) states that all other agents not affected by the blocking pair have the same matches under the new matching μ' as under the old matching μ .

Remark 3. *Any individually irrational matching can be transformed into an individually rational matching by having agents dump any unacceptable matches.*

We now demonstrate that starting from an arbitrary matching we can achieve a stable matching by sequentially satisfying a finite number of blocking pairs.

Lemma 3. *When agent preferences are consistent with subjective homophily and satisfy approximate symmetry, for any matching μ , there exists a finite sequence of matchings $(\mu_1, \mu_2, \dots, \mu_T)$, such that $\mu_1 = \mu$, μ_T is pairwise-stable, and for each $t = 1, 2, \dots, T - 1$, there is a blocking pair for μ_t such that μ_{t+1} is obtained from μ_t by satisfying that blocking pair.*

Proof. We provide a constructive algorithm that will transform the current matching μ into a stable matching in a finite number of steps:

Step 1: Let $\mu_1 = \mu$. Consider the first entry of $\bar{\sigma}$ that corresponds to a pair of agents $\{x_i, x_j\}$ that are not matched under μ_1 . This represents the first potential blocking pair. If these agents do not form

a blocking pair, then either x_i or x_j must be matched to capacity and does not wish to dump any of his current matches. As subjective homophily outputs are increasing, no future blocking pair will arise involving these two agents together. Remove any subjective homophily outputs corresponding to this agent from $\bar{\sigma}$. Define $\mu_2 = \mu_1$. If, however, $\{x_i, x_j\}$ does constitute a blocking pair, we have two possible cases:

Case 1. If $|\mu_1(x_i)| < c_i$ and $|\mu_1(x_j)| < c_j$, we match x_i and x_j . Since neither x_i nor x_j is currently matched at full capacity, no other agents are affected. Call the resulting matching μ_2 .

Case 2. If $|\mu_1(x_i)| = c_i$ or $|\mu_1(x_j)| = c_j$, then either x_i or x_j are at full capacity under the current matching, and must dump their least preferred agent to satisfy the blocking pair. We satisfy the blocking pair and any dumped agents gain one unit of excess capacity. Call the resulting matching μ_2 .

Define $\bar{\sigma}_1$ as the vector of subjective homophily function outputs remaining following Step 1.

Step k : Consider the first entry of $\bar{\sigma}_{k-1}$ corresponding to a pair of agents $\{x_i, x_j\}$ that are not matched according to the matching μ_{k-1} . This represents a potential blocking pair. If these agents do not form a blocking pair, then either x_i or x_j must be matched to capacity and does not wish to dump any of his current matches. As subjective homophily function outputs are increasing, no future blocking pairs involving this agent can arise. Remove any subjective homophily function outputs corresponding to this agent from $\bar{\sigma}_{k-1}$. Define $\mu_k = \mu_{k-1}$. If, however, $\{x_i, x_j\}$ does constitute a blocking pair, we have the same two possible cases as above and proceed accordingly.

This algorithm terminates in a finite number of iterations resulting in the matching, μ_T . The proof of stability follows the same argument as given in the proof of Lemma 2. \square

The critical step in the above proof is that blocking pairs can be satisfied sequentially based on subjective homophily. Any time an agent is dumped when a blocking pair is satisfied, the dumped agent has a greater subjective homophily function output value than the newly matched agent. This means that a dumped agent will not create any new instability among the matches that have been generated in previous steps of the algorithm.

Having proved the above lemma, the random paths to pairwise-stability result is an immediate consequence of the standard Markov-chain argument, summarized as follows: starting from an arbitrary matching μ , a random process can generate a sequence of matchings by satisfying a single randomly chosen blocking pair. The probability of any one blocking pair being chosen is positive for all such blocking pairs for a given matching. The following proposition results from the fact that for any matching, every blocking pair has a positive probability of being chosen.

Proposition 1. If agent preferences are consistent with subjective homophily and satisfy approximate symmetry, then a decentralized process of allowing randomly chosen blocking pairs to match will converge to a pairwise-stable fixtures matching with probability one.

5. Strategic Consideration

We now provide a strategic consideration in light of the above results. We will focus on the following direct mechanism. Let S_i be the set of all possible subjective homophily functions of agent x_i and let $S = \prod_{x_i \in \mathcal{X}} S_i$. We consider a direct mechanism $v : \sigma \rightarrow \mathcal{M}$, where \mathcal{M} represents all possible matchings, as defined earlier. Since this is a direct mechanism, each agent x_i reports a subjective homophily function, $\tilde{\sigma}_i$, that may or may not represent the true preference. Let $\sigma_i \in S_i$ represent the true subjective homophily function of agent x_i , $\sigma = (\sigma_1, \sigma_2, \dots, \sigma_n) \in S$. A direct mechanism is *strategy-proof* if for all $\sigma \in S$, $\tilde{\sigma}_i \in S_i$, $v_i(\sigma) \succeq_i v_i(\tilde{\sigma}_i, \sigma_{-i})$, where σ_{-i} represents the subjective homophily functions of all other agents except x_i and preferences over matchings are derived from subjective homophily in the following manner: if μ is a matching where $\mu(x_i)$ represents the matches of $x_i \in \mu$ and $|\mu(x_i)| \leq c_i$, for any agents x_j and x_k we have that:

1. $\mu(x_i) \cup x_j \succ_i \mu(x_i) \cup x_k \Leftrightarrow \sigma_i(j) < \sigma_i(k)$;
2. $\mu(x_i) \cup x_j \approx_i \mu(x_i) \cup x_k \Leftrightarrow \sigma_i(j) = \sigma_i(k)$; and
3. $\mu(x_i) \cup x_j \succeq_i \mu(x_i) \Leftrightarrow x_j \succeq_i \emptyset$.

In words, this means that an agent's preference over overall matchings is determined by his preference over individual agents as matches. For any two matchings $\bar{\mu}$ and $\hat{\mu}$ where the set of matches for agent x_i differ by only one agent, x_i will prefer the matching where his set of matches contains the more preferred agent. This is an example of responsiveness in preferences, following the definition of responsiveness used by Sotomayor [20].

Let ν be a mechanism for selecting a pairwise-stable matching. We have the following result:

Proposition 2. *If agent preferences are consistent with subjective homophily and satisfy approximate symmetry, then any direct mechanism for selecting a pairwise stable matching does not satisfy strategy-proofness.*

Proof. We show the existence of a preference profile such that $\nu_i(\tilde{\sigma}_i, \sigma_{-i}) \succ_i \nu(\sigma)$ for some $x_i \in \mathcal{X}$ and $\tilde{\sigma}_i \in S_i$. Consider the case of five agents, where $c_1 = 2$ and $c_j = 1$ for $j = 2, 3, 4, 5$, with preferences defined by the following subjective homophily functions: $\sigma_i(5) = 10 \forall i \neq 5$ and $\sigma_i(j) = 1 \forall i \neq j$ otherwise. Therefore, in this example, the first four agents all rank x_5 last, but otherwise each agent is indifferent between all other agents, the first agent has a capacity of 2 units while the remaining four agents each have capacities of one unit. If the agents report their subjective homophily functions truthfully, that is, each agent x_i submits σ_i , the resulting matching will terminate with agent x_4 matched to agent x_5 . The matching is pairwise-stable as a result of Lemma 1. Consider the following misreport of agent x_4 when the other agents report truthfully. If agent x_4 reports $\tilde{\sigma}_4(1) = 0.5$, this will result in agent x_4 being matched with agent x_1 . Now agent x_3 is matched with x_5 rather than x_1 . However, since x_1, x_2 , and x_4 are indifferent between their current matches and agent x_3 , the resulting matching is pairwise-stable and is strictly preferred by agent x_4 . Therefore, in this example, x_4 has an incentive to deviate from the strategy profile σ . This implies that the mechanism ν is not strategy-proof. \square

In the above example, the algorithm of Lemma 1 results in one of the agents being matched with the least preferred fifth agent, depending on the manner in which the agents are numbered. Thus, for this preference profile there is a profitable deviation for that agent in manipulating his reported subjective homophily function. This is only feasible in the case of weak preferences where ties are admitted. In the strict preferences case, there exists a unique pairwise-stable matching and approximate symmetry rules out these manipulations, leading to the following corollary:

Corollary 1. *If agent preferences are strict, consistent with subjective homophily, and satisfy approximate symmetry, then a direct mechanism for selecting a pairwise-stable matching is strategy-proof.*

Proof. Suppose all agents truthfully report their subjective homophily functions to obtain the matching $\mu = \nu(\sigma)$. Further suppose that there is an agent x_i who misreports to obtain a more favorable matching, μ' . This implies that $\mu'(x_i) \succ_i \mu(x_i)$, that is, the set of matches that x_i attains under the misreport is preferred to the outcome when reporting truthfully. This implies that there must exist an agent, x_j with whom x_i is matched under μ' but not under μ . However, by Lemma 1, if x_i and x_j are not matched under truthful reporting, this means that x_j 's matches under μ must be preferred by x_j to her matches under μ' . That is, for x_i and x_j to be matched under the new matching μ' , there must exist an agent x_k with whom x_j is matched under μ , but x_k is replaced by x_i under the new matching μ' . However, given that x_i and x_j are not matched under μ and that preferences are strict, this implies that x_j must prefer x_k to x_i . Therefore, the matching μ' resulting from agent x_i 's misreport will not be pairwise-stable and will therefore result in the unraveling of the matching. \square

6. Other Applications

An immediate application of Proposition 1 is that by demonstrating that a given class of preferences is consistent with the subjective homophily function formulation, we can guarantee the existence of pairwise-stable fixture matchings and the attendant random paths to pairwise-stability result, extending results from the one-sided one-to-one Stable Roommates problem to the one-sided many-to-many Stable Fixtures Problem. We now provide two examples of preference domains that are consistent with the subjective homophily function derivation.

The first application of our results is extending the results of Bartholdi and Trick [17]. Bartholdi and Trick considered a preference restriction for the Stable Roommates problem where agent attributes can be represented by points in a metric space, every agent strictly prefers agents who are more similar according to the metric, and every agent prefers having a roommate to not having one. They prove that this preference restriction guarantees the existence of Stable Roommate matchings.

Corollary 2. *If agents can be represented as points in a metric space, every agent strictly prefers agents closer to him to those farther away, and strictly prefers having a match to not, then there exists pairwise-stable fixture matchings. A decentralized process of allowing randomly chosen blocking pairs to match will converge to a pairwise-stable fixtures matching with probability one.*

Proof. Let $d(i, j)$ represent the distance between agents $x_i, x_j \in \mathcal{X}$. For all $x_i, x_j \in \mathcal{X}$, define $s_i(j) = d(i, j)$. By definition of a metric space, $d(i, j) = d(j, i)$ for all $x_i, x_j \in \mathcal{X}$. Thus approximate symmetry is satisfied and a pairwise-stable matching exists. \square

Another domain of interest is Dichotomous preferences. Under Dichotomous preferences, each agent partitions the set of all agents into two groups (see Bogomolnaia and Moulin [21] for further discussion of dichotomous preferences).

Definition 12 (Dichotomous Preferences). *A preference profile is Dichotomous if every agent classifies all agents into two groups in such a way that within each group he is indifferent among members.*

Corollary 3. *If the preference profile is Dichotomous, there exist pairwise-stable fixture matchings. A decentralized process of allowing randomly chosen blocking pairs to match will converge to a pairwise-stable fixtures matching with probability one.*

Proof. Assume that each agent $x_i \in \mathcal{X}$ partitions the set of agents into two sets, agents who are acceptable as matches and agents who are not acceptable. For all $x_i, x_j \in \mathcal{X}$, define

$$s_i(j) = \begin{cases} 1 & \text{if } \{x_i, x_j\} \text{ is an acceptable pair,} \\ \infty & \text{if } \{x_i, x_j\} \text{ is not an acceptable pair.} \end{cases}$$

Thus Dichotomous preferences are consistent with subjective homophily satisfying approximate symmetry, and therefore pairwise-stable fixture matchings exist and the attendant random paths result holds. \square

7. Discussion and Concluding Remarks

We have demonstrated that for a psychologically appealing class of preferences, a decentralized matching process will converge to a pairwise-stable matching with probability one by satisfying random blocking pairs from any unstable matching. We further demonstrated that in the absence of a purely strict preference ordering for this class of preferences, a direct mechanism is not strategy-proof. This may provide normative evidence to favor the decentralized process of matching in these matching markets rather than a centralized mechanism that may be subject to manipulation by the agents. These results represent an attempt to extend previous work on both one- and two-sided matching

models from the one-to-one to the many-to-many one-sided case, and constitute a step towards better understanding one-sided many-to-many matching models which remain to be studied in further detail.

Abraham et al.'s [18] study of globally ranked pairs in the Stable Roommates problem represents in one sense an extension and generalization of Bartholdi and Trick's [17] analysis of the roommates problem. Globally ranked pairs are consistent with the notion of subjective homophily. Indeed, Abraham et al. [18] consider the case of globally-acyclic preferences, which is related to the No Odd Rings condition of Chung [4]. Subjective homophily and approximate symmetry can be considered a special case of globally acyclic preferences. However, Abraham et al. [18] focus on the Stable Roommates problem, which is one particular instance of the Stable Fixtures Problem. Lemma 1 can be viewed in some sense as an extension of Abraham et al.'s [18] Stable Roommates results to the Stable Fixtures setting.

The random paths analysis of the one-sided many-to-many setting of this paper mirror results found for two-sided settings: Ackerman et al. [13] and Cseh and Skutella [14] look at correlated markets in the context of two-sided matching markets. Correlated markets, in a graph-theoretic context, are related to the approximate symmetry assumption of this paper in that more preferred edges have lower weights. However, in the correlated markets studied by Ackerman et al. [13] and Cseh and Skutella [14], the same weight for an edge cannot appear more than once, while this can occur in our setting given the weak preferences case of subjective homophily and approximate symmetry, making this a slightly more general preference structure than correlated markets.

One-sided many-to-many matching is of theoretical interest because it provides a general framework that can encapsulate many of the most commonly studied matching markets. Understanding the interaction between preferences and stability in this type of unified framework that nests Stable Roommates, Stable Marriage, and College Admissions as special cases is of theoretical and also practical interest. For example, it is conceivable to consider markets where firms act as both employers (hiring independent consultants or signing contracts with suppliers) while simultaneously being employed to provide particular services. When the separation between workers and firms is not clearly delineated, this type of one-sided many-to-many matching model may be of interest.

The scheduling of American college football games is yet another possible application of a fixtures problem, and importantly one where preferences may exhibit a certain degree of subjective homophily. Such athletic contests represent a kind of one-sided many-to-many matching problem with the teams as agents: Team A may play both Team B and Team C, but this does not necessarily imply that Teams B and C have to play one another. Given that there are typically only twelve regular season games played by any one college football team (Wischnowsky [22]), the fixed number of games played in a season can be viewed as the capacity of a team. In the United States, there are too many college football programs and too few games for any one team to face all possible rivals. For this reason, there is contentious debate about how teams are ranked after the regular season, as this determines which teams end up competing in the invitational tournament that determines the national champion. At the end of the college football season, the College Football Playoff Selection Committee chooses what it deems to be the four top teams in the country to compete in a playoff to determine the national champion.⁵ There are a variety of criteria that are used for selecting the four best teams. Since not all teams play one another, simply having the most wins is not sufficient to guarantee a playoff spot. The quality of wins also matters in this context. Indeed, strength of schedule is one of the primary selection criteria [23]. A top team that has faced and defeated lesser opponents will not be viewed as favorably as a rival that has beaten more challenging teams. This means that teams have an incentive to schedule and defeat opponents of similar or better quality, rather than easily defeating lesser competition. This is arguably an example of subjective homophily in preferences. These matching markets also

⁵ Information regarding the selection criteria and process is available from the official College Football Playoff website: <https://collegefootballplayoff.com/sports/2016/10/24/selection-committee-protocol.aspx> [23].

represent a potentially rich area for the exploration of strategic considerations in examining wider classes of preferences.

Sufficient conditions that guarantee stability in the Roommates Problem have been found, and a natural direction for future research is looking into general sufficient conditions that guarantee the existence of Stable Fixture matchings. The subjective homophily function with approximate symmetry satisfies Chung's "No Odd Rings" sufficient condition for the existence of Stable Roommates matchings, and an open question is whether and how his results may be further generalized to the Stable Fixtures problem. This has the potential to deepen our understanding of the relationships between one-sided and two-sided matching markets and one-to-one, many-to-one, and many-to-many matchings. This paper provides a sufficient condition for stable matchings and random paths to pairwise-stability under a restricted, psychologically appealing class of preferences. Examining other preference profiles and whether they can guarantee stability is a promising potential direction for future research that may help us understand general sufficient conditions for the existence of stable matchings in the Stable Fixtures problem.

Further work remains to be done to study more general conditions for the existence of random paths to pairwise-stable matchings in the Stable Fixtures problem. Kojima and Ünver [12] proved that for a two-sided many-to-many matching model, as long as one side has responsive preferences while the other side has substitutable preferences, a random path to a pairwise-stable matching always exists. The subjective distance metric preferences are responsive, and responsive preferences satisfy substitutability. Understanding the interplay of these types of preference domains in one-sided many-to-many matching models will be of interest in developing new results on random paths to pairwise-stable matchings.

Another promising direction for future research is to consider modeling the Stable Fixtures Problem using linear programming techniques. Abeledo and Rothblum [24] demonstrated how the existence of stable matchings in the Stable Marriage and Stable Roommates problems can be determined using linear inequalities and integer programming techniques. Chung [4] notes that his no odd rings condition can be derived using the techniques of Abeledo and Rothblum. This may prove fruitful in further examining the Stable Fixtures Problem.

Irving developed a general algorithm for finding solutions to the Stable Fixtures problem, but the strategy-proofness of this algorithm, whether agents can misreport their preferences to attain a more preferred matching, remains to be studied. The strategic implications of subjective homophily based preferences represent one small step in this direction, and more work remains to be done in examining strategy-proofness for a wider class of preferences in these kinds of markets.

Notions of Stability in one-sided many-to-many matching problems may also hold promise for future research. Echenique and Oviedo [19] discuss various notions of stability in two-sided many-to-many matching markets. In this paper, we focus on examining pairwise-stability, but there are other notions of stability that are not equivalent to pairwise stability in many-to-many markets, such as group stability, setwise stability, and core stability. Exploring these stability concepts in the one-sided many-to-many matching framework is a promising future avenue to explore.

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