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Article

A Stochastic Maximum Principle for Markov Chains of Mean-Field Type

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Abstract: We derive sufficient and necessary optimality conditions in terms of a stochastic maximum principle (SMP) for controls associated with cost functionals of mean-field type, under dynamics driven by a class of Markov chains of mean-field type which are pure jump processes obtained as solutions of a well-posed martingale problem. As an illustration, we apply the result to generic examples of control problems as well as some applications.

Keywords: mean-field; nonlinear Markov chain; backward SDEs; optimal control; stochastic maximum principle

MSC: 60H10; 60H07; 49N90

1. Introduction

The goal of this paper is to find sufficient and necessary optimality conditions in terms of a stochastic maximum principle (SMP) for a set of admissible controls \bar{u} which minimize payoff functionals of the form

$$J(u) := E^u \left[\int_0^T f(t, x, E^u[\kappa_f(x(t))], u(t)) dt + h(x(T), E^u[\kappa_h(x(T))]) \right],$$

w.r.t. admissible controls u , for some given functions f, h, κ_f and κ_h , under dynamics driven by a pure jump process x with state space $I = \{0, 1, 2, 3, \dots\}$ whose jump intensity under the probability measure P^u is of the form

$$\lambda_{ij}^u(t) := \lambda_{ij}(t, x, E^u[\kappa_\ell(x(t))], u(t)), \quad i, j \in I,$$

for some given functions λ and κ_ℓ , as long as the intensities are predictable. Due to the dependence of the intensities on the mean of (a function of) $x(t)$ under P^u , the process x is commonly called a nonlinear Markov chain or Markov chain of mean-type, although it does not satisfy the standard Markov property, as explained in the seminal paper by McKean [1] for diffusion processes. The dependence of the intensities on the whole path x over the time interval $[0, T]$, makes the jump process cover a large class of real-world applications. For instance, in queuing theory it is desirable that the intensities are functions of $\int_0^t x(s) ds$, $\sup_{0 \leq s \leq t} x(s)$, or $\inf_{0 \leq s \leq t} x(s)$.

The Markov chain of mean-field type is obtained as the limit of a system of weakly interacting Markov chains $(x^{l,N}, l = 1, \dots, N)$ as the size N becomes large. That is,

$$\lambda_{ij}(t, x, E[\kappa_\ell(x(t))]) = \lim_{N \rightarrow \infty} \lambda_{ij}(t, x^{l,N}, \frac{1}{N} \sum_{l=1}^N \kappa_\ell(x^{l,N}(t))), \quad l \in \{1, \dots, N\}, i, j \in I.$$

Such a weak interaction is usually called a mean-field interaction. It occurs when the jump intensities of the Markov chains depend on their empirical mean. When the system's size grows to infinity, the sequence of N -indexed empirical means, which describes the states of the system, converges to the expectation $E[\kappa_\ell(x(t))]$ of $x(t)$, which evolves according to a McKean–Vlasov equation (or nonlinear Fokker–Planck). A more general situation is when the jump intensities of the obtained nonlinear Markov chain depend on the marginal law $P \circ x^{-1}(t)$ of $x(t)$. To keep the content of the paper as simple as possible, we do not treat this situation.

Markov chains of mean-field type have been used as models in many different fields, such as chemistry, physics, biology, economics, epidemics, etc. (e.g., [2–6]). Existence and uniqueness results with bounded and unbounded jump intensities were proven in [7,8], respectively. We refer to [9] for existence and uniqueness of solutions to McKean–Vlasov equations with unbounded jump intensities, and to [10,11] for results related to the law of large numbers for unbounded jump mean-field models, and large deviations for corresponding empirical processes.

The present work is a continuation of [12], where the authors proved the existence and uniqueness of this class of processes in terms of a martingale problem, and derived sufficient conditions (cf. Theorem 4.6 in [12]) for the existence of an optimal control which minimizes $J(u)$ for a rather general class of (unbounded) jump intensities. Since the suggested conditions are rather difficult to apply in concrete situations (see Remark 4.7 and Example 4.8 in [12]), we aim in this paper to investigate whether the SMP can yield optimality conditions that are tractable and easy to verify.

While in the usual strong-type control problems the dynamics are given in terms of a process X^u which solves a stochastic differential equation (SDE) on a given probability space (Ω, \mathcal{F}, Q) , the dynamics in our formulation are given in terms of a family of probability measures $(P^u, u \in \mathcal{U})$, where x is the coordinate process (i.e., it does not change with the control u). This type of formulation is usually called a weak-type formulation for control problems.

The main idea in the martingale and dynamic programming approaches to optimal control problems for jump processes (without mean-field coupling) suggested in previous work, including the following first papers on the subject [13–16] (the list of references is far from being exhaustive), is to use the Radon–Nikodym density process L^u of P^u w.r.t. some reference probability measure P as dynamics and recast the control problem to a standard one. In this paper, we apply the same idea and recast the control problem to a mean-field-type control problem to which an SMP can be applied. By a Girsanov-type result for pure jump processes, the density process L^u is a martingale and solves a linear SDE driven by some accompanying P -martingale M . The adjoint process associated to the SMP solves a (Markov chain) backward stochastic differential equation (BSDE) driven by the P -martingale M , whose existence and uniqueness can be derived using the results by Cohen and Elliott [17,18]. For some linear and quadratic cost functionals, we explicitly solve these BSDEs and derive a closed form of the optimal control.

In Section 2, we briefly recall the basic stochastic calculus for pure jump processes that we will use in the sequel. In Section 3, we derive sufficient and necessary optimality conditions for the control problem. As already mentioned, the SMP optimality conditions are derived in terms of a mean-field stochastic maximum principle, where the adjoint equation is a Markov chain BSDE. In Section 3, we illustrate the results using two examples of optimal control problems that involve two-state chains and linear quadratic cost functionals. We also consider an optimal control of a mean-field version of the Schlögl model for chemical reactions. We consider linear and quadratic cost functionals in all examples for the sake of simplicity and also because, in these cases, we obtain the optimal controls in closed form.

The obtained results can easily be extended to pure jump processes taking values on more general state spaces such as $I = \mathbb{Z}^d$, $d \geq 1$.

2. Preliminaries

Let $I := \{0, 1, 2, \dots\}$ equipped with its discrete topology and σ -field and let $\Omega := \mathcal{D}([0, T], I)$ be the space of functions from $[0, T]$ to I that are right-continuous with left limits at each $t \in [0, T)$ and are left-continuous at time T . We endow Ω with the Skorohod metric d_0 so that (Ω, d_0) is a complete separable metric (i.e., Polish) space. Given $t \in [0, T]$ and $\omega \in \Omega$, put $x(t, \omega) \equiv \omega(t)$ and denote by $\mathcal{F}_t^0 := \sigma(x(s), s \leq t)$, $0 \leq t \leq T$, the filtration generated by x . Denote by \mathcal{F} the Borel σ -field over Ω . It is well-known that \mathcal{F} coincides with $\sigma(x(s), 0 \leq s \leq T)$.

To x we associate the indicator process $I_i(t) = \mathbf{1}_{\{x(t)=i\}}$ whose value is 1 if the chain is in state i at time t and 0 otherwise, and the counting processes $N_{ij}(t)$, $i \neq j$, independent of $x(0)$, such that

$$N_{ij}(t) = \#\{\tau \in (0, t] : x(\tau^-) = i, x(\tau) = j\}, \quad N_{ij}(0) = 0,$$

which counts the number of jumps from state i into state j during the time interval $(0, t]$. Obviously, since x is right-continuous with left limits, both I_i and N_{ij} are right-continuous with left limits. Moreover, by the relationship

$$x(t) = \sum_i iI_i(t), \quad I_i(t) = I_i(0) + \sum_{j:j \neq i} (N_{ji}(t) - N_{ij}(t)), \tag{1}$$

the state process, the indicator processes, and the counting processes carry the same information, which is represented by the natural filtration $\mathbb{F}^0 := (\mathcal{F}_t^0, 0 \leq t \leq T)$ of x . Note that (1) is equivalent to the following useful representation:

$$x(t) = x(0) + \sum_{i,j:i \neq j} (j - i)N_{ij}(t). \tag{2}$$

Let $G = (g_{ij}, i, j \in I)$, where g_{ij} are constant entries, be a Q -matrix:

$$g_{ij} > 0, \quad i \neq j, \quad \sum_{j:j \neq i} g_{ij} < +\infty, \quad g_{ii} = - \sum_{j:j \neq i} g_{ij}. \tag{3}$$

By Theorem 4.7.3 in [19], or Theorem 20.6 in [20] (for the finite state-space), given the Q -matrix G and a probability measure ζ over I , there exists a unique probability measure P on (Ω, \mathcal{F}) under which the coordinate process x is a time-homogeneous Markov chain with intensity matrix G and starting distribution ζ (i.e., such that $P \circ x^{-1}(0) = \zeta$). Equivalently, P solves the martingale problem for G with initial probability distribution ζ , meaning that for every f on I , the process defined by

$$M_t^f := f(x(t)) - f(x(0)) - \int_{(0,t]} (Gf)(x(s)) ds \tag{4}$$

is a local martingale relative to $(\Omega, \mathcal{F}, \mathbb{F}^0)$, where

$$Gf(i) := \sum_j g_{ij}f(j) = \sum_{j:j \neq i} g_{ij}(f(j) - f(i)), \quad i \in I,$$

and

$$Gf(x(s)) = \sum_{i,j:i \neq j} I_i(s)g_{ij}(f(j) - f(i)). \tag{5}$$

By Lemma 21.13 in [20], the compensated processes associated with the counting processes N_{ij} , defined by

$$M_{ij}(t) = N_{ij}(t) - \int_{(0,t]} I_i(s^-)g_{ij} ds, \quad M_{ij}(0) = 0, \tag{6}$$

are zero mean, square integrable, and mutually orthogonal P -martingales whose predictable quadratic variations are

$$\langle M_{ij} \rangle_t = \int_{(0,t]} I_i(s^-) g_{ij} ds. \tag{7}$$

Moreover, at jump times t , we have

$$\Delta M_{ij}(t) = \Delta N_{ij}(t) = I_i(t^-) I_j(t). \tag{8}$$

Thus, the optional variation of M

$$[M](t) = \sum_{0 < s \leq t} |\Delta M(s)|^2 = \sum_{0 < s \leq t} \sum_{i,j:i \neq j} |\Delta M_{ij}(s)|^2$$

is

$$[M](t) = \sum_{0 < s \leq t} \sum_{i,j:i \neq j} I_i(s^-) I_j(s). \tag{9}$$

We call $M := \{M_{ij}, i \neq j\}$ the accompanying martingale of the counting process $N := \{N_{ij}, i \neq j\}$ or of the Markov chain x .

Denote by $\mathbb{F} := (\mathcal{F}_t)_{0 \leq t \leq T}$ the completion of $\mathbb{F}^0 = (\mathcal{F}_t^0)_{t \leq T}$ with the P -null sets of Ω . Hereafter, a process from $[0, T] \times \Omega$ into a measurable space is said to be predictable (resp. progressively measurable) if it is predictable (resp. progressively measurable) w.r.t. the predictable σ -field on $[0, T] \times \Omega$ (resp. \mathbb{F}).

For a real-valued matrix $m(t) := (m_{ij}(t), i, j \in I)$ indexed by $I \times I$, we let

$$\|m\|_g^2(t) := \sum_{i,j:i \neq j} |m_{ij}(t)|^2 g_{ij} \mathbf{1}_{\{x(t^-)=i\}} < \infty. \tag{10}$$

Consider the local martingale

$$W(t) = \int_0^t Z(s) dM(s) := \sum_{i,j:i \neq j} \int_0^t Z_{ij}(s) dM_{ij}(s). \tag{11}$$

Then, the optional variation of the local martingale W is

$$[W](t) = \sum_{0 < s \leq t} |Z(s) \Delta M(s)|^2 = \sum_{0 < s \leq t} \sum_{i,j:i \neq j} |Z_{ij}(s) \Delta M_{ij}(s)|^2, \tag{12}$$

and its compensator is

$$\langle W \rangle_t = \int_{(0,t]} \|Z(s)\|_g^2 ds. \tag{13}$$

Provided that

$$E \left[\int_{(0,T]} \|Z(s)\|_g^2 ds \right] < \infty, \tag{14}$$

W is a square-integrable martingale and its optional variation satisfies

$$E [[W](t)] = E \left[\sum_{0 < s \leq t} |Z(s) \Delta M(s)|^2 \right] = E \left[\int_{(0,t]} \|Z(s)\|_g^2 ds \right]. \tag{15}$$

Moreover, the following Doob's inequality holds:

$$E \left[\sup_{0 \leq t \leq T} \left| \int_0^t Z(s) dM(s) \right|^2 \right] \leq 4E \left[\int_{(0,T]} \|Z(s)\|_g^2 ds \right]. \tag{16}$$

3. A Stochastic Maximum Principle

We consider controls with values in some subset U of \mathbb{R}^d and let \mathcal{U} be the set of \mathbb{F} -progressively measurable processes $u = (u(t), 0 \leq t \leq T)$ with values in $U \subset \mathbb{R}^d$. \mathcal{U} is the set of admissible controls. For $u \in \mathcal{U}$, let P^u be the probability measure on (Ω, \mathcal{F}) under which the coordinate process x is a jump process with intensities

$$\lambda_{ij}^u(t) := \lambda_{ij}(t, x, E^u[\kappa_\ell(x(t))], u(t)), \quad i, j \in I, \quad 0 \leq t \leq T, \tag{17}$$

where for each $i, j \in I$,

$$\begin{aligned} \lambda_{ij} &: [0, T] \times \Omega \times \mathbb{R} \times U \mapsto \mathbb{R}, \\ \kappa_\ell &: I \mapsto \mathbb{R}. \end{aligned}$$

The cost functional associated to P^u is of the form

$$J(u) := E^u \left[\int_0^T f(t, x, E^u[\kappa_f(x(t))], u(t)) dt + h(x(T), E^u[\kappa_h(x(T))]) \right], \tag{18}$$

where

$$\begin{aligned} f &: [0, T] \times \Omega \times \mathbb{R} \times U \mapsto \mathbb{R}, \\ h &: I \times \mathbb{R} \mapsto \mathbb{R}, \\ \kappa_f &: I \mapsto \mathbb{R}, \\ \kappa_h &: I \mapsto \mathbb{R}. \end{aligned}$$

In this section, we propose to characterize minimizers \bar{u} of J , that is, $\bar{u} \in \mathcal{U}$ satisfying

$$J(\bar{u}) = \min_{u \in \mathcal{U}} J(u) \tag{19}$$

in terms of a stochastic maximum principle (SMP). We first state and prove the sufficient optimality conditions. Then, we state the necessary optimality conditions.

Let P be the probability measure on (Ω, \mathcal{F}) under which x is a time-homogeneous Markov chain such that $P \circ x^{-1}(0) = \xi$ and with Q -matrix $(g_{ij})_{ij}$ satisfying (3). Then, by a Girsanov-type result for pure jump processes (e.g., [20,21]), it holds that

$$dP^u := L^u(T) dP, \tag{20}$$

where, for $0 \leq t \leq T$,

$$L^u(t) := \prod_{\substack{ij \\ i \neq j}} \exp \left\{ \int_{(0,t]} \ln \frac{\lambda_{ij}^u(s)}{g_{ij}} dN_{ij}(s) - \int_0^t (\lambda_{ij}^u(s) - g_{ij}) I_i(s) ds \right\}, \tag{21}$$

which satisfies

$$L^u(t) = 1 + \int_{(0,t]} L^u(s^-) \sum_{i,j:i \neq j} I_i(s^-) \ell_{ij}^u(s) dM_{ij}(s), \tag{22}$$

where $\ell_{ij}^u(s) := \ell_{ij}(t, x, E^u[\kappa_\ell(x(s))], u(s))$ is given by the formula

$$\ell_{ij}^u(s) = \begin{cases} \lambda_{ij}^u(s)/g_{ij} - 1 & \text{if } i \neq j, \\ 0 & \text{if } i = j, \end{cases} \tag{23}$$

and $(M_{ij})_{ij}$ is the P -martingale given in (6). Moreover, the accompanying martingale $M^u = (M_{ij}^u)_{ij}$ satisfies

$$M_{ij}^u(t) = M_{ij}(t) - \int_{(0,t]} \ell_{ij}^u(s) L_i(s^-) g_{ij} ds. \tag{24}$$

Noting that

$$J(u) = E \left[L^u(T) \int_0^T f(t, x, E^u[\kappa_f(x(t))], u(t)) dt + L^u(T) h(x(T), E^u[\kappa_h(x(T))]) \right].$$

Integrating by parts and taking expectation, we obtain

$$J(u) := E \left[\int_0^T L^u(t) f(t, x, E[L^u(t)\kappa_f(x(t))], u(t)) dt + L^u(T) h(x(T), E[L^u(T)\kappa_h(x(T))]) \right]. \tag{25}$$

We recast our problem of controlling a Markov chain through its intensity matrix to a standard control problem which aims at minimizing the cost functional (25) under the dynamics given by the density process L^u which satisfies (22), to which the mean-field stochastic maximum principle in [22] can be applied. The corresponding optimal dynamics are given by the probability measure \bar{P} on (Ω, \mathcal{F}) defined by

$$d\bar{P} = L^{\bar{u}}(T) dP, \tag{26}$$

where $L^{\bar{u}}$ is the associated density process. $(L^{\bar{u}}, \bar{u})$ is called an optimal pair associated with (19).

For $w \in \{y, \bar{y}, u\}$, ψ_w denotes the partial derivative of the function $\psi(y, \bar{y}, u)$ w.r.t. w . for $\alpha = \{\ell, f\}$, we set

$$\alpha(t) := \alpha(t, x, E[L^u(t)\kappa_\alpha(x(t))], u(t)), \quad \bar{\alpha}(t) := \alpha(t, x, E[L^{\bar{u}}(t)\kappa_\alpha(x(t))], \bar{u}(t)),$$

and we define

$$h(T) := h(x(T), E[L^u(T)\kappa_h(x(T))]), \quad \bar{h}(T) := h(x(T), E[L^{\bar{u}}(T)\kappa_h(x(T))]).$$

To the admissible pair of processes $(L^{\bar{u}}, \bar{u})$, we associate the solution (p, q) (if it exists) of the following linear BSDE of mean-field type, known as first-order adjoint equation:

$$\begin{cases} dp(t) = - \left\{ \langle \bar{\ell}(t), q(t) \rangle_g - \bar{f}(t) + \kappa_\ell(x(t)) E[L^{\bar{u}}(t) (\langle \bar{\ell}_{\bar{y}}(t), q(t) \rangle_g) \right. \\ \qquad \qquad \qquad \left. - \kappa_f(x(t)) E[L^{\bar{u}}(t) \bar{f}_{\bar{y}}(t)] \right\} dt + q(t) dM(t), \\ p(T) = -\bar{h}(T) - \kappa_h(x(T)) E[L^{\bar{u}}(T) \bar{h}_{\bar{y}}(T)]. \end{cases} \tag{27}$$

In the next proposition we give sufficient conditions on $f, h, \ell, \kappa_\ell, \kappa_f$, and κ_h that guarantee the existence of a unique solution to the BSDE (27).

Proposition 1. Assume that

(A1) For each $i, j \in I, i \neq j$,

(a) $(\bar{y}, u) \mapsto \lambda_{ij}(\cdot, \cdot, \bar{y}, u)$ is differentiable,

(b) there exists a positive constant C_λ s.t. P-a.s. for all $(t, \bar{y}, u) \in [0, T] \times \mathbb{R} \times U$,

$$\lambda_{ij}(t, \omega, \bar{y}, u) + \left| \frac{\partial}{\partial \bar{y}} \lambda_{ij}(t, \omega, \bar{y}, u) \right| \leq C_\lambda.$$

(A2) The functions f, h, κ_f , and κ_h are bounded. f and h are differentiable in \bar{y} with bounded derivatives.

Then, the BSDE (27) admits a solution (p, q) consisting of an adapted process p which is right-continuous with left limits and a predictable process q which satisfies

$$E \left[\sup_{t \in [0, T]} |p(t)| + \int_{(0, T]} \|q(s)\|_g^2 ds \right] < +\infty. \tag{28}$$

This solution is unique up to indistinguishability for p and equality $dP \times g_{ij} I_i(s^-) ds$ -almost everywhere for q .

Remark 1. (i) Assumptions (A1) and (3) imply that there exists a positive constant C s.t. for all $t \in [0, T]$

$$\|\ell^u(t)\|_g + \|\ell_{\bar{y}}^u(t)\|_g \leq C \quad P\text{-a.s.} \tag{29}$$

(ii) By Theorem T11 (chapter VII) in [21], the uniform boundedness of $(\lambda_{ij})_{ij}, i \neq j$ implies that for each $u \in U$

$$\tilde{C} := \sup_{0 \leq t \leq T} E[(L^u(t))^2] < \infty. \tag{30}$$

Proof. Assumptions (A1) and (A2) make the driver of the BSDE (27) Lipschitz continuous in q . The proof is similar to that of Theorem 3.1 for the Brownian motion-driven mean-field BSDE derived in [23] by considering the following norm:

$$\|(p, q)\|_\beta^2 := E \int_0^T e^{\beta t} (|p(t)|^2 + \|q(t)\|_g^2) dt,$$

where $\beta > 0$, along with Itô–Stieltjes formula for purely discontinuous semi-martingales. For the sake of completeness, we give a proof in Appendix A. \square

Remark 2. (i) The boundedness on f and h and their derivatives is strong and can be considerably weakened using standard truncation techniques.

(ii) If $\ell_{\bar{y}} = 0$ (i.e., the intensity does not contain any mean-field coupling), the BSDE (27) becomes standard. Thanks to Theorem 3.10 in [12], it is solvable only by imposing similar conditions to (H1)–(H3) therein.

(iii) If $\ell_{\bar{y}} \neq 0$ (i.e., the intensity is of mean-field type), we do not know whether we can relax the imposed boundedness of ℓ, κ_ℓ and $\ell_{\bar{y}}$, because without this condition the standard comparison theorem for Markov chain BSDEs simply does not generally apply for such drivers.

Let $(L^{\bar{u}}, \bar{u})$ be an admissible pair and (p, q) be the associated first-order adjoint process solution of (27).

For $v \in U$, we introduce the Hamiltonian associated to our control problem

$$H(t, v) := L^{\bar{u}}(t) \left(\langle \ell(t, x, E[L^{\bar{u}}(t)\kappa_\ell(x(t))], v), q(t) \rangle_g - f(t, x, E[L^{\bar{u}}(t)\kappa_f(x(t))], v) \right). \tag{31}$$

Next, we state the SMP sufficient and necessary optimality conditions, but only prove the sufficient optimality case, as the necessary optimality conditions result is tedious and more involved but by now “standard” and can be derived following the same steps of [22,24,25].

In the next two theorems, we assume that (A1) and (A2) of Proposition 1 hold.

Theorem 1 (Sufficient optimality conditions). *Let $(L^{\bar{u}}, \bar{u})$ be an admissible pair and (p, q) be the associated first-order adjoint process which satisfies (27) and (28). Assume*

- (A4) *The set of controls U is a convex body (i.e., U is convex and has a nonempty interior) of \mathbb{R}^d , and the functions ℓ and f are differentiable in u .*
- (A5) *The functions $(y, \bar{y}, u) \mapsto y\ell(\cdot, \cdot, \bar{y}, u)$ and $(y, \bar{y}, u) \mapsto -yf(\cdot, \cdot, \bar{y}, u)$ are concave in (y, \bar{y}, u) for a.e. $t \in [0, T]$, P -almost surely.*
- (A6) *The function $(y, \bar{y}) \mapsto yh(\cdot, \bar{y})$ is convex.*

If the admissible control \bar{u} satisfies

$$H(t, \bar{u}(t)) = \max_{v \in U} H(t, v), \quad \text{a.e. } t \in [0, T], \quad P\text{-a.s.}, \tag{32}$$

then the pair $(L^{\bar{u}}, \bar{u})$ is optimal.

Proof. We want to show that if the pair $(L^{\bar{u}}, \bar{u})$ satisfies (32), then

$$J(u) - J(\bar{u}) = E \left[\int_0^T (L^u(t)f(t) - L^{\bar{u}}(t)\bar{f}(t))dt + L^u(T)h(T) - L^{\bar{u}}(T)\bar{h}(T) \right] \geq 0.$$

Since $(y, \bar{y}) \mapsto yh(\cdot, \bar{y})$ is convex, we have

$$\begin{aligned} E[L^u(T)h(T) - L^{\bar{u}}(T)\bar{h}(T)] &\geq E[(\bar{h}(T) + \kappa_h(T)E[L^{\bar{u}}(T)\bar{h}_{\bar{y}}(T)])(L^u(T) - L^{\bar{u}}(T))] \\ &= -E[p(T)(L^u(T) - L^{\bar{u}}(T))]. \end{aligned}$$

Integrating by parts, using (27), we obtain

$$\begin{aligned} E[p(T)(L^u(T) - L^{\bar{u}}(T))] &= E \left[\int_0^T (L^u(t^-) - L^{\bar{u}}(t^-))dp(t) + p(t^-)d(L^u(t) - L^{\bar{u}}(t)) \right. \\ &\quad \left. + d[L^u - L^{\bar{u}}, p](t) \right] \\ &= -E \left[\int_0^T \left(\langle \bar{\ell}(t), q(t) \rangle_g - \bar{f}(t) + \kappa_\ell(x(t))E[L^{\bar{u}}(t)(\langle \bar{\ell}_{\bar{y}}(t), q(t) \rangle_g] \right. \right. \\ &\quad \left. \left. - \kappa_f(x(t))E[L^{\bar{u}}(t)\bar{f}_{\bar{y}}(t)] \right) (L^u(t) - L^{\bar{u}}(t)) - \langle L^u(t)\ell(t) - L^{\bar{u}}(t)\bar{\ell}(t), q(t) \rangle_g \right) dt \right]. \end{aligned}$$

We introduce the following ‘‘Hamiltonian’’ function:

$$\mathcal{H}(t, x, y, \bar{y}, u, z) := y\langle \ell(t, x, \bar{y}, u), z \rangle_g - yf(t, x, \bar{y}, u). \tag{33}$$

Furthermore, for u and \bar{u} in \mathcal{U} , we set

$$\begin{cases} \mathcal{H}(t) := L^u(t) \left(\langle \ell(t, x, E[L^u(t)\kappa_\ell(x(t))], u(t)), q(t) \rangle_g - f(t, x, E[L^u(t)\kappa_f(x(t))], u(t)) \right), \\ \bar{\mathcal{H}}(t) := L^{\bar{u}}(t) \left(\langle \ell(t, x, E[L^{\bar{u}}(t)\kappa_\ell(x(t))], \bar{u}(t)), q(t) \rangle_g - f(t, x, E[L^{\bar{u}}(t)\kappa_f(x(t))], \bar{u}(t)) \right). \end{cases} \tag{34}$$

Since $(y, \bar{y}, u) \mapsto y\ell(\cdot, \bar{y}, u)$ and $-yf(\cdot, \bar{y}, u)$ are concave, we have

$$\mathcal{H}(t) - \bar{\mathcal{H}}(t) \leq \bar{\mathcal{H}}_y(t)(L^u(t) - L^{\bar{u}}(t)) + \bar{\mathcal{H}}_{\bar{y}}(t)(E[\kappa_\ell(x(t))](L^u(t) - L^{\bar{u}}(t))) + \bar{\mathcal{H}}_u(t) \cdot (u(t) - \bar{u}(t)).$$

Since, by (32), $\bar{\mathcal{H}}_u(t) = H_u(t, \bar{u}(t)) = 0$ a.e. $t \in [0, T]$, we obtain

$$\begin{aligned} E[\mathcal{H}(t) - \bar{\mathcal{H}}(t)] &\leq E \left[\left\{ \langle \bar{\ell}(t), q(t) \rangle_g - \bar{f}(t) + \kappa_\ell(x(t))E[L^{\bar{u}}(t)(\langle \bar{\ell}_{\bar{y}}(t), q(t) \rangle_g] \right. \right. \\ &\quad \left. \left. - \kappa_f(x(t))E[L^{\bar{u}}(t)\bar{f}_{\bar{y}}(t)] \right\} (L^u(t) - L^{\bar{u}}(t)) \right] \end{aligned}$$

for a.e. $t \in [0, T]$.

Therefore,

$$E[L^u(T)h(T) - L^{\bar{u}}(T)\bar{h}(T)] \geq E \left[\int_0^T (\mathcal{H}(t) - \bar{\mathcal{H}}(t) - \langle L^u(t)\ell(t) - L^{\bar{u}}(t)\bar{\ell}(t), q(t) \rangle_g) dt \right].$$

Hence,

$$J(u) - J(\bar{u}) \geq E \left[\int_0^T (\mathcal{H}(t) - \bar{\mathcal{H}}(t) + L^u(t)f(t) - L^{\bar{u}}(t)\bar{f}(t) - \langle L^u(t)\ell(t) - L^{\bar{u}}(t)\bar{\ell}(t), q(t) \rangle_g) dt \right] = 0.$$

□

Theorem 2 (Necessary optimality conditions (Verification Theorem)). *If $(L^{\bar{u}}, \bar{u})$ is an optimal pair of the control problem (19) and there is a unique pair of \mathcal{F} -adapted processes (p, q) associated to $(L^{\bar{u}}, \bar{u})$ which satisfies (27) and (28), then*

$$H(t, \bar{u}(t)) = \max_{v \in U} H(t, v), \quad \text{a.e. } t \in [0, T], \quad P\text{-a.s.}$$

Remark 3. *Unfortunately, the sufficient optimality conditions can rarely be satisfied in practice because the convexity conditions imposed on the involved coefficients are not always satisfied, even for the simplest examples: assume ℓ and f without mean-field coupling and linear in the control u . Then, none of the functions $(y, u) \mapsto y\ell(\cdot, u)$ and $(y, u) \mapsto -yf(\cdot, u)$ are concave in (y, u) . However, the verification theorem in terms of necessary optimality conditions holds for a fairly general class of functions with sufficient smoothness. Hence, if we can solve the associated BSDEs, the necessary optimality conditions result can be useful.*

4. Numerical Examples

In this section we first solve the adjoint equation associated to an optimal control problem associated with a standard two-state Markov chain, then we extend the problem to a two-state Markov chain of mean-field type. As mentioned in Remark 3, whether sufficient or necessary conditions may apply of course depends on the smoothness of the involved functions. Not all the functions involved in the next examples satisfy the convexity conditions imposed in Theorem 1.

Example 1. *Optimal Control of a Standard Two-State Markov Chain.*

We study the optimal control of a simple Markov chain x whose state space is $\mathcal{X} = \{a, b\}$, where $(0 \leq a < b)$ are integers, and its jump intensity matrix is

$$\lambda^u(t) = \begin{bmatrix} -\alpha & \alpha \\ u(t) & -u(t) \end{bmatrix},$$

where α is a given positive constant intensity and u is the control process assumed to be nonnegative, bounded, and predictable. Let P be the probability measure under which the chain x has intensity matrix

$$G = \begin{bmatrix} -g_{ab} & g_{ab} \\ g_{ba} & -g_{ba} \end{bmatrix}, \quad g_{ab}, g_{ba} > 0.$$

Further, let $L^u(t) = \frac{dP^u}{dP} \Big|_{\mathcal{F}_t}$ be the density process given by (22), where ℓ is defined by

$$\ell_{ij}^u(t) = \begin{cases} \lambda_{ij}^u(t)/g_{ij} - 1 & \text{if } i \neq j, \\ 0 & \text{if } i = j. \end{cases}$$

The control problem we want to solve consists of finding the optimal control \bar{u} that minimizes the linear-quadratic cost functional

$$J(u) = E^u \left[\frac{1}{2} \int_0^T u^2(t) dt + h(x(T)) \right], \quad h(b) \geq h(a). \tag{35}$$

Given a control $v \in U$, consider the Hamiltonian

$$H(t, L^{\bar{u}}(t), q(t), v) := L^{\bar{u}}(t) (\langle \ell^v(t), q(t) \rangle_g - \frac{1}{2} v^2) =: H(t, v),$$

where

$$\langle \ell^v(t), q(t) \rangle_g = q_{ab}(t)(\alpha - g_{ab})I_a(t^-) + q_{ba}(t)(v - g_{ba})I_b(t^-).$$

By the first-order optimality conditions, an optimal control \bar{u} is a solution of the equation $\frac{\partial H(t,v)}{\partial v} = 0$, which implies

$$0 = \left\langle \frac{\partial \ell^v}{\partial v}(t), q(t) \right\rangle_g - v = q_{ba}(t)I_b(t^-) - v.$$

The optimal control is thus

$$\bar{u}(t) = q_{ba}(t)I_b(t^-), \tag{36}$$

where for each t , $q_{ba}(t) \geq 0$, since $\bar{u}(t) \geq 0$.

It remains to identify $q_{ba}(t)$. Consider the associated adjoint equations given by

$$\begin{cases} dp(t) = - \left\{ \langle \ell^{\bar{u}}(t), q(t) \rangle_g - \frac{1}{2} \bar{u}^2(t) \right\} dt + q_{ab}(t) dM_{ab}(t) + q_{ba}(t) dM_{ba}(t), & 0 \leq t < T, \\ p(T) = -h(x(T)). \end{cases}$$

In view of (36), the driver reads

$$\langle \ell^{\bar{u}}(t), q(t) \rangle_g - \frac{1}{2} \bar{u}^2(t) = q_{ab}(t)I_a(t^-)(\alpha - g_{ab}) + q_{ba}(t)I_b(t^-) \left\{ \frac{1}{2} q_{ba}(t)I_b(t^-) - g_{ba} \right\}. \tag{37}$$

The adjoint equation becomes

$$dp(t) = q_{ab}(t) \left\{ -(\alpha - g_{ab})I_a(t^-) dt + dM_{ab}(t) \right\} + q_{ba}(t) \left\{ -\left(\frac{1}{2} q_{ba}(t) - g_{ba}\right)I_b(t^-) dt + dM_{ba}(t) \right\}.$$

Now, considering the probability measure \tilde{P} under which x is a Markov chain whose jump intensity matrix is

$$\tilde{G}(t) = \begin{bmatrix} -\alpha & \alpha \\ \frac{1}{2} q_{ba}(t) & -\frac{1}{2} q_{ab}(t) \end{bmatrix},$$

the processes defined by

$$\begin{cases} d\tilde{M}_{ab}(t) = dM_{ab}(t) - (\alpha - g_{ab})I_a(t^-) dt, \\ d\tilde{M}_{ba}(t) = dM_{ba}(t) - \left(\frac{1}{2} q_{ba}(t) - g_{ba}\right)I_b(t^-) dt \end{cases}$$

are \tilde{P} -martingales having the same jumps as the martingales M_{ij} :

$$\Delta \tilde{M}_{ab}(t) = \Delta M_{ab}(t) = I_a(t^-)I_b(t), \quad \Delta \tilde{M}_{ba}(t) = \Delta M_{ba}(t) = I_b(t^-)I_a(t) \tag{38}$$

and

$$dp(t) = q_{ab}(t)d\tilde{M}_{ab}(t) + q_{ba}(t)d\tilde{M}_{ba}(t). \tag{39}$$

This yields

$$\Delta p(t) = q_{ab}(t)I_a(t^-)I_b(t) + q_{ba}(t)I_b(t^-)I_a(t). \tag{40}$$

Integrating (39) and then taking conditional expectation yields

$$p(t) = -\tilde{E}[h(x(T))|\mathcal{F}_t].$$

Therefore,

$$\Delta p(t) = -\Delta\tilde{E}[h(x(T))|\mathcal{F}_t]. \tag{41}$$

Under the probability measure \tilde{P}

$$\begin{aligned} h(x(T)) = & h(x(t)) + \int_t^T \left\{ \alpha(h(b) - h(a))I_a(s^-) + \frac{1}{2}q_{ba}(s)(h(a) - h(b))I_b(s^-) \right\} ds \\ & + \int_t^T (h(b) - h(a))d\tilde{M}_{ab}(s) + \int_t^T (h(a) - h(b))d\tilde{M}_{ba}(s). \end{aligned}$$

Taking conditional expectation, we obtain

$$\tilde{E}[h(x(T))|\mathcal{F}_t] = h(x(t)) + \int_t^T \tilde{E} \left[\alpha(h(b) - h(a))I_a(s^-) + \frac{1}{2}q_{ba}(s)(h(a) - h(b))I_b(s^-) \middle| \mathcal{F}_t \right] ds$$

and

$$\Delta\tilde{E}[h(x(T))|\mathcal{F}_t] = \Delta h(x(t)) = -(h(b) - h(a))I_a(t^-)I_b(t) - (h(a) - h(b))I_b(t^-)I_a(t),$$

which in view of (41) implies that

$$q_{ab}(t) = h(a) - h(b), \quad q_{ba}(t) = h(b) - h(a). \tag{42}$$

Therefore,

$$\bar{u}(t) = (h(b) - h(a))I_b(t^-) = h(b)I_b(t^-) - h(a) + h(a)I_a(t^-) = h(x(t^-)) - h(a),$$

which yields the following explicit form of the optimal control:

$$\bar{u}(t) = h(x(t^-)) - h(a).$$

In the next two examples we highlight the effect of the mean-field coupling in both the jump intensity and the cost functional on the optimal control.

Example 2. Mean-Field Optimal Control of a Two-State Markov Chain.

We consider the same chain as in the first example but with the following mean-field type jump intensities, ($t \in [0, T]$),

$$\lambda^u(t) = \begin{bmatrix} -\alpha & \alpha \\ u(t) + E^u[x(t^-)] & -u(t) - E^u[x(t^-)] \end{bmatrix}, \quad \alpha > 0, \quad u(t) + E^u[x(t^-)] \geq 0,$$

and want to minimize the cost functional

$$J(u) = E^\mu \left[\frac{1}{2} \int_0^T u^2(t) dt \right] + Var^\mu(x(T)), \tag{43}$$

where $Var^\mu(x(T))$ denotes the variance of $x(T)$ under the probability P^μ defined by

$$Var^\mu(x(T)) := E^\mu \left[(x(T) - E^\mu[x(T)])^2 \right].$$

Given a control $v \in U$, consider the Hamiltonian

$$H(t, v) := L^{\bar{u}}(t) (\langle \ell^v(t), q(t) \rangle_g - \frac{1}{2} v^2),$$

where

$$\langle \ell^v(t), q(t) \rangle_g = q_{ab}(t)(\alpha - g_{ab})I_a(t^-) + q_{ba}(t)(v + E^{\bar{u}}[x(t^-)] - g_{ba})I_b(t^-).$$

Performing similar calculations as in Example 1, we find that the optimal control is given by

$$\bar{u}(t) = q_{ba}(t)I_b(t^-). \tag{44}$$

We will now identify q_{ba} . The associated adjoint equation is given by

$$\begin{cases} dp(t) = - \left\{ \langle \ell^{\bar{u}}(t), q(t) \rangle_g - \frac{1}{2} \bar{u}^2(t) + x(t)E^{\bar{u}}[\bar{H}_{\bar{y}}(t)] \right\} dt + q_{ab}(t)dM_{ab}(t) + q_{ba}(t)dM_{ba}(t), \\ p(T) = - (x(T) - E^{\bar{u}}[x(T)])^2. \end{cases}$$

In view of (44), the driver reads

$$\begin{aligned} \langle \ell^{\bar{u}}(t), q(t) \rangle_g - \frac{1}{2} \bar{u}^2(t) + x(t)E^{\bar{u}}[\bar{H}_{\bar{y}}(t)] &= q_{ba}(t) \left(\frac{1}{2} q_{ba}(t) + E^{\bar{u}}[x(t^-)] - g_{ba} \right) I_b(t^-) \\ &\quad + q_{ab}(t)(\alpha - g_{ab})I_a(t^-) + x(t)E^{\bar{u}}[q_{ba}(t)I_b(t^-)]. \end{aligned}$$

The adjoint equation becomes

$$\begin{aligned} dp(t) &= q_{ab}(t) \{ dM_{ab}(t) - (\alpha - g_{ab})I_a(t^-)dt \} - x(t)E^{\bar{u}}[q_{ba}(t)I_b(t^-)]dt \\ &\quad + q_{ba}(t) \left\{ dM_{ba}(t) - \left(\frac{1}{2} q_{ba}(t) + E^{\bar{u}}[x(t^-)] - g_{ba} \right) I_b(t^-)dt \right\}. \end{aligned} \tag{45}$$

Consider the probability measure \tilde{P} , under which x is a Markov chain whose jump intensity matrix

$$\tilde{G}(t) = \begin{bmatrix} -\alpha & \alpha \\ \frac{1}{2} q_{ba}(t) + E^{\bar{u}}[x(t^-)] & -\frac{1}{2} q_{ba}(t) - E^{\bar{u}}[x(t^-)] \end{bmatrix}, \quad \frac{1}{2} q_{ba}(t) + E^{\bar{u}}[x(t^-)] \geq 0.$$

This change of measure yields the \tilde{P} -martingales

$$\begin{cases} d\tilde{M}_{ab}(t) = dM_{ab}(t) - (\alpha - g_{ab})I_a(t^-)dt, \\ d\tilde{M}_{ba}(t) = dM_{ba}(t) - \left(\frac{1}{2} q_{ba}(t) + E^{\bar{u}}[x(t^-)] - g_{ba} \right) I_b(t^-)dt \end{cases}$$

and

$$dp(t) = -x(t)E^{\bar{u}}[q_{ba}(t)I_b(t^-)]dt + q_{ab}(t)d\tilde{M}_{ab}(t) + q_{ba}(t)d\tilde{M}_{ba}(t). \tag{46}$$

This yields

$$\Delta p(t) = q_{ab}(t)I_a(t^-)I_b(t) + q_{ba}(t)I_b(t^-)I_a(t). \tag{47}$$

Integrating (46), then taking conditional expectation yields

$$p(t) = -\tilde{E}[(x(T) - E^{\tilde{\mu}}[x(T)])^2 | \mathcal{F}_t] + \tilde{E}\left[\int_t^T x(s) E^{\tilde{\mu}}[q_{ba}(t) I_b(s^-)] ds | \mathcal{F}_t\right].$$

Therefore,

$$\Delta p(t) = -\Delta \tilde{E}[(x(T) - E^{\tilde{\mu}}[x(T)])^2 | \mathcal{F}_t]. \tag{48}$$

Next, we compute the right hand side of (48), then we identify q_{ba} by matching. Set $\tilde{\mu}(t) := E^{\tilde{\mu}}[x(t)]$ and $\phi(t, x(t)) := (x(t) - \tilde{\mu}(t))^2$. Under \tilde{P} , Dynkin's formula yields

$$\phi(T, x(T)) = \phi(t, x(t)) + \int_t^T \left(\frac{\partial \phi}{\partial s} + \tilde{G}\phi\right)(s, x(s)) ds + \tilde{M}_T^\phi - \tilde{M}_t^\phi.$$

Taking conditional expectation yields

$$\tilde{E}[\phi(T, x(T)) | \mathcal{F}_t] = \phi(t, x(t)) + \int_t^T \tilde{E}\left[\left(\frac{\partial \phi}{\partial s} + \tilde{G}\phi\right)(s, x(s)) \mid \mathcal{F}_t\right] ds$$

and

$$\Delta \tilde{E}[\phi(T, x(T)) | \mathcal{F}_t] = \Delta \phi(t, x(t)),$$

where

$$\begin{aligned} \Delta \phi(t, x(t)) &= \sum_{i,j: i \neq j} (\phi(t, j) - \phi(t, i)) I_i(t^-) I_j(t) \quad (i, j \in \{a, b\}) \\ &= \left((b^2 - a^2) - 2\tilde{\mu}(t)(b - a)\right) I_a(t^-) I_b(t) + \left((a^2 - b^2) - 2\tilde{\mu}(t)(a - b)\right) I_b(t^-) I_a(t). \end{aligned}$$

Therefore,

$$\begin{aligned} \Delta p(t) &= -\Delta \tilde{E}[(x(T) - \tilde{\mu}(T))^2 | \mathcal{F}_t] \\ &= \left((a^2 - b^2) + 2\tilde{\mu}(t)(b - a)\right) I_a(t^-) I_b(t) + \left((b^2 - a^2) + 2\tilde{\mu}(t)(a - b)\right) I_b(t^-) I_a(t). \end{aligned} \tag{49}$$

Matching (47) with (49) yields

$$\begin{cases} q_{ab}(t) = (a^2 - b^2) + 2\tilde{\mu}(t)(b - a), \\ q_{ba}(t) = (b^2 - a^2) + 2\tilde{\mu}(t)(a - b). \end{cases}$$

Hence,

$$\tilde{u}(t) = \left((b^2 - a^2) + 2\tilde{\mu}(t)(a - b)\right) I_b(t^-).$$

Noting that $a \leq \tilde{\mu}(t) \leq b$, to guarantee that both $\lambda^{\tilde{u}}(t)$ and $\tilde{G}(t)$ above are indeed intensity matrices, it suffices to impose that

$$0 \leq \tilde{\mu}(t) \leq \frac{1}{2}(a + b). \tag{50}$$

We further characterize the optimal control $\tilde{u}(t)$ by finding $\tilde{\mu}(t)$ which satisfies (50). Indeed, under $P^{\tilde{\mu}}$, x has the representation

$$\begin{aligned} x(t) &= x(0) + \int_0^t \{\alpha(b - a) I_a(s^-) + \tilde{\mu}(s)(a - b) I_b(s^-) + \tilde{u}(s)(a - b) I_b(s^-)\} ds \\ &\quad + \int_0^t (b - a) dM_{ba}^{\tilde{\mu}}(s) + \int_0^t (a - b) dM_{ab}^{\tilde{\mu}}(s). \end{aligned}$$

Taking the expectation under $P^{\bar{\mu}}$ yields

$$\bar{\mu}(t) = \bar{\mu}(0) + E^{\bar{\mu}}\left[\int_0^t \{\alpha(b-a)I_a(s^-) + \bar{\mu}(s)(a-b)I_b(s^-) + (a-b)\bar{u}(s)I_b(s^-)\} ds\right]. \tag{51}$$

In particular, the mapping $t \rightarrow \mu(t)$ is absolutely continuous. Using the fact that $(a-b)I_b(t^-) = a - x(t^-)$ and $(b-a)I_a(t^-) = b - x(t^-)$, Equation (51) becomes

$$\begin{aligned} \bar{\mu}(t) &= \bar{\mu}(0) + \int_0^t \{\alpha(b - \bar{\mu}(s)) + \bar{\mu}(s)(a - \bar{\mu}(s))\} ds + \int_0^t \{E^{\bar{\mu}}[(a-b)\bar{u}(s)I_b(s^-)]\} ds \\ &= \bar{\mu}(0) + \int_0^t \{\alpha(b - \bar{\mu}(s)) + \bar{\mu}(s)(a - \bar{\mu}(s))\} ds \\ &\quad + \int_0^t \left\{E^{\bar{\mu}}\left[(a-b)I_b(s^-) \left((b^2 - a^2) + 2\bar{\mu}(s)(a-b)\right)\right]\right\} ds \\ &= \bar{\mu}(0) + \int_0^t \{\alpha(b - \bar{\mu}(s)) + \bar{\mu}(s)(a - \bar{\mu}(s))\} ds \\ &\quad + \int_0^t \left\{(b^2 - a^2)(a - \bar{\mu}(s)) + 2(a-b)\bar{\mu}(s)(a - \bar{\mu}(s))\right\} ds \\ &= \bar{\mu}(0) + \int_0^t \left\{(\alpha b + a(b^2 - a^2)) + (2(b-a) - 1)\bar{\mu}^2(s) + (3a^2 + a(1 - 2b) - b^2)\bar{\mu}(s)\right\} ds, \end{aligned}$$

with

$$\begin{cases} A := 2(b-a) - 1, \\ B := 3a^2 + a(1 - 2b) - b^2, \\ C := \alpha b + a(b^2 - a^2). \end{cases}$$

Thus, in view (50), $\bar{\mu}$ should satisfy the following constrained Riccati equation:

$$\begin{cases} \dot{\bar{\mu}}(t) = A\bar{\mu}^2(t) + B\bar{\mu}(t) + C, \\ \bar{\mu}(0) = m_0, \\ 0 \leq \bar{\mu}(t) \leq \frac{1}{2}(a+b), \end{cases} \tag{52}$$

where m_0 is a given initial value. As is well-known, without the imposed constraint on $\bar{\mu}$, the Riccati equation admits an explicit solution that may explode in finite time unless the involved coefficients a, b, α and m_0 evolve within certain ranges. With the imposed constraint on $\bar{\mu}$, these ranges may become tighter. Below, we illustrate this with a few cases. As shown in Tables 1–5 below, for low values of α , the ODE (52) can be solved for any time. How low the intensity should be mainly depends on the size of b and $b - a$. The larger b is, the wider is the range for α for which the ODE is solvable. In particular, when $a = 0$ and $b = 1$, (52) is solvable for any time when $\alpha = 0.1, 0.2$. For greater values of α , the ODE violates the constraint proportionally “faster”.

The results also show that the initial conditions may affect the time horizon T . Starting with values reasonably close to $\frac{a+b}{2}$, the ODE (52) is solvable only for relatively shorter time horizons than when we start with values reasonably close to zero.

Table 1. Solvability of the ODE (52) with the states $a = 0$ and $b = 1$.

a	b	α	$T_{m_0=0}$	$T_{m_0=0.25}$
0	1	0.1	.	.
0	1	0.2	.	.
0	1	0.3	5.145	3.762
0	1	0.4	2.355	1.481
0	1	0.5	1.571	0.928
0	1	0.6	1.870	0.676
0	1	0.7	0.955	0.532
0	1	0.8	0.800	0.439
0	1	0.9	0.689	0.373
0	1	1	0.605	0.325
0	1	5	0.104	0.053
0	1	10	0.051	0.026

Table 2. Solvability of the ODE (52) with the states $a = 1$ and $b = 2$.

a	b	α	$T_{m_0=0.25}$	$T_{m_0=1}$
1	2	0.1	.	.
1	2	0.2	.	.
1	2	0.3	.	.
1	2	0.4	2.644	2.153
1	2	0.5	1.429	1.001
1	2	0.6	1.073	0.692
1	2	0.7	0.878	0.535
1	2	0.8	0.750	0.438
1	2	0.9	0.659	0.371
1	2	1	0.589	0.322
1	2	5	0.121	0.053
1	2	10	0.062	0.026

Table 3. Solvability of the ODE (52) with the states $a = 2$ and $b = 3$.

a	b	α	$T_{m_0=0.25}$	$T_{m_0=2}$
2	3	0.1	.	.
2	3	0.2	.	.
2	3	0.3	.	.
2	3	0.4	.	.
2	3	0.5	1.206	0.761
2	3	0.6	0.899	0.494
2	3	0.7	0.746	0.373
2	3	0.8	0.648	0.302
2	3	0.9	0.578	0.254
2	3	1	0.524	0.220
2	3	5	0.131	0.035
2	3	10	0.070	0.018

Table 4. Solvability of the ODE (52) with the states $a = 0$ and $b = 2$.

a	b	α	$T_{m_0=0}$	$T_{m_0=0.75}$
0	2	0.1	.	.
0	2	0.2	.	.
0	2	0.3	.	.
0	2	0.4	.	.
0	2	0.5	.	.
0	2	0.6	.	.
0	2	0.7	5.593	1.433
0	2	0.8	2.227	0.636
0	2	0.9	1.470	0.418
0	2	1	1.111	0.312
0	2	5	0.112	0.029
0	2	10	0.053	0.014

Table 5. Solvability of the ODE (52) with the states $a = 0$ and $b = 3$.

a	b	α	$T_{m_0=0}$	$T_{m_0=1}$
0	3	0.1	.	.
0	3	0.2	.	.
0	3	0.3	.	.
0	3	0.4	.	.
0	3	0.5	.	.
0	3	0.6	.	.
0	3	0.7	.	.
0	3	0.8	.	.
0	3	0.9	.	.
0	3	1	.	.
0	3	5	0.126	0.043
0	3	10	0.056	0.019

Example 3. Mean-Field Schlöggl Model.

We suggest to solve a control problem associated with a mean-field version of the Schlöggl model (cf. [3,9,10,26]) where the intensities are of the form

$$\lambda_{ij}^u(t, x, u(t)) := \begin{cases} v_{ij}(t) & \text{if } j \neq i - 1, \\ u(t) + \beta E^u[x(t)] & \text{if } j = i - 1 \end{cases} \tag{53}$$

for some predictable and positive control process u , where $\beta > 0$ and $(\alpha_{ij})_{ij}$ is a deterministic Q-matrix for which there exists $N_0 \geq 1$ such that $\alpha_{ij} = 0$ for $|j - i| \geq N_0$ and $\alpha_{ij} > 0$ for $|j - i| < N_0$.

We consider the following mean field-type cost functional

$$J(u) = E^u \left[\int_0^T \frac{1}{2} u^2(t) dt + x(T) \right]. \tag{54}$$

Given a control $v > 0$, the associated Hamiltonian reads

$$H(t, v) := L^u(t) \left(\sum_{i,j,j \neq i-1} \{I_i(t^-)(\alpha_{ij} - g_{ij})q_{ij}(t)\} + \sum_i \{I_i(t^-)(v + \beta_1 E^u[x(t)] - g_{i,i-1})q_{i,i-1}(t)\} - \frac{v^2}{2} \right). \tag{55}$$

The first-order optimality conditions yield

$$\bar{u}(t) = \sum_i I_i(t^-)q_{i,i-1}(t). \tag{56}$$

Next, we write the associated adjoint equation and identify q_{ii-1} .

$$\begin{cases} dp(t) = -\sum_i q_{ii-1}(t) \left\{ I_i(t^-) \left(\frac{1}{2} q_{ii-1}(t) + \beta E^{\bar{u}}[x(t)] - g_{ii-1} \right) dt - dM_{ii-1}(t) \right\} \\ \quad - \sum_{ij, j \neq i, i-1} q_{ij}(t) \left\{ I_i(t^-) (v_{ij} - g_{ij}) dt - dM_{ij}(t) \right\} - \beta x(t) E^{\bar{u}} \left[\sum_i q_{ii-1}(t) I_i(t^-) \right] dt, \\ p(T) = -x(T). \end{cases}$$

Consider the probability measure \tilde{P} , under which x is a pure jump process whose jump intensity matrix is

$$\tilde{G}_{ij}(t) = \begin{cases} \alpha_{ij} & \text{if } j \neq i - 1, \\ \frac{1}{2} q_{ij}(t) + \beta E^{\bar{u}}[x(t)] & \text{if } j = i - 1. \end{cases}$$

The adjoint equations become

$$\begin{cases} dp(t) = -\beta x(t) E^{\bar{u}} \sum_i [q_{ii-1}(t) I_i(t^-)] dt + \sum_i q_{ii-1}(t) d\tilde{M}_{ii-1}(t) + \sum_{i \neq j, j \neq i-1} q_{ij}(t) d\tilde{M}_{ij}(t), \\ p(T) = -x(T), \end{cases}$$

where \tilde{M}_{ij} , $i \neq j$ are mutually orthogonal \tilde{P} -martingales.

Thus

$$p(t) = -\tilde{E}[x(T)|\mathcal{F}_t] + \beta \int_t^T \tilde{E} \left[x(s) E^{\bar{u}} \left[\sum_i q_{ii-1} I_i(s^-) \right] | \mathcal{F}_t \right] ds,$$

and

$$\Delta p(t) = \Delta \tilde{E}[-x(T)|\mathcal{F}_t]. \tag{57}$$

Following the same steps leading to (42), from (57), we obtain $q_{ij}(t) = i - j$, thus $q_{ii-1}(t) = 1$, $i = 1, 2, \dots$

Therefore,

$$\bar{u}(t) = \sum_{i \geq 1} I_i(t^-) = 1 - I_0(t^-).$$

Final remark. There are more real-world Markov chain mean-field control problems that we can apply our results to, such as the mean-field model proposed in [27] for malware propagation over computer networks where the nodes/devices interact through network-based opportunistic meetings (e.g., via internet, email, USB keys). Markov chains of mean-field type can be used to model the state of the devices, which can be either Susceptible/Honest (H), Infected (I), or Dormant (D). The fact that the jump intensities may depend nonlinearly on the mean-field coupling makes it more involved to have closed-form expressions even when considering linear quadratic cost functionals.

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Appendix A

Proof of Proposition 1. Below, C_ϕ denotes the bound of $\phi \in \{\lambda, \kappa, h, f\}$.

Uniqueness.

Assume (p, q) and (\bar{p}, \bar{q}) are two solutions of (27) which satisfy (28). Set $y(t) := p(t) - \bar{p}(t)$, $z(t) := q(t) - \bar{q}(t)$ and

$$\gamma(s) := \langle \bar{\ell}(s), z(s) \rangle_g + \kappa_\ell(x(s))E[L^{\bar{u}}(s)\langle \bar{\ell}_{\bar{y}}(s), z(s) \rangle_g],$$

we have

$$y(t) = \int_t^T \gamma(s)ds - \int_t^T z(s)dM(s),$$

where

$$E\left[\sum_{t < s \leq T} |z(s)\Delta M(s)|^2\right] = E\left[\int_t^T \|z(s)\|_g^2 ds\right]. \tag{A1}$$

We can apply the Itô–Stieltjes chain rule to obtain

$$|y(t)|^2 = 2 \int_t^T y(s^-)\gamma(s)ds - 2 \int_t^T y(s^-)z(s)dM(s) - \sum_{t < s \leq T} |\Delta p(s) - \Delta \bar{p}(s)|^2.$$

Since $\Delta p(s) = q(s)\Delta M(s)$, $\Delta \bar{p}(s) = \bar{q}(s)\Delta M(s)$, we have

$$|y(t)|^2 = 2 \int_t^T y(s^-)\gamma(s)ds - 2 \int_t^T y(s^-)z(s)dM(s) - \sum_{t < s \leq T} |z(s)\Delta M(s)|^2.$$

Now, since (y, z) satisfies (28), by Doob’s inequality the stochastic integral $\int_0^t y(s^-)z(s)dM(s)$ is in fact a uniformly integrable martingale. Therefore,

$$\begin{aligned} E[|y(t)|^2] &= 2 \int_t^T E[y(s^-)\gamma(s)]ds - E\left[\sum_{t < s \leq T} |z(s)\Delta M(s)|^2\right] \\ &= 2 \int_t^T E[y(s^-)\gamma(s)]ds - \int_t^T E[\|z(s)\|_g^2]ds. \end{aligned} \tag{by (A1)}$$

We can now just use Young’s inequality

$$2ab \leq \epsilon a^2 + \frac{b^2}{\epsilon}$$

to get

$$2y(s^-)\gamma(s) \leq \epsilon |y(s^-)|^2 + \frac{\gamma^2(s)}{\epsilon}.$$

We have

$$\begin{aligned} \gamma^2(s) &\leq 2(|\langle \bar{\ell}(s), z(s) \rangle_g|^2 + |\kappa_\ell(x(s))|^2(E[L^{\bar{u}}(s)\langle \bar{\ell}_{\bar{y}}(s), z(s) \rangle_g])^2) \\ &\leq 2(\|\bar{\ell}(s)\|_g^2 \|z(s)\|_g^2 + C_\kappa(E[L^{\bar{u}}(s)\|\bar{\ell}_{\bar{y}}(s)\|_g\|z(s)\|_g])^2) \\ &\leq 2C(\|z(s)\|_g^2 + C_\kappa E[(L^{\bar{u}}(s))^2]E[\|z(s)\|_g^2]) \tag{by (29) and Cauchy–Schwarz} \\ &\leq (2C + 2C_\kappa C\tilde{C})(\|z(s)\|_g^2 + E[\|z(s)\|_g^2]) \tag{by (30)}. \end{aligned}$$

By setting $\hat{C} := 2(C + C_\kappa C\tilde{C})$, we have

$$\gamma^2(s) \leq \hat{C}(\|z(s)\|_g^2 + E[\|z(s)\|_g^2]). \tag{A2}$$

Therefore,

$$E[|y(t)|^2] \leq E[\int_t^T (\epsilon|y(s)|^2 + (\frac{2\hat{C}}{\epsilon} - 1)\|z(s)\|_g^2)ds],$$

choose ϵ s.t. $\frac{2\hat{C}}{\epsilon} - 1 = -\frac{1}{2}$, i.e., $\epsilon = 4\hat{C}$, to get

$$E[|y(t)|^2] + \frac{1}{2}E[\int_t^T \|z(s)\|_g^2 ds] \leq \epsilon E[\int_t^T |y(s)|^2 ds],$$

and by Gronwall’s inequality we obtain $y = 0$ P -a.s. Moreover,

$$E[\int_t^T \|z(s)\|_g^2 ds] \leq 2\epsilon E[\int_t^T |y(s)|^2 ds] = 0,$$

implying that $z = 0$ $dP \times g_{ij}dt$.

Existence.

We use the Picard iteration as follows: set $\zeta := -\bar{h}(T) - k_h(x(T))E[L^{\bar{u}}(T)\bar{h}_y(T)]$ and

$$F(q(t), \tilde{q}(t)) := \langle \bar{\ell}(t), q(t) \rangle_g - \bar{f}(t) + \kappa_\ell(x(t))E[L^{\bar{u}}(t)\langle \bar{\ell}_y(t), \tilde{q}(t) \rangle_g] - \kappa_f(x(t))E[L^{\bar{u}}(t)\bar{f}_y(t)].$$

By assumption (A2), $|\zeta| \leq C$ P -a.s. Moreover, set $(p^0(t), q^0(t)) = (0, 0)$ and consider the BSDE

$$p^{n+1}(t) = \zeta + \int_t^T F(q^{n+1}(s), q^n(s))ds - \int_t^T q^{n+1}(s)dM(s).$$

Given $q^n; (p^{n+1}, q^{n+1})$ solves a standard linear BSDE whose existence and uniqueness is guaranteed (thanks to (A1) and (A2)) by Theorem 5.1 of [17]. We have

$$p^{n+1}(t) - p^n(t) = \int_t^T F(q^{n+1}(s), q^n(s)) - F(q^n(s), q^{n-1}(s))ds - \int_t^T q^{n+1}(s) - q^n(s)dM(s). \tag{A3}$$

In particular,

$$E[\sup_{0 \leq t \leq T} |p^n(t)| + \int_0^T \|q^n(s)\|_g^2 ds] \leq \infty. \tag{A4}$$

Applying a similar estimate as (A2) we obtain

$$|F(q^{n+1}(t), q^n(t)) - F(q^n(t), q^{n-1}(t))|^2 \leq \hat{C}\|q^{n+1}(t) - q^n(t)\|_g^2 + \hat{C}E[\|q^n(t) - q^{n-1}(t)\|_g^2].$$

Set $y^n(t) := p^n(t) - p^{n-1}(t)$, $z^n(t) := q^n(t) - q^{n-1}(t)$ and apply the Itô–Stieltjes chain rule to $e^{\beta t}|y^n(t)|^2$ to get

$$E[e^{\beta t}|y^{n+1}(t)|^2] + E[\int_t^T \beta e^{\beta s}|y^{n+1}(s)|^2 ds] + E[\int_t^T e^{\beta s}\|z^{n+1}(s)\|_g^2 ds] = E[\int_t^T e^{\beta s}|y^{n+1}(s)|(F(q^{n+1}(s), q^n(s)) - F(q^n(s), q^{n-1}(s)))ds].$$

Applying Young's inequality, we obtain

$$\begin{aligned} E[e^{\beta t} |y^{n+1}(t)|^2] + E\left[\int_t^T \beta e^{\beta s} |y^{n+1}(s)|^2 ds\right] + E\left[\int_t^T e^{\beta s} \|z^{n+1}(s)\|_g^2 ds\right] \\ \leq \epsilon E\left[\int_t^T e^{\beta s} |y^{n+1}(s)|^2 ds\right] + \frac{\hat{C}}{\epsilon} E\left[\int_t^T e^{\beta s} \|z^n(s)\|_g^2 ds\right] \\ + \frac{\hat{C}}{\epsilon} E\left[\int_t^T e^{\beta s} \|z^{n+1}(s)\|_g^2 ds\right] \end{aligned}$$

or

$$\begin{aligned} E[e^{\beta t} |y^{n+1}(t)|^2] + (\beta - \epsilon) E\left[\int_t^T \beta e^{\beta s} |y^{n+1}(s)|^2 ds\right] + \left(1 - \frac{\hat{C}}{\epsilon}\right) E\left[\int_t^T e^{\beta s} \|z^{n+1}(s)\|_g^2 ds\right] \\ \leq \frac{\hat{C}}{\epsilon} E\left[\int_t^T e^{\beta s} \|z^n(s)\|_g^2 ds\right]. \end{aligned}$$

Choose $\epsilon = 3\hat{C}$, $\beta = 3\hat{C} + 1$ to get

$$\begin{aligned} E\left[\int_0^T e^{\beta s} (|y^{n+1}(s)|^2 + \|z^{n+1}(s)\|_g^2) ds\right] \leq \frac{1}{2} E\left[\int_0^T e^{\beta s} \|z^n(s)\|_g^2 ds\right] \\ \leq \frac{1}{2} E\left[\int_0^T e^{\beta s} (|y^n(s)|^2 + \|z^n(s)\|_g^2) ds\right]. \end{aligned}$$

This shows that (p^n, q^n) is a Cauchy sequence w.r.t the norm $\|(y, z)\|_\beta = E[\int_0^T e^{\beta s} (|y(s)|^2 + \|z(s)\|_g^2) ds]$. Taking the limit $(p = \lim_{n \rightarrow \infty} p^n, q = \lim_{n \rightarrow \infty} q^n)$ in (A3), it holds that (p, q) solves (27) as a fixed point and thus satisfies (28) (cf. [23]). \square

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