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**Working Paper**

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CESifo Working Paper, No. 8141

**Provided in Cooperation with:**

Ifo Institute – Leibniz Institute for Economic Research at the University of Munich

*Suggested Citation:* Simula, Laurent; Trannoy, Alain (2020) : Gini and Optimal Income Taxation by Rank, CESifo Working Paper, No. 8141, Center for Economic Studies and Ifo Institute (CESifo), Munich

This Version is available at:

<https://hdl.handle.net/10419/216537>

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# Gini and Optimal Income Taxation by Rank

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## **Impressum:**

CESifo Working Papers

ISSN 2364-1428 (electronic version)

Publisher and distributor: Munich Society for the Promotion of Economic Research - CESifo GmbH

The international platform of Ludwigs-Maximilians University's Center for Economic Studies and the ifo Institute

Poschingerstr. 5, 81679 Munich, Germany

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Editor: Clemens Fuest

<https://www.cesifo.org/en/wp>

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# Gini and Optimal Income Taxation by Rank

## Abstract

We solve the non-linear income tax program for a rank-dependent social welfare function à la Yaari, expressing the trade-off between size and inequality using the Gini or related families of positional indices. The key idea is that when agents optimize and absent bunching, ranks in the actual and optimal allocations become an invariant dimension. This allows us to obtain optimal marginal tax rates as a function of ranks, and numerically illustrate the relationship between ranks and taxes. For singles without children, the actual US tax schedule seems to indicate a distaste for differences in the upper part of the distribution.

JEL-Codes: D630, D820, H210.

Keywords: Gini, optimal taxation, income taxation, ranks.

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March 3, 2020

We thank Rolf Aaberge, Nathaniel Hendren, Thomas Piketty, Emmanuel Saez and participants in the Oslo Workshop on "Inequality: Measurement, Evolution, Mechanisms, and Policies" for helpful comments and discussion. A part of this research was realized while Laurent Simula was visiting the department of economics, Harvard University. We acknowledge financial support from IDEX-University of Lyon within the Programme Investissements d'Avenir (ANR-16-IDEX-0005).

# I. INTRODUCTION

“The questions therefore arise what principles should govern an optimum income tax; what such a tax schedule would look like; and what degree of inequality would remain once it was established.” (Mirrlees, 1971)

As emphasized by Mirrlees, the trade-off between efficiency and the degree of inequality is at the core of optimal nonlinear income taxation. However, quite surprisingly, the optimal taxation and inequality measurement literatures have so far remained largely disconnected. From the very beginning, optimal income tax models have primarily relied on social welfare as measured by a concave transformation of individual utilities. Atkinson (1970) connected inequality and social welfare by means of the introduction of the equally distributed equivalent level of income (see also Blackorby and Donaldson (1978); Blackorby et al. (1999)). However, this theoretical relationship has hardly been used in practice to infer the level of optimal inequalities (see Sattinger (2017)). Moreover, it sets aside the notion of position, i.e. the ranking of individuals, and more specifically the Gini inequality index and its extensions, to focus on income levels. Yet, the Gini index and its graphical representation through the Lorenz curve are by far the most used tools to measure inequalities.

This article adopts a rank-based approach to the optimal nonlinear income tax problem. Ranks play a key part when thinking about inequalities. Chetty et al. (2014) have recently shown how ranks help obtain clear-cut results with regards to intergenerational mobility. In the same perspective, we show that a rank-dependent social welfare function à la Yaari (1987, 1988) allows us to obtain very simple tax rate formulas, and empirical results clearly expressing the trade-off between the “size of the cake” and its uneven sharing, as measured by the Gini coefficient of utility levels, or its extensions to the S-Gini family (Donaldson and Weymark, 1980; Weymark, 1981; Bossert, 1990) and “A-family” (Aaberge, 2000, 2009; Aaberge et al., 2020).

We exploit the key feature that in a social optimum respecting incentive-compatibility constraints, ranks in terms of skills, indirect utilities, gross incomes and net incomes are all the same, as emphasized by Trannoy (2019). Furthermore, absent bunching, a mild assumption in view of empirical findings (Saez, 2010; Bastani and Selin, 2014), these ranks are the same as in the actual allocation providing agents maximize their utility. Ranks thus provide a fruitful invariant with respect to which tax rates can be expressed. Therefore, far to being farfetched, rank-dependent social welfare functions appear to be a relevant criterion to assess optimality of an incentive-compatible income tax. Social welfare is then equal to a weighted sum of indirect utilities –with weights solely depending on ranks– or equivalently using its “abbreviate” form as the mean indirect utility multiplied by one minus the corresponding inequality index.

Relying on the Gini social welfare function introduced by Sen (1974), we first investigate the situation in which the inequality index is the popular Gini coefficient.<sup>1</sup> In that case, individual weights in the social objective vary linearly with the rank. We also consider extensions to the S-Gini and A-families, two single-parameter families sharing the Gini social welfare function as unique common element. In the former, individual weights are convex with respect to the rank, leading to downward positional sensitivity; they are concave in the latter, leading to upward positional sensitivity (Aaberge, 2009; Aaberge et al., 2020). This implies that Pigou-Dalton transfers should be prioritized when taking place at the bottom for the S-Gini family, and at the

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<sup>1</sup>Our approach remains welfarist. See Prete et al. (2016) for an analysis using the Gini index in a non-welfarist setting.

top for the A-family.

Following Piketty (1997), Diamond (1998) and Saez and Stantcheva (2016), we assume away income effects on taxable income. We show that optimal marginal tax rates can be expressed in a very simple way as functions of ranks. On this basis, we establish that marginal tax profiles depend on ranks in a decreasing or U-shaped manner, for all members of the S-Gini and A-families. We even obtain a closed-form expression for Gini tax liabilities, providing the elasticity of taxable income is constant and ranks follow a Pareto distribution. We then numerically illustrate the relationship between ranks and taxes. We parameterize the formulas based on US data for singles without children and an elasticity of taxable income in the 0.1-0.5 range. Although the Gini social objective function appears on paper as a natural intermediate case between the Rawlsian maximin and pure utilitarianism, we find that the Gini optimal tax schedule is much closer to the former. Moreover, the schedules obtained for the different members of the S-Gini class are rather close. On the contrary, the A-family offers much more dispersed views regarding how marginal and average tax rates should vary with ranks. It seems that the relative concern for top people matters more for the design of the optimal non-linear income tax schedule than the concern for bottom people. Furthermore, we show –admittedly for a specific population– that the US very unlikely optimizes with the Gini coefficient in mind, but with a member of the A-family, which expresses a particular concern for inequalities at the top. Still, the approach delivers optimal marginal tax rates about ten points higher than the actual ones for top-income earners for a constant elasticity of taxable-income equal to 0.5 and, therefore, a significantly more progressive tax schedule at the top. This calls for complementary and more comprehensive simulation exercises, for the US but also for other developed countries.

Our approach illustrates the fruitfulness of the “generalized social weights approach” to income taxation introduced by Saez and Stantcheva (2016), on which we build extensively, even though the latter article only introduced ranks in relation to equality of opportunity principles.

The article is organized as follows. Section 2 recalls the concepts of rank-dependent social welfare functions and their connection with inequality indexes. On this basis, Section 3 formulates the optimal income tax problem in terms of ranks. Section 4 develops the main theoretical results. Section 5 empirically examines the link between ranks and optimal taxes. Section 6 concludes.

## II. RANK-DEPENDENT SOCIAL WELFARE AND INEQUALITY

This section introduces rank-dependent social welfare functions and shows how they connect with inequality indexes.

### II.1. *Rank-Dependent Social Welfare*

We consider a population of individuals, heterogeneous with respect to a variable  $x$ . For simplicity, we assume that the latter is uni-dimensional and smoothly distributed according to the cumulative distribution function  $F(x)$ , with support  $X \subseteq \mathbb{R}_+$ . We call  $f(x)$  the corresponding probability density function (pdf). This continuous setting is a good approximation of a large discrete population, makes the analysis more straightforward, and will later allow us to make

the connection with marginal tax rates, which are not well-defined in a discrete setting (cf. Weymark (1987) or Simula (2010)). The average value of  $x$  within the population is  $\mu = \int_X xf(x)dx$ .

We define the quantile function as  $F^{-1}(p) = x$ , where  $p \in [0, 1]$  stands for the rank or "position", and introduce weights to capture the social planner's aversion to inequality. The marginal weights are denoted  $\lambda(p)$  and the cumulated weights  $\Lambda(p) = \int_0^p \lambda(\pi)d\pi$ . In the whole article, we focus on weights consistent with second-order stochastic dominance, belonging to the set:

$$\mathcal{L} = \{\forall p \in (0, 1), \lambda(p) > 0 \text{ and } \lambda'(p) < 0; \Lambda(0) = \lambda(1) = 0; \Lambda(1) = 1\}. \quad (1)$$

The assumption that  $\lambda(p)$  is positive and decreasing means that every individual counts, but to a lower extent the higher the rank. The other assumptions are normalizations. On this basis, *rank-dependent social welfare* (Yaari, 1987, 1988) is defined as:

$$\mathcal{W} = \int_0^1 \lambda(p)F^{-1}(p)dp. \quad (2)$$

## II.2. From Rank-Dependent Social Welfare to Inequality Indexes

The distribution  $F$  is egalitarian if and only if  $\mathcal{W} = \mu$ . Otherwise, there is a positive gap between welfare  $\mathcal{W}$  and the equality benchmark, denoted  $\Delta \equiv \mu - \mathcal{W}$ . Dividing by  $\mu$ , we obtain the mean-invariant inequality index  $I \equiv \Delta/\mu$ . Combining the definitions of  $\Delta$  and  $I$ , we can rewrite social welfare (2) in abbreviated form:

$$\mathcal{W} = \mu(1 - I). \quad (3)$$

This expression illustrates the close connection between rank-dependent social welfare functions and inequality indexes:  $\mathcal{W}$  is equal to the egalitarian benchmark deflated by inequality as measured by  $I$ .

Specifying the weights  $\Lambda(p)$ , we focus on two important families of rank-dependent social welfare functions:

- The S-Gini family (Donaldson and Weymark, 1980) for  $\Lambda(p) = 1 - (1 - p)^\delta$  and  $\delta \geq 2$ .
- The "A" family introduced by Aaberge (2000)<sup>2</sup> for  $\Lambda(p) = (\delta p - p^\delta)/(\delta - 1)$  and  $\delta \geq 2$ . When  $\delta \rightarrow 1$ ,  $\Lambda(p) = p(1 - \log(p))$  which corresponds to the rank-dependent welfare function  $\mathcal{W}$  in which  $I$  is the Bonferroni index of inequality.<sup>3</sup>

For both families, the weights coincide when  $\delta = 2$ , with  $\Lambda(p) = p(2 - p)$ . In that case, the inequality measure  $I$  is the Gini coefficient and  $\mathcal{W}$  the Gini social welfare function introduced by Sen (1974). The marginal weights  $\lambda(p) = 2(1 - p)$  are then linear with respect to rank. By contrast, for any  $\delta > 2$ , the weights  $\lambda(p)$  are convex when considering the S-Gini family, and concave for the A family. Figure I shows the weights for both families, the common Gini case, and the Bonferroni case. The left panel shows individual weights  $\lambda(p)$ . Because they all belong

<sup>2</sup>Aaberge (2000) refers to this class as the Lorenz family for  $\delta = 2, 3, \dots$ . It corresponds to the integer subfamily of the "illfare-ranked single-series Ginis" discussed by Donaldson and Weymark (1980) and Bossert (1990). We refer to the latter as the "A" family. Aaberge (2009) and Aaberge et al. (2020) have indeed shown the usefulness of this family for analyzing inequality when Lorenz curves intersect.

<sup>3</sup>Let  $\mu(x) = (\int_0^x x dF(x)) / F(x)$  and  $r(x) = (\mu - \mu(x)) / \mu$ . The Bonferroni inequality index is  $\int_X r(x)dF(x)$ .

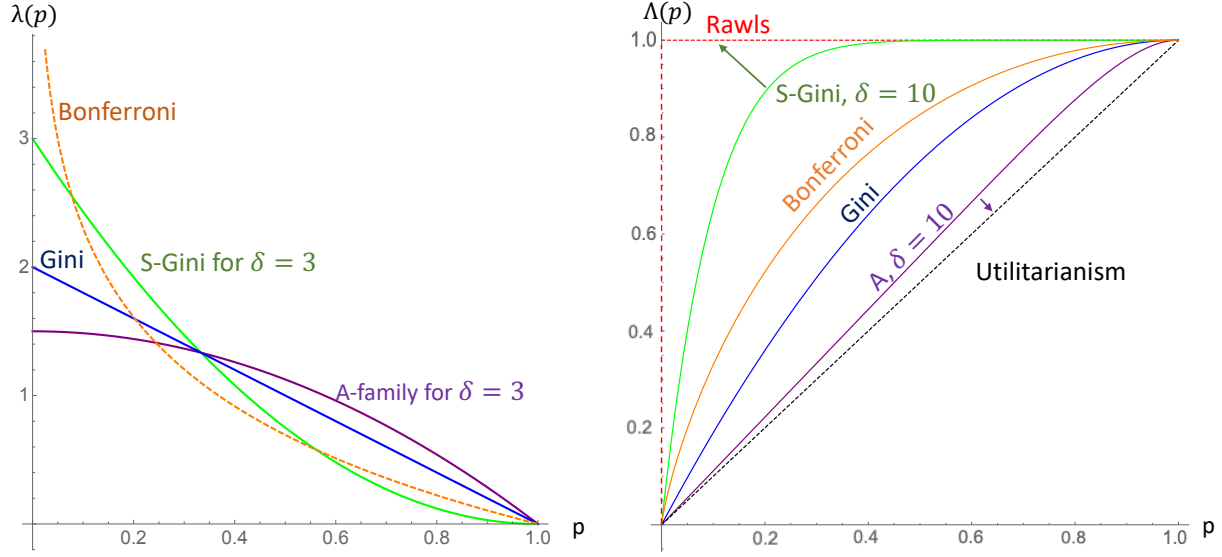


FIGURE I: WEIGHTS TO RANKS

to  $\mathcal{L}$ , the area below each curve is equal to one. Looking at the different curves, let us consider a fixed transfer taking place between two agents with equal difference in ranks. For the S-Gini family, convexity implies that the equalizing effect of the transfer becomes larger the lower the ranks considered. The focus is thus on poverty. On the contrary, for the A family, the higher the ranks the stronger the equalizing effect. The emphasis is layed on inequalities at the top of the distribution. The right panel shows cumulated weights  $\Lambda(p)$  for various values of  $\delta$ . It illustrates the following convergence pattern when  $\delta$  goes up: the S-Gini and A-families tend to two important benchmarks, the Rawlsian maximin on the one hand and pure utilitarianism on the other hand. It also shows that, for every interior  $p$ , the cumulated weights are larger in the Bonferroni case than in the Gini one, implying that the former stresses poverty more than the latter.

### III. OPTIMAL NON-LINEAR INCOME TAXATION AND INEQUALITY MEASUREMENT

We now introduce taxes and transfers, and thus the main elements needed to investigate the trade-off between equity and efficiency, in the tradition of Mirrlees (1971) and Diamond (1998). We consider a population with a continuum of individuals and size normalized to one. This population is heterogeneous with respect to productivity per unit of effort,  $\theta$ , which belongs to a subset  $[\underline{\theta}, \bar{\theta}]$  of  $\mathbb{R}_+$ .  $\bar{\theta}$  can be finite or tend to  $+\infty$ . Productivity is privately known to each agent, and only its cumulated distribution  $H(\theta)$  (with pdf  $h(\theta) = H'(\theta) > 0$ ) is common knowledge. Every individual derives utility from consumption  $c$  and dis-utility from earning income  $z$ , with a utility representation:

$$u(c, z; \theta) = c - v(z; \theta). \quad (4)$$



The function  $v$  is common to all individuals. It is increasing and convex in  $z$ , decreasing in  $\theta$ , and with a negative cross-derivative. The quasi-linear form (4) rules out income effects on earnings, and greatly simplifies the analysis by allowing us to focus on the distortive impact of substitution effects. The government sets an income tax  $T(z)$  as a function of earnings only, so that  $c = z - T(z)$ . Individuals choose  $z$  to maximize  $u = z - T(z) - v(z; \theta)$ . We call  $z(\theta)$  the solution and  $V(\theta)$  the corresponding indirect utility.

Because of asymmetric information, the government must account for incentive-compatibility constraints when designing the income tax  $T(z)$ . As is well-known (see, e.g., Salanié (2011)), these constraints are equivalent to (i)  $V'(\theta) = -v'_\theta(z(\theta); \theta)$  and (ii) non-decreasing  $z(\theta)$  (equivalent to non-decreasing  $c(\theta) = z(\theta) - T(z(\theta))$ ). The first condition implies that the higher the productivity the larger the indirect utility. Consequently, the ranks in terms of productivity and the ranks in terms of indirect utility are the same. Absent bunching, the latter are also the same as for gross income and net income. In the theoretical derivations, we assume that there is no bunching, and check it ex post in the numerical simulations of Section V.

In reference to the previous Section, we let  $x = V$ . Because of the above remark about ranks, we note that  $p = F(V(\theta)) = H(\theta)$ , which also implies:  $F^{-1}(p) = V(\theta)$ . Consequently, the rank-dependent social welfare function (2) can be rewritten as follows:

$$\mathcal{W} = \int_{\underline{\theta}}^{\bar{\theta}} \lambda(F(V(\theta)))V(\theta)dF(V(\theta)) = \int_{\underline{\theta}}^{\bar{\theta}} \lambda(H(\theta))V(\theta)dH(\theta). \quad (5)$$

The government's problem is to design the tax function  $T(z)$  maximizing (5) subject to the conditions for incentive compatibility and budget-balancedness. We call  $E$  the exogenous amount of expenditures to finance if any. In this form, we see the clear connection with the problem investigated by Saez and Stantcheva (2016), with "generalized welfare weights" which in the present article only depend on productivity. What is novel is that we offer a new perspective, connecting optimal taxation and inequality measurement thanks to the rank-dependent formulation of the social objective. Such an analysis was not previously made in the literature.

### **Remarks**

The rank-dependent social welfare functions we consider depend on a single parameter  $\delta$ . One may therefore wonder whether the solutions to the above optimal income tax problem may not be obtained as the solution to the maximization of Atkinson social welfare function,

$$\mathcal{W}_{Atk} = \int_{\underline{\theta}}^{\bar{\theta}} \frac{1}{1-\rho} V(\theta)^{1-\rho} dH(\theta),$$

for a given value of the society's aversion to inequality  $\rho$ . This would be the case if an allocation was satisfying the optimality conditions for the optimal income tax problem under the  $\mathcal{W}$  and  $\mathcal{W}_{Atk}$  objective functions. For that to be the case, Euler equations should be the same, implying  $\lambda(H(\theta)) = V(\theta)^{-\rho}$  and thus  $V(\theta) = [\lambda(p)]^{-1/\rho}$ . Therefore, to get a similar solution, the indirect utility path  $V(\theta)$  should be completely exogenous, independent in particular of individual preferences and elasticities of taxable income, which is impossible. This establishes the specificity of the rank-dependent approach within the class of welfare functions depending on a one-dimensional parameter.

In addition, the rank-dependent social welfare functions we consider allow a transparent formulation of the optimal income tax problem, because social welfare is simply obtained as the product of  $\mu$  and  $1 - I$ , as already noted in (2). This is not the case if we instead use  $\mathcal{W}_{Atk}$ . The closest formulation would involve the equally distributed equivalent income EDEI and Atkinson inequality index  $I_{Atk} = 1 - EDEI/\mu$ . On this basis,  $\mathcal{W}_{Atk}$  can be rewritten as:

$$\mathcal{W}_{Atk} = \frac{1}{1-\rho} [\mu(1 - I_{Atk})]^{1-\rho},$$

which however is a nonlinear transformation of the product of  $\mu$  and  $1 - I_{Atk}$ .

## IV. OPTIMAL MARGINAL TAX RATES BY RANK

Optimal tax formulas are usually presented as a function of earnings in the optimum (see, e.g., Saez (2001); Piketty and Saez (2013); Saez and Stantcheva (2016)) which are endogenous to the optimal schedule itself. We derive formulas in terms of rank, a dimension which is completely exogenous to the optimal tax policy as long as the actual allocation is incentive compatible.

### *General Case*

At a given rank  $p$ , we denote the marginal tax rate by  $MTR(p)$  and define the elasticity of taxable income with respect to the retention rate  $1 - MTR(p)$  as:

$$\epsilon(p) = \frac{1 - MTR(p)}{z(p)} \frac{\partial z(p)}{\partial(1 - MTR(p))}. \quad (6)$$

We also introduce the elasticity of taxable income with respect to productivity  $\theta$ :

$$\epsilon_{\theta}(p) = \frac{\theta(p)}{z(p)} \frac{\partial z(p)}{\partial \theta(p)}, \quad (7)$$

with  $\theta(p) = H^{-1}(p)$ . Behavioral responses are summarized by the ratio  $e(p) = \epsilon(p)/\epsilon_{\theta}(p)$ , referred to below as the "behavioral term".<sup>4</sup> When the disutility of effort is isoelastic, given by  $v(z;\theta) = (z/\theta)^{1+1/\epsilon}$ , the behavioral term  $e(p)$  and the "usual" elasticity of taxable income  $\epsilon(p)$  are both constant, equal to  $\epsilon$ .

In addition, it is useful to introduce the (relative) average social marginal weight for individuals with rank above  $p$ ,

$$G(p) = \frac{\int_p^1 \lambda(\pi) d\pi}{\int_p^1 d\pi} = \frac{1 - \Lambda(p)}{1 - p},$$

and the local Pareto parameter of the skill distribution at rank  $p$ ,

$$\alpha(p) = \frac{\theta(p)h(\theta(p))}{1 - p}.$$

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<sup>4</sup>Our derivation of optimal marginal tax rates is done for the classic case where there are only supply-side responses as in Mirrlees (1971) and Saez (2001). It is straightforward to extend the analysis to capture other responses, such as tax avoidance and bargaining, as in Piketty et al. (2014)

Absent bunching, the optimal marginal tax rates are given by Proposition 2 in Saez and Stantcheva (2016), which remains valid for rank-dependent welfare functions:

$$MTR(p) = \frac{1 - G(p)}{1 - G(p) + e(p)\alpha(p)} \text{ for } 0 < p < 1. \quad (8)$$

Alternatively, we can rearrange (8) into an ABC-formula.

PROPOSITION 1. For the rank-dependent social welfare function  $\mathcal{W}$ , optimal marginal tax rates satisfy:

$$\frac{MTR(p)}{1 - MTR(p)} = \underbrace{\frac{1}{e(p)}}_A \underbrace{\frac{1}{\alpha(p)}}_B \underbrace{\frac{\Lambda(p) - p}{1 - p}}_C \text{ for } 0 < p < 1. \quad (9)$$

This formula is extremely simple and straightforward to implement. Compared to Diamond's (1998) formula, the A and B factors are just the same but rewritten in terms of ranks: they reflect efficiency and demographic considerations respectively (see Piketty (1997) or Piketty and Saez (2013)).<sup>5</sup> By contrast, the C-factor summarizing ethical considerations directly depends on ranks and on the parameter  $\delta$  only; and in particular is independent of the marginal utility of income. Under pure utilitarianism, it is equal to zero, so that  $MTR(p) = 0$  for every rank  $p$ . The Rawlsian maximin, often used as a benchmark in the literature (Piketty, 1997; Boadway and Jacquet, 2008), is easily obtained as a sub-case of our rank dependent setting. The only rank which then matters is the lowest; hence,  $\lambda(p) = 0$  but at  $\underline{\theta}$ , implying  $G(p) = 0$  in Formula (8) and  $C = 1$  in Formula (9).

Providing the actual tax schedule is incentive-compatible and absent bunching, the distributions of ranks in terms of productivity, gross earning, net earning and indirect utility are all the same than those in the optimal allocation. Consequently, the B- and C-factors are exogenous to the optimal policy. The only factor which may differ is A, which captures efficiency and therefore evaluates behavioral responses along the optimal schedule. If these responses are structural, A is also exogenous, and the optimal tax in our rank-dependent setting depends on the primitives of the model and not on any endogenous variable.

only depends on the actual distribution of  $p$ .

Figure II shows values of the C-factor by rank  $p$ , for various situations: S-Gini and A families with  $\delta = \{2, 3, 9, 15, 21\}$  together with the Bonferroni, Rawlsian and utilitarian cases. It is useful to recall that  $MTR$  increases with the ethical C-factor, everything else being equal, and that the highest incentive-compatible marginal tax rates are obtained for the Rawlsian maximin. For the Gini social welfare function, C is simply equal to the rank  $p$ . The contrasting shapes of C echo our above remark on the main relative focuses on poverty and inequalities at the top, for the S-Gini and A families respectively: in the S-Gini case, C tends to increase the marginal tax burden in the left part of the distribution to increase collected taxes and thus the transfer to the poorest; in the A-family case, most of the increase takes place for large ranks. When  $\delta$  becomes arbitrarily large and for all interior  $p$ , the C-factor converges to its Rawlsian value for the S-Gini family and to its utilitarian value for the A-family.<sup>6</sup> The Bonferroni case appears as

<sup>5</sup>The A factor in Diamond's (1998) formula is written in terms of elasticity of labor supply. Proposition (9) uses the equivalent formulation in terms of elasticity of taxable income.

<sup>6</sup>When  $\delta \rightarrow \infty$ ,  $MTR(p)$  is zero for every  $p < 1$  and then equal to the Rawlsian one at the limit, when  $p \rightarrow 1$ . In the limit, the graph of the C-Factor is thus inverted-L shaped.

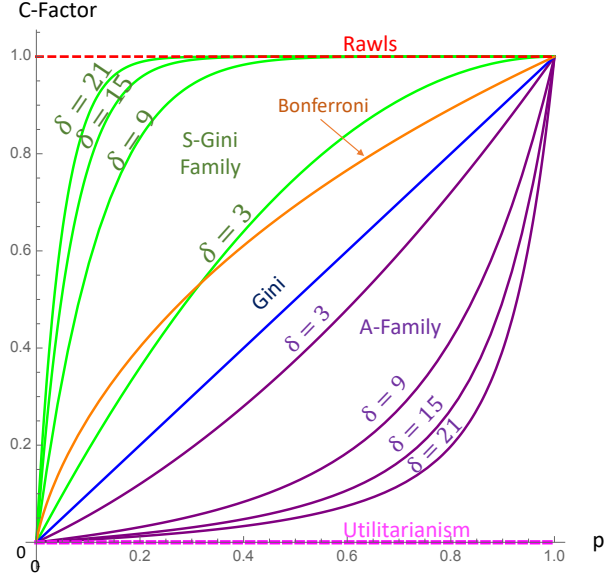


FIGURE II: C-FACTOR IN ABC TAX FORMULA (FOR INTERIOR RANKS)

an intermediate situation; if we compare it to S-Gini with  $\delta = 3$ , the C-factor is larger for  $p$  between 0 and  $p \approx 0.32$ , and lower for larger values of  $p$ .

### Heuristic Derivation

It is insightful to rely on a small tax reform perturbation to contrast Formula (8) with that obtained in the standard "welfarist" approach. To this aim, let us consider a given tax schedule and increase marginal tax rates by an amount  $\Delta$  on a small interval  $[\bar{p}, \bar{p} + dp]$ . This gives rise to the following effects:

- **Behavioral effect:** an agent with rank  $p$  in the interval responds to the rise in the marginal tax rate by a substitution effect. Given the definition of  $\epsilon$ , the latter reduces her taxable income by:

$$dz(p) = \frac{z(p)}{1 - MTR(p)} \epsilon(p) \Delta.$$

This decreases the taxes she pays by an amount:

$$dT(p) = MTR(p) \cdot dz(p) = \frac{MTR(p)}{1 - MTR(p)} z(p) \epsilon(p) \Delta.$$

The mass of taxpayers in the interval is  $dp = h(\theta) d\theta$ . Hence, using (7),

$$d\theta(p) = \frac{\theta(p) dz(p)}{z(p) \epsilon_\theta(p)},$$

the total behavioral response amounts to  $dp \cdot dT(p)$ , i.e.,

$$dB = \frac{MTR(p)}{1 - MTR(p)} \frac{\epsilon(p)}{\epsilon_\theta(p)} h(\theta(p)) \theta(p) dz(p) \Delta.$$

- **Mechanical effect:** every agent with rank above  $\bar{p}$  faces a lump-sum increase in taxes, equal to  $dz(p)\Delta$ . Collected taxes increase by

$$dM = dz(p)[1 - p]\Delta.$$

- **Welfare effect:** the lump-sum increase in tax liabilities faced by agents with rank above  $\bar{p}$  induces a negative income effect, which reduces their indirect utility  $V(\theta(p))$  by  $dz(p)\Delta$ . Social welfare thus varies by:

$$dW = - \int_{\bar{p}}^1 \lambda(p)h(\theta(p))\Delta dz(p).$$

A small tax reform perturbation around the optimal schedule has no first-order effect. Therefore, optimal marginal tax rates must verify  $dB + dM + dW = 0$ , and thus the formula in Proposition 1.

The welfare effect  $dW$  is negative as in the standard welfarist framework where the social objective function is the sum of a concave transformation  $\Phi(V(\theta(p)))$  of individual utilities. In the latter, a lump-sum increase in tax liabilities at skill  $\theta(p)$  reduces indirect utility  $V(\theta(p))$ , thus increasing the social marginal utility  $\Phi'(V(\theta(p)))$ . This "feedback" effect does not arise in the rank-dependent approach: because the lump-sum increase in tax liabilities has no effect on ranks  $p$ , there is no variation in  $\lambda(p)$ .

### *Tax Rates at the Top and at the Bottom*

To investigate tax rates at the top, we consider a constant behavioral term  $\bar{e}$  and a rank  $\hat{p}$  above which  $H$  is well-approximated by a Pareto distribution (Saez, 2001).<sup>7</sup> The latter implies that  $\alpha(p)$  is constant above  $\hat{p}$ , equal to  $\bar{\alpha}$ . We consider Bonferroni as well as the S-Gini and A-families for finite values of  $\delta$ . In all cases, the increase in the C-factor emphasized above implies higher marginal tax rates for larger ranks. We therefore generalize the results of Diamond (1998) over increasing marginal tax rates at the top to a broad class of alternative social welfare functions. Moreover, we are able to rank the marginal tax rates above  $\hat{p}$  on the basis of Figure II: for example, the marginal tax rates are always larger for the S-Gini family of social welfare functions than for the A-one; also, at a given rank, the marginal tax rate increases with  $\delta$  for the S-Gini family and decreases with it for the A-family. When  $p \rightarrow 1$ , they all converge to the Rawlsian one, where  $MTR = 1/(1 + \bar{e} \cdot \bar{\alpha})$ .

We now turn our focus to the bottom of the distribution, assuming again that the behavioral term  $e(p)$  is constant. The ranking of the C-Factors as shown in Figure II is preserved when  $p$  tends to 0. Indexing the marginal tax rates by their family, we obtain:

$$1 = MTR_R(0) > MTR_{SG}(0) > MTR_G(0) > MTR_A(0) > 0.$$

In addition, the higher  $\delta$  in the S-Gini family, the closer  $MTR_{SG}(0)$  to 1; and the higher  $\delta$  in the A-family, the closer  $MTR_A(0)$  to 0. Therefore, while marginal tax rates for all families tend to converge at the top (for a Pareto distribution), there is a grading of marginal tax rates at the bottom.

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<sup>7</sup>The Pareto distribution is unbounded with  $\bar{\theta} \rightarrow \infty$ .

## The Gini Benchmark

The optimal tax formulas can be written in a strikingly simple form for the Gini social welfare function. In that case,  $G(p) = 1 - p$ . Therefore, applying Formulas (8) and (9), we obtain:

PROPOSITION 2. For the Gini social welfare function, optimal marginal tax rates satisfy:

$$MTR(p) = \frac{p}{p + e(p)\alpha(p)} \text{ and } \frac{MTR(p)}{1 - MTR(p)} = \frac{p}{e(p)\alpha(p)} \text{ for } 0 < p < 1. \quad (10)$$

When behavioral responses are structural (in the sense that  $p \rightarrow e(p)$  is fixed at every  $p$ ), the ratio of the marginal tax rate to the retention tax rate for the Gini social welfare function is equal to that of the Rawlsian maximin, multiplied by  $p$ . As a result, for rank intervals where the Rawlsian tax schedule is marginally progressive (i.e.,  $MTR(p)$  increasing in  $p$ ), so is the Gini schedule. If we further assume that  $e(p)$  and  $\alpha(p)$  are constant above productivity  $\theta(\hat{p})$ , respectively equal to  $\bar{e}$  and  $\bar{\alpha}$ , the tax liability  $T(p)$  is obtained by integration over ranks, which by essence are uniformly distributed over  $[0, 1]$ . Hence,

$$T(p) = p - \bar{e}\bar{\alpha} \log(p + \bar{e}\bar{\alpha}) + \hat{T},$$

where  $\hat{T}$  is a constant. If  $\hat{E}$  is the tax receipts to be levied from agents above  $\hat{p}$ , we obtain

$$\hat{T} = \bar{e}\bar{\alpha} [\log(1 + \bar{e}\bar{\alpha}) - 1] + (\bar{e}\bar{\alpha})^2 [\log(1 + \bar{e}\bar{\alpha}) - \log(\bar{e}\bar{\alpha})] + \hat{E} - \frac{1}{2}.$$

If the whole distribution of productivity is Pareto and the tax policy purely redistributive ( $E = \hat{E} = 0$ ), we obtain a full characterization of the tax function  $T(p)$ , with increasing marginal tax rates.

## V. NUMERICAL ILLUSTRATION

We numerically illustrate the link between ranks  $p$  and optimal taxes. We consider that the disutility of effort is isoelastic, with  $v(z; \theta) = (z/\theta)^{(1+1/\epsilon)}$ . In that case, the elasticity of taxable income  $\epsilon(p)$  is constant, equal to  $e(p)$ . The calibration is the same as in Lehmann et al. (2014) for a closed economy, with the distribution of productivity obtained by inversion from the CPS data (2007) for singles without children, extended by a Pareto tail. In that sense, we “replicate” a US economy populated with this specific set of agents.

The elasticity  $\epsilon$  is assumed to be in the range 0.25 – 0.5 (Saez et al., 2012). Regarding the amount of exogenous expenditures to be financed ( $E$ ), we consider two scenarios: it is equal to zero in the first one, where the tax policy is purely redistributive; it is equal to the average of the tax liabilities actually paid by the agents in our sample in the second one ( $E = \hat{T}_{actual}$ ). The chosen scenario is without impact on the marginal tax rates, but affects average tax rates and therefore net earnings and indirect utilities. The first situation puts the emphasis on the normative dimension of the simulation exercise and is typically used as a benchmark in the optimal taxation literature. The second one allows us to more directly relate the “optimal” and “actual” schedules, given that little vertical redistribution actually takes place within singles without children, as illustrated by the shape of the observed average tax rates in Figure IV.

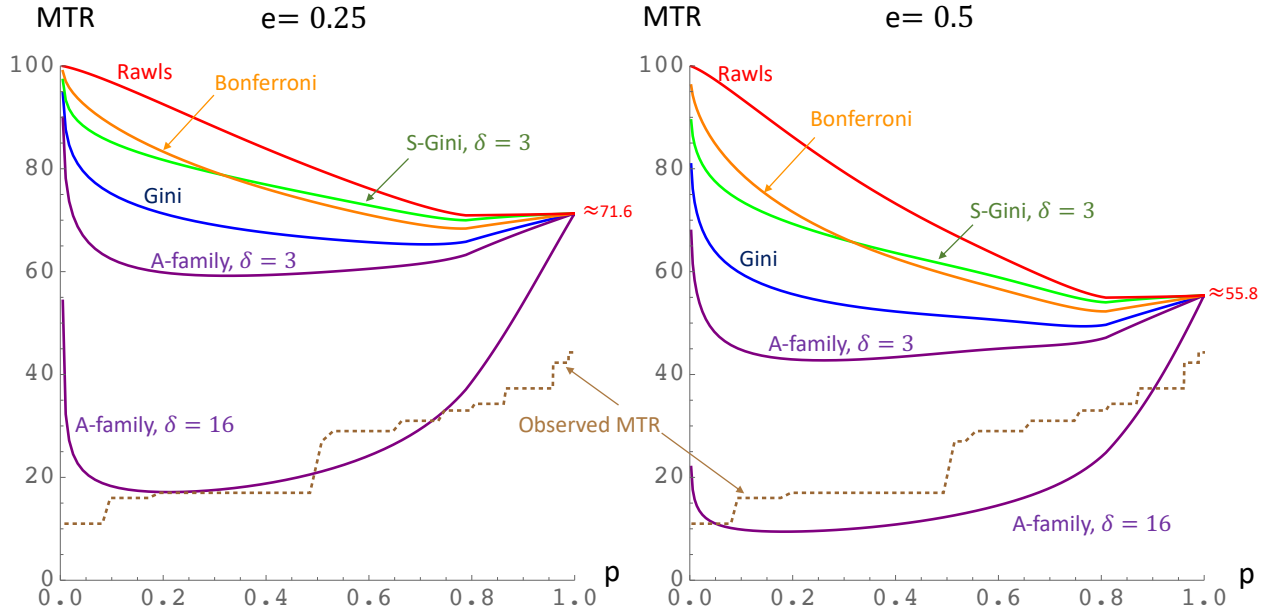


FIGURE III: OPTIMAL MARGINAL TAX RATES (IN %) BY RANK

Based on this numerical exercise, we hope to generate interest for “alternative” welfare functions which may be closer to the social planners’ actual preferences, as emphasized below, and pave the way to more precise and complete simulations, with better calibrations, which however are beyond the scope of the present insight paper.

Figure III shows optimal marginal tax rates for the the different families of rank-dependent social welfare functions considered above. The left panel is obtained for an elasticity  $e = 0.25$ ; the right panel for  $e = 0.5$ . We also report the “observed” marginal tax rates. On both panels and for the S-Gini, Gini and Bonferroni cases, the optimal marginal tax rates are decreasing for about 80% of the ranks, and then slightly increasing, converging to the Rawlsian marginal tax rate at the very top. When  $\delta$  increases, the marginal tax rates for the S-Gini family rapidly become very close to those obtained under the Rawlsian maximin. The pattern looks very different for the A-family. The marginal tax rates are really U-shaped, with a marked decline at the bottom, getting steeper the larger  $\delta$ , and then a sharp increase for ranks above .8. For example, when  $\delta = 16$  and  $e = 0.25$ , the marginal tax rates are multiplied by about two in the top 20% interval. On both panels, we see that the A-family generates the optimal tax profiles which are the closest to the observed one. When  $e = 0.5$  and  $\delta = 16$ , the optimal marginal tax rates are increasing for more than 90% of the ranks, with an initial decline that would probably be phased out if participation margins were accounted for; for the top 10% however, the optimal tax rates become significantly larger than the observed ones, with a difference going up to 11 percent points.

Figure IV shows the optimal and observed average tax rates when  $e = 0.25$ , for  $E = 0$  in the left panel and  $E = \bar{T}_{actual}$  in the right panel. Let us first focus on the Rawlsian, S-Gini and Bonferroni families. The average tax rates are close at ranks where they are positive. When the tax policy is purely redistributive as in the left panel, the average tax rates are negative for the bottom 75% and sharply increasing thereafter, reaching 71.3% at the top. When exogenous

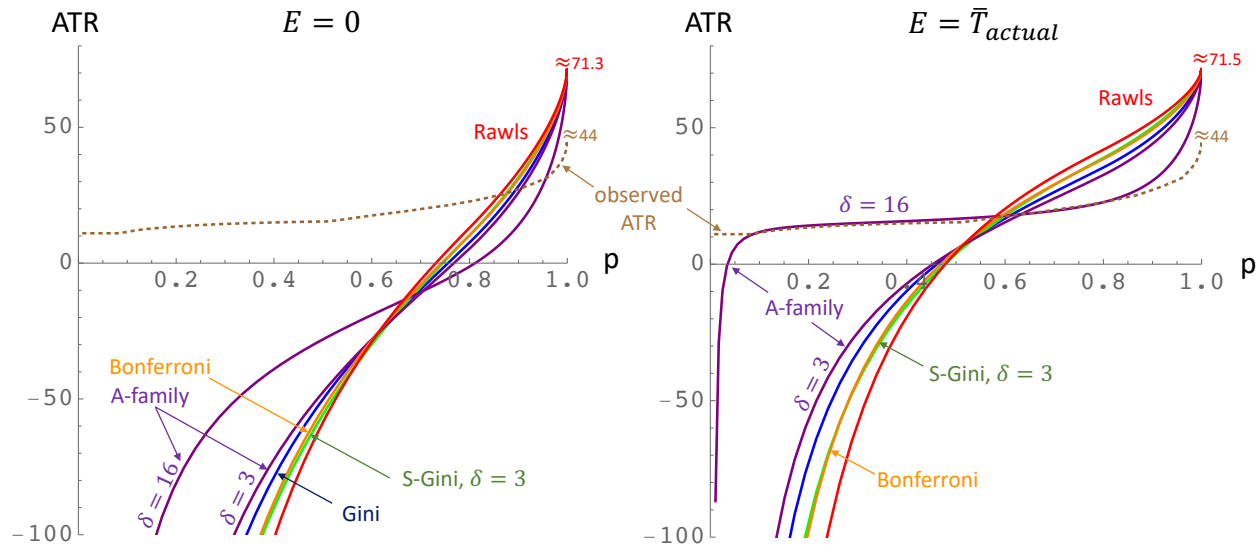


FIGURE IV: AVERAGE TAX RATES IN % BY RANK FOR  $e = 0.25$

taxes have to be levied as in the right panel, the “break even” point is shifted to the left, with positive tax rates for about 56% of the ranks. The differences are larger at ranks for which the average tax rate is negative. We now consider the A-family, which spans a much broader part of the space, compared to the other families. When  $\delta = 16$  and  $E = \bar{T}_{actual}$ , the optimal and actual schedules are extremely close for 75% of the ranks. As emphasized above, differences in the bottom 10% would probably disappear if participation margins were accounted for. The main discrepancy is therefore at the top, especially in the last decile, with observed rates up to 27.5 percent points lower than the optimal levels.

As emphasized above, Mirrlees (1971) wondered “what degree of inequality would remain once [an optimal income tax] was established”. We provide a graphical answer, assuming  $E = 0$ . We want to illustrate the trade-off between equity and efficiency for the various conceptions of equality considered above. The value of the inequality index  $I$  in  $\mathcal{W}$  as shown in Equation (3) will vary for each of these conceptions. To make a graphical representation, we must therefore choose one particular index as a common metric. We choose the popular Gini index. This is why Figure V shows equity measured as one minus the Gini coefficient of indirect utility, as a function of efficiency measured as the average indirect utility. We show two “frontiers”, corresponding to an elasticity  $e$  of 0.25 and 0.5 respectively. Each of them starts with the Rawlsian maximin on the upper left and ends up with the laissez-faire/pure utilitarianism on the lower right.<sup>8</sup> The first part of each frontier corresponds to the S-Gini family (in green) and the second part to the A-family (in purple). When indicated, the numerical labels correspond to the values of  $\delta$ . We note that the two frontiers are concave, and become roughly linear in the A-family part for large values of  $\delta$ . This implies “marginal decreasing returns”, especially in the left part, where aggregate resources have to be largely reduced in order to set slight gains in equity. We also see that the A-family spans much more diverse situations in terms of inequality than the S-Gini family. When  $e = 0.5$ , one minus the Gini coefficient of indirect utility varies by about .03 between the Rawls and Gini situations; and by 0.28 between Gini and Utilitarianism.

<sup>8</sup>Given that individuals have quasilinear preferences, the laissez-faire and pure utilitarianism coincide.



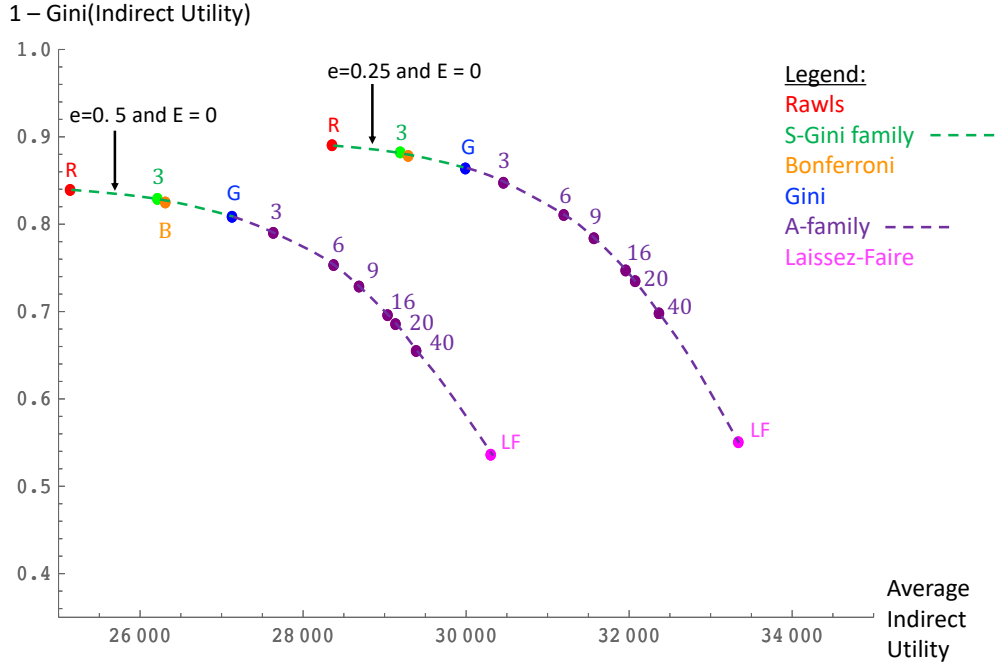


FIGURE V: EQUITY VS EFFICIENCY TRADE-OFF ( $E = 0$ )

Alternatively, we could show very similar graphs with one minus the Gini coefficient of net income as a function of average gross income (cf. Appendix). When  $E = \bar{T}_{actual}$ ,  $e = 0.25$  and  $\mathcal{W}$  is the Gini social welfare function, the Gini coefficient of net income amounts to 0.252 (against 0.481 in the actual distribution) and the mean gross income to \$42,916 (against \$51,410).

## VI. CONCLUSION

This article can be viewed as going one step further in the list of studies which, starting from Mirrlees (1971), have attempted to get more readable results for optimal non-linear income taxes. The main milestones on this road were Piketty (1997), Diamond (1998), Saez (2001) and Saez and Stantcheva (2016). Our contribution is to capture the ethical advantages of redistributing across ranks, a dimension which is completely exogenous to the optimal tax policy as long as agents optimize and there is no bunching. This allows us to express optimal tax rates in an extremely simple way.

The Rawlsian maximin is often used as a benchmark in the optimal taxation literature. We provide alternative benchmarks, based on ranks, and especially the Gini welfare function emphasized by Sen (1974), in which the average multiplied by one minus the Gini coefficient is maximized. We show that the S-Gini family spans the space between Gini and the Rawlsian maximin, while the A-family spans the gap between Gini and pure utilitarianism.

Optimal income taxation has often been criticized on the ground that it generates schedules with little resemblance to the actual ones. Our numerical simulations show that social welfare functions from the A-family are more likely to generate marginally increasing tax schedules for the entire population, as typically observed in reality (when focusing on the “payment”

side of the tax-and-transfer system). Moreover, for a specific population, we show that the US very unlikely optimizes with the Gini coefficient in mind, but rather with a member of the A-family, laying emphasis on differences occurring in the upper part of the distribution; however, top tax rates appear to be significantly larger than in the actual schedule. This calls for complementary and more comprehensive simulation exercises, for the US but also for other developed countries.

## Appendix (Meant to be Online)

Figure VI shows the trade-off between equity and efficiency with respect to two observable dimensions, gross income and net income.

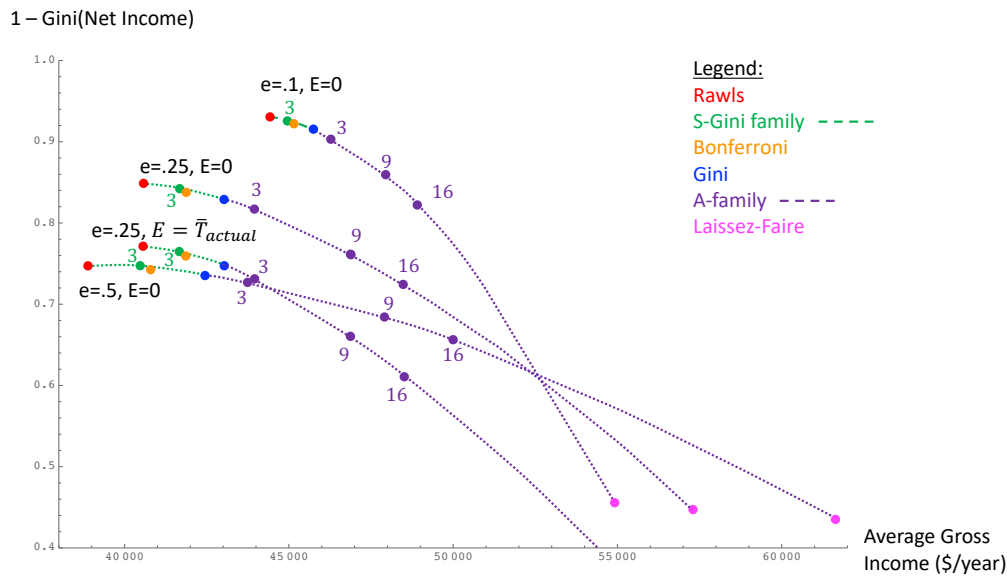


FIGURE VI: EQUITY VS EFFICIENCY TRADE-OFF

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