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Supplement to “A persistence-based Wold-type decomposition for stationary time series”

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A. ONLINE SUPPLEMENT

A.1 Proofs

The notation employed here is taken from Section 2.1. Lemma A.1 is preparatory for the proof of Theorem 1.

LEMMA A.1. *Let ε be a unit variance white noise. The Hilbert space $\mathcal{H}_t(\varepsilon)$ decomposes into the orthogonal sum $\mathcal{H}_t(\varepsilon) = \bigoplus_{j=1}^{\infty} \mathbf{R}^{j-1} \mathcal{L}_t^{\mathbf{R}}$, where*

$$\mathbf{R}^{j-1} \mathcal{L}_t^{\mathbf{R}} = \left\{ \sum_{k=0}^{+\infty} b_k^{(j)} \varepsilon_{t-k2^j}^{(j)} \in \mathcal{H}_t(\varepsilon) : b_k^{(j)} \in \mathbb{R} \right\}$$

and, for any $j \in \mathbb{N}$ and $t \in \mathbb{Z}$, $\varepsilon_t^{(j)}$ is given by equation (6).

PROOF. $\mathcal{H}_t(\varepsilon)$ is a Hilbert subspace of $L^2(\Omega, \mathcal{F}, \mathbb{P})$, equipped with the inner product $\langle A, B \rangle = \mathbb{E}[AB]$ for all $A, B \in L^2(\Omega, \mathcal{F}, \mathbb{P})$. We begin with showing that the scaling operator \mathbf{R} is well-defined, linear, and isometric on $\mathcal{H}_t(\varepsilon)$.

Consider any $X = \sum_{k=0}^{\infty} a_k \varepsilon_{t-k}$ in $\mathcal{H}_t(\varepsilon)$, that is, $\|X\|^2 = \sum_{p=0}^{\infty} a_p^2 < +\infty$. Then

$$\|\mathbf{R}X\|^2 = \frac{1}{2} \sum_{k=0}^{+\infty} a_{\lfloor \frac{k}{2} \rfloor}^2 = \frac{1}{2} \sum_{p=0}^{+\infty} a_{\lfloor \frac{2p}{2} \rfloor}^2 + \frac{1}{2} \sum_{p=0}^{+\infty} a_{\lfloor \frac{2p+1}{2} \rfloor}^2 = \sum_{p=0}^{+\infty} a_p^2 = \|X\|^2$$

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and this quantity is finite. As a result, \mathbf{R} is a well-defined (and bounded) operator. The linearity of \mathbf{R} is immediate. To prove that \mathbf{R} is isometric, take any $X = \sum_{k=0}^{\infty} a_k \varepsilon_{t-k}$, $Y = \sum_{h=0}^{\infty} b_h \varepsilon_{t-h}$ in $\mathcal{H}_t(\boldsymbol{\varepsilon})$. By the white-noise properties of $\boldsymbol{\varepsilon}$,

$$\begin{aligned} \langle \mathbf{R}X, \mathbf{R}Y \rangle &= \sum_{k=0}^{+\infty} \frac{a_{\lfloor \frac{k}{2} \rfloor}}{\sqrt{2}} \frac{b_{\lfloor \frac{k}{2} \rfloor}}{\sqrt{2}} = \frac{1}{2} \sum_{p=0}^{+\infty} a_{\lfloor \frac{2p}{2} \rfloor} b_{\lfloor \frac{2p}{2} \rfloor} + \frac{1}{2} \sum_{p=0}^{+\infty} a_{\lfloor \frac{2p+1}{2} \rfloor} b_{\lfloor \frac{2p+1}{2} \rfloor} \\ &= \sum_{p=0}^{+\infty} a_p b_p = \langle X, Y \rangle. \end{aligned}$$

As a result, \mathbf{R} is an isometry on $\mathcal{H}_t(\boldsymbol{\varepsilon})$ and the abstract Wold theorem (i.e., Theorem 1.1 in Nagy, Foias, Bercovici, and Kérchy (2010)) applies.

The abstract Wold theorem supplies the orthogonal decomposition $\mathcal{H}_t(\boldsymbol{\varepsilon}) = \hat{\mathcal{H}}_t(\boldsymbol{\varepsilon}) \oplus \tilde{\mathcal{H}}_t(\boldsymbol{\varepsilon})$, where

$$\hat{\mathcal{H}}_t(\boldsymbol{\varepsilon}) = \bigcap_{j=0}^{+\infty} \mathbf{R}^j \mathcal{H}_t(\boldsymbol{\varepsilon}), \quad \tilde{\mathcal{H}}_t(\boldsymbol{\varepsilon}) = \bigoplus_{j=1}^{+\infty} \mathbf{R}^{j-1} \mathcal{L}_t^{\mathbf{R}}$$

and $\mathcal{L}_t^{\mathbf{R}} = \mathcal{H}_t(\boldsymbol{\varepsilon}) \ominus \mathbf{R}\mathcal{H}_t(\boldsymbol{\varepsilon})$ is called *wandering subspace*.

In particular, we show that $\hat{\mathcal{H}}_t(\boldsymbol{\varepsilon})$ is the null subspace. Indeed, the subspaces $\mathbf{R}^j \mathcal{H}_t(\boldsymbol{\varepsilon})$ are made of linear combinations of innovations ε_t with coefficients equal to each others 2^j -by- 2^j :

$$\mathbf{R}^j \mathcal{H}_t(\boldsymbol{\varepsilon}) = \left\{ \sum_{k=0}^{+\infty} c_k^{(j)} \left(\sum_{i=0}^{2^j-1} \varepsilon_{t-k2^j-i} \right) \in \mathcal{H}_t(\boldsymbol{\varepsilon}) : c_k^{(j)} \in \mathbb{R} \right\}.$$

As a result, $\hat{\mathcal{H}}_t(\boldsymbol{\varepsilon})$ can just include variables as $\sum_{h=0}^{\infty} c \varepsilon_{t-h}$ with $c \in \mathbb{R}$. These elements belong to $\mathcal{H}_t(\boldsymbol{\varepsilon})$, hence $\sum_{k=0}^{\infty} c^2$ is finite and this is possible just in case $c = 0$. As a result, $\hat{\mathcal{H}}_t(\boldsymbol{\varepsilon}) = \{0\}$ and $\mathcal{H}_t(\boldsymbol{\varepsilon}) = \tilde{\mathcal{H}}_t(\boldsymbol{\varepsilon})$.

We now focus on the subspace $\tilde{\mathcal{H}}_t(\boldsymbol{\varepsilon})$. As the orthogonal complement of $\mathbf{R}\mathcal{H}_t(\boldsymbol{\varepsilon})$ is the kernel of the adjoint operator \mathbf{R}^* (see, e.g., Theorem 1, Section 6.6 in Luenberger (1968)), we determine \mathbf{R}^* . Specifically, $\mathbf{R}^* : \mathcal{H}_t(\boldsymbol{\varepsilon}) \rightarrow \mathcal{H}_t(\boldsymbol{\varepsilon})$ is such that

$$\mathbf{R}^* : \sum_{k=0}^{+\infty} a_k \varepsilon_{t-k} \mapsto \sum_{k=0}^{+\infty} \frac{a_{2k} + a_{2k+1}}{\sqrt{2}} \varepsilon_{t-k}.$$

Indeed, \mathbf{R}^* is well-defined and the relation $\langle \mathbf{R}X, Y \rangle = \langle X, \mathbf{R}^*Y \rangle$ holds for any $X = \sum_{h=0}^{\infty} b_h \varepsilon_{t-h}$, $Y = \sum_{k=0}^{\infty} a_k \varepsilon_{t-k}$ in $\mathcal{H}_t(\boldsymbol{\varepsilon})$, due to the white noise nature of $\boldsymbol{\varepsilon}$:

$$\begin{aligned} \langle \mathbf{R}X, Y \rangle &= \sum_{h=0}^{+\infty} \sum_{k=0}^{+\infty} \frac{b_{\lfloor \frac{h}{2} \rfloor}}{\sqrt{2}} a_k \langle \varepsilon_{t-h}, \varepsilon_{t-k} \rangle = \sum_{k=0}^{+\infty} b_{\lfloor \frac{k}{2} \rfloor} \frac{a_k}{\sqrt{2}} = \sum_{k=0}^{+\infty} b_k \frac{a_{2k} + a_{2k+1}}{\sqrt{2}} \\ &= \sum_{h=0}^{+\infty} \sum_{k=0}^{+\infty} b_h \frac{a_{2k} + a_{2k+1}}{\sqrt{2}} \langle \varepsilon_{t-h}, \varepsilon_{t-k} \rangle = \langle X, \mathbf{R}^*Y \rangle. \end{aligned}$$

As for the kernel of \mathbf{R}^* , we prove that

$$\ker(\mathbf{R}^*) = \left\{ \sum_{k=0}^{+\infty} d_k^{(1)} (\varepsilon_{t-2k} - \varepsilon_{t-2k-1}) \in \mathcal{H}_t(\boldsymbol{\varepsilon}) : d_k^{(1)} \in \mathbb{R} \right\}.$$

Take any element of $\mathcal{H}_t(\boldsymbol{\varepsilon})$ of the kind $X = \sum_{k=0}^{\infty} d_k^{(1)} (\varepsilon_{t-2k} - \varepsilon_{t-2k-1})$ for some square-summable sequence of real numbers $\{d_k^{(1)}\}_k$. Such X can be rewritten as $X = \sum_{h=0}^{\infty} a_h \varepsilon_{t-h}$ with $a_{2k+1} = -a_{2k}$ for all $k \in \mathbb{N}_0$, that is $a_{2k} + a_{2k+1} = 0$. Therefore, by the expression of \mathbf{R}^* , we realize that $\mathbf{R}^*X = 0$. Thus,

$$\left\{ \sum_{k=0}^{+\infty} d_k^{(1)} (\varepsilon_{t-2k} - \varepsilon_{t-2k-1}) \in \mathcal{H}_t(\boldsymbol{\varepsilon}) : d_k^{(1)} \in \mathbb{R} \right\} \subset \ker(\mathbf{R}^*). \quad (23)$$

Conversely, consider $X = \sum_{h=0}^{\infty} a_h \varepsilon_{t-h}$ in $\ker(\mathbf{R}^*)$. Since $\mathbf{R}^*X = 0$ in the L^2 -norm, $\sum_{k=0}^{\infty} (a_{2k} + a_{2k+1})^2 = 0$. As a consequence, $a_{2k+1} = -a_{2k}$ for any $k \in \mathbb{N}_0$ and we can write $X = \sum_{k=0}^{\infty} d_k^{(1)} (\varepsilon_{t-2k} - \varepsilon_{t-2k-1})$ with $d_k^{(1)} = a_{2k}$. As a result, also the converse inclusion in (23) holds and

$$\mathcal{L}_t^{\mathbf{R}} = \ker(\mathbf{R}^*) = \left\{ \sum_{k=0}^{+\infty} b_k^{(1)} \varepsilon_{t-2k} \in \mathcal{H}_t(\boldsymbol{\varepsilon}) : b_k^{(1)} \in \mathbb{R} \right\}.$$

In addition,

$$\mathbf{R}\mathcal{L}_t^{\mathbf{R}} = \left\{ \sum_{k=0}^{+\infty} b_k^{(2)} \varepsilon_{t-4k} \in \mathcal{H}_t(\boldsymbol{\varepsilon}) : b_k^{(2)} \in \mathbb{R} \right\}$$

and, for any $j \in \mathbb{N}$,

$$\mathbf{R}^{j-1}\mathcal{L}_t^{\mathbf{R}} = \left\{ \sum_{k=0}^{+\infty} b_k^{(j)} \varepsilon_{t-k2j} \in \mathcal{H}_t(\boldsymbol{\varepsilon}) : b_k^{(j)} \in \mathbb{R} \right\}.$$

As the case with $j \in \mathbb{N}$ can be derived by induction, we focus on $\mathbf{R}\mathcal{L}_t^{\mathbf{R}}$ and show that

$$\mathbf{R}\mathcal{L}_t^{\mathbf{R}} = \left\{ \sum_{k=0}^{+\infty} d_k^{(2)} (\varepsilon_{t-4k} + \varepsilon_{t-4k-1} - \varepsilon_{t-4k-2} - \varepsilon_{t-4k-3}) \in \mathcal{H}_t(\boldsymbol{\varepsilon}) : d_k^{(2)} \in \mathbb{R} \right\}. \quad (24)$$

Consider any $Y \in \mathbf{R}\mathcal{L}_t^{\mathbf{R}}$. Since Y is the image of some $X \in \mathcal{L}_t^{\mathbf{R}}$, there exists a square-summable sequence of real numbers $\{d_k^{(1)}\}_k$ such that

$$X = \sum_{k=0}^{+\infty} d_k^{(1)} (\varepsilon_{t-2k} - \varepsilon_{t-2k-1}), \quad Y = \sum_{k=0}^{+\infty} \frac{d_k^{(1)}}{\sqrt{2}} (\varepsilon_{t-4k} + \varepsilon_{t-4k-1} - \varepsilon_{t-4k-2} - \varepsilon_{t-4k-3}).$$

As a result, $\mathbf{R}\mathcal{L}_t^{\mathbf{R}}$ is included in the set in (24). Vice versa, take any $Y = \sum_{k=0}^{\infty} d_k^{(2)} (\varepsilon_{t-4k} + \varepsilon_{t-4k-1} - \varepsilon_{t-4k-2} - \varepsilon_{t-4k-3})$ for some square-summable sequence of coefficients $\{d_k^{(2)}\}_k$. Then Y belongs to $\mathbf{R}\mathcal{L}_t^{\mathbf{R}}$, too, because it is the image of $X = \sum_{k=0}^{\infty} \sqrt{2}d_k^{(2)} (\varepsilon_{t-2k} - \varepsilon_{t-2k-1})$, which belongs to $\mathcal{L}_t^{\mathbf{R}}$. Therefore, the characterization in (24) is assessed. \square

Proof of Theorem 1

PROOF. By applying the classical Wold decomposition to the zero-mean, weakly stationary purely nondeterministic process \mathbf{x} , we find that x_t belongs to the Hilbert space $\mathcal{H}_t(\boldsymbol{\varepsilon})$, where $\boldsymbol{\varepsilon}$ is the unit variance white noise of classical Wold innovations of \mathbf{x} . Importantly, $\mathcal{H}_t(\boldsymbol{\varepsilon})$ orthogonally decomposes as in Lemma A.1. By denoting $g_t^{(j)}$ the orthogonal projections of x_t on the subspaces $\mathbf{R}^{j-1}\mathcal{L}_t^{\mathbf{R}}$, we find that $x_t = \sum_{j=1}^{\infty} g_t^{(j)}$, where the equality is in the L^2 -norm. Then, by using the characterizations of subspaces $\mathbf{R}^{j-1}\mathcal{L}_t^{\mathbf{R}}$, for any scale $j \in \mathbb{N}$ we find a square-summable sequence of real coefficients $\{\beta_k^{(j)}\}_k$ such that equation (9) holds. As a result, we are allowed to decompose the variable x_t as in equation (5).

We now show (i). As we can see in equation (6), the process $\boldsymbol{\varepsilon}^{(j)}$ is an MA($2^j - 1$) with respect to the fundamental innovations $\boldsymbol{\varepsilon}$. In addition, the subprocess $\{\boldsymbol{\varepsilon}_{t-k2^j}^{(j)}\}_{k \in \mathbb{Z}}$ is weakly stationary. Indeed, since $\boldsymbol{\varepsilon}$ is a unit variance white noise, for any $k \in \mathbb{Z}$,

$$\mathbb{E}[(\boldsymbol{\varepsilon}_{t-k2^j}^{(j)})^2] = \frac{1}{2^j} \mathbb{E} \left[\left(\sum_{i=0}^{2^{j-1}-1} \boldsymbol{\varepsilon}_{t-k2^j-i} - \sum_{i=0}^{2^{j-1}-1} \boldsymbol{\varepsilon}_{t-k2^j-2^{j-1}-i} \right)^2 \right] = \frac{1}{2^j} \sum_{i=0}^{2^j-1} \mathbb{E}[\boldsymbol{\varepsilon}_i^2] = 1.$$

Thus, $\mathbb{E}[(\boldsymbol{\varepsilon}_{t-k2^j}^{(j)})^2]$ is finite and it does not depend on k . Moreover, $\mathbb{E}[\boldsymbol{\varepsilon}_{t-k2^j}^{(j)}] = 0$ for any $k \in \mathbb{Z}$ and the expectation does not depend on k . Finally, we analyze cross-moments in the support $S_t^{(j)} = \{t - k2^j : k \in \mathbb{N}_0\}$. By taking $h \neq k$,

$$\begin{aligned} & \mathbb{E}[\boldsymbol{\varepsilon}_{t-h2^j}^{(j)} \boldsymbol{\varepsilon}_{t-k2^j}^{(j)}] \\ &= \frac{1}{2^j} \mathbb{E} \left[\left(\sum_{i=0}^{2^{j-1}-1} \boldsymbol{\varepsilon}_{t-h2^j-i} - \sum_{i=0}^{2^{j-1}-1} \boldsymbol{\varepsilon}_{t-h2^j-2^{j-1}-i} \right) \right. \\ & \quad \cdot \left. \left(\sum_{l=0}^{2^{j-1}-1} \boldsymbol{\varepsilon}_{t-k2^j-l} - \sum_{l=0}^{2^{j-1}-1} \boldsymbol{\varepsilon}_{t-k2^j-2^{j-1}-l} \right) \right] \\ &= \frac{1}{2^j} \left\{ \sum_{i=0}^{2^{j-1}-1} \sum_{l=0}^{2^{j-1}-1} \mathbb{E}[\boldsymbol{\varepsilon}_{t-h2^j-i} \boldsymbol{\varepsilon}_{t-k2^j-l}] - \sum_{i=0}^{2^{j-1}-1} \sum_{l=0}^{2^{j-1}-1} \mathbb{E}[\boldsymbol{\varepsilon}_{t-h2^j-i} \boldsymbol{\varepsilon}_{t-k2^j-2^{j-1}-l}] \right. \\ & \quad \left. - \sum_{i=0}^{2^{j-1}-1} \sum_{l=0}^{2^{j-1}-1} \mathbb{E}[\boldsymbol{\varepsilon}_{t-h2^j-2^{j-1}-i} \boldsymbol{\varepsilon}_{t-k2^j-l}] + \sum_{i=0}^{2^{j-1}-1} \sum_{l=0}^{2^{j-1}-1} \mathbb{E}[\boldsymbol{\varepsilon}_{t-h2^j-2^{j-1}-i} \boldsymbol{\varepsilon}_{t-k2^j-2^{j-1}-l}] \right\}. \end{aligned}$$

Since $h \neq k$, the sets of indices $\{h2^j, \dots, h2^j + 2^j - 1\}$ and $\{k2^j, \dots, k2^j + 2^j - 1\}$ are disjoint and so the last sums are null. Therefore, $\mathbb{E}[\boldsymbol{\varepsilon}_{t-h2^j}^{(j)} \boldsymbol{\varepsilon}_{t-k2^j}^{(j)}] = 0$ for all $h \neq k$.

As a result, $\{\boldsymbol{\varepsilon}_{t-k2^j}^{(j)}\}_{k \in \mathbb{Z}}$ is weakly stationary on $S_t^{(j)}$ and it is a unit variance white noise.

We now turn to (ii). For any fixed scale $j \in \mathbb{N}$, since the variables $\boldsymbol{\varepsilon}_{t-k2^j}^{(j)}$ are orthonormal when k varies, the component $g_t^{(j)}$ has a unique representation as in equation (8). Thus, the coefficients $\beta_k^{(j)}$ are uniquely defined, and clearly, $\sum_{j=1}^{\infty} \sum_{k=0}^{\infty} (\beta_k^{(j)})^2$ is finite.

In order to find the explicit expression of coefficients $\beta_k^{(j)}$, we exploit the orthogonal decompositions of $\mathcal{H}_t(\boldsymbol{\varepsilon})$ at different scales $J \in \mathbb{N}$:

$$\mathcal{H}_t(\boldsymbol{\varepsilon}) = \mathbf{R}^J \mathcal{H}_t(\boldsymbol{\varepsilon}) \oplus \bigoplus_{j=1}^J \mathbf{R}^{j-1} \mathcal{L}_t^{\mathbf{R}}.$$

We call $\pi_t^{(j)}$ the orthogonal projection of x_t on the subspace $\mathbf{R}^j \mathcal{H}_t(\boldsymbol{\varepsilon})$ and we proceed inductively.

We start by the first decomposition $x_t = \pi_t^{(1)} + g_t^{(1)}$ coming from scale $J = 1$, namely $\mathcal{H}_t(\boldsymbol{\varepsilon}) = \mathbf{R} \mathcal{H}_t(\boldsymbol{\varepsilon}) \oplus \mathcal{L}_t^{\mathbf{R}}$. By the definitions of elements in $\mathbf{R} \mathcal{H}_t(\boldsymbol{\varepsilon})$ and $\mathcal{L}_t^{\mathbf{R}}$ described in Lemma A.1, we set

$$\begin{aligned} \pi_t^{(1)} &= \sum_{k=0}^{+\infty} \gamma_k^{(1)} \frac{\varepsilon_{t-2k} + \varepsilon_{t-(2k+1)}}{\sqrt{2}} = \sum_{k=0}^{+\infty} c_k^{(1)} (\varepsilon_{t-2k} + \varepsilon_{t-(2k+1)}), \\ g_t^{(1)} &= \sum_{k=0}^{+\infty} \beta_k^{(1)} \varepsilon_{t-2k} = \sum_{k=0}^{+\infty} d_k^{(1)} (\varepsilon_{t-2k} - \varepsilon_{t-2k-1}) \end{aligned}$$

for some sequences of coefficients $\{c_k^{(1)}\}_k$ and $\{d_k^{(1)}\}_k$, or equivalently $\{\gamma_k^{(1)}\}_k$ and $\{\beta_k^{(1)}\}_k$, to determine in order to have $x_t = \pi_t^{(1)} + g_t^{(1)}$, where we set $\sqrt{2}c_k^{(1)} = \gamma_k^{(1)}$ and $\sqrt{2}d_k^{(1)} = \beta_k^{(1)}$. The expressions above may be rewritten as

$$x_t = \sum_{k=0}^{+\infty} \{(c_k^{(1)} + d_k^{(1)})\varepsilon_{t-2k} + (c_k^{(1)} - d_k^{(1)})\varepsilon_{t-2k-1}\}.$$

However, from the classical Wold decomposition of \mathbf{x} ,

$$x_t = \sum_{k=0}^{+\infty} \{\alpha_{2k} \varepsilon_{t-2k} + \alpha_{2k+1} \varepsilon_{t-2k-1}\}$$

with the same fundamental innovations ε_t . By the uniqueness of writing of the classical Wold decomposition, the two expressions for x_t must coincide. As a result, $c_k^{(1)}$ and $d_k^{(1)}$ are the solutions of the linear system

$$\begin{cases} c_k^{(1)} + d_k^{(1)} = \alpha_{2k}, \\ c_k^{(1)} - d_k^{(1)} = \alpha_{2k+1}, \end{cases}$$

that is,

$$c_k^{(1)} = \frac{\alpha_{2k} + \alpha_{2k+1}}{2}, \quad d_k^{(1)} = \frac{\alpha_{2k} - \alpha_{2k+1}}{2}$$

and, in particular, we find

$$\gamma_k^{(1)} = \frac{\alpha_{2k} + \alpha_{2k+1}}{\sqrt{2}}, \quad \beta_k^{(1)} = \frac{\alpha_{2k} - \alpha_{2k+1}}{\sqrt{2}}.$$

Next, we focus on the scale $J = 2$. We exploit the decomposition of the space $\mathbf{R}\mathcal{H}_t(\boldsymbol{\varepsilon}) = \mathbf{R}^2\mathcal{H}_t(\boldsymbol{\varepsilon}) \oplus \mathbf{R}\mathcal{L}_t^{\mathbf{R}}$ that implies the relation $\pi_t^{(1)} = \pi_t^{(2)} + g_t^{(2)}$. We follow the same track as in the previous case, by using the features of elements in $\mathbf{R}^2\mathcal{H}_t(\boldsymbol{\varepsilon})$ and in $\mathbf{R}\mathcal{L}_t^{\mathbf{R}}$ and, finally, by comparing the expression of $\pi_t^{(2)} + g_t^{(2)}$ with the (unique) writing of $\pi_t^{(1)}$ that we found before. Since

$$\pi_t^{(2)} = \sum_{k=0}^{+\infty} \gamma_k^{(2)} \frac{\varepsilon_{t-4k} + \varepsilon_{t-(4k+1)} + \varepsilon_{t-(4k+2)} + \varepsilon_{t-(4k+3)}}{2}, \quad g_t^{(2)} = \sum_{k=0}^{+\infty} \beta_k^{(2)} \varepsilon_{t-4k},$$

by solving a simple linear system we discover that

$$\gamma_k^{(2)} = \frac{\alpha_{4k} + \alpha_{4k+1} + \alpha_{4k+2} + \alpha_{4k+3}}{2}, \quad \beta_k^{(2)} = \frac{\alpha_{4k} + \alpha_{4k+1} - \alpha_{4k+2} - \alpha_{4k+3}}{2}.$$

At the generic scale $J = j$, we retrieve the expressions of $\beta_k^{(j)}$ and $\gamma_k^{(j)}$ of equation (7) and (11), where $\pi_t^{(j)}$ is defined in equation (10).

Finally, we show (iii). First, when t is fixed, $\mathbb{E}[g_t^{(j)} g_t^{(l)}] = 0$ for all $j \neq l$ because $g_t^{(j)}$ and $g_t^{(l)}$ are, respectively, the projections of x_t on the subspaces $\mathbf{R}^{j-1}\mathcal{L}_t^{\mathbf{R}}$ and $\mathbf{R}^{l-1}\mathcal{L}_t^{\mathbf{R}}$ that are orthogonal by construction. Now, consider any $g_{t-m2j}^{(j)}$ with $m \in \mathbb{N}_0$. Clearly, $g_{t-m2j}^{(j)}$ belongs to $\mathbf{R}^{j-1}\mathcal{L}_{t-m2j}^{\mathbf{R}}$ but, by the definition of $g_t^{(j)}$, we can write

$$g_{t-m2j}^{(j)} = \sum_{k=0}^{+\infty} \beta_k^{(j)} \varepsilon_{t-(m+k)2j} = \sum_{K=0}^{+\infty} \beta_K^{(j)} \varepsilon_{t-K2j},$$

where $\beta_K^{(j)} = 0$ if $K = 0, \dots, m-1$ and $\beta_K^{(j)} = \beta_k^{(j)}$ if $K = m+k$ for some $k \in \mathbb{N}_0$. As a result, $g_{t-m2j}^{(j)}$ belongs to $\mathbf{R}^{j-1}\mathcal{L}_t^{\mathbf{R}}$, too. Similarly, at scale l , taken any $n \in \mathbb{N}_0$, it is easy to see that $g_{t-n2l}^{(l)}$ belongs to $\mathbf{R}^{l-1}\mathcal{L}_t^{\mathbf{R}}$. Hence, the orthogonality of such subspaces guarantees that $\mathbb{E}[g_{t-m2j}^{(j)} g_{t-n2l}^{(l)}] = 0$ for all $j \neq l$ and $m, n \in \mathbb{N}_0$.

As for the general requirement on $\mathbb{E}[g_{t-p}^{(j)} g_{t-q}^{(l)}]$ for any $j, l \in \mathbb{N}$ and $p, q, t \in \mathbb{Z}$,

$$\begin{aligned} \mathbb{E}[g_{t-p}^{(j)} g_{t-q}^{(l)}] &= \sum_{k=0}^{+\infty} \sum_{h=0}^{+\infty} \beta_k^{(j)} \beta_h^{(l)} \mathbb{E}[\varepsilon_{t-p-k2j}^{(j)} \varepsilon_{t-q-h2l}^{(l)}] \\ &= \frac{1}{\sqrt{2^{j+l}}} \sum_{k=0}^{+\infty} \sum_{h=0}^{+\infty} \beta_k^{(j)} \beta_h^{(l)} \sum_{u=0}^{2^j-1} \sum_{v=0}^{2^l-1} \{ \mathbb{E}[\varepsilon_{t-p-k2j-u} \varepsilon_{t-q-h2l-v}] \\ &\quad - \mathbb{E}[\varepsilon_{t-p-k2j-u} \varepsilon_{t-q-h2l-2^{l-1}-v}] - \mathbb{E}[\varepsilon_{t-p-k2j-2^{j-1}-u} \varepsilon_{t-q-h2l-v}] \\ &\quad + \mathbb{E}[\varepsilon_{t-p-k2j-2^{j-1}-u} \varepsilon_{t-q-h2l-2^{l-1}-v}] \} \end{aligned}$$

and so

$$\begin{aligned} \mathbb{E}[g_{t-p}^{(j)} g_{t-q}^{(l)}] &= \frac{1}{\sqrt{2^{j+l}}} \sum_{k=0}^{+\infty} \sum_{h=0}^{+\infty} \beta_k^{(j)} \beta_h^{(l)} \sum_{u=0}^{2^j-1} \sum_{v=0}^{2^l-1} \{ \gamma(p-q+k2^j+u-h2^l-v) \\ &\quad - \gamma(p-q+k2^j+u-h2^l-2^{l-1}-v) \} \end{aligned}$$

$$\begin{aligned}
& -\gamma(p - q + k2^j + 2^{j-1} + u - h2^l - v) \\
& + \gamma(p - q + k2^j + 2^{j-1} + u - h2^l - 2^{l-1} - v)\},
\end{aligned}$$

where coefficients $\beta_k^{(j)}$, $\beta_h^{(l)}$ do not depend on t and γ denotes the autocovariance function of ε . After the summations over u, v and k, h , the one remaining variables are $j, l, p - q$. In other words, $\mathbb{E}[g_{t-p}^{(j)}g_{t-q}^{(l)}]$ depends at most on $j, l, p - q$. \square

Proof of Theorem 2

PROOF. First, we show that any process $\mathbf{g}^{(j)}$ is well-defined. Indeed, by using the moving average representation of each $g_t^{(j)}$ with respect to the innovations on the support $S_t^{(j)}$ and the definition of detail processes $\varepsilon^{(j)}$, we have

$$g_t^{(j)} = \sum_{k=0}^{+\infty} \beta_k^{(j)} \varepsilon_{t-k2^j}^{(j)} = \sum_{k=0}^{+\infty} \sum_{i=0}^{2^j-1} \beta_k^{(j)} \delta_i^{(j)} \varepsilon_{t-k2^j-i} = \sum_{h=0}^{+\infty} \beta_{\lfloor \frac{h}{2^j} \rfloor}^{(j)} \delta_{h-2^j \lfloor \frac{h}{2^j} \rfloor}^{(j)} \varepsilon_{t-h}, \quad (25)$$

where $h = k2^j + i$, $k = \lfloor \frac{h}{2^j} \rfloor$ and $i = h - 2^j \lfloor \frac{h}{2^j} \rfloor$. Condition (13) ensures the square-summability of the coefficients and so each $\mathbf{g}^{(j)}$ is well-defined.

In addition, the process \mathbf{x} is well-defined because of Conditions (13) and (14). According to the latter, the components $g_t^{(j)}$ are orthogonal to each others at different scales for fixed $t \in \mathbb{Z}$. Therefore,

$$\mathbb{E}[x_t^2] = \mathbb{E}\left[\left(\sum_{j=1}^{+\infty} g_t^{(j)}\right)^2\right] = \sum_{j=1}^{+\infty} \mathbb{E}[(g_t^{(j)})^2] = \sum_{j=1}^{+\infty} \sum_{h=0}^{+\infty} (\beta_{\lfloor \frac{h}{2^j} \rfloor}^{(j)} \delta_{h-2^j \lfloor \frac{h}{2^j} \rfloor}^{(j)})^2,$$

which is finite because of (13). In consequence, the process \mathbf{x} is well-defined.

Now we show that \mathbf{x} is weakly stationary, with zero mean. We already observed that $\mathbb{E}[x_t^2]$ is finite and not dependent on t . In addition, since the processes $\mathbf{g}^{(j)}$ have zero mean, $\mathbb{E}[x_t] = 0$ for any $t \in \mathbb{Z}$. Finally, take $p \neq q$. Then

$$\mathbb{E}[x_{t-p}x_{t-q}] = \mathbb{E}\left[\left(\sum_{j=1}^{+\infty} g_{t-p}^{(j)}\right)\left(\sum_{l=1}^{+\infty} g_{t-q}^{(l)}\right)\right] = \sum_{j=1}^{+\infty} \sum_{l=1}^{+\infty} \mathbb{E}[g_{t-p}^{(j)}g_{t-q}^{(l)}].$$

As $\mathbb{E}[g_{t-p}^{(j)}g_{t-q}^{(l)}]$ are dependent at most on j, l and $p - q$ (see, e.g., the computations in the proof of Theorem 1), $\mathbb{E}[x_{t-p}x_{t-q}]$ depends at most on the difference $p - q$. As a result, \mathbf{x} is weakly stationary, with zero mean.

By taking the sum over scales $j \in \mathbb{N}$ in equation (25), we obtain the decomposition of x_t with respect to the process ε stated in equation (16). Clearly, \mathbf{x} is purely non-deterministic. \square

PROPOSITION A.1. The time series

$$\mathbf{R}x_t = \sum_{k=0}^{+\infty} \frac{\alpha_{\lfloor \frac{k}{2} \rfloor}}{\sqrt{2}} \varepsilon_{t-k} \quad \text{and} \quad \mathbf{R}_x x_t = \frac{1}{\sqrt{2}}(x_t + x_{t-1})$$

have spectral density functions, respectively,

$$f_{\mathbf{R}}(\lambda) = 2 \cos^2\left(\frac{\lambda}{2}\right) f_x(2\lambda) \quad \text{and} \quad f_{\mathbf{R}_x}(\lambda) = 2 \cos^2\left(\frac{\lambda}{2}\right) f_x(\lambda),$$

where $f_x(\lambda)$ is the spectral density function of x_t .

PROOF. Define the time-invariant linear filter $A(\mathbf{L}) = \sum_{h=0}^{\infty} \alpha_h \mathbf{L}^h$, so that $x_t = A(\mathbf{L})\varepsilon_t$. Since $\sum_{h=0}^{\infty} |\alpha_h| < +\infty$ and the spectral density function of ε_t is $f_\varepsilon(\lambda) = 1/2\pi$,

$$\begin{aligned} f_x(\lambda) &= |A(e^{-i\lambda})|^2 f_\varepsilon(\lambda) = \left| \sum_{h=0}^{+\infty} \alpha_h e^{-ih\lambda} \right|^2 \frac{1}{2\pi} \\ &= \frac{1}{2\pi} \left\{ \left(\sum_{h=0}^{+\infty} \alpha_h \cos(h\lambda) \right)^2 + \left(\sum_{h=0}^{+\infty} \alpha_h \sin(h\lambda) \right)^2 \right\} \\ &= \frac{1}{2\pi} \sum_{h=0}^{+\infty} \sum_{k=0}^{+\infty} \alpha_h \alpha_k \cos(\lambda(k-h)). \end{aligned}$$

First, consider $\mathbf{R}x_t$. As $\sum_{k=0}^{\infty} |\alpha_{\lfloor \frac{k}{2} \rfloor}| = 2 \sum_{h=0}^{\infty} |\alpha_h| < +\infty$, we have

$$\begin{aligned} f_{\mathbf{R}}(\lambda) &= \left| \sum_{k=0}^{+\infty} \frac{\alpha_{\lfloor \frac{k}{2} \rfloor}}{\sqrt{2}} e^{-ik\lambda} \right|^2 \frac{1}{2\pi} = \frac{1}{2\pi} \sum_{h=0}^{+\infty} \sum_{k=0}^{+\infty} \frac{\alpha_{\lfloor \frac{h}{2} \rfloor} \alpha_{\lfloor \frac{k}{2} \rfloor}}{2} \cos(\lambda(k-h)) \\ &= \frac{1}{2\pi} \sum_{h=0}^{+\infty} \sum_{k=0}^{+\infty} \alpha_h \alpha_k \left\{ \cos(2\lambda(k-h)) + \frac{\cos(\lambda(2k-2h+1)) + \cos(\lambda(2k-2h-1))}{2} \right\} \\ &= \frac{1}{2\pi} \sum_{h=0}^{+\infty} \sum_{k=0}^{+\infty} \alpha_h \alpha_k \cos(2\lambda(k-h)) \{1 + \cos(\lambda)\} = 2 \cos^2\left(\frac{\lambda}{2}\right) f_x(2\lambda). \end{aligned}$$

Now consider $\mathbf{R}_x x_t$. The spectral density function in the claim follows from

$$f_{\mathbf{R}_x}(\lambda) = \left| \frac{1}{\sqrt{2}}(e^0 + e^{-i\lambda}) \right|^2 f_x(\lambda) = \frac{1}{2} \{ (1 + \cos(\lambda))^2 + \sin^2(\lambda) \} f_x(\lambda). \quad \square$$

A.2 Forecasting from the persistence-based decomposition

We provide the formulas for conditional expectation and variance of a process $\mathbf{x} = \{x_t\}_{t \in \mathbb{Z}}$ that has classical and extended Wold decompositions given by equations (4) and (5), respectively. We consider the filtration generated by the white noise $\varepsilon = \{\varepsilon_t\}_{t \in \mathbb{Z}}$ assuming that the innovations ε_t are independent.

Fix $p \in \mathbb{N}$. The conditional expectation at time t of x_{t+p} is characterized by an offset of the classical Wold coefficients, namely $\mathbb{E}_t[x_{t+p}] = \sum_{h=0}^{\infty} \alpha_{h+p} \varepsilon_{t-h}$. Notably, such offset is inherited by the extended Wold decomposition of $\mathbb{E}_t[x_{t+p}]$:

$$\mathbb{E}_t[x_{t+p}] = \sum_{j=1}^{+\infty} \sum_{k=0}^{+\infty} \beta_{k,p}^{(j)} \varepsilon_{t-k2j}^{(j)},$$

where, for any $j \in \mathbb{N}$ and $k \in \mathbb{N}_0$,

$$\beta_{k,p}^{(j)} = \frac{1}{\sqrt{2^j}} \left(\sum_{i=0}^{2^{j-1}-1} \alpha_{k2^j+i+p} - \sum_{i=0}^{2^{j-1}-1} \alpha_{k2^j+2^{j-1}+i+p} \right).$$

Therefore, once the extended Wold decomposition of x_t is known, p -step ahead forecasts do not require a large additional effort because they are driven by the detail processes $\varepsilon^{(j)}$, too, and coefficients $\beta_{k,p}^{(j)}$ are easily computed.

As to the conditional variance, the properties of the classical Wold decomposition ensure that $\text{Var}_t(x_{t+p}) = \alpha_0^2 + \dots + \alpha_{p-1}^2$. By Theorem 2 the coefficients α_h can be obtained from the scale-specific $\beta_k^{(j)}$ and so $\text{Var}_t(x_{t+p})$ can be derived directly from them. For example, $\text{Var}_t(x_{t+1}) = \alpha_0^2 = (\sum_{j=1}^{\infty} \beta_0^{(j)} / \sqrt{2^j})^2$.

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