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Vesa Ronkainen

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FINANSSIVALVONTA
FINANSINSPEKTIONEN
FINANCIAL SUPERVISORY AUTHORITY



EUROJÄRJESTELMÄ
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Abstract

This work studies and develops tools to quantify and manage the risks and uncertainty relating to the pricing of annuities in the long run. To this end, an idealized Monte-Carlo simulation model is formulated, estimated and implemented, which enables one to investigate some typical pension and life insurance products. The main risks in pension insurance relate to investment performance and mortality/longevity development. We first develop stochastic models for equity and bond returns. The S&P 500 yearly total return is modeled by an uncorrelated and Normally distributed process to which exogenous Gamma distributed negative shocks arrive with Geometrically distributed interarrival times. This regime switching jump model takes into account the empirical observations of infrequent exceptionally large losses. The 5-year US government bond yearly total return is modeled as an ARMA(1,1) process after suitably log-transforming the returns. This model is able to generate long term interest rate cycles and allows rapid year-to-year corrections in the returns. We also address the parameter uncertainty in these models.

We then develop a stochastic model for mortality. The chosen mortality forecasting model is the well-known model of Lee and Carter (1992), in which we use the Bayesian MCMC methods in the inference concerning the time index. Our analysis with a local version of the model showed that the assumptions of the Lee-Carter model are not fully compatible with Finnish mortality data. In particular we found that mortality has been lower than average for the cohort born in wartime. However, because the forecasts of these two models were not significantly different, we chose the more parsimonious Lee-Carter model. Although our main focus is on the total population data, we also analysed the data for males and females separately. Finally we build a flexible model for the dependence structure that allows us to generate stochastic scenarios in which mortality and economic processes are either uncorrelated, correlated or shock-correlated.

By using the simulation model to generate stochastic pension cash-flows, we are then able to analyse the financing of longevity risk in pension insurance and the resulting risk management issues. This is accomplished via three case studies. Two of these concentrate on the pricing and solvency questions of a pension portfolio. The first study covers a single cohort of different sizes, and the second allows for multiple cohorts of annuitants. The final case study discusses

individual pension insurance from the customer and long-term points of view.

Realistic statistical long-term risk measurement is the key theme of this work, and so we compare our simulation results with the Value-at-Risk or VaR approach. The results show that the limitations of basic VaR approach must be carefully accounted for in applications. The VaR approach is the most commonly used risk measurement methodology in insurance and finance applications. For instance, it underlies the solvency capital requirement in Solvency II, which we also discuss in this work.

Key words: equities, stocks, jump model, bond, longevity, Lee-Carter model, stochastic mortality, cohort mortality, dependence model, asymmetric dependence, parameter uncertainty, stochastic annuity, pension, cohort size, solvency, internal model

JEL classification codes: G12, J11

Tiivistelmä

Väitöskirjatyössä tutkitaan ja kehitetään työkaluja eläkevakuutusten hinnoitteluun liittyvien riskien ja epävarmuuden kvantifioimiseksi ja hallitsemiseksi pitkällä aikavälillä. Tätä varten määritellään, estimoidaan ja implementoidaan idealisoitu Monte-Carlo simulointimalli, jonka avulla voidaan tutkia joitakin tyypillisiä eläke- ja henkivakuutustuotteita. Keskeisimmät riskit eläkevakuutuksessa aiheutuvat sijoitusten tuottojen ja kuolevuuden/pitkäikäisyyden kehityksestä. Ensin työssä kehitetään stokastiset mallit osakkeiden ja obligaatioiden tuotoille. S&P 500 indeksin vuosittainen kokonaistuotto mallinnetaan korreloimattomalla normaali-jakautuneella prosessilla, johon saapuu eksogeenisiä Gamma-jakautuneita negatiivisia shokkeja Geometrisen jakauman mukaisin odotusajoin. Tämä tilaa vaihtava hyppymalli ottaa huomioon harvinaiset erityisen suuret tappiot, joita empiirisesti on havaittu. USA:n 5 vuoden valtionobligaatioiden vuosittainen kokonaistuotto mallinnetaan ARMA(1,1) -prosessilla, jota sovelletaan sopivalla log-transformaatiolla muunnettuihin log-tuottoihin. Tämä malli kykenee tuottamaan pitkiä syklejä ja korjaamaan nopeasti vuosituottoja vastakkaiseen suuntaan. Myös mallien parametripävarmuus otetaan huomioon.

Seuraavaksi kehitetään stokastinen malli kuolevuudelle. Kuolevuuden ennusteisiin on valittu tunnettu Lee-Carter malli (1992), jossa aika-indeksin tilastollinen päättely tehdään Bayesilaisella MCMC metodilla. Mallista kehitetyn lokaalin version avulla tehty analyysi osoittaa, että Lee-Carter malli ei täysin sovi suomalaiseseen kuolevuusaineistoon. Erityisesti havaittiin, että kuolevuus on ollut keskimääräistä pienempää sota-aikana syntyneellä kohortilla. Koska kyseisten mallien tuottamat ennusteet eivät kuitenkaan merkittävästi poikkea toisistaan, päädyttiin käyttämään yksinkertaisempaa Lee-Carter mallia. Vaikka työssä pääpaino on väestön kokonaiskuolevuudessa, analysoitiin aineistoa myös erikseen miehille ja naisille. Lopuksi kehitetään joustava malli riippuvuuksien rakenteelle, jonka avulla voidaan generoida stokastisia skenaarioita, joissa kuolevuus ja taloudelliset prosessit ovat joko korreloimattomia, korreloituneita tai shokkikorreloituneita.

Simulointimallia käyttäen voidaan generoida stokastisia eläkkeiden kassavirtoja ja analysoida pitkäikäisyysriskin rahoitusta eläkevakuutuksessa ja siihen liittyviä riskinhallintakysymyksiä. Tätä tarkastelua tehdään kolmessa tapaustutkimuksessa. Kaksi niistä keskittyy eläkevakuutusportfolion hinnoittelu- ja vaka-

varaisuuskysymyksiin. Ensimmäisessä käsitellään yhtä annuiteettikohorttia, jonka koko vaihtelee, ja toisessa tarkastellaan useampia kohortteja. Viimeisessä tapauksessa keskitytään yksilölliseen eläkevakuutukseen asiakkaan ja pitkän aikavälin näkökulmista.

Riskien realistinen tilastollinen kvantifointi pitkällä aikavälillä on työn keskeinen tavoite ja sen vuoksi simulointituloksia verrataan ns. Value-at-Risk eli VaR-menetelmään. Tulokset osoittavat, että yksinkertaisen VaR-lähestymistavan rajoitukset on otettava huolellisesti huomioon menetelmää sovellettaessa. VaR-metodi on rajoituksistaan huolimatta yleisimmin käytetty riskinmittausmenetelmä vakuutus- ja finanssisovelluksissa. Siihen perustuu esimerkiksi Solvenssi II:n vakavaraisuuspääomavaatimus, jota myös käsitellään työssä.

Avainsanat: osakkeet, hyppymalli, obligaatio, pitkäikäisyys, Lee-Carter malli, stokastinen kuolevuus, kohorttikuolevuus, riippuvuusmalli, epäsymmetrinen riippuvuus, parametripävarmuus, stokastinen annuiteetti, eläke, kohortin koko, solvenssi, sisäinen malli

JEL luokitus: G12, J11

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Vesa Ronkainen

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1 Introduction

1.1 Motivation

The importance of mortality/longevity modeling and forecasting is nowadays globally acknowledged due to the increasing financial burden that lower mortality means for national social and insurance systems (cf. Holzmann and Palmer, 2006). Moreover, there is strong evidence that the traditional deterministic approaches of demographers and actuaries have not proven adequate for longevity risk measurement and management. This is evidenced in several studies in a number of countries (e.g. Keilman, 1990, Alho, 1990 and Keilman et al., 2008). Even more alarming is that the forecast errors have been invariably on the wrong side, i.e. the forecasts have generally underestimated the improvements in life expectancy. Moreover, the experts' judgements in forecasting have made things even worse. The following quotation of Alho et al. (2011), sec. 3.2., makes these points clear:

‘There appears to have been widely held support of ‘diminishing returns’ for mortality. Ever-increasing resources were thought to be needed to achieve improvements similar to those in the past... In fact, the conclusion to be drawn is that simply allowing past declines to continue would have made mortality forecasts more accurate... This is an observation that speaks in favor of trend-based statistical modelling’.

We have chosen to forecast mortality with the well-known statistical model by Lee and Carter (1992) in which we use the Bayesian MCMC methods in the inference concerning the time index. The analysis based on our local version of the model showed that the assumptions of the Lee-Carter model are not fully compatible with Finnish mortality data. In particular, we found that mortality has been lower than average for the cohort born in wartime. However, because the forecasts of these two models do not differ significantly, we prefer the more parsimonious Lee-Carter model. Although our main focus is on the total population data, we have also analysed the data for males and females separately.

It is not enough, however, to study only longevity when dealing with pension insurance risk management. The other side of the coin relates to the financing of longevity risk. Insurance companies must estimate their liabilities and have sufficient assets to cover

them. Additionally, they are required to have capital or solvency buffers for the risks they face. Equity investments, which play an important role in life and pension insurance operations due to the anticipation of higher returns from equities than from bonds, are prone to white noise random variation. However, in our data we also observe shocks or exceptionally large losses. We address this by using a Bernoulli-mixture model where exogenous Gamma-distributed negative shocks impact on an uncorrelated process of equity returns. The 5-year US government bond yearly total return is modeled as ARMA(1,1), after suitably log-transforming the returns. This model is able to generate long term interest rate cycles and allows for rapid changes of direction in returns. We also address the parameter uncertainty in these models and build a dependence model that enables generation of stochastic scenarios in which mortality and economic processes are either uncorrelated, correlated or shock-correlated.

The simulation model is then used to analyze three case studies. Two cases concentrate on pricing and solvency questions for a pension portfolio, and the third deals with individual pension insurance from the customer and long-term points of view. Our case studies also point out some weaknesses of Solvency II and areas where internal models could be especially useful. Based on our studies we believe that pension insurance risk modeling is better done using a simulation model than by a formula-based approach. Our model takes a long-term risk management view of pension insurance, and can be used to supplement insurance companies' own risk and solvency assessments.

In general, risk management issues are becoming better recognized in the international insurance and accounting regulations. In the EU, the rules for insurance companies are to be implemented via the new Solvency II Directive (2009/138/EC) in around 2014, and the other rules are included in the IFRS accounting standards. We next discuss the above-mentioned topics in more detail and draw some implications for this study.

1.2 Pension insurance and risk management

The general idea of a pension insurance scheme – whether a pension insurance contract in a life insurance company, a pension rule in

a pension fund, or a state pension based on national legislation – is to transfer the premiums or contributions and asset returns into pensions. The process is inherently stochastic, which means that few if any aspects of the system can be known in advance with certainty. We must thus face up to uncertainty, which calls for the toolkit for risk management.¹ Generally speaking, risk management includes the identification, analysis and evaluation of all risks, and finding the best methods of dealing with them. Risk management actions may include the elimination, sharing, mitigation, or buffering of risks. We give examples of these concepts below. Moreover, a risk may be diversifiable or nondiversifiable. In the latter case, pooling arrangements are ineffective for reducing risk to the participants in the pool.

In the pension insurance context it is generally the case that not all the risks can be eliminated from the pension system, but they can be shared or mitigated in various ways among the stakeholders. We distinguish three stakeholders: sponsor, beneficiary and insurer. The sponsor has agreed to pay the insurance premium(s) for the pension insurance contract. The insurer is in charge of providing the pension insurance services according to the contract, which include setting the premiums adequately and managing the assets prudently, so that the pensions can be paid as agreed. The beneficiary, after retiring, gets the pension benefits when the savings are transformed into an annuity. He can also be the sponsor, depending on the pension scheme in question. If a death benefit is included, there can be several beneficiaries. We give some examples to illustrate these concepts and risk analyses.

In a Defined Benefit (DB) scheme, there is a fixed rule for calculating the benefits. The sponsor then faces the risk that the contribution level of the pension scheme may become too high or too volatile. One important means of managing this risk is a Defined Contribution (DC) scheme where the contribution rule is fixed in advance. It is an example of a risk sharing arrangement between the sponsor and the beneficiary which leads to a significant risk transfer to the beneficiary.

In a funded pension system premiums have to be invested until they are paid out as pensions. In DB schemes this introduces a

¹Vaughan and Vaughan (2008) define (p. 2–3) risk as a condition in which there is a possibility of an adverse deviation from an expected or desired outcome. The existence of risk creates uncertainty on the part of individuals when that risk is recognized.

market risk for the insurer because asset returns are uncertain. To manage these risks, the insurer – an insurance company or a pension fund – must hold adequate reserves and capital resources. A risk management tool that many life insurance companies have recently been using extensively, e.g. in the Nordic countries, is to transfer market risk to beneficiaries via Unit Linked insurance. Unit Linked insurance is often in essence a mutual fund re-packaged as insurance.

During retirement, the annuitants may face uncertainty as to the level and variability of their pension. This in turn depends on the pension rule applied. We assume whole-life annuities, i.e. we do not study temporary annuities in this work. Therefore the main source of uncertainty for the annuitant is in the pension index rule.

In whole life annuities either the insurer or the sponsor or both face longevity risk. Insurance companies and pension funds can mitigate longevity risk via reinsurance arrangements or mortality bonds etc investment operations (cf. McWilliam, 2011). Insurers that offer both life insurances and annuities benefit from this product diversification because mortality risk and longevity risk then partially offset.

The three alternative ways to manage longevity risk in the national pension systems are to adjust the retirement age or the premiums upwards or the pension level downwards. A Finnish example that shows how longevity risk can explicitly and transparently be taken into account via an annuity longevity adjustment factor is discussed in Alho and Spencer (2005), sec. 11.1. Different pension indexation rules can be studied in the same context (cf. Alho et al., 2011).

There are plenty of options for the design of a pension scheme: DB, DC, Notional DC (NDC), full funding, partial funding, pay-as-you-go (PAYG), etc. These concepts are discussed e.g. in Holzmann and Palmer (2006). Every pension scheme can be characterised by its risk profile vis-à-vis sponsor, insurer and beneficiary, if appropriate stochastic modeling and forecasting tools are available. For example one way of illustrating the uncertainty and fair sharing of risk from the point of view of both sponsor and beneficiary is to present both contributions and pensions and their uncertainty in the same graph and ask which policies would be acceptable to both parties (cf. the concept of viability region in Alho et al., 2005).

We give some examples of pension insurance risk management from the point of view of the sponsor and insurer in the first two case studies of Chapter 6 where we discuss premium and solvency buffer issues. The third case study recognizes that a pension insurance contract is also a risk management tool for an individual acting as a

sponsor, i.e. when purchasing and paying premiums on an individual pension insurance. It allows him or her to manage investment and/or longevity risk in exchange for the insurance premium. An insurance company and pension fund of sufficient size can pool the variance of individual life times (diversifiable or idiosyncratic risk), although the aggregate uncertainty of increasing longevity will remain and must be managed. Moreover, the forecasting horizon has a major impact on the aggregate or systemic risk, which is nondiversifiable. We address these questions in the final case study.

1.3 Solvency II

A new insurance supervisory system called Solvency II will replace the current rules in the EU in 2014. At the same time important global development of insurance regulation is being carried out by the International Association of Insurance Supervisors (IAIS). In this section we briefly review the key new features of Solvency II. The fundamental differences between Solvency II and Solvency I are summarized in the following two tables.

Table 1.1 Solvency II vs Solvency I valuation principles

Balance sheet	Solvency II	Solvency I
Assets	market value	book value
Liabilities	BE + RM	prudent deterministic value

The short-hand notations for valuation in Table 1.1 are defined as follows. BE = Best Estimate = present value of expected future cash-flows,² RM = risk margin. The prudent deterministic value is the result of traditional actuarial valuation methods, where best estimates and risk margins are not separated and the level of prudence is not defined for the calculation of technical provisions. Book value refers to historical cost based accounting, where the market values of assets can differ from the values shown on the balance sheet or the book values. In Solvency II the value of insurance liabilities or technical provisions is calculated using market consistent methods

²We note that Solvency II directive (2009/138/EC), Article 77.2, uses the expected present value of cash-flows, while in a more recent document (QIS5 specification) the present value of expected cash-flows is used. These definitions give in general different results when a stochastic interest rate model is used.

and assumptions.³ Modeling is necessary because directly observable market prices for insurance liabilities are not generally available.

Table 1.2 Solvency II vs Solvency I capital requirements

Solvency II	Solvency I
risk-based VaR	constant

In Table 1.2 solvency capital requirements are compared. VaR denotes the Value-at-Risk or quantile risk measure, which is discussed in a separate section below.

We conclude this brief comparison by noting that the key new features of Solvency II are 1) more consistent, economic, transparent and harmonized valuation, 2) risk-based capital requirements, and 3) more harmonized supervision. These measures, together with a number of qualitative tools included in Solvency II (SII), are aimed at better risk management and supervision of insurance companies, for the benefit of the European internal market of insurance services (cf. Linder and Ronkainen, 2004, for an early discussion of the goals of SII, and Sandström, 2006, for the background of SII).

Consequently, SII builds on a market consistent balance sheet, where Own Funds (OF) comprise the difference between assets and liabilities. How OF or Net Asset Value (NAV) changes under various risk scenarios is then analyzed for each risk type (market, underwriting, credit and operational risk). Finally, the results are aggregated via a number of correlation matrices to arrive at the Solvency Capital Requirement (SCR). The tentative standardized calculation rules for SCR, which aim at limiting the probability of OF getting negative in 1 year to 0.5 percent, are given in QIS5 Technical Specifications. These specifications, which are available at

³Regarding the concept of market consistency we note the following quote of the European actuaries' parent organization Groupe Consultatif on their website www.gcactuaries.org/market_consistency/index.html: The term 'market consistent' has become increasingly popular as a description of liability valuations or, more generally, of cash flow valuations. However, there is no widely accepted definition of the term. A valuation algorithm is a method for converting projected cash flows into a present value. A valuation algorithm may be specified with reference to a set of calibration assets. We say a valuation is market consistent if it replicates the market prices of the calibration assets to within an acceptable tolerance. Market consistent valuation can take many forms. Two different models could produce different liability valuations and yet still both are market consistent.

www.eiopa.eu, include the detailed calculation rules. We summarize briefly some key notions.

SII technical provisions generally consist of the best estimate and the risk margin. The risk margin in SII approximates the amount that would have to be paid in addition to the BE in order to transfer the portfolio to another insurer. It is calculated by determining the cost of providing an amount of Own Funds equal to the SCR necessary to support the insurance obligations over their lifetime, given the Cost-of-Capital rate (currently 6 percent). In practice many simplifying assumptions are needed to determine the RM. However, under certain conditions that relate to the replicability of the cash flows underlying the insurance obligations, BE and RM are not valued separately but as a whole.

In Solvency II the yield curve derived from the swap market⁴ for various maturities is used for discounting the cash flows, to which recently an illiquidity premium and an equilibrium modification for the very long term rates has been added (cf. QIS5 Technical Specification at www.eiopa.eu). Therefore interest rates are different for different maturities and are observed directly from the market. In Solvency I, on the other hand, (initially) prudently set constant discount interest rates are used in life insurance when calculating the technical provisions (cf. Table 1.1).

In calculating the SCR in the SII standard approach, the best estimate or expected value assumptions are changed, typically by a given percentage amount upwards and downwards, to determine the prediction interval at time $t+1$ for the various risks and their so-called ΔNAV impact, i.e. the change of Own Funds. For instance, the interest rate risk, which is very important in life and pension insurance, would be handled as follows. The yield curve for various maturities at time t , YC_t , is given, as are the upward stress YC_t^{up} and downward stress YC_t^{down} . To determine ΔNAV for upward stress we revalue the assets and liabilities using YC_t^{up} in the discounting instead of the current market yield curve YC_t . Another calculation would be done for the downward stress, and the worse of the two stresses would be chosen. Along the same lines we would determine the ΔNAV impact of other risks, and finally aggregate the individual risk charges via correlation matrices to get the total solvency capital

⁴Interest rate swap is an agreement to exchange fixed-rate payments for floating-rate payments. Swap curve can differ substantially from the government bond yield curve during financial crises, cf. CEIOPS/EIOPA (2010).

requirement SCR, taking into account any risk mitigation tools that have been used.

Alternatively, the insurance company may develop its own stochastic model for calculating the SCR or part of it. Insurance companies have plenty of freedom when developing their internal models. However, these models have to be first approved by the supervisory authorities and fulfill a number of requirements. For a broader discussion, see Solvency II Directive 2009/138/EC, and for a review see e.g. Cruz (2009), Ronkainen et al. (2007), or Ronkainen et al. (2009). Next we review the VaR concept in more detail.

1.4 Value-at-Risk (VaR)

Realistic statistical modeling of equity and bond returns and mortality is the key theme of this work. We will also frequently make use of the Value-at-Risk or VaR approach, which is the most commonly used risk measurement methodology in insurance and finance applications. For instance it underlies the capital requirement SCR in Solvency II. The basic idea of VaR is to give a reserve amount that is sufficient to compensate potential adverse changes in prices with a high probability.

A standard reference in a large literature on VaR is the RiskMetrics Technical Document (Fourth Edition, 1996), available at www.riskmetrics.com. Measurement of the market risk of an asset portfolio of value W_t at t consists of modeling the asset return processes and their dependence structure over the period $(t, t+s]$. From the resulting probability distribution of profit and loss, one then chooses the value that corresponds to the chosen quantile point $1-\alpha \in (0, 1)$. In other words we require that

$$Pr(W_{t+s} - W_t + VaR_\alpha < 0) = 1 - \alpha. \quad (1.1)$$

In practice VaR is typically assumed independent of t and the multivariate Normal distribution is used. In the bi-variate case, which is sufficient for our purposes, we thus assume $X = [X_1, X_2]' \sim N(M, \Sigma)$, where $M = [\mu_1, \mu_2]'$ is the mean vector, and Σ the variance-covariance matrix with elements $\sigma_{ij}, i, j \in \{1, 2\}$. Then σ_{11} and σ_{22} are the variances of X_1 and X_2 , and $\sigma_{12} = \sigma_{21} = \rho\sqrt{\sigma_{11}\sigma_{22}}$ their covariance with correlation coefficient ρ .

Using X_1 and X_2 to describe the yearly returns on equity and bond investments with the fraction $h \in [0, 1]$ of equities in the portfolio, we can calculate the VaR for 1-year horizon with the confidence level α as follows. The variance of the portfolio is

$$\sigma^2 = h^2\sigma_{11} + (1-h)^2\sigma_{22} + 2h(1-h)\rho\sqrt{\sigma_{11}\sigma_{22}}, \quad (1.2)$$

so that

$$VaR_\alpha(X) = k_\alpha\sigma - (h\mu_1 + (1-h)\mu_2), \quad (1.3)$$

when k_α corresponds to the chosen $1 - \alpha$ quantile of the Normal distribution.

The more general case of 3 or more random variables can be handled in the same fashion. Risks other than market risks can also be addressed via the VaR approach, as is done for the SCR in Solvency II. However, many simplifying assumptions are typically used in the VaR process, which may significantly affect the accuracy of the risk measurement, as we will see in the following chapters. The VaR approach has serious structural limitations in its basic form. Firstly, it assumes Normally distributed random variables, and secondly, it only allows for linear correlation in the dependence structure. However, with more advanced modeling of joint probability distribution these shortcomings can be circumvented. There are also other smaller problems with the VaR risk measure, and so other risk measures have been developed. We return to this matter in the case studies. For a broader discussion on VaR and its limitations and generalizations see e.g. Malz (2011) and the references therein.

1.5 Insurance modeling

Insurance and financial modeling is clearly becoming increasingly important, as Solvency II will for the first time allow insurance companies to use their own stochastic models to calculate the regulatory capital requirement. These internal models must first be approved by the supervisors, and various tests and standards are applied. These concern e.g. the statistical quality, calibration and validation of the model (cf. the references mentioned above). Modeling skills and knowledge of good modeling practices are

therefore necessary for the insurance companies and for the supervisors.

Various types of stochastic models are currently employed in the insurance sector for many purposes such as economic capital calculations, profit calculations for shareholders and product lines, asset modeling, liability valuation, and asset and liability or ALM analyses of the whole balance sheet (see e.g. IAA, 2010, for further discussion).

The goal of this work is to develop a realistic but yet parsimonious stochastic model which enables quantification and forecasting of the chosen key risks and their interactions in pension insurance. As a first step we simplify the risk profile of the pension insurer to include just 3 major risks: mortality risk and the risks of equity and bond investments. These risks are typically very important in life and pension insurance (cf. EIOPA report on the fifth Quantitative Impact Study for Solvency II, available at <https://eiopa.europa.eu/>). There are other risks that confront pension insurers, e.g. credit risk and the risks of expenses and policyholder behaviour, which we do not analyse in this study.

We take a long term (several decades) view regarding risk management. This is necessary in order to quantify mortality improvements, which happen only gradually. A long term view is also needed in order to analyse the risks marked by low frequency but large impact (equity shocks) or long term cycles (interest rates). The question of modeling horizon is of fundamental importance, and it has several important implications.

Firstly, we want to use the longest relevant data series available that are of good quality. This has lead us to use the US market index data from 1925 to 2006 for the equity and mid-term government bond models. Our assumption is that these data are representative of the underlying stochastic processes for equity and bond prices. If this is the case, the description of uncertainties relating to asset management of a pension insurer can in principle be done by stochastic models based on these two data series. In practice this is an idealized assumption. The data series alone do not provide sufficient information for our long term forecasting purposes. Therefore a full data-based approach is not possible. This will be seen in particular in developing and applying the interest rate model, where we have to make assumptions about stationarity and stylized facts of the process. Another practical limitation concerns the number of modeled assets. In practice insurers usually invest in several asset classes. Therefore

either the model parameters have to be adjusted or new models have to be developed for assets not included in our model. For the mortality model we use the data from the Human Mortality Database (cf. www.mortality.org). This data covers over 100 years for Finland. We use approximately the last 50 years of the data in order to focus on more recent developments in mortality.

Secondly, we use yearly data to capture the features that are relevant for long-term forecasting. Use of yearly data leads us to apply discrete time series models, which we specify, estimate and validate according to the modeling steps discussed in Box et al. (1994), and depicted in Figure 1.1.

An important but a difficult part of modeling is the dependence structure, which we address below. Currently ad hoc approaches are often used in practice. Furthermore, the theoretical tools for handling situations of multivariate non-normal and time-varying dependences, are not yet well developed. Our solution to this problem is to allow a flexible dependence structure between the chosen 3 risks for stochastic scenario simulations. We allow the risks to be uncorrelated, correlated or shock-correlated.

The final part of the study deals with applications. We illustrate how the model can be used for analysing practical problems faced by insurance companies, pension funds, and pension product developers. The first case study concentrates on the single premium calculation and the risk and solvency assessment of a whole life unit annuity for a cohort of 65 year old persons, where the cohort size varies. The second study is an extension of the previous case to the situation of large multiple cohorts. The final case study discusses individual pension insurance from the customer and long-term points of view.

Our discussion follows the outline above. In other words we start with the equity index model, then move to the bond index model, then to the mortality model, and finally we add the dependence structure before concluding with the case studies and discussion.

As a final introductory remark we note that in insurance and risk management applications also various types of models are frequently applied other than those that we have chosen to study. Those models typically use more frequently collected data and are formulated in continuous time. The model parameters can be chosen to fit the current prices of various securities and derivatives, instead of being estimated statistically from historical data. This applies in particular to the market consistent valuation of liabilities according to Solvency II. An important part of the market consistent valuation is an interest

rate model that includes the whole yield curve (cf. IAA, 2010, or Panjer and Boyle, 1998). In contrast, we only model the 5-year bond total return index, which is also used in discounting the cash-flows. However, one has to remember that all models give a simplified image of reality, which is much too complex to be described exactly. Therefore expert views, parsimony and ease of use are important factors when choosing and using risk models.

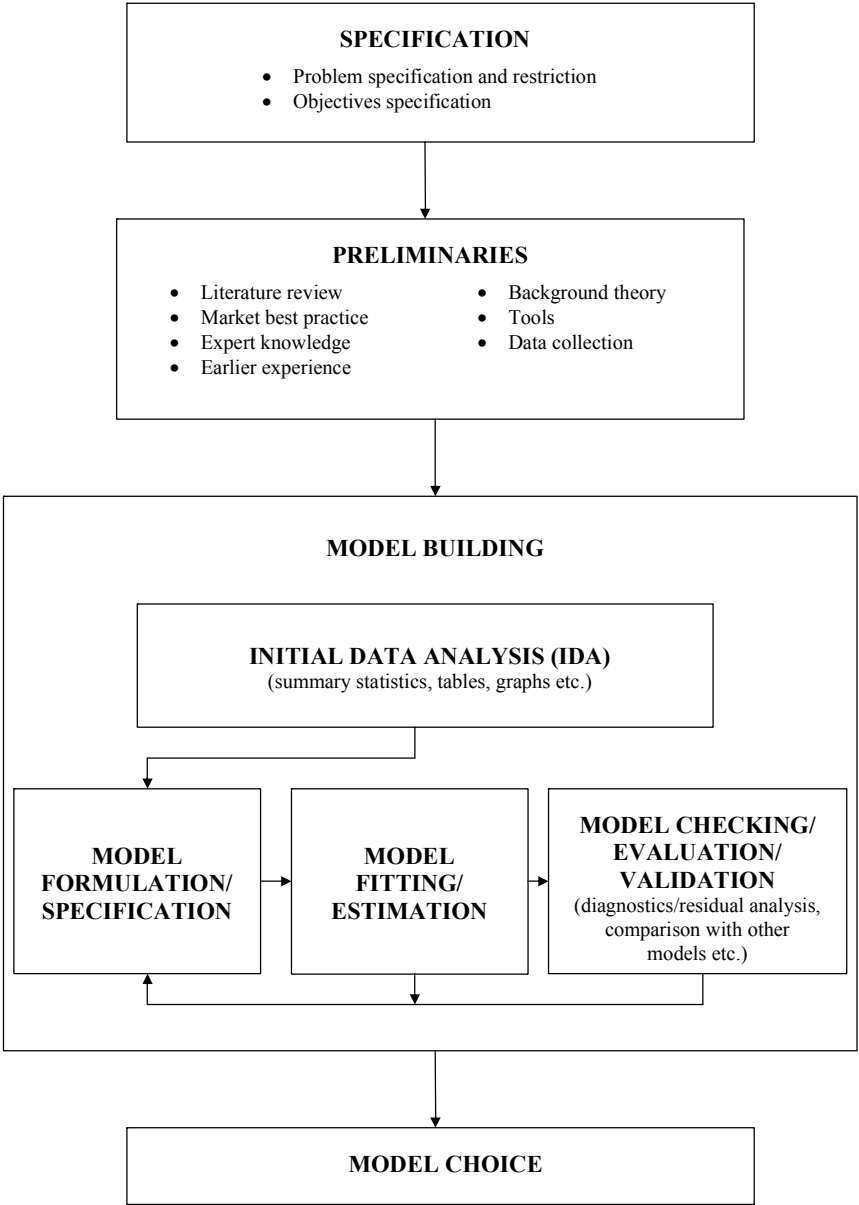


Figure 1.1 Box-Jenkins modeling steps

2 Equity index model

We take a statistical point of view to risk management and assume that uncertain future is well described by history. Therefore our goal in equity index modeling is to be able to generate long term simulations (up to 80 years) that have approximately similar distributional and dynamic features that can be observed from the chosen reference data set for equity returns.⁵ The reference data should form a suitable basis for long-term forecasting and risk management and should adequately approximate a well-diversified equity portfolio of an insurance company and its clients (in case of unit linked business).

In this chapter⁶ we first analyse the S&P 500 equity market data and review the most common models for the equity returns. Subsequently we develop a jump model for the equity returns and estimate it using Maximum Likelihood and Markov Chain Monte-Carlo methods. In the final section we present forecasts generated by the model.

2.1 Data on equity returns

2.1.1 General characteristics of long-term equity returns

Our data series for the equity returns is the S&P 500 Total Return Index as at year-end $t=0, \dots, 81$, where $t=0$ corresponds to year 1925, as given in Table 5-1 on pages 102–103 of Morningstar (2007). This index, denoted SP500, is expressed in nominal values (starting at 1.00 at the end of 1925 and reaching 3077.33 at the end of 2006), and it includes the effect of reinvested dividends. The SP500 consists of 500 large U.S. stocks weighted by their monthly market values.⁷

⁵We restrict our attention solely to equity market data and do not consider any other explanatory economic variables.

⁶This chapter is based on Ronkainen and Alho (2009).

⁷For more details see Chapter 3 in Morningstar (2007).

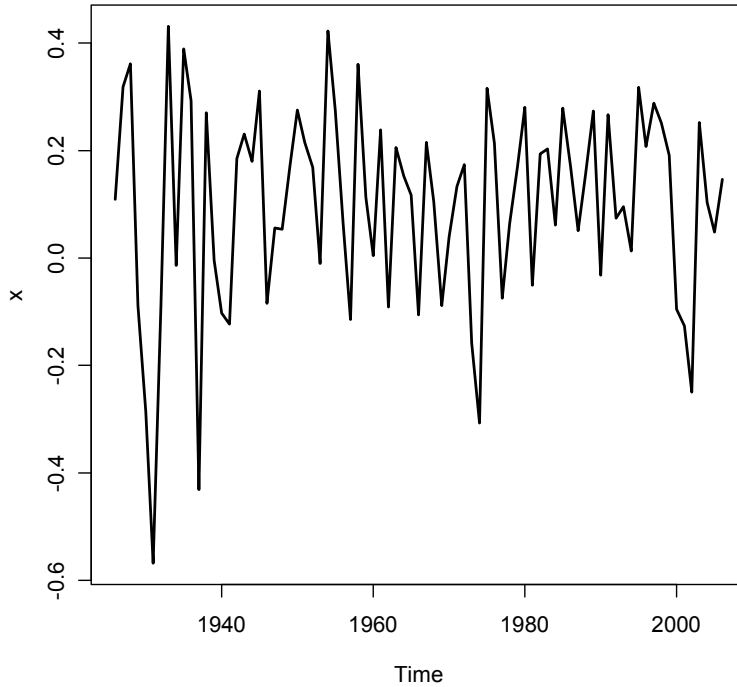


Figure 2.1 SP500 log-returns for 1926–2006

Figure 2.1 shows the yearly log-return series of SP500, defined as the difference in successive values of the natural logarithm of the index. A histogram of the log-returns is given in Figure 2.2. The summary statistics of SP500 log-returns are as follows: the sample mean is 0.099, standard deviation 0.192, skewness -0.853, and kurtosis 3.893. From the histogram and the skewness statistic we observe that the returns are skewed to the left, and kurtosis indicates a higher probability of extreme values than in a Normal distribution.

The autocorrelations and partial autocorrelations of SP500 log-return series are given in Figures 2.4 and 2.5, and the autocorrelations of the squared log-returns in Figure 2.3. From these data we observe that autocorrelation is weak for the log-returns but more significant for the squared log-returns.

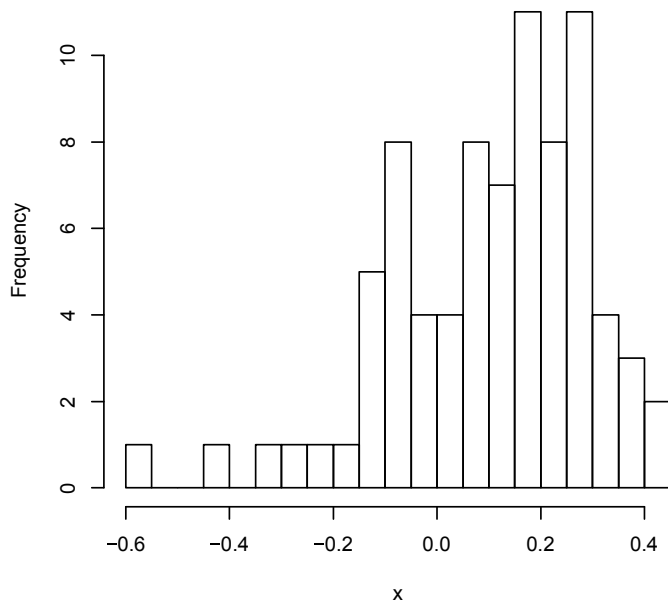


Figure 2.2 Histogram of SP500 log-returns for 1926-2006

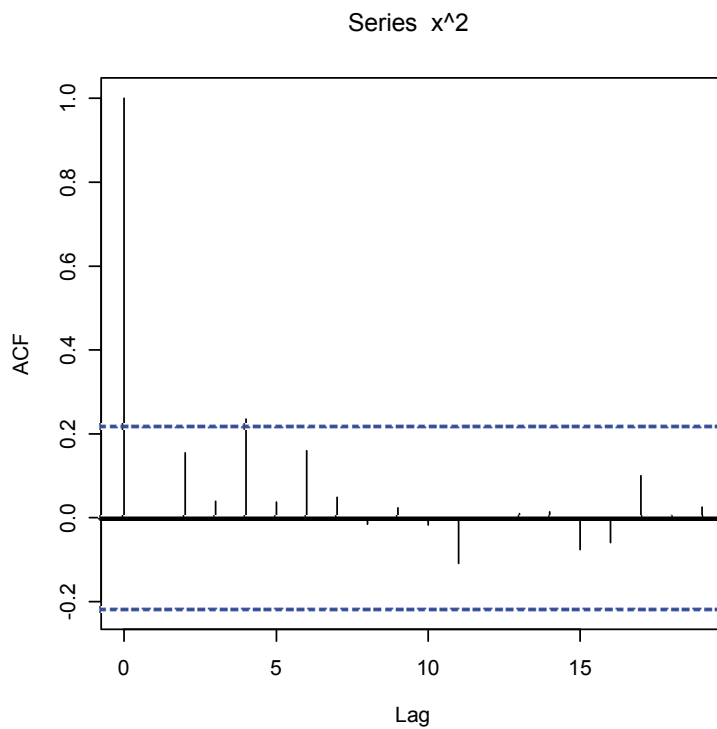


Figure 2.3 ACF of squared SP500 log-returns for 1926–2006

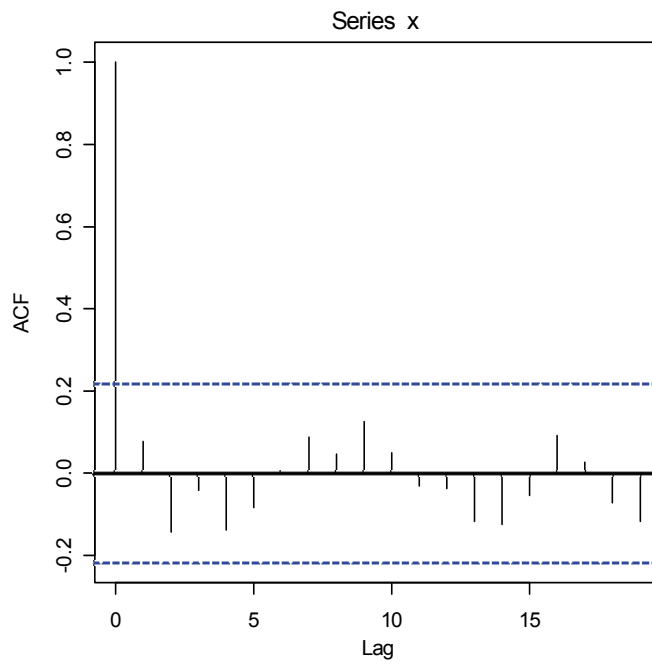


Figure 2.4 ACF of log-returns of SP500 for 1926–2006

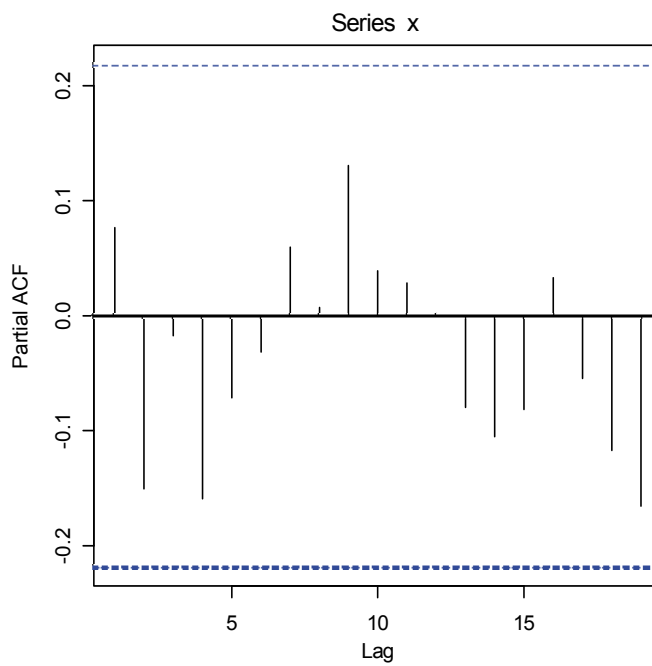


Figure 2.5 PACF of log-returns of SP500 for 1926–2006

Koskela et al. (2008) have analysed in chapter 4 the data in detail using ARIMA and GARCH models. Their findings can be summarized as follows:

1. For the yearly data from 1925-2006 the following ARCH(1) model gives the best fit of ARIMA and GARCH models, as judged by the information criteria AICC or AIC or BIC:⁸

$$p_t - p_{t-1} = 0.1163 + \epsilon_t, \quad (2.1)$$

where p_t is the log of SP500 value for $t = 0, \dots, 81$, $\epsilon_t = z_t \sqrt{h_t}$ with independent $z_t \sim N(0, 1)$, and $h_t = 0.0183 + 0.5829\epsilon_{t-1}^2$.

2. The yearly data from 1955-2006 appear to be uncorrelated and the best model is:

$$p_t - p_{t-1} = 0.1002 + \epsilon_t, \quad (2.2)$$

where $\epsilon_t \sim N(0, 0.0234)$.⁹

We note that the sample period matters. From 1955 onwards an uncorrelated model gives a good description of the data. On the other hand, by analysing the series with some of the data deleted, we find that it is approximately the first 10 years, i.e. the period from 1926 to 1935, that caused the ARCH(1) model to be chosen in point 1 above. However, one serious problem with the ARCH(1) model is its symmetry: it treats losses and profits in the same way. This is not what we observe in the data (cf. Figures 2.1 and 2.2 and the skewness statistic). Asymmetry and fat tails are well-known empirical findings for equity returns, although no commonly agreed scientific theory exists to explain these stylized facts (cf. Kaliva, 2011, Chen et al. 2001).

We conclude that it is necessary to take into account the possibility of very bad losses for risk management purposes, but in our view an asymmetric model is a better alternative than a symmetric ARCH model. We now turn to models that are able to describe downward jumps.

⁸These terms are defined for instance in Brockwell and Davis (2002) on page 173.

⁹For the monthly data from 1955-2006 a GARCH(1,1) model is the best.

2.1.2 Review of jump models for equity returns

Infrequent equity market crashes can be modeled by a jump process. The classical example of this approach is the model of Merton (1976), which is specified in continuous time. This model adds log-normal jumps to a diffusion process according to a Poisson process. Maximum Likelihood-based comparative analyses of this and some other jump model classes has been carried out for weekly and monthly equity market data from June 1973 to December 1983 in Jorion (1988). His analysis concludes that a simple diffusion model is chosen for monthly stock returns over a jump diffusion, a jump-ARCH and an ARCH model, and that for weekly data the jump-diffusion model is a significant improvement over the simple diffusion model.

A simplified modeling approach in discrete time has been suggested for stock returns by Ball and Torous (1983). In their approach the Bernoulli process is used for the jump times instead of the Poisson process, and the resulting model is a Bernoulli mixture of Gaussian densities for the daily stock returns.

Ramezani and Zeng (1998) apply a continuous time asymmetric jump-diffusion process to equity prices. Their model assumes that good news and bad news arrive according to two Poisson processes, and that the jump sizes are Pareto and Beta distributed. Another, more recent jump model specification of this type is the double exponential Poisson jump diffusion, first proposed by Kou (2002) for option pricing applications. In the context of modeling the default risk in corporate bonds when the asset values may have jumps, Hilberink and Rogers (2002) model only the negative jumps with an Exponential distribution.

Our model, developed below, observes discrete time for the yearly equity log-returns, in contrast to the above mentioned models, which are designed for shorter term applications and use more frequently sampled data. As in Ball and Torous (1983), we use the Bernoulli process for the jump times. We apply a similar idea as in Hilberink and Rogers (2002) in that we only consider negative returns in the jump term. However, in our model there is a coefficient to eliminate the effect of jump years from the normal years (cf. $(1 - J_t)$ in (2.3)), which is not used in the above mentioned models. Moreover, our model is formulated for Gamma-distributed jumps.

2.2 Model specification and preliminary estimation

2.2.1 Definition of the model

Denote the log of SP500 year-end index as p_t , $t = 0, 1, \dots, T=81$, where $t=0$ corresponds to year 1925. The yearly log-returns from 1926 to 2006 are $x_t = p_t - p_{t-1}$. We specify the jump model as follows:

$$x_t = (1 - J_t)(\mu + \sigma\epsilon_t) - J_t Y_t, \quad (2.3)$$

where μ is the mean and σ is the standard deviation of no-jump years, and $\epsilon_t \sim i.i.d.^{10} N(0, 1)$. For the jump process we assume that $J_t \sim i.i.d. Ber(q)$, $0 < q < 1$, and $Y_t \sim i.i.d. Gam(\alpha, \beta)$, $\alpha, \beta > 0$. Moreover, we assume that these three random variables are independent.

Because the yearly returns x_1, \dots, x_T are assumed independent, we can write the likelihood function as

$$L(\theta) = \prod_{t=1}^T f(x_t; \theta), \quad (2.4)$$

where

$$f(x_t; \theta) = (1 - q) \frac{1}{\sqrt{2\pi}\sigma} e^{-(x_t - \mu)^2 / 2\sigma^2} + q \frac{\beta^\alpha}{\Gamma(\alpha)} (-x_t)^{\alpha-1} e^{\beta x_t} 1_{\{x_t < 0\}}. \quad (2.5)$$

Here $1_{\{x_t < 0\}} = 1$ when the yearly return is negative, and $1_{\{x_t < 0\}} = 0$ otherwise, as we only wish to model the negative jumps by the last term in (2.3). In addition we have used the independency of jump times and jump sizes, and the product rule of probability with the fact that the jump probability is $Pr(J_t = 1) = q$. Thus our model can be described as a Bernoulli-mixture of $N(\mu, \sigma^2)$ and $Gam(\alpha, \beta)$ distributions.

In this model the mean is

$$E[x_t] = (1 - q)\mu - q\alpha/\beta. \quad (2.6)$$

By taking expectations from the square of (2.3) we get

$$E[x_t^2] = (1 - q)(\mu^2 + \sigma^2) + q\alpha(1 + \alpha)/\beta^2, \quad (2.7)$$

and using $Var[x_t] = E[x_t^2] - E[x_t]^2$ we find that the variance is

¹⁰Independently and identically distributed.

$$Var[x_t] = (1-q)(\mu^2 + \sigma^2) + q\alpha(1+\alpha)/\beta^2 - \{(1-q)\mu - q\alpha/\beta\}^2. \quad (2.8)$$

2.2.2 Maximum likelihood estimation

The log-likelihood function l corresponding to (2.4) is

$$l(\theta) = \sum_{t=1}^T \ln \left[(1-q) \frac{1}{\sqrt{2\pi}\sigma} e^{-(x_t-\mu)^2/2\sigma^2} + q \frac{\beta^\alpha}{\Gamma(\alpha)} (-x_t)^{\alpha-1} e^{\beta x_t} 1_{\{x_t < 0\}} \right]. \quad (2.9)$$

We maximized (2.9) directly by R's unrestricted optimization function *optim()* via the Nelder-Mead simplex method.¹¹

The $Gam(\alpha, \beta)$ distribution is able to produce a rich variety of functional shapes. This feature, together with the observed bimodality of the empirical distribution of SP500 log-returns (cf. Figure 2.2), motivated us to analyse several specifications for the equity return model. The summary results of these calculations are given in Table 2.1.

Model 1 is based on the maximum likelihood estimates (MLE) with unrestricted α parameter. This model is highly bimodal, as seen in Figure 2.7. This does not seem reasonable. Namely, the probability density function has two spikes in its graph: a local maximum at -0.0227, and a local minimum at zero, and its first derivative is discontinuous at these points. We conclude that bimodality is not easily explained, it brings undesirable features to the density without introducing any apparent advantages, and thus it should not form a basic feature of the model. Indeed, this bimodality is not observed in the mid-year index data. The same conclusions regarding the bimodality problem apply to more parsimonious Model 6, where $\alpha=1$, corresponding to the Exponential distribution (cf. Figure 2.6).

To make our model more plausible, we change α and carry out the Maximum Likelihood estimation. We note that when α is increased, the bimodality decreases. In Figure 2.8 α is 2, and in Figure 2.9 it is 3. These models have less bimodality but still seem implausible. When α is between 3.5 and 4, the bimodality problem gradually disappears, as seen in Figures 2.10 and 2.11.

¹¹Using the version 2.6.0.

Table 2.1 Comparison of 6 models with Gamma jumps

Variables	Model 1	Model 2	Model 3	Model 4	Model 5	Model 6
μ	0.177	0.171	0.162	0.152	0.131	0.178
σ	0.121	0.125	0.131	0.138	0.153	0.121
q	0.229	0.208	0.176	0.143	0.071	0.235
α	1.41	2	3	3.5	4	1
β	8.65	11.4	15.45	16.17	12.79	6.29
l	24.77	24.5	23.58	23.04	22.86	24.36

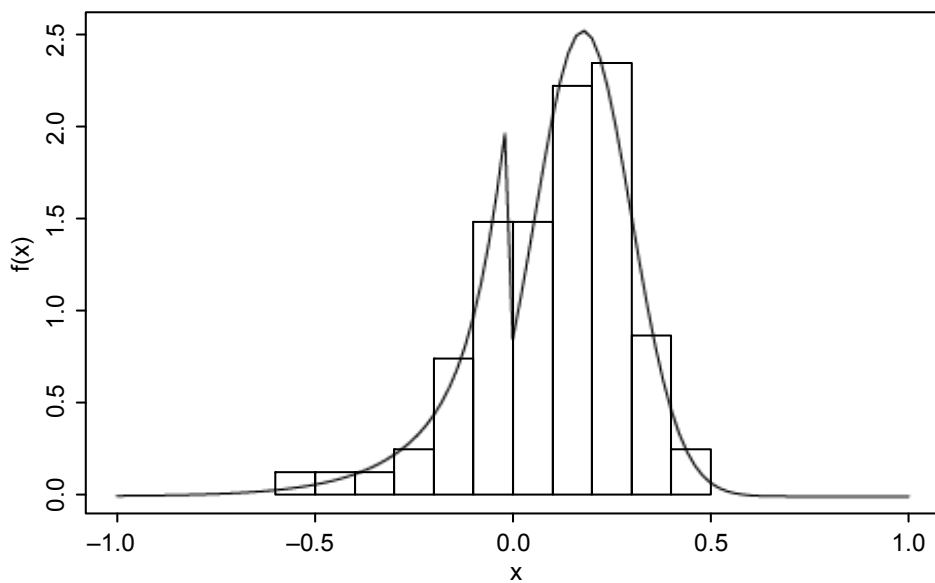


Figure 2.6 Log-returns of Model 6 when $\alpha = 1$

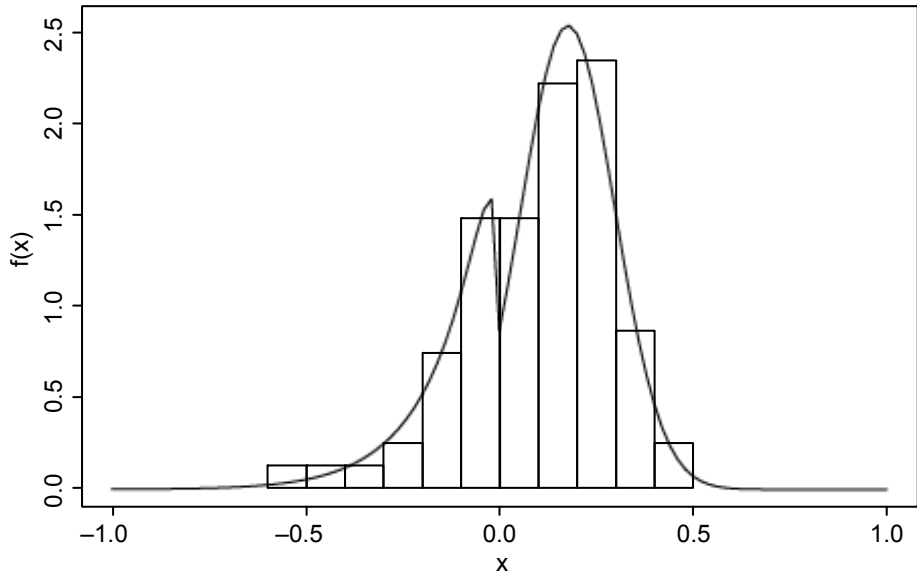


Figure 2.7 Log-returns of Model 1 when $\alpha = 1.41$

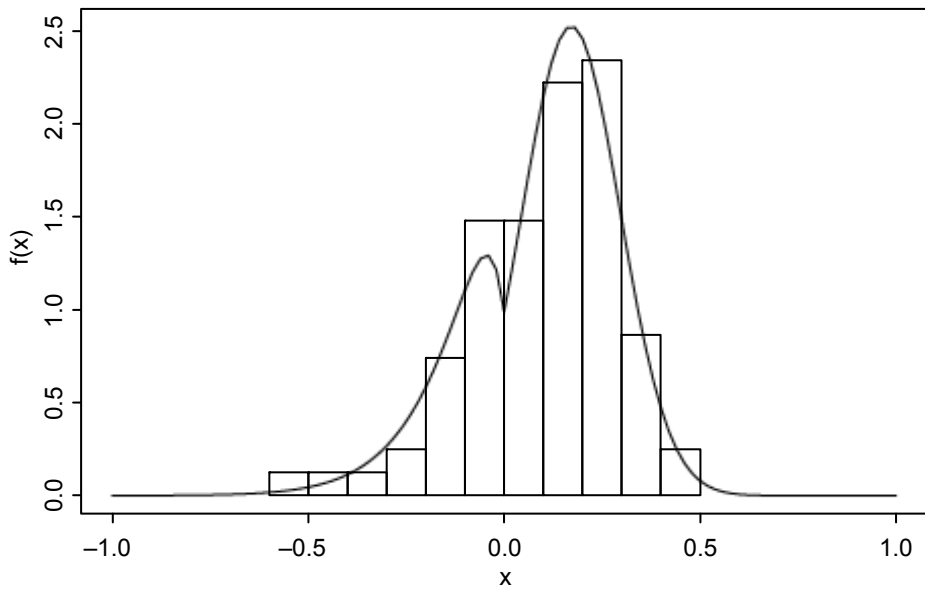


Figure 2.8 Log-returns of Model 2 when $\alpha = 2$

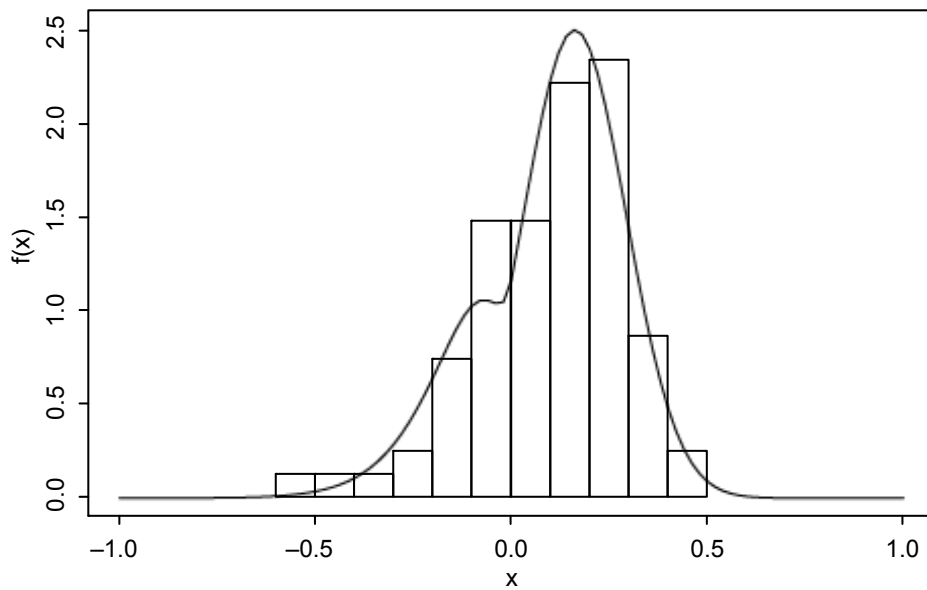


Figure 2.9 Log-returns of Model 3 when $\alpha = 3$

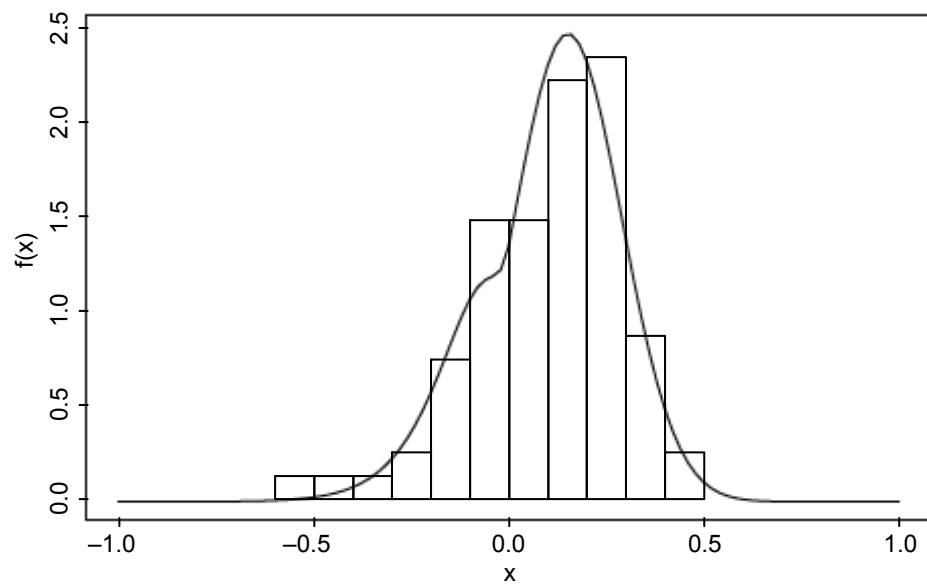


Figure 2.10 Log-returns of Model 4 when $\alpha = 3.5$

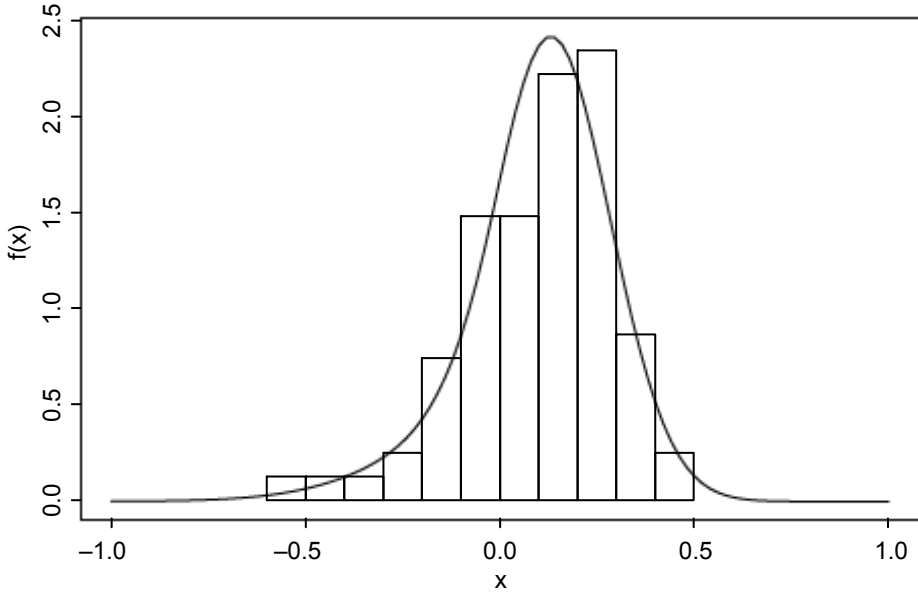


Figure 2.11 Log-returns of Model 5 when $\alpha = 4$

In conclusion, the MLE result (Model 1), which has the best fit according to likelihood value, as well as the simpler Exponential model (Model 6), have the serious problem of bimodality. We thus analysed above several specifications for the model, and we cannot say with certainty which would be the best specification (cf. the discussion on parameter uncertainty below). However, it seems to us that the Gamma distribution with α between 3.5 and 4 would provide a reasonable class of models. Our preferred choice of $\alpha=4$ (Model 5) also takes into account our original idea that the jumps should be relatively rare, as then $q=0.071$, so that on average there would be a negative jump of equity returns once every 14 years. This feature of rare but large negative jumps is in our view desirable. It allows us to model catastrophic losses for risk management purposes although the returns come from the Normal distribution independently most of the time. Our model is thus able to take into account the main features of the data analysis, it has a natural interpretation, and the ML estimation is not very complicated.

In the final part of our MLE procedure we calculated the confidence intervals for the parameters using the profile likelihood method, which is based on the likelihood ratio test and its asymptotic distribution. In this approach we search for the lower and upper bound for each parameter such that $2(l(\hat{\theta}) - l^*(\hat{\theta}))$ is approximately 3.84, i.e. the 5th percentile of the Chi-square distribution with 1

degree of freedom. Here the log-likelihood function l is evaluated at the maximum point $\hat{\theta}$, and the other term, $l^*(\theta)$, is calculated by optimizing the log-likelihood for the remaining parameters while keeping one parameter fixed. By gradually changing the fixed value and re-running the optimization, we obtain the lower and upper bounds of the confidence interval. This process is applied in turn to each term of the parameter vector. The optimization algorithm is the same as that for the Maximum Likelihood estimation.

To determine the profile likelihood 95 percent confidence intervals for the chosen model (Model 5), we first fixed $\alpha=4$, and then calculated the following confidence intervals for the remaining parameters: $\mu \in [0.09, 0.19]$, $\sigma \in [0.11, 0.19]$, $q \in [0.01, 0.27]$, $\beta \in [5.2, 26]$. We note from the confidence interval sizes that the parameter uncertainty is high. This is not particularly surprising considering the complex nature of equity returns and their jump process, which depend not only on the economic climate but also on other exogenous factors and human behavior. Although the joint analysis of parameter uncertainty is difficult and implies more complex modeling, there are methods available for that, and we consider them next.

2.3 Parameter uncertainty via Markov Chain Monte-Carlo

Our goal is to simulate future equity returns so as to include the parameter uncertainty in the calculations. Bayesian approach offers a natural solution as it treats the parameters of the model as random variables, each having a prior distribution. Posterior distribution is the joint distribution of the parameters, conditioned on the observed data. From this joint density we can sample parameters for instance by the so-called Markov Chain Monte-Carlo (MCMC) method. We implement MCMC via the Gibbs sampler, which uses conditional distributions of the parameters to specify the Markov Chain with the target joint density as its stationary distribution. For a comprehensive discussion of MCMC see Gelman et al. (2004) and Gilks et al. (1996), and for a more accessible introduction see Greenberg (2008).

We specify the Gibbs algorithm for our model in 2 steps as follows:

1. First we assume the jumps $J = J_t$ and the data $X = x_t$, $t=1, \dots, T$, are known and express the conditional joint density

$L|J$ of the model. Then the conditional distributions for the remaining parameters, $L(\beta|J, X, \mu, \sigma^2, q)$, $L(1/\sigma^2|J, X, \mu, \beta, q)$, $L(\mu|J, X, \sigma^2, \beta, q)$, and $L(q|J, X, \mu, \sigma^2, \beta)$, are derived by picking only those terms that include the parameter in question (the other terms are constants and can be neglected in the Gibbs algorithm).

2. In the second step we generate new jumps when all the other parameters are known.

We assume the following independent priors: $q \sim \text{Beta}(a_q, b_q)$, $\mu \sim N(0, \sigma_\mu^2)$, $\tau = 1/\sigma^2 \sim \text{Gam}(a_\tau, b_\tau)$ and $\beta \sim \text{Gam}(a_\beta, b_\beta)$. The parameters should be chosen to allow for a sufficiently flat and wide distribution. Based on both visual and empirical analysis we proceed as follows:

- $a_q = b_q = 1$, which leads to a non-informative uniform distribution on $(0,1)$.
- $\sigma_\mu = 0.8$, which is 4 times the observed standard deviation (0.192) of log-returns.
- $a_\tau = 3, b_\tau = 0.05$, which gives a mean of 0.15 and standard deviation of 0.05 for σ , while its MLE was 0.15 and the 95 % profile likelihood confidence interval was $[0.11, 0.19]$.
- $a_\beta = 1.5, b_\beta = 0.1$, which gives a mean of 15.0 and a standard deviation of 12.2, while the MLE was 12.8 and the 95 % profile likelihood confidence interval was $[5.2, 26.0]$.

From the arguments above we conclude that the chosen priors are sufficiently non-informative (flat) for our purposes.¹²

The conditional joint density for the observations and parameters is

¹²Testing with other reasonable parameters did not significantly change the MCMC results.

$$\begin{aligned}
L|J &= \prod_{J_t=1} \frac{\beta^4}{\Gamma(4)} e^{\beta x_t} (-x_t)^3 1_{\{x_t < 0\}} \\
&\times \prod_{J_t=0} \frac{\tau^{1/2}}{\sqrt{2\pi}} e^{-\frac{\tau}{2}(x_t - \mu)^2} \\
&\times \frac{1}{\sqrt{2\pi}\sigma_\mu} e^{-\mu^2/2\sigma_\mu^2} \\
&\times \frac{\tau^{a_\tau-1} e^{-\tau b_\tau} b_\tau^{a_\tau}}{\Gamma(a_\tau)} \\
&\times \frac{\beta^{a_\beta-1} e^{-\beta b_\beta} b_\beta^{a_\beta}}{\Gamma(a_\beta)}
\end{aligned} \tag{2.10}$$

Note that $\alpha=4$ by our earlier assumption, and that the form of this likelihood is simpler and much better suited for simulation than (2.4) because here we use the conditional $L|J$, i.e. we assume J is known.

Now the conditional distributions required for the Gibbs simulation can be derived as follows.

$$\begin{aligned}
L(\beta|J, X, \mu, \sigma^2, q) &\propto \left\{ \prod_{J_t=1} \beta^4 e^{\beta x_t} 1_{\{x_t < 0\}} \right\} \beta^{a_\beta-1} e^{-\beta b_\beta} \\
&= \beta^{4n_1+a_\beta-1} \exp \left\{ -\beta(b_\beta - \sum_{J_t=1} x_t 1_{\{x_t < 0\}}) \right\}.
\end{aligned} \tag{2.11}$$

Thus, $\text{Gam}(4n_1 + a_\beta, b_\beta - \sum_{J_t=1} x_t 1_{\{x_t < 0\}})$ is the posterior of β .

$$\begin{aligned}
L(\tau|J, X, \mu, \beta, q) &\propto \left\{ \prod_{J_t=0} \tau^{1/2} e^{-\frac{\tau}{2}(x_t - \mu)^2} \right\} \tau^{a_\tau-1} e^{-\tau b_\tau} \\
&= \tau^{n_0/2+a_\tau-1} \exp \left\{ -\tau(b_\tau + \frac{1}{2} \sum_{J_t=0} (x_t - \mu)^2) \right\}.
\end{aligned} \tag{2.12}$$

Thus, $\text{Gam}(n_0/2 + a_\tau, b_\tau + \frac{1}{2} \sum_{J_t=0} (x_t - \mu)^2)$ is the posterior of τ .

$$\begin{aligned}
L(\mu|J, X, \sigma^2, \beta, q) &\propto \left\{ \prod_{J_t=0} e^{-\frac{\tau}{2}(x_t-\mu)^2} \right\} e^{-\mu^2/2\sigma_\mu^2} \\
&= \exp \left\{ -\frac{\tau}{2} \sum_{J_t=0} (x_t - \mu)^2 - \frac{\mu^2}{2\sigma_\mu^2} \right\}.
\end{aligned} \tag{2.13}$$

Thus, $N((\frac{1}{\sigma_\mu^2} + n_0\tau)^{-1}\tau \sum_{J_t=0} x_t, (\frac{1}{\sigma_\mu^2} + n_0\tau)^{-1})$ is the posterior of μ .

$$L(q|J, X, \mu, \sigma^2, \beta) \propto q^{a_q-1}(1-q)^{b_q-1}q^{n_1}(1-q)^{n_0}. \tag{2.14}$$

Thus, $Beta(a_q + n_1, b_q + n_0)$ is the posterior of q .

In the equations above, n_0 and n_1 denote the number of no-jump ($J_t = 0$) and jump ($J_t = 1$) years respectively. The relation (2.13) follows from an analogous result on page 76 in Gilks et al. (1996), or by the completing the square technique.¹³

In the second step we update the jump process as follows. For $x_t < 0$, $t=1, \dots, T$, we obtain by (2.5)

$$\begin{aligned}
P(J_t = 1 | \mu, \tau, \beta, q, x_t) & \\
&= \frac{q\beta^4 e^{\beta x_t} (-x_t)^3 / \Gamma(4)}{(1-q)^{\frac{\tau^{1/2}}{\sqrt{2\pi}}} \exp \left\{ -\frac{\tau}{2}(x_t - \mu)^2 \right\} + q\beta^4 e^{\beta x_t} (-x_t)^3 / \Gamma(4)}
\end{aligned} \tag{2.15}$$

Using these probabilities with the most recent parameters, we generate a new jump process realisation from the Bernoulli distribution for each $x_t < 0$. After that, we start a new round of iterations (step 1 \rightarrow step 2 etc) until the number of iterations is adequate. In the first iteration round we use the MLE results as initial values.

2.4 Simulation of future equity returns

In this section we generate forecasts from our model with both the MLE and MCMC parameters derived above, and compare the results.

¹³See e.g. the opening page of Finney (2001).

The method for generating equity returns is as follows. We take one parameter vector at a time from the MCMC sample (after the burn-in period) and plug the parameters into the basic equation (2.3) as constants; for the MLE-based forecast we use the MLE parameters. Using 750 000 sampled observations (corresponding to a forecast for 75 years repeated 10 000 times), we get the empirical distributions of the equity returns, as shown in Figure 2.12. As initial values for the MCMC iteration we used the MLE results; other values were also tested but they did not change the outcome.

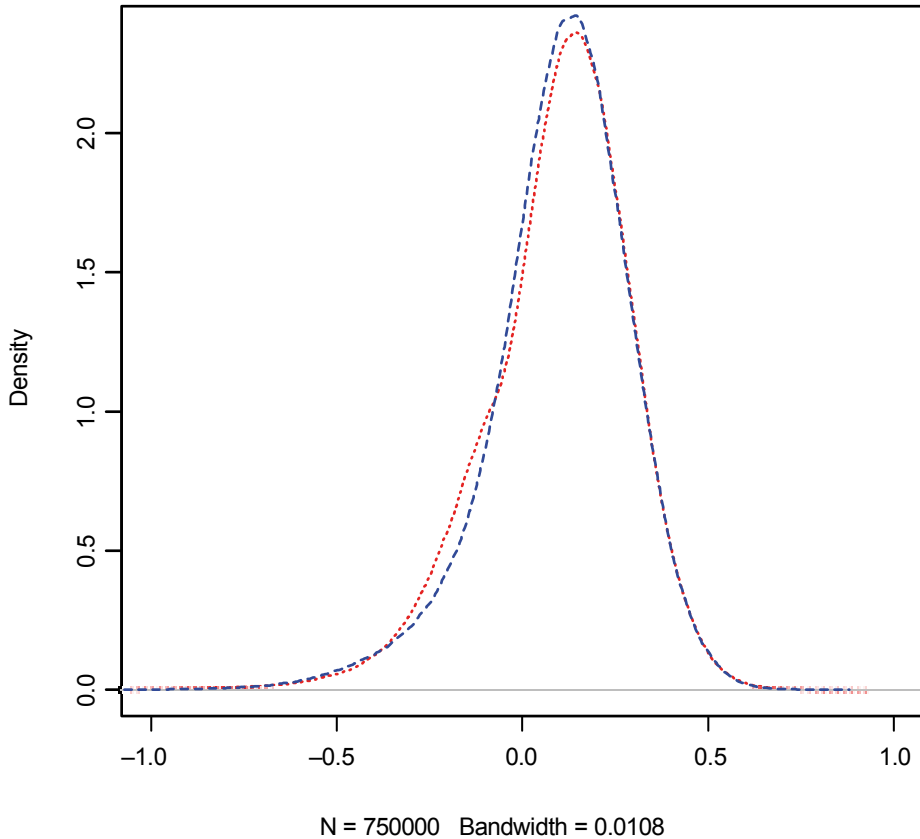


Figure 2.12 Comparison of MLE (dashed) and MCMC (dotted) based equity return simulations

We note from the kernel density plot in Figure 2.12 that the two methods give rather similar general distributional results. However, in the area where the returns are between -0.4 and 0, we see a systematic difference, as the MCMC method gives more probability mass there. In fact this seems to result in a return distribution that resembles

the case where the value of α is between 3.5 and 4, as we suggested earlier. These findings are made more explicit in Table 2.2, where we have included the MLE figures of Model 4 (where $\alpha=3.5$), and both the MLE and the mean MCMC estimates for our preferred model 5 (where $\alpha=4$). We note that the jump probability q and *Gamma* rate parameter β are higher in the MCMC estimation, which implies that jumps occur more often and are smaller. We also included in the table the MLE 95 % confidence intervals calculated earlier by the profile likelihood method, and the 2.5th and 97.5th quantiles of the MCMC results for Model 5. The confidence intervals are very similar. We also note asymmetry except for the σ parameter.

Table 2.2 Comparison of Model 4 and the MLE and mean MCMC parameters of Model 5, and the lower and upper 95% MLE confidence intervals (CI) and the respective MCMC quantiles for Model 5

Variables	Model 4	MLE	MCMC	CI-left	CI-right	2.5%	97.5%
μ	0.152	0.131	0.14	0.09	0.19	0.09	0.19
σ	0.138	0.153	0.15	0.11	0.19	0.12	0.19
q	0.143	0.071	0.11	0.01	0.27	0.02	0.25
β	16.17	12.79	15.3	5.2	26	6.6	25.1

We conclude that both the MLE and MCMC methods are suitable for the estimation and produce consistent results. The MCMC parameters partially cancel the jump features that we subjectively preferred when choosing $\alpha=4$. We have plotted in Figure 2.13 simulated μ against σ , in 2.14 μ against β , and in 2.15 q against β , from the posterior distributions. The posterior correlations of simulated model parameters are listed in Table 2.3.

Table 2.3 Posterior correlations of model parameters

Variables	μ	σ	q	β
μ	1	-0.50	0.58	0.47
σ		1	-0.58	-0.40
q			1	0.60
β				1

From these statistics we conclude that the model parameters generally are correlated. For instance we note from the high correlation (0.6) between q and β that the more frequent the jumps,

the smaller their sizes (cf. (2.6)). We also observe that μ , q , and β are pairwise positively correlated, while σ is negatively correlated with μ , q and β .

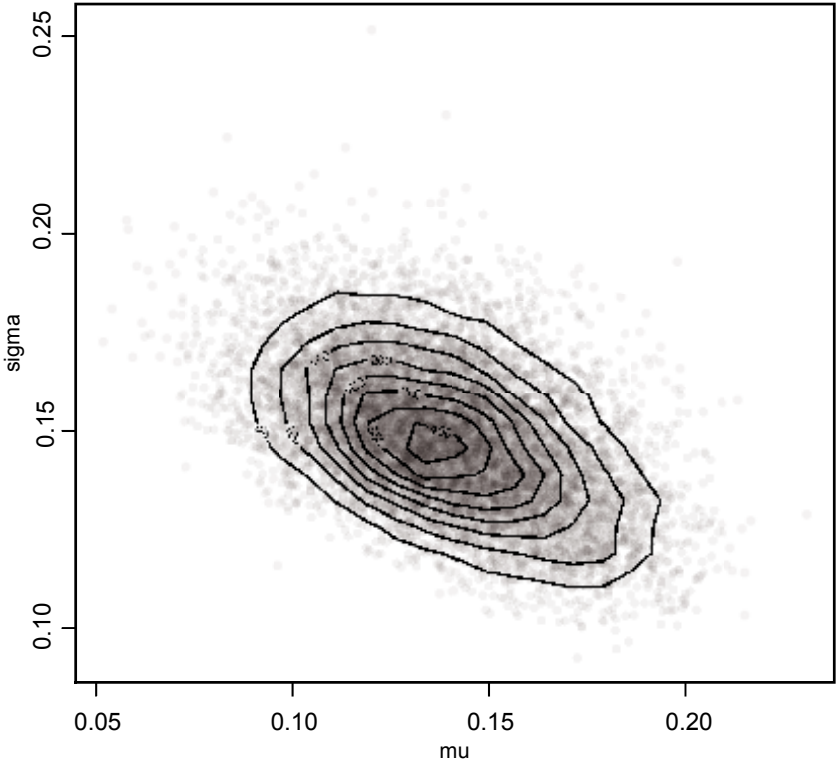


Figure 2.13 Simulated μ against σ

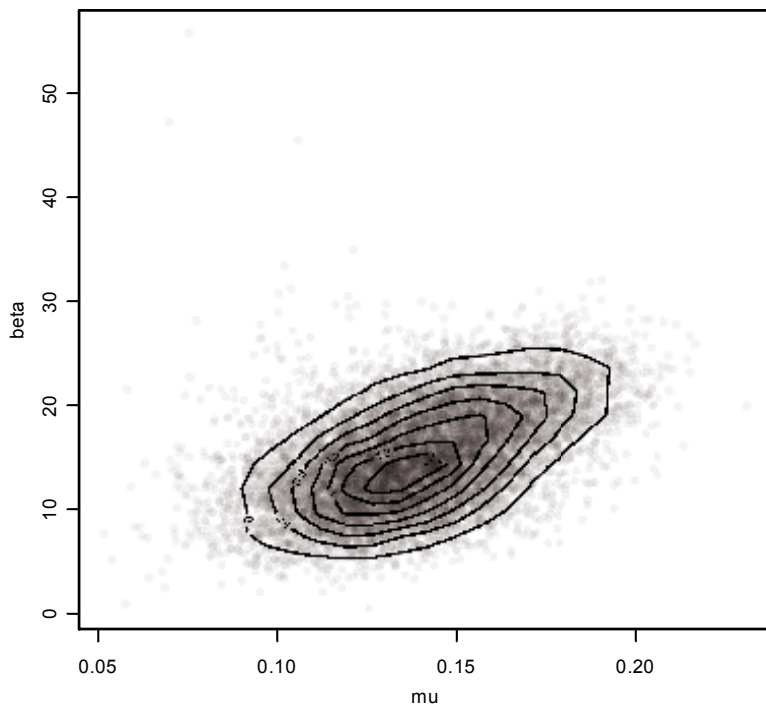


Figure 2.14 Simulated μ against β

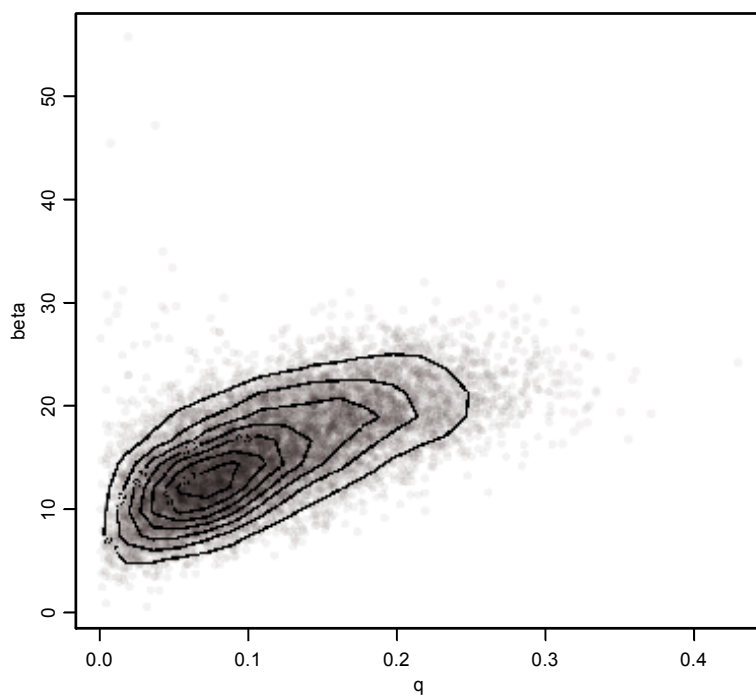


Figure 2.15 Simulated q against β

In Figure 2.16 we compare the MCMC results to the histogram of empirical log-returns of SP500. We observe the smoothness and wider range of values of the modeled returns, which are desirable simulation characteristics for risk management purposes. Table 2.4 provides some 1-year VaR_α measures, i.e. the maximum losses of a unit investment with a given confidence level α , for the price index I_1^{eq} , where $I_1^{eq} = \exp(x_1)$, and x_1 was simulated $3000 \times 80 = 240000$ times using both the MLE and MCMC parameters.

We compare the Normal assumption to the MLE and MCMC results, where $N(0.119, 0.209^2)$ is based on the MCMC simulation results for the price index, and $N(0.099, 0.19^2)$ corresponds to the historical log-return data, which might be the most commonly used option in practice. Thus rows 1 and 2 are based on the Normal assumption and the empirical and modeled data respectively. By comparing these figures to the third row (MLE) we note that the methods give different results due to the fat tail of the return distribution. Finally, by comparing the final rows MLE and MCMC, we observe a further increase in VaR due to the parameter uncertainty. This shows more clearly when comparing $VaR_{0.999}$, which is 0.53 and 0.57 for the MLE and MCMC cases respectively. We also note that the range of the three MCMC and MLE results is wider than for the Normal case.

Table 2.4 1-year VaR_α values, i.e. the maximum losses of a unit investment with a given confidence level α , for the equity prices based on $N1 = N(0.099, 0.19^2)$ and $N2 = N(0.119, 0.209^2)$ assumptions, and on the MLE and MCMC simulations

α	0.95	0.99	0.995
N1	0.19	0.29	0.32
N2	0.20	0.31	0.32
MLE	0.22	0.38	0.43
MCMC	0.23	0.38	0.44

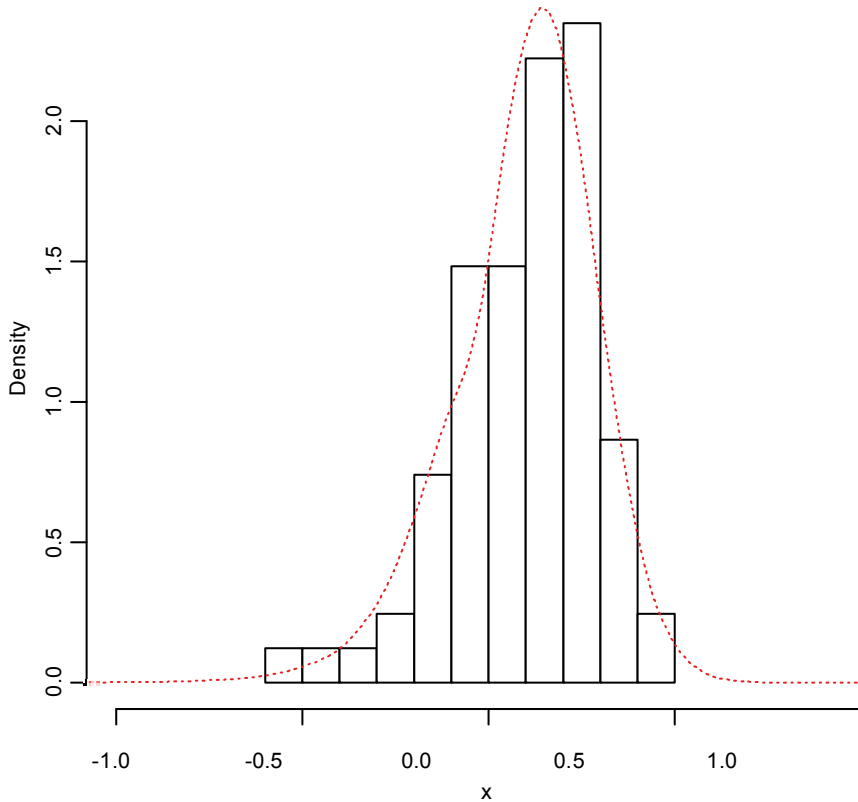


Figure 2.16 Comparison of the histogram of empirical log-returns and the MCMC based simulations

3 Bond index model

We take a statistical point of view to risk management and assume that uncertain future is well described by history. Thus our goal in interest rate index modeling is to be able to generate long term simulations, up to 80 years, with approximately similar distributional and dynamic features that can be observed from the chosen reference data set for government bond returns.¹⁴ The reference data should form a suitable basis for long-term forecasting and risk management, and should give an approximation to a bond portfolio of an insurance company and its clients (in case of unit linked business).

¹⁴We restrict our attention to interest rates and do not consider any other explanatory economic variables.

In this chapter we first analyse the chosen mid-term US government bond market data and briefly review the most common approaches for interest rate modeling. We then develop an ARMA(1,1) model for the log of yearly bond returns and estimate it by the Maximum Likelihood and parametric bootstrap methods. We also present simulations of the model.

3.1 Medium term bond index data

As a benchmark index for our interest rate model we chose the 5-year US government bond total return index at the year-end $t=0, \dots, 81$, where $t=0$ corresponds to year 1925, as given in Table 5–1 on pages 102–103 of Morningstar (2007). This index (5yB) is expressed in nominal values, and it started at 1.00 at the end of 1925 and reached 64.64 at the end of 2006. The index is calculated according to the following principles:

- One-bond portfolios are used. The bond chosen each year is the shortest noncallable bond with maturity not less than 5 years.
- The bond is held for the calendar year, and the total return is computed, which includes both capital appreciation and income return.

For more details see chapter 3 in Morningstar (2007).

As is typical in financial modeling, we start with the relevant index, 5yB, take the natural logarithm, and difference the data once. This gives us the yearly log-return on the index. The transformed time series is shown below in Figure 3.1. The summary statistics of 5yB are as follows: the sample mean is 0.051, the median 0.040, and the standard deviation 0.052. The difference between the two central location statistics reveal that the returns are skewed to the right, which is even more clearly seen from the histogram in Figure 3.2. In the time series graph we observe different regimes or long-term cycles. This is confirmed by the slowly decreasing autocorrelations in Figure 3.3. On the other hand the partial autocorrelation in Figure 3.4 is significant only at lag 2.

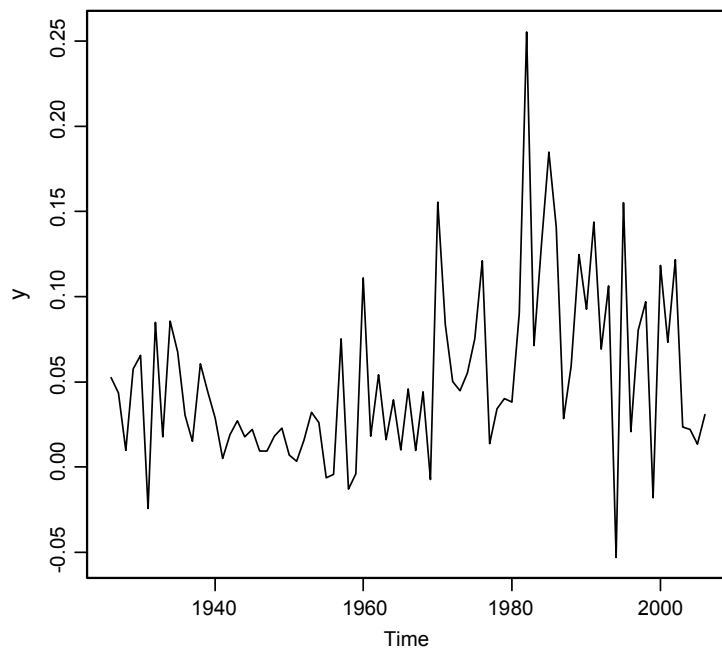


Figure 3.1 Log-returns on 5yB

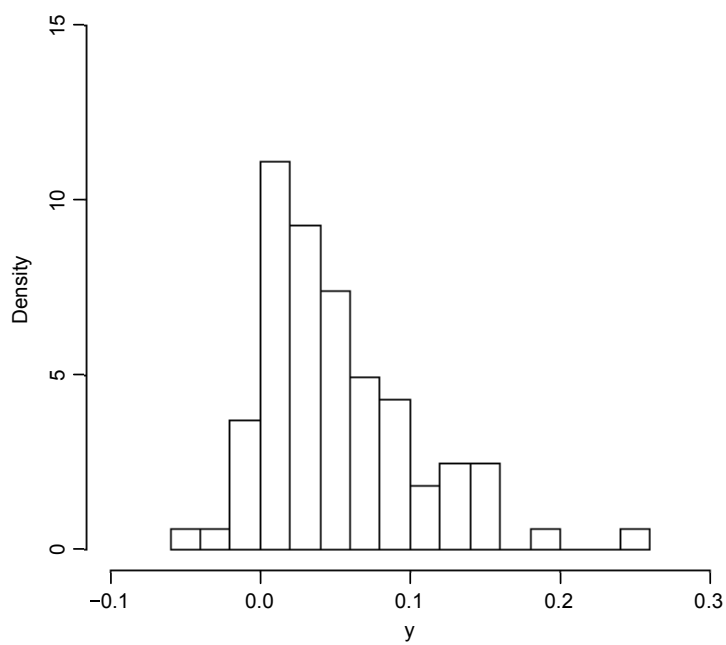


Figure 3.2 Log-returns on 5yB

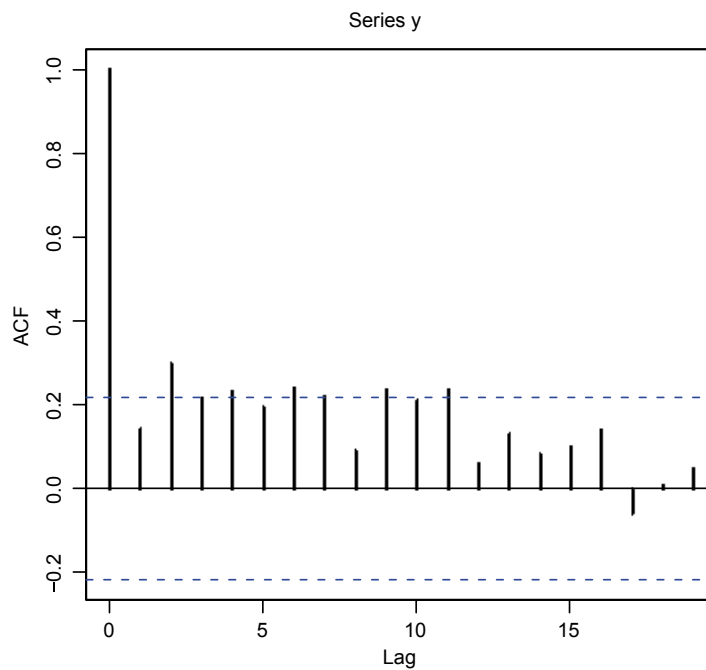


Figure 3.3 ACF of log-returns on 5yB

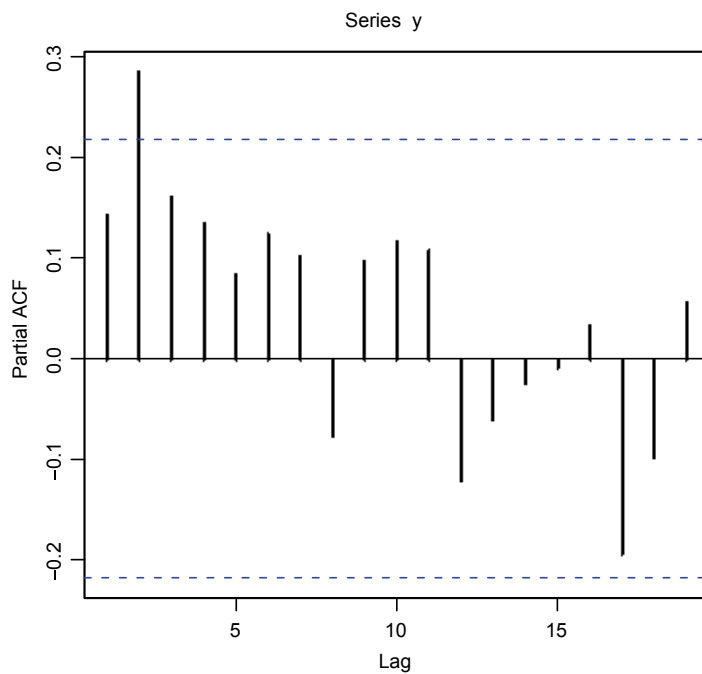


Figure 3.4 PACF of log-returns on 5yB

3.2 Review of interest rate modeling approaches

3.2.1 Interest rates and the yield curve

Interest rates define the cost of demanding or borrowing loanable funds for different maturities at a given point in time. This term structure of interest rates is conveniently presented graphically as the yield curve by plotting the observed market yields against the maturities. Typically a very low default risk is traditionally assumed for highly rated government bonds whereas a risk premium for default risk is added for corporate bonds. We do not model default risk in this work.

Economists have tried to develop theories to explain the behavior of the yield curve. Among the most common are the following (Cairns, 2004, sec. 1.6):

1. *Expectations Theory* argues that the annualized one-year forward rate of interest for delivery over the period S to $S+1$ in the future is equal to the expected value of the actual one-year rate of interest at time S .
2. *Liquidity Preference Theory* is based on the argument that investors usually prefer short to long term investments and thus require a risk premium to offset the higher risk in a longer-term bond.
3. *Market Segmentation Theory* assumes that each investor prefers a set of bonds and maturity dates that are most suitable for his own purposes (cf. the Asset Liability Management of life insurance companies and the discussion in Introduction and Case Study 1).
4. *Arbitrage-Free Pricing Theory* builds on the assumption that the market for interest rate securities is free of arbitrage. This theory underlies most interest rate models in the finance literature.

In practice several theories are needed to explain the various observed shapes of the yield curve, depending on the situation (see

e.g. Fabozzi, 2002, chapter 4). For example, Theory 2 (unlike Theory 1) is able to explain why the yield curve usually is upward sloping. Theory 3 allows short, medium and long term interest rates to change in unrelated ways.

One factor that partly explains the observed yield curve shape is the interest rate policy of central banks such as the Fed and the ECB. A long historical time series will include several different economic, interest rate, currency etc policies and regimes that influence the data (see James and Webber, 2000, chapter 2, for a discussion on interest rates in history). For instance, in our data we observe that the first half of the observations differ from the second half. During the later period the returns are higher and more volatile. Moreover, there are some outliers in the data. For instance the minimum value of -0.053 in 1994 is explained by the fact that the Fed doubled the short term interest rate from 3 to 6 percent in a short span of time, in response to faster growth and the threat of higher inflation. Because the interest rates of different maturities are typically highly correlated,¹⁵ the rise of the short rate had similar effects to the other parts of the yield curve. For the 5-year maturity bonds, the rise in discount rates implied a decrease in their market value, which explains the negative value of log-return in our series 5yB.¹⁶

Despite the fact that there are many economic factors that influence interest rates, we will base our model on the 5yB data alone. We have also chosen not to model the whole yield curve but the price index of a dynamically updated bond portfolio. This is much simpler and we consider it adequate as it allows the quantification of bond portfolio uncertainty, and we are mostly interested in longevity risk management issues in pension insurance. However, for each application the pros and cons of prospective interest rate models should be analysed and prioritized.

¹⁵to the extent that the so called 1-factor interest rate models may assume nearly perfect correlation

¹⁶The so-called modified duration of bonds in 5yB is approximately 5, which means that when the interest rate level or yield increases by 1 percentage point, the value of the bond decreases by approximately 5 percentages, and vice versa. For definitions and details see e.g. Jarrow, 2002, Chapter 2.

3.2.2 Interest rate models

A large number of interest rate models is actively used by market players because different applications seem to require different types of models. The majority of models are expressed by a stochastic differential equation (SDE) in continuous time. A brief review of some common models is given in Koskela et al. (2008). Comprehensive references are e.g. James and Webber (2000), Cairns (2004), and Fabozzi (2002). The first book also includes a brief introduction to stochastic calculus, which is needed for manipulating SDEs.

According to Cairns (2004), the following characteristics are desirable but not essential for the development of an interest rate term-structure model:

1. Interest rates should be positive.
2. The short rate,¹⁷ should be autoregressive.
3. We should get simple formulas for bond prices and for the prices of some derivatives.
4. The model should be flexible enough and produce dynamics which are realistic with respect to historical and current market prices.

The first point is often observed by choosing a model which only allows for positive values (such as the log-Normal distribution). Negative real interest rates are possible but negative nominal rates would violate the no-arbitrage condition, as cash would then earn more than those interest rate securities.

For most applications a one-factor time-homogeneous short rate model is not sufficient (Cairns, 2004, chapter 6). Therefore various multifactor models have been developed. For instance a two-factor model would include separate equations for the short rate and for some longer term rate.

Time series models in discrete time are one possible modeling approach although less commonly used in financial applications. For example in Campbell et al. (1997) some short rate models are expressed and estimated via time series analysis. Multivariate models are needed if several maturities in the yield curve are to be modelled

¹⁷Usually approximated by the 1-month or 3-month rate.

separately. For our chosen data series, 5yB, a univariate time series model is sufficient.

For long-term insurance applications the first globally used model was the multivariate time series model of Wilkie (Wilkie, 1984). His approach has been summarized e.g. in Daykin et al. (1994), sec. 8.4. More recently the so-called Economic Scenario Generator simulation models have been developed by consultants, typically based on SDEs and tailored for market consistent valuation of assets and liabilities (cf. IAA, 2010).

Regarding points 3 and 4 we note that the question of how the parameters of the SDE model are estimated is fundamental. Two main approaches are a) using historical data and statistical estimation, and b) calibrating the model to fit the current market prices. Often for risk measurement purposes a) is used, and b) for market consistent valuation (cf. Fabozzi, 2002, Chapter 2). Also mixtures of a) and b) can be used for both purposes.

For risk management applications there are a number of factor models for bonds, which focus on parsimonious sets of drivers of interest rate risk. These models may (to a varying degree) build on the market, economic or statistical point of view. Fabozzi (2002) in chapter 9 reviews some factor model classes and cites as the main benefit of factor models that they simplify the risk measurement (e.g. the VaR calculation) of complex interest rate security portfolios. This goal is relevant for us as well when we search for a parsimonious model that is capable of producing the main features observed in the initial data analysis above. Via the iterative model development cycle of Figure 1.1, we discovered that the following ARMA(1,1) type of time series model is able to describe the data well after a log-transformation.

3.3 Model specification and estimation

3.3.1 Model specification

Starting at the year-end values B_t of 5yB, where $t=0,1,\dots,81$, and $t=0$ corresponds to year 1925, we can write the yearly log-returns from 1926 to 2006 as

$$r_t = \ln B_t - \ln B_{t-1}. \quad (3.1)$$

Choose $a > 0$. The model is then specified for

$$y_t = \ln(r_t + a) \quad (3.2)$$

as

$$y_t - \beta_y = \phi(y_{t-1} - \beta_y) + u_t - \theta_y u_{t-1}, \quad (3.3)$$

where $u_t \sim i.i.d.N(0, \sigma_y^2)$, $0 < \theta_y < \phi < 1$.

We have restricted the stationarity (I_ϕ) and invertibility (I_θ) areas to cover only half of their full range (-1,1). Moreover, for acceptable parameter values ($I_{\phi,\theta}$) we require that $\theta_y < \phi$. These restrictions on the parameter space are based on the initial data analysis. They guarantee exponentially decaying autocorrelations and partial autocorrelations (cf. Box et al., 1994, p. 82 and the discussion on parameter uncertainty below).

3.3.2 Maximum likelihood estimation

We start the estimation by preliminary selecting $a=0.1$. This parameter must be greater than the negative of the sample minimum, -0.053, for the logarithm to be defined. This leaves us unknown $\Psi = (\beta_y, \phi, \sigma_y^2, \theta_y)$ to be estimated, which we carry out by means of the Maximum Likelihood estimation method implemented in the time series package *arima* of the R language.¹⁸

Next we allow a to vary and re-run the *arima* estimation for $a=0.07, \dots, 0.2$. The results show no significant changes in the AR and MA parameters, but they do influence the mean, the variance and the log-likelihood: the larger the constant, the smaller the variance and the higher the log-likelihood value. However, after de-transformation, larger values of a lead to an increasingly more symmetric return distribution. Therefore we resort to the method of moments or moment fitting and choose the value $a=0.091$, which gives the best fit of simulation results to the empirical data in terms of skewness; cf. Table 3.1.

The final ML estimation results with $a=0.091$ and the standard errors in parentheses are as follows: $\beta_y = -2.01$ (0.086), $\phi = 0.93$ (0.055), $\theta_y = 0.83$ (0.079), $\sigma_y^2 = 0.114$. We note that both the AR and MA coefficient are fairly close to 1 in absolute value. This indicates that

¹⁸Version 2.8.1.

the stationarity of yearly bond returns is not guaranteed by the data alone as the next section also confirms. However, it is not plausible to have a non-stationary interest rate process over a long period, as there are various economic controls in place to stabilize interest rates. Several studies based on international panel data (e.g. Wu and Zhang, 1996, and Constantini and Lupi, 2007) support the assumption of stationarity, as pointed out by Risku and Kaliva (2009) (p.16). We assume stationarity as stylized fact 1.

Residuals in Figure 3.5 raise no other concerns except for the outlier of 1994, which we explained earlier. Figure 3.6 shows a histogram of the residuals and Figure 3.7 shows a Q-Q plot. A simulation using these parameters and de-transformation for 100 000 times gives the return distribution of Figure 3.8, where the empirical histogram of 5yB is also displayed. The summary statistics for this simulation and for the original data are given in Table 3.1. The autocorrelations of the simulation are in Figure 3.9 and the partial autocorrelations in Figure 3.10.

We can conclude that the fit of the model is good. Although ARMA(2,1) is the best of the ARMA(p,q) class of models according to the AIC and AICC information criteria,¹⁹ we prefer parsimony over slightly improved fit. Due to the AR-term the model is able to generate long-term cycles, and due to the negative MA-term it is able to rapidly correct high and low returns to the other direction, both of which are features - stylized facts 2 and 3 - that we empirically observed in the data. Note that the range of the simulations is wider than in the data and that negative values are not uncommon. In the data we observe negative returns approximately once every 10 years, which is our stylized fact 4.

Table 3.1 Simulated returns (MLE) vs empirical returns of 5yB

Variables	Simulation	Empirical
min	-0.064	-0.053
max	0.54	0.26
mean	0.051	0.051
median	0.043	0.040
sd	0.053	0.052
skew	1.16	1.14

¹⁹Definitions are given in Brockwell and Davis (2002), p. 173, for instance.

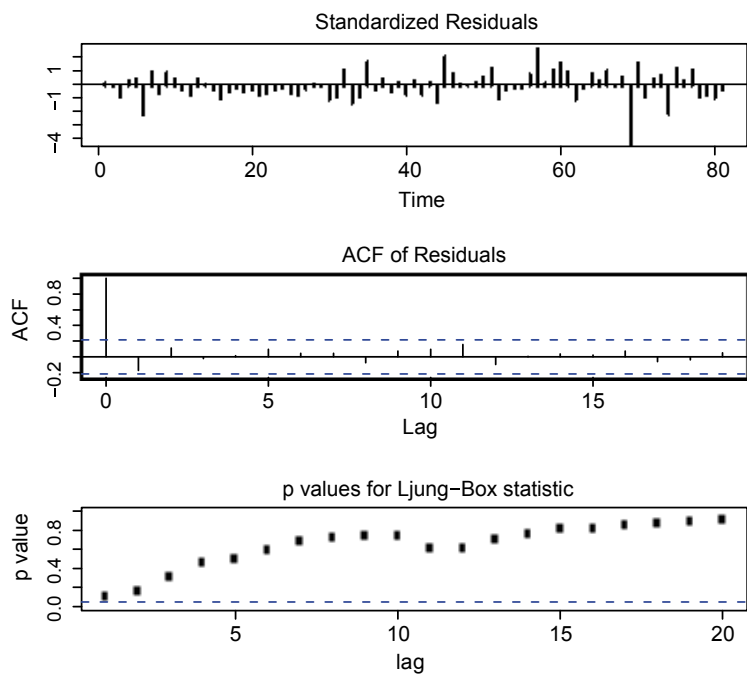


Figure 3.5 Residual analysis

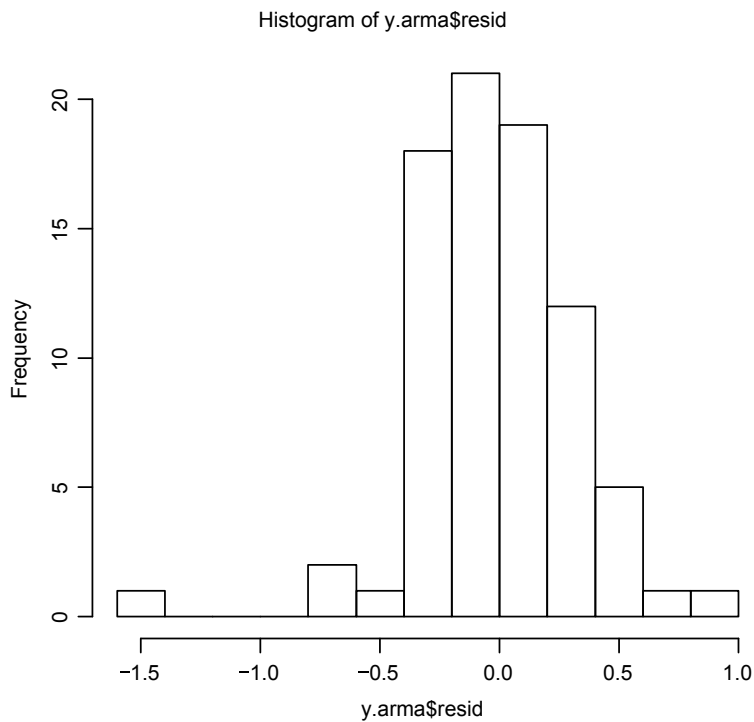


Figure 3.6 Residual histogram

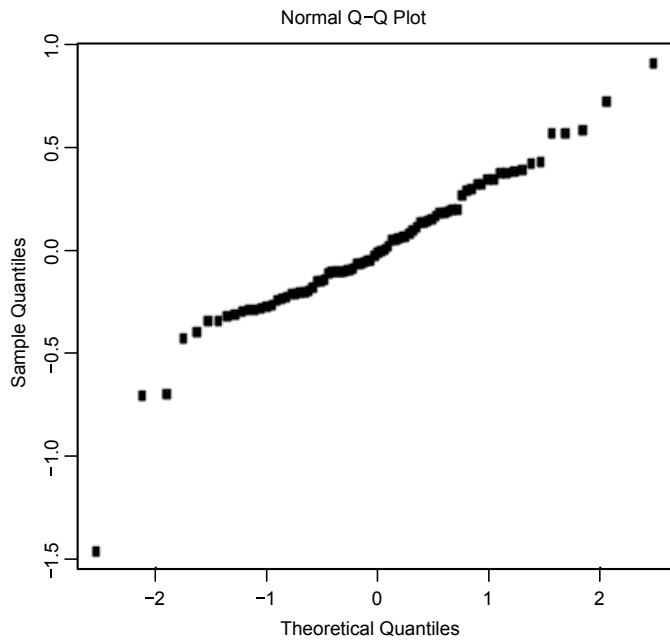


Figure 3.7 Residual Q-Q plot

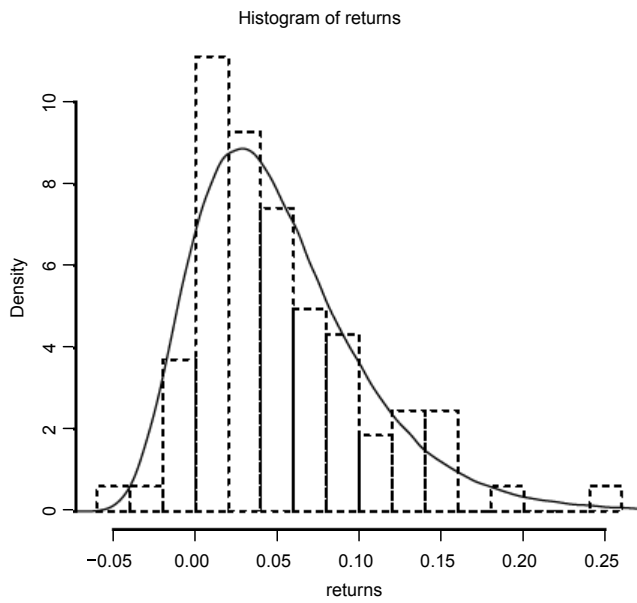


Figure 3.8 Simulated and empirical returns

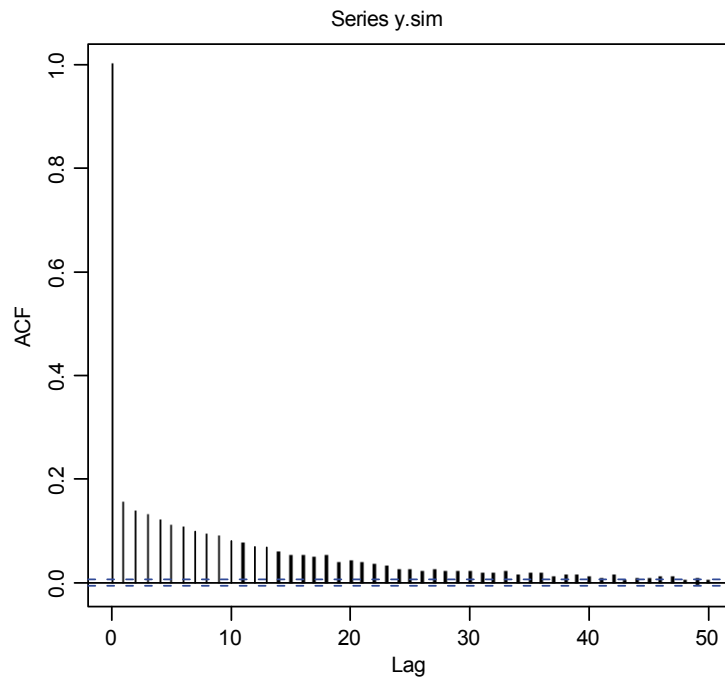


Figure 3.9 ACF of the ARMA(1,1) simulations

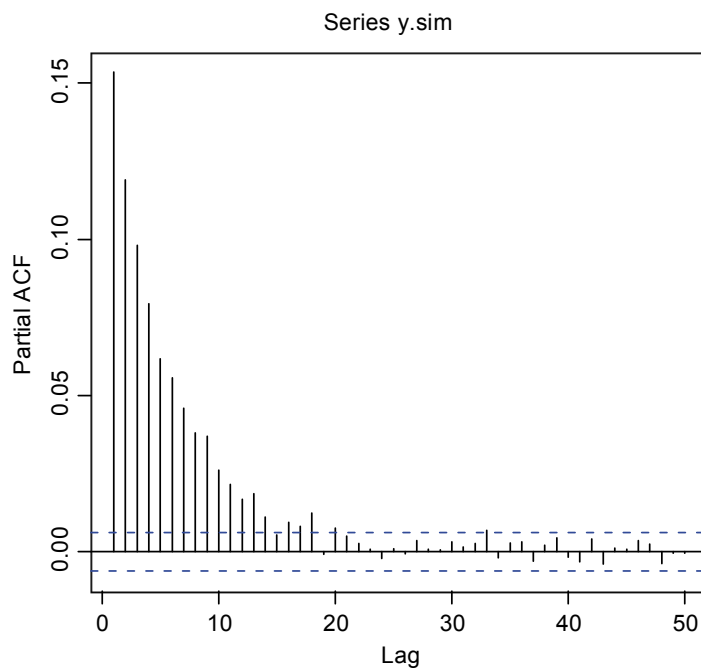


Figure 3.10 PACF of the ARMA(1,1) simulations

3.4 Parameter uncertainty

Model's limitations or model error is an important consideration in any modeling or forecasting exercise. Parameter uncertainty is a significant element of model error and therefore point estimates should be supplemented with confidence intervals. We have studied this issue from three points of view: a) asymptotic inference, b) Markov Chain Monte-Carlo (MCMC) approach, and c) parametric bootstrapping. In a) one uses the result that the MLEs are asymptotically Normally distributed. However, in our case the sample size is rather small (81 observations), and there is a significant probability that the AR-parameter ϕ may go outside the stationary region (0,1), as its upper bound is only 1.2 standard errors from the ML value $\phi=0.93$. Therefore we resorted to the Bayesian MCMC approach along the lines of Chib and Greenberg (1994). We wrote the full conditional distributions for our model explicitly, and used uniform priors for ϕ and θ_y in their acceptable region $I_{\phi,\theta}$. For the other parameters we used uninformative priors. However, our algorithm did not converge under these assumptions.

Therefore we ultimately chose to take into account parameter uncertainty using the parametric Bootstrap method to generate additional data samples by our simulation model. Another approach would bootstrap residuals of the estimated model. Time series bootstrap is more complicated than the conventional bootstrap based on independent random sampling, and there are still questions that are not well known (relating e.g. to unit roots). We refer to Horowitz (2001) for further discussion and references.

We generated new simulated returns from the model equation using the ML estimates and randomly generated innovation series u_t , $t=1, \dots, 2 \times 10^6$. The starting value of the process was the mean $y_1=-2.01$. From the generated long series we took samples of 81 consecutive observations and accepted those samples where the newly estimated ϕ and θ_y were in the acceptable region $I_{\phi,\theta}$. New parameters Ψ were estimated using the *arima* function in R. When running the new estimation, if an R error occurred, i.e. the parameters were outside the stationary or invertibility areas or unidentified (when $\phi=\theta_y$, cf. Box et al., 1994, p. 266), we started the estimation again from

a further point in time. From the resulting 16904 simulations we accepted 10054 belonging to $I_{\phi,\theta}$. This gave us the simulated sample distributions for the parameters. The histograms for each parameter are presented in Figures 3.11–3.14. Two-dimensional contour plots of the parameters are given in Figures 3.15–3.20. Additionally, in Figure 3.21 all the simulations are presented for ϕ and θ_y . We note that prior information is indeed necessary because the simulated data alone are not sufficiently informative to identify an appropriate model for our application. We continue this discussion in the next section.

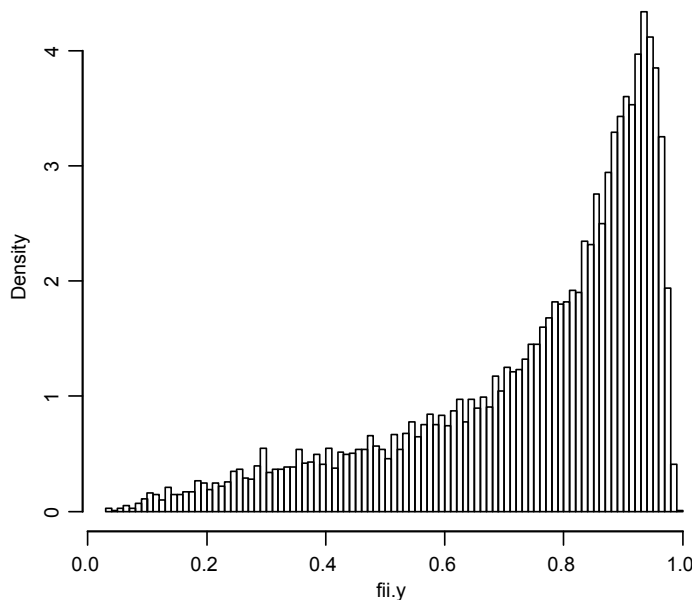


Figure 3.11 Bootstrap results for ϕ

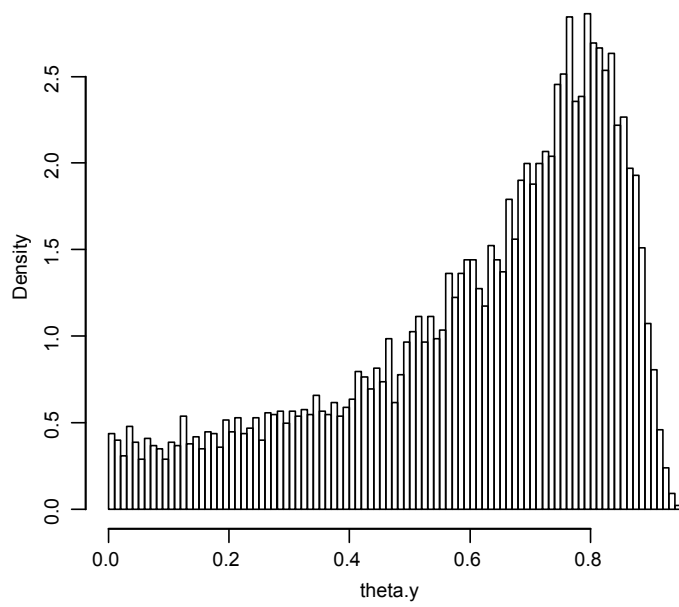


Figure 3.12 Bootstrap results for θ

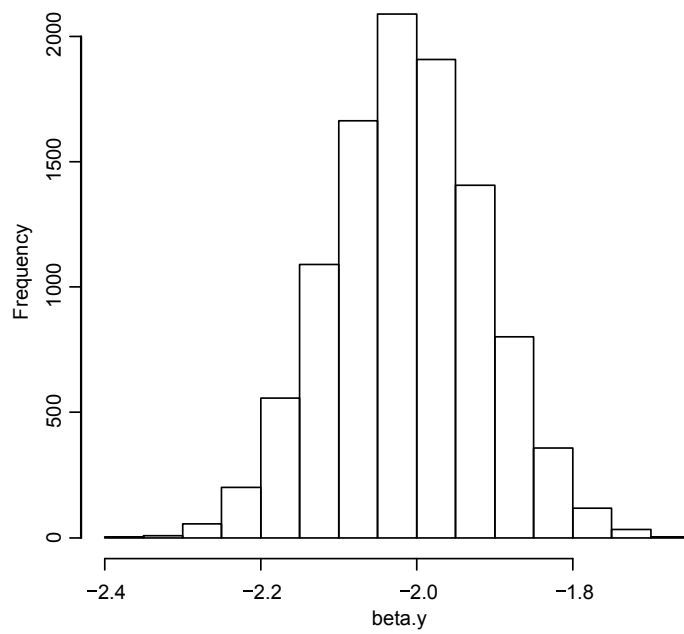


Figure 3.13 Bootstrap results for β

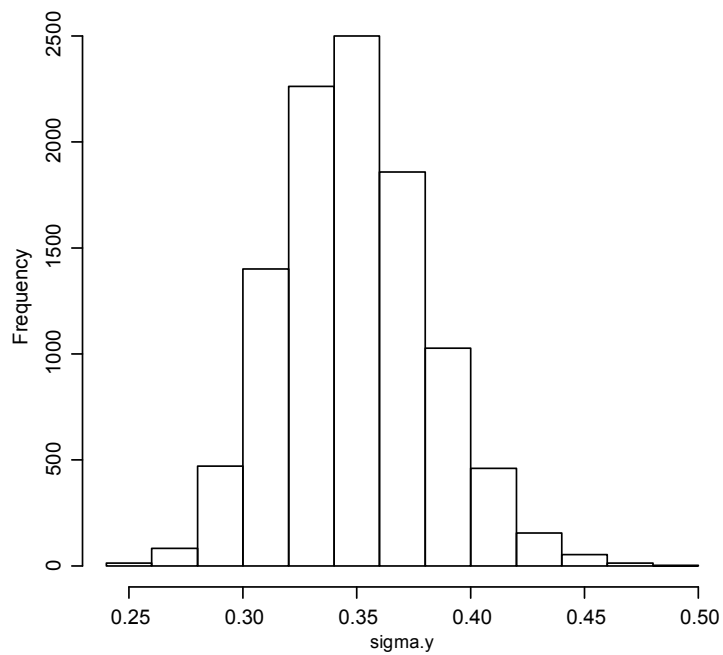


Figure 3.14 Bootstrap results for σ

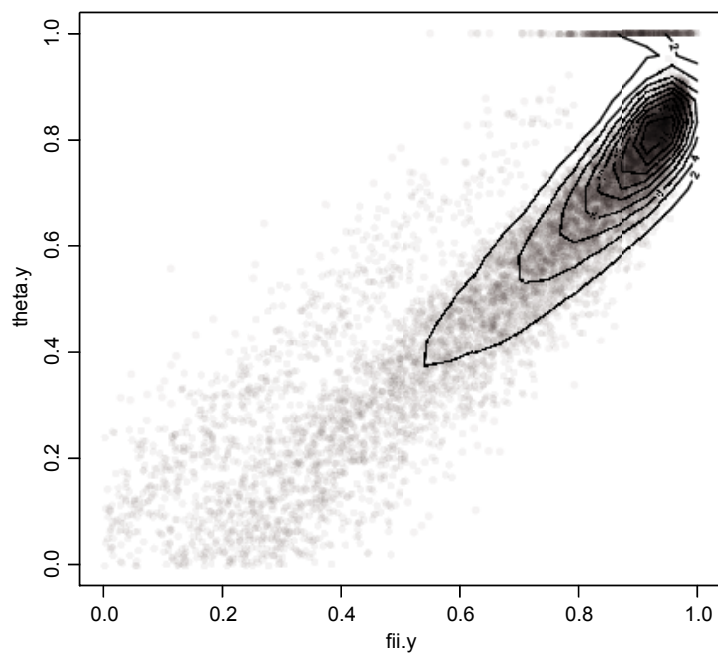


Figure 3.15 Bootstrap results, ϕ vs θ

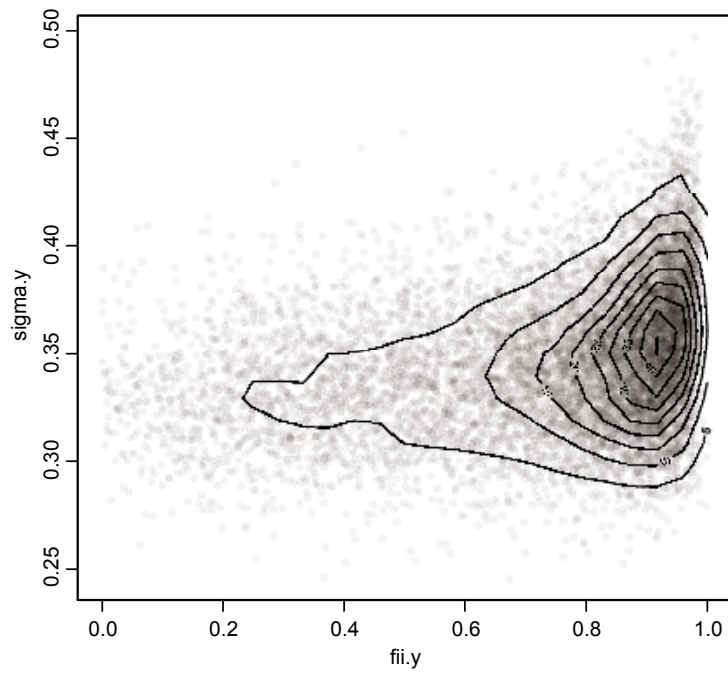


Figure 3.16 Bootstrap results, ϕ vs σ

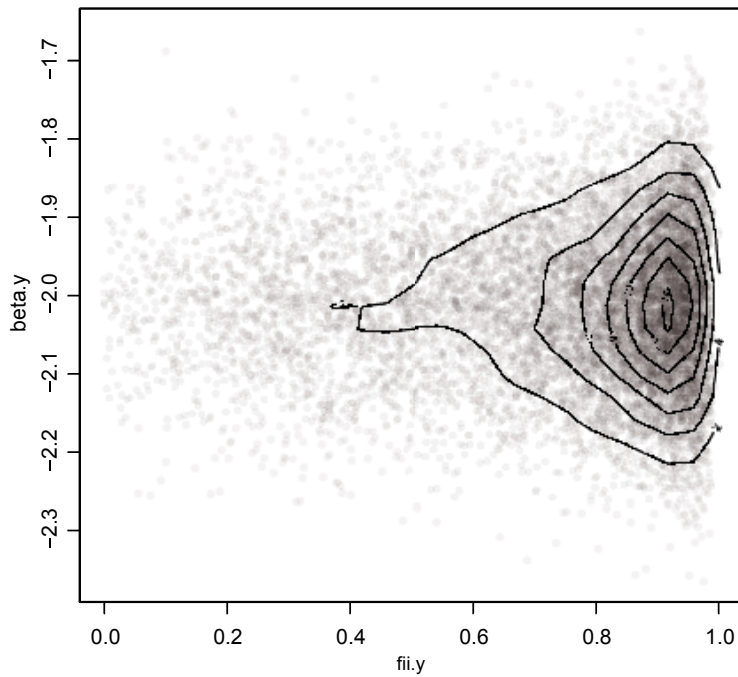


Figure 3.17 Bootstrap results, ϕ vs β

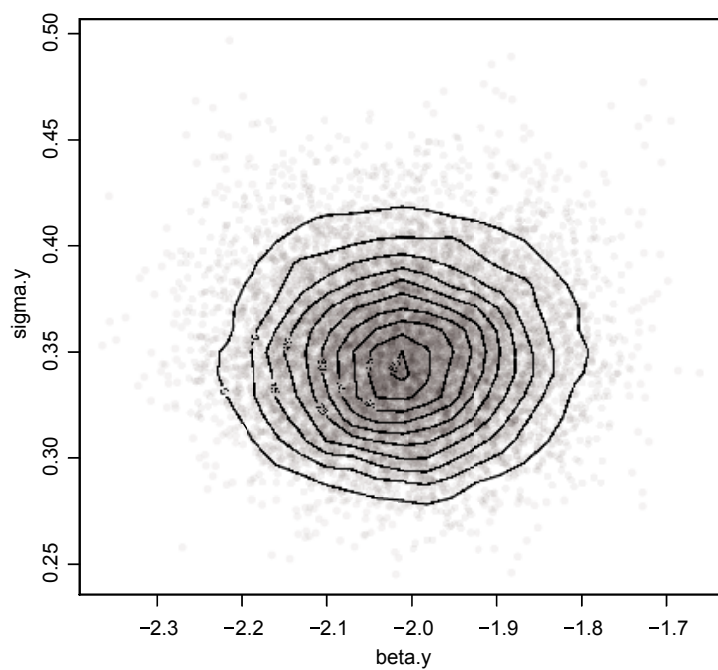


Figure 3.18 Bootstrap results, β vs σ

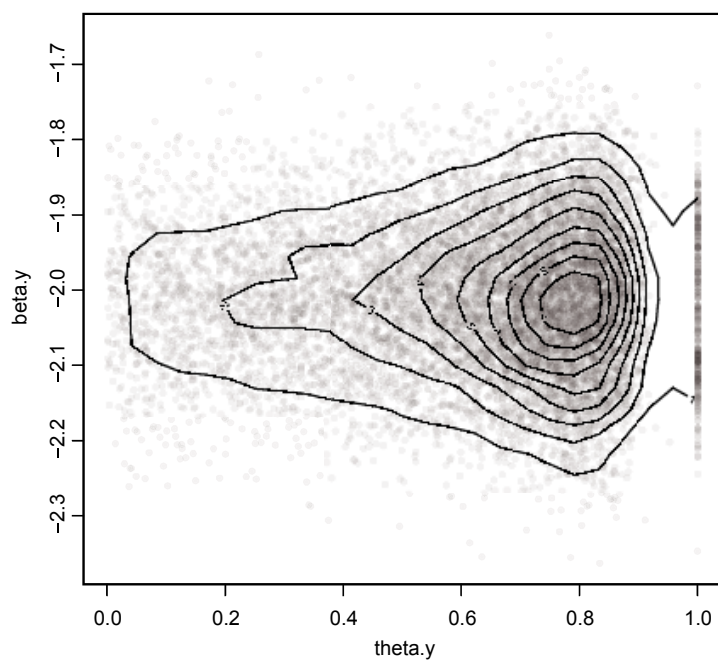


Figure 3.19 Bootstrap results, θ vs β

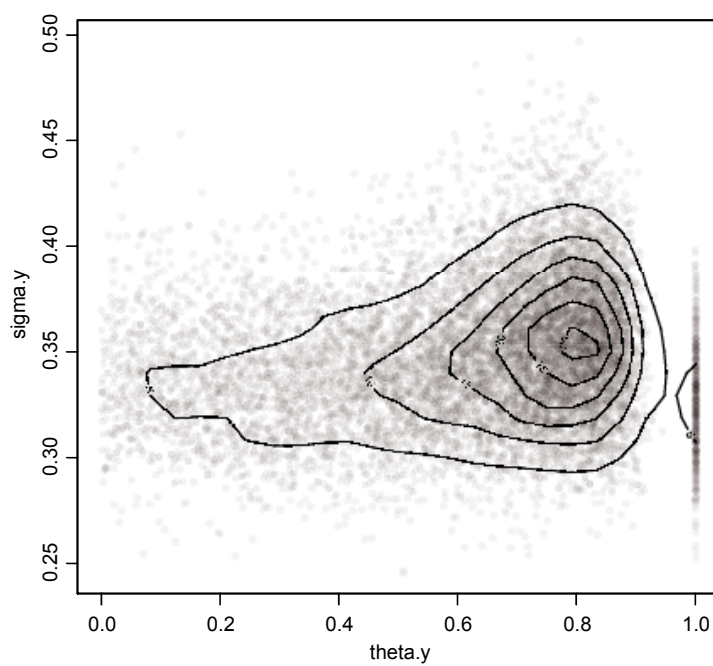


Figure 3.20 Bootstrap results, θ vs σ

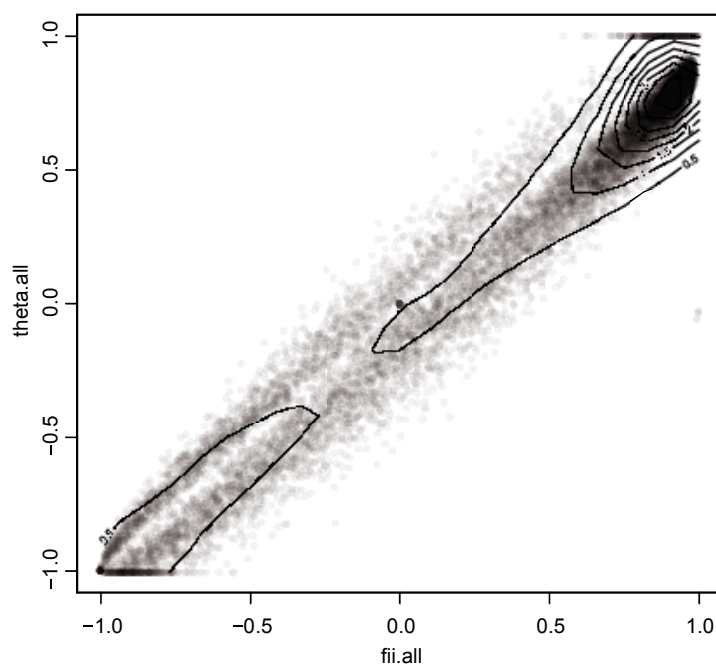


Figure 3.21 Unrestricted bootstrap, ϕ vs θ .

3.4.1 Simulations

In practical application we choose parameter vectors $\Psi=(\beta_y, \phi, \theta_y, \sigma_y)$ from the bootstrap sample, one at a time, and simulate new observations for the chosen forecasting horizon from the resulting ARMA(1,1) model. This is repeated as many times as necessary. We used R function *arima.sim*²⁰ in the simulation. Finally, we de-transform the results, to get the simulated bond log-return series. However, compared to the MLE simulations above, this approach leads to more volatile results, where the standard deviation and skewness in particular are higher than in the original data and negative values are more common, as seen in Table 3.2. This shows the significant uncertainty underlying the bond return process. Moreover, our statistical model does not include the policy rules used by the decision makers to guide and stabilize interest rates. We therefore resort to the stylized facts and restrict the simulations further as follows.

When the parameter uncertainty is taken into account in the simulations, we also get some less likely (from economic point of view) realisations. E.g. when the MA parameter θ_y is small, a rapid correction from negative to positive return is less likely. This may allow several successive negative values to emerge in the simulation, which is not observed in the data and might not happen in reality. Another stylized fact of the data is the long-term cycles. This is consistent with a large AR parameter ϕ . Therefore we restricted the AR and MA parameters and also the standard deviation in order to get the portion of negative values more in line with the empirical data and our stylized fact.

The chosen restrictions are $0.6 < \phi < 0.95, 0.6 < \theta_y < \phi, \sigma_y < 0.34$, which leads to the acceptance of approximately 22 percent of the bootstrap parameters. The simulation results based on the restricted parameters (see Table 3.2) give a better statistical fit and lead to plausible simulation scenarios. Still, in our view, the uncertainty of the model is adequately reflected in the simulations, which can be seen when comparing the chosen area to the MLE parameters $(\phi, \theta_y)=(0.93, 0.83)$ and their standard errors (0.06, 0.08).

²⁰R version 2.12.2.

For the standard deviation, the restriction is severe due to the goal of restricting the portion of negative returns. Still they emerge more often in the simulations than in the data (13 percent vs 10 percent), which causes significant interest rate risk for the insurer.

We will use the interest rate model in our case studies to model the bond portfolio returns and to discount the cash flows. We also base our pension indexation rule on the modeled interest rates. Therefore the indexation and discounting of pensions largely offset each other, which does not happen, if the pensions are not index-linked. Another feature we will study is a floor in the pension rule, which does not allow the pensions to decrease when the bond returns are negative. This causes interest rate risk for the insurer. We illustrate these questions in the first case study.

Table 3.2 Simulated returns, 3000 series of 80 years, using bootstrap parameters (BS1) and with the restriction (BS2) vs empirical returns of 5yB. Fraction of negative returns is given in neg.ratio

Variables	BS1	BS2	Empirical
mean	0.053	0.051	0.051
median	0.043	0.043	0.040
sd	0.057	0.051	0.052
skew	1.55	1.16	1.14
neg.ratio	0.15	0.13	0.10

In Table 3.3 we include some 1-year VaR_α metrics for the bond price index I_t^B , i.e. the maximum losses on a unit investment with given confidence level α . We compare the Normal assumption to the MLE and bootstrap results, where $I_1^B = \exp(r_t)$ and r_t was simulated for 80 years 3000 times. Parameters for the Normal-distribution are based on the restricted bootstrap simulation results for the price index in $N(0.054, 0.055^2)$ and on the historical log-return data in $N(0.051, 0.052^2)$. We note that the VaR figures are greater in the Normal case than in either the MLE or bootstrap case. We also note that the results are very different from the equity returns. Now the VaR figures are of a much smaller magnitude due to the different shape of the log-return distribution.

Table 3.3 1-year VaR_α values, i.e. the maximum losses on a unit investment with a given confidence level α , for the bond price index based on $N1 = N(0.051, 0.052^2)$ and $N2 = N(0.054, 0.055^2)$ and MLE parameters and bootstrap parameters. BS1 is without and BS2 with the parameter restriction.

VaR	0.95	0.99	0.995
N1	0.034	0.068	0.079
N2	0.036	0.072	0.084
MLE	0.016	0.032	0.037
BS1	0.019	0.036	0.041
BS2	0.015	0.031	0.036

Finally, we note that because our goal is a parsimonious statistical model reflecting the main features of the chosen data, there are certain limitations in our approach. Stylized facts are a case in point. Another one is the log-transformation (3.2). The model specification implies that log-returns on bonds are bounded from below by $-a$ or equivalently that bond prices are bounded away from zero by a positive constant.

4 Mortality model

4.1 Introduction

Mortality forecasting is about modeling the stochastic lifetimes of people. Several types of models are commonly used in practice, depending on the application (cf. Booth and Tickle, 2008, Cairns et al., 2008, Keilman, 2003, and McWilliam, 2011). The key questions to consider in making modeling choices include the following:

- micro level (individuals) or macro level (groups of individuals)
- static or dynamic approach
- explanatory factors (e.g. age, sex, cause of death)
- forecasting horizon
- availability and choice of data

In actuarial practice models have often been used that build on an analytical static formula for the force of mortality or the hazard rate (e.g. the Makeham or Gompertz law; see e.g. Gerber, 1990, sec. 2.3). The main reason for their use is simplicity. In addition life and pension insurance portfolios are usually large enough to diversify away the idiosyncratic risk. However, aggregate uncertainty, i.e. the systematic risk of mortality change, remains. This is typically accounted for by implicit safety margins based on ad hoc approaches. The current international trend in life insurance modeling is towards the dynamic stochastic approach. A standard approach nowadays in stochastic mortality forecasting is to use the Lee-Carter model (Lee and Carter, 1992). We first describe and implement this model for the Finnish data. In the estimation phase we make use of the MCMC method to take into account parameter uncertainty. Subsequently we develop a local version of the Lee-Carter model, that is, a local bilinear model, and compare the estimation and forecasting results of the two approaches.

4.2 Data

We use the annual age-specific death rates for the entire Finnish population for the years 1955 to 2008 from the Human Mortality Database (HMD, available at www.mortality.org). This choice puts more focus on recent developments, as it excludes the early part of the available data in the HMD, which starts from 1878. Our data also exclude the war years and the period of rapid mortality decline of younger generations in Finland, around 1950. The general picture of Finnish mortality is such that more recent data leads to lower forecasted mortality. Our data choice is also supported by the Finnish insurance industry's internal research report by Kuusela and Kukkala (2010) where different HMD data periods and the resulting Lee-Carter forecasts are analyzed and compared. We note that the choice of data includes inevitably subjectivity and expert judgement. However, we allow certain parameter values to vary in the forecasts by applying the MCMC method to the mortality index.

Our focus is on the long term trend and uncertainty of mortality forecasts, and thus the population data is a suitable basis for our study. Insurance portfolios, however, are selected sub-populations of the entire population and they typically have a lower mortality due to

various socioeconomic factors and insurance underwriting practices. This topic has been analysed recently in Sweden (DUS, 2007, Chapter 4). There it is concluded on page 65 that mortality for the voluntary insurance sub-population was about 30 percent lower than population mortality for women and 40 percent lower for men up to 95 years. However, the data period was only from 2001 until 2005. In general, the smaller amounts of data in sub-populations, such as those of small insurance companies, necessarily raise the question of their suitability as such for long term forecasting. One way to get the best of both worlds is to aggregate the data of all insurance companies and then carry out the forecasting, as was done in Sweden and is currently being done in Finland.

In the HMD the death or mortality rates are available for ages $x=0,1,2, \dots, 109,110+$. The yearly death rate $M_x = D_x/E_x$, where D_x = number of deaths and E_x = exposure-to-risk,²¹ is calculated in the HMD period life table according to the formula

$$M_x = \frac{D_L + D_U}{(P(x, t) + P(x, t + 1))/2 + (D_L - D_U)/6}, \quad (4.1)$$

where D_L is the number of deaths in the lower and D_U in the upper Lexis triangle and $P(x, t)$ is the population aged x at the start of calendar year t . These and other key concepts that are needed in the construction of life tables are explained in detail in the Methods Protocol for the HMD or Alho and Spencer (2005).

For infant mortality rate the HMD formula is modified as

$$M_0 = \frac{D_0}{\frac{1}{3}B(t-1) + \frac{2}{3}B(t)}, \quad (4.2)$$

where $B(t)$ denotes the number of births occurring in year t .

The modeling of older ages also requires special measures. We rely again on the methods developed for the HMD life tables, the main features of which we review here. For details we refer to the HMD Methods Protocol.

The HMD uses a combination of methods for deriving age-specific estimates of population size on January 1 of each year. For most of the age range, they use either linear interpolation of population estimates from other sources or intercensal survival methods. At older ages (80+) they generally use population estimates computed using special methods, except for Finland and the other Nordic countries, where reliable official population estimates are available.

²¹The number of life-years exposed to risk of death

Period life tables are computed by converting death rates to probabilities of death (see (4.12)). Before this conversion, death rates for ages 80 and above are smoothened by fitting a logistic function separately for males and females. A cohort life table depicts the life history of a specific group of individuals. A period life table represents the mortality conditions at a specific moment in time, which however are not known with certainty. At older ages where this inherent randomness is most noticeable, the HMD smoothenes the observed values in order to obtain a better representation of the underlying mortality conditions.

4.3 Review of the Lee-Carter model

The model developed by Lee and Carter (1992) is defined for the death rate for age x in year t , $m_{x,t}$, as follows

$$\log[m_{x,t}] = a_x + b_x k_t + \epsilon_{x,t}, \quad (4.3)$$

where a_x and b_x are age-specific constants, k_t is a time-varying index of mortality, and $\epsilon_{x,t} \sim i.i.d.N(0, \sigma_\epsilon^2)$ is the error term. The identifying constraints are chosen as $\sum_x b_x = 1$ and $\sum_t k_t = 0$, with $k_t > 0$ for $t=1955$.

In other words the Lee-Carter model expresses the log of age-specific death rates as a linear function of an unobserved time-dependent mortality intensity index k_t , where a_x defines the age-profile and b_x is a weighing factor.

Lee and Carter suggested use of the singular value decomposition (SVD) to find a least squares solution to the underdetermined model equation. This method is based on matrix theory (see e.g. page 427 in Horn and Johnson, 1985), and its implementation for the Lee-Carter model is explained in Alho et al. (2011). The method can be implemented as follows. We first subtract the averages over time (row-averages) of the matrix $\log[m_{x,t}]$ (see Figure 4.1), where $m_{x,t}$ comes as in (4.1) for $t=1955, \dots, 2008$ for the Finnish total population data. Then we use the R-function *svd* to find the first left and right vectors and the leading value of the SVD. b_x in Figure 4.2 is the first left vector when divided by the normalizing constant, which makes the components sum to unity. k_t , which is the product of the latter two terms in SVD, is divided by the normalizing constant in Figure 4.3.

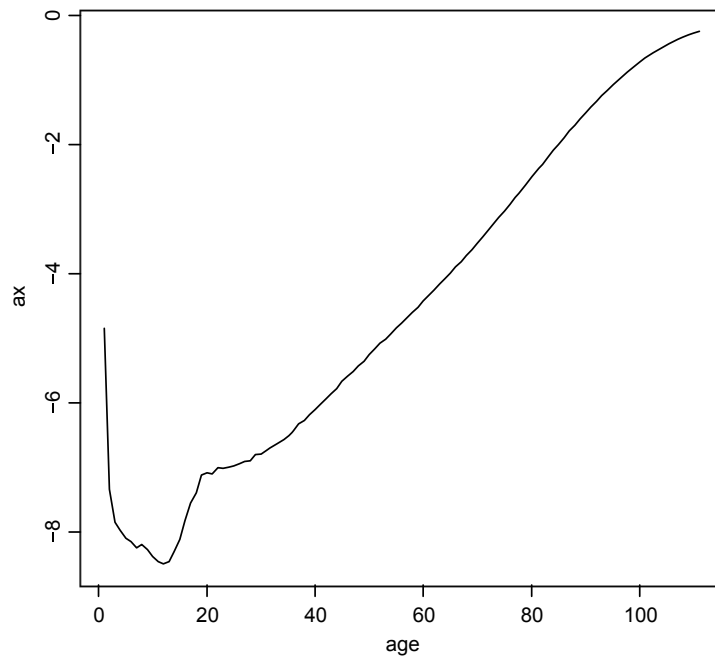


Figure 4.1 Lee-Carter ageprofile a_x for the total population in Finland in 1955–2008

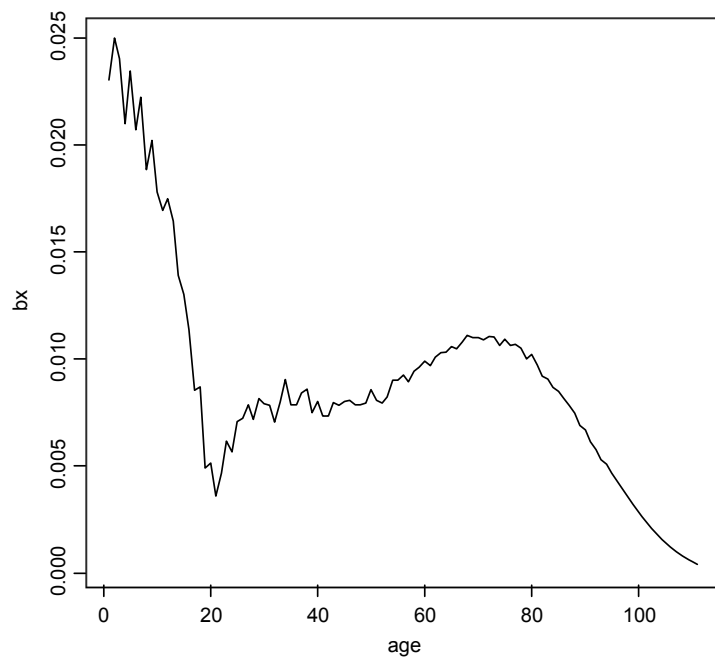


Figure 4.2 Lee-Carter weighing factors b_x for the total population in Finland in 1955–2008

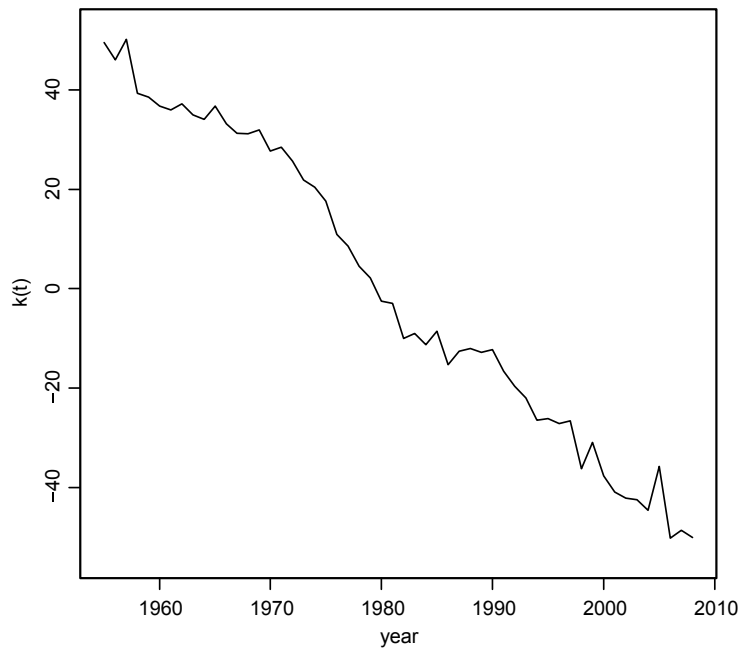


Figure 4.3 Mortality index k_t for 1955–2008

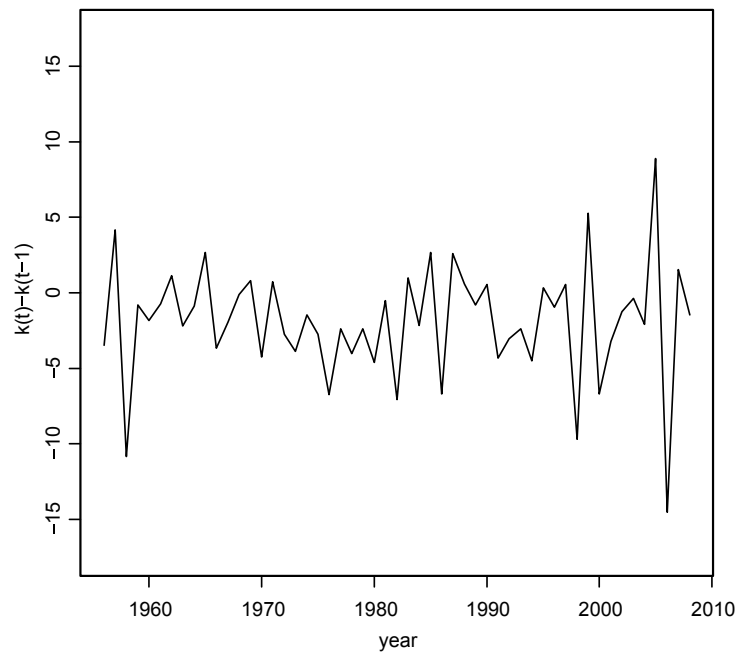


Figure 4.4 Differences $k_t - k_{t-1}$

Next the mortality index k_t is modeled using time series methods. This index is approximately a decreasing line during the sample period in Finland (Figure 4.3), and it can be modeled as a random walk with drift. This is the same approach that Lee and Carter used although the U.S. data included more deviation from the linear trend. In our case the summary of the once-differenced data (Figure 4.4) is as follows: Data Points = 53, Sample Mean = -1.880 (s.e. 0.530), Sample Variance = 14.88, Min = -14.510, Max = 8.883, Median = -1.818. Consequently the R *arima* software gives the following estimated model for the yearly mortality index

$$k_t = k_{t-1} - 1.88 + e_t, \quad (4.4)$$

where $e_t \sim i.i.d.N(0, 14.88)$.

In order to validate the model we carried out the following analysis. The Lee-Carter model assumes that mortality develops in a way where the common mortality index k_t is multiplied by the age-dependent weighing factor b_x . The accuracy of this assumption for our data is assessed in Figure 4.5, which gives $\log(m_{x,t}) - \log(m_{x,t+1})$ for age-cohorts 30–49, 50–64, 65–79 and 80–99 during the period from 1950 to 2007. The mean values for these cohorts are respectively 0.0193, 0.0165, 0.0188 and 0.0137. We note that the shapes of the log-mortality change-curves are not indentical and that the means and the standard deviations for the different age groups differ. We will address this problem below in detail, but first we analyse parameter uncertainty in the Lee-Carter model.

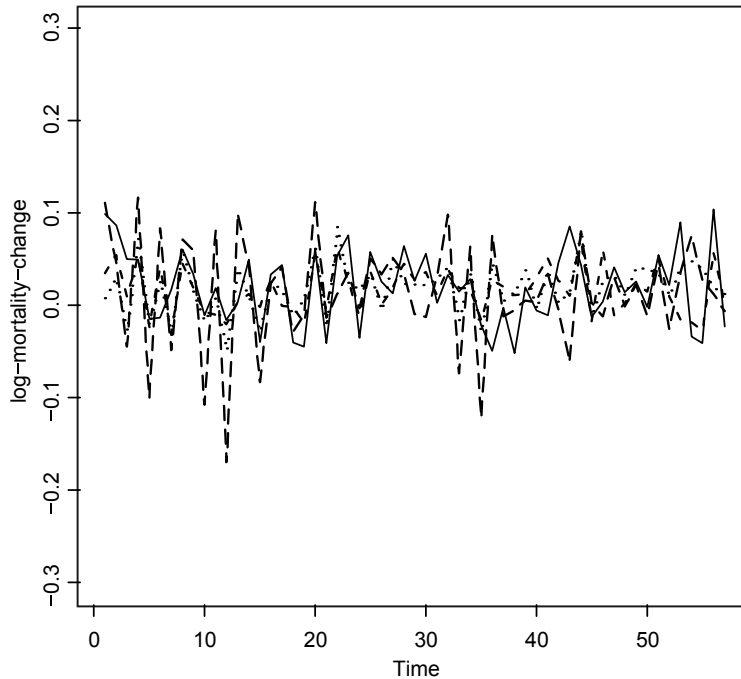


Figure 4.5 Difference of log-mortality $\log(m_{x,t}) - \log(m_{x,t+1})$ for ages 30–49(solid), 50–64(dashed), 65–79(dotted), 80–99(longdashed) during 1950–2007.

4.4 Parameter uncertainty in the Lee-Carter model

We take into account the parameter uncertainty of the Lee-Carter model by specifying a Markov-Chain Monte-Carlo algorithm for the random walk with drift model that was chosen for the mortality index k_t . The algorithm below can be found in Gilks et al. (1996) on p.75–77 and in Alho and Spencer (2005) in sec. 9.2.2. Lee and Carter also studied the sources of error for the other parameters of the model (cf. Appendix B of their paper). Their conclusion was that the error in forecasting the mortality index, which they evaluated using the Normal assumption, dominates the total forecasting error for long horizons. Therefore this approach is sufficient for our purposes. As in the previous section we use the HMD yearly data from 1955 to 2008 for the age groups 0,1,2,...,109,110+.

Assuming for $t = 1, \dots, n = 53$ that the prior distributions are sufficiently uninformative conjugate distributions,²²

$$\begin{aligned} z_t &= k_{t+1} - k_t \sim N(\mu_m, 1/\tau_m), \\ \mu_m &\sim N(b, S^2) = N(-1.9, 16), \\ \tau_m &\sim \text{Gam}(\alpha_m, \beta_m) = \text{Gam}(3, 45), \end{aligned} \tag{4.5}$$

that the z_t are conditionally independent given the parameters, and that μ_m and τ_m are independent, the joint density for the observations and parameters is

$$\begin{aligned} L &= \prod_{t=1}^n P(z_t | \mu_m, \tau_m) P(\mu_m) P(\tau_m) \\ &= \frac{\tau_m^{n/2}}{2\pi^{-(n+1)/2}} e^{-\frac{\tau_m}{2} \sum_{t=1}^n (z_t - \mu_m)^2} e^{-0.5S^{-2}(\mu_m - b)^2} \tau_m^{\alpha_m - 1} e^{-\tau_m \beta_m}. \end{aligned} \tag{4.6}$$

In deriving the full conditional distribution for each parameter, we pick only those terms that include the parameter in question. Using the analogous results in Gilks et al. (1996) or completing the square, we find that the posterior for μ_m is

$$N\left(\frac{\tau_m}{S^{-2} + n\tau_m} \sum_t z_t, (S^{-2} + n\tau_m)^{-1}\right), \tag{4.7}$$

and for τ_m

$$\text{Gam}(\alpha_m + n/2, \beta_m + 0.5 \sum_t (z_t - \mu_m)^2). \tag{4.8}$$

The Gibbs-algorithm samples from the full conditional distributions above in a stepwise fashion using the results of the previous iteration as parameters. The starting value for τ_m can be taken from the ML estimation. The iteration procedure is repeated until a sufficient sample is obtained.

The results of the procedure are given in Figures 4.6, 4.7 and 4.8, where the conditional posterior distributions of the parameters and their dependence are shown.

²²i.e. flat and wide enough in light of preliminary estimation results above for (4.4) to not affect the numerical results

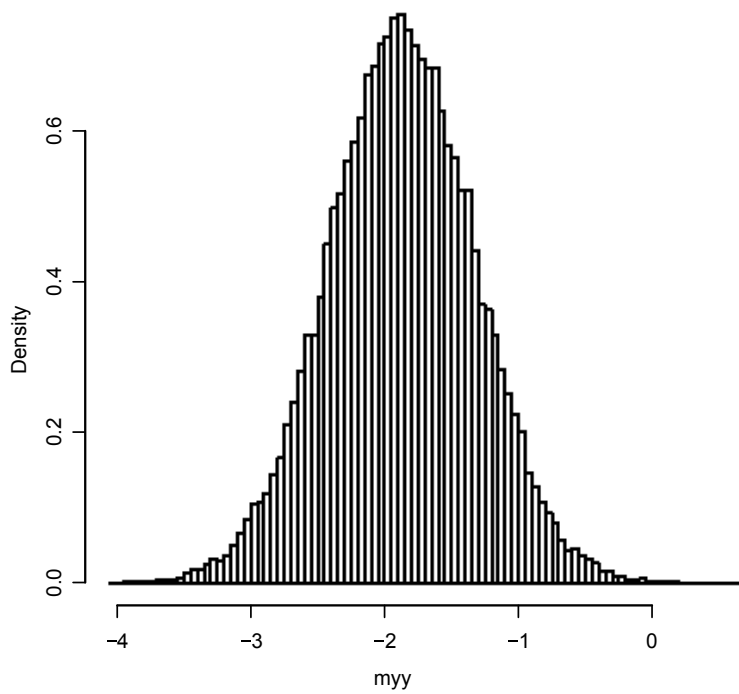


Figure 4.6 Posterior distribution of μ_m

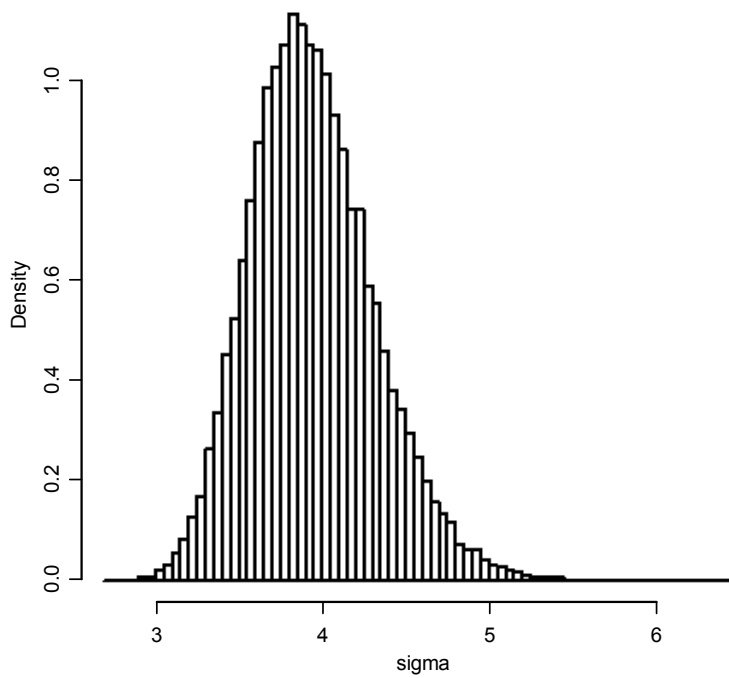


Figure 4.7 Posterior distribution of $\sigma_m = 1/\tau_m^{0.5}$

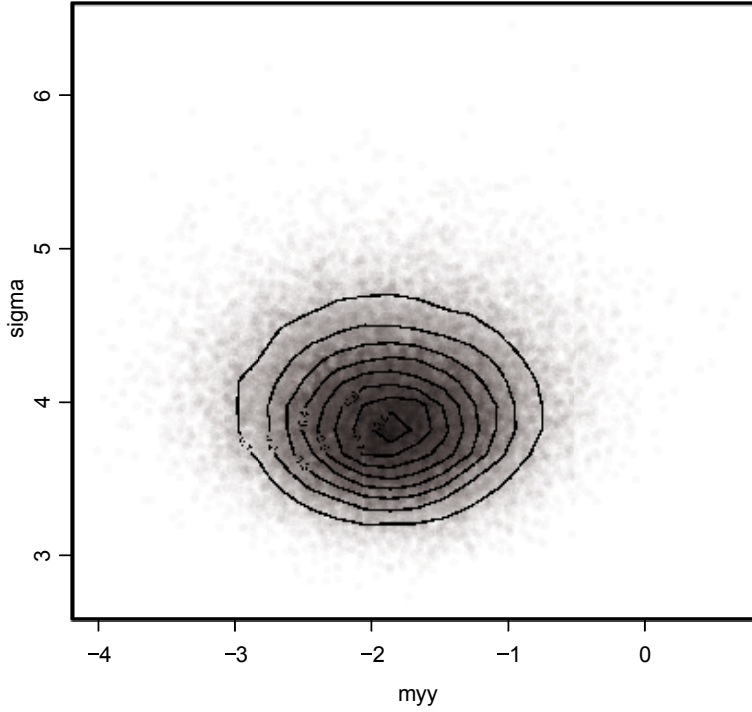


Figure 4.8 The posterior of μ_m vs the posterior of σ_m

To simulate forecasts from the model, we proceed as follows:

- Choose (μ_m, τ_m) from the MCMC sample.
- Generate an i.i.d. sequence $e_t \sim N(0, 1/\tau_m)$ of the desired length.
- Calculate $k_t = k_{t-1} + \mu_m + e_t$ to get the simulated forecast for k_t . We choose $k_0 = -50$ so that the jump-off error in the start of forecast is minimized. Lee and Miller (2001) have shown that jump-off bias is avoided by constraining the model so that k_t passes through zero in the jump-off year (see also Booth and Tickle, 2008).
- Calculate $m_{x,t} = \exp(a_x + b_x k_t)$ to get the simulated forecast for $m_{x,t}$. We smoothen a_x and b_x by the *smooth.spline* function in R in order to reduce the unwanted yearly variation in b_x ; see Figure 4.2.

- Choose the next pair (μ_m, τ_m) and repeat the procedure for a new forecast.

We can use the simulation algorithm to study the mortality forecasting uncertainty for different horizons. Table 4.1 contains the simulated total population death rates $m_{65,t}$ for different forecasting horizons $t=(10,20,30,40)$. We observe the increase in forecasting uncertainty for the longer horizons when comparing the summary statistics. For instance, when the forecasting horizon changes from 10 to 40 years, the mean decreases from 0.91 to 0.53 percent, whereas the standard deviation increases from 0.13 percent to 0.19 percent. This is a key feature in longevity forecasting, which makes the pricing and risk management of annuities challenging.

It is interesting to compare our simulations to the official deterministic forecast done in 2009 by Statistics Finland. In that study the expected life time of a 65-year old male (female) in 2010 is 17.6 (21.5) years, the average being 19.6 years.²³ In our forecast the corresponding figure for the total population is 20.2; see Figure 6.2. Our forecast is thus more optimistic.

Table 4.1 Quantiles and some percentiles of 3000 simulations for $100m_{65,t}$ with different forecasting horizons t

t	1%	Q1	Q2	mean	Q3	99%	sd
0				1.09			
10	0.64	0.82	0.90	0.91	0.99	1.26	0.13
20	0.44	0.64	0.74	0.75	0.85	1.22	0.16
30	0.31	0.50	0.61	0.63	0.73	1.14	0.18
40	0.22	0.40	0.50	0.53	0.63	1.10	0.19

4.5 Gender-specific mortality

Remaining life expectancy at age 65 for males and females in Finland according to the HMD life table for 2008 are 17.3 and 21.0 years respectively, while for the total population it is 19.5 years. We now analyse the effect that gender has on mortality by carrying out the

²³The official forecast is available at <http://tilastokeskus.fi>, and the lifetimes are from Kuusela and Kukkala (2010).

Lee-Carter calculations above separately for males and females using the HMD data from 1955 to 2008. This is an important and topical issue for the European life and pension insurance industry because a recent decision by the EU Court prohibits the use of gender as a pricing factor for insurance risks since 2013 (Test-Achats judgement, C-236/09). We use the same HMD data set as before except, leaving out ages 0,1,...,19, as there are very few deaths in this age range (except for infants).

Age-specific parameters a_x , b_x and k_t resulting from the SVD estimation are shown in Figures 4.9, 4.10 and 4.11. We note that the mortality age profiles according to a_x differ, men having higher mortality except at the very highest ages. Age-dependent weighing factors b_x for the mortality index k_t are also different for males and females. For men in the age range of 40-65 the mortality gains are much higher, but the situation changes up to age 90, after which there are no big differences. When comparing the dynamic factor k_t that controls the mortality change, we note that the last 10 years or so have been rather similar for both sexes. Before that, there are differences and changes. Men's k_t has been higher since the 1970s, in contrast to the prior situation. However, if one takes 1970 as the starting and 2008 as the end point and connects these points, the lines are almost identical. This could indicate that mortality intensity is changing in the long run similarly for both sexes. However, the correlation of the once differenced k_t series for males and females in our data is 0.47. Moreover, we note that it is the product $b_x k_t$ that drives future mortality changes. Calculating the difference of these matrix products for males and females, we get the surface shown in Figure 4.14. We observe that the mortality changes have not been the same during the observation period from 1955 to 2008, except for the highest ages. The pattern has also changed during the period. We note that the yearly change in mortality for males has been greater than for females during the last 10 years, in particular for ages around 20 and around 70 years, as the peaks in the figure show.

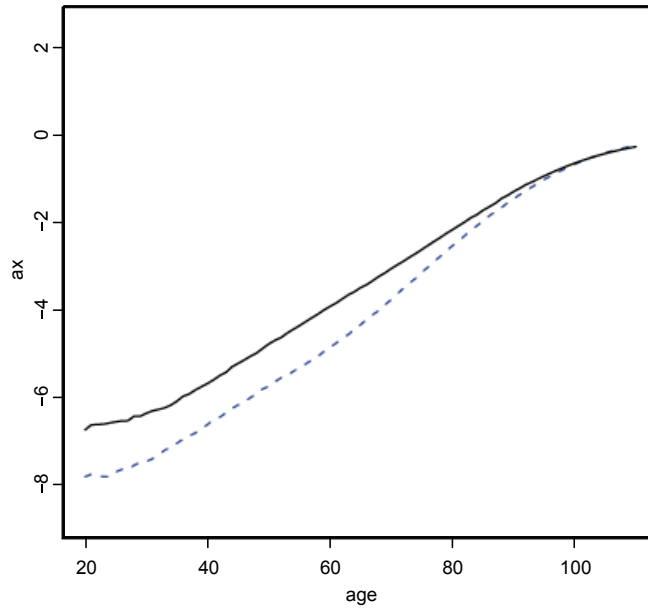


Figure 4.9 Lee-Carter ageprofiles a_x for males (solid) and females in Finland in 1955–2008

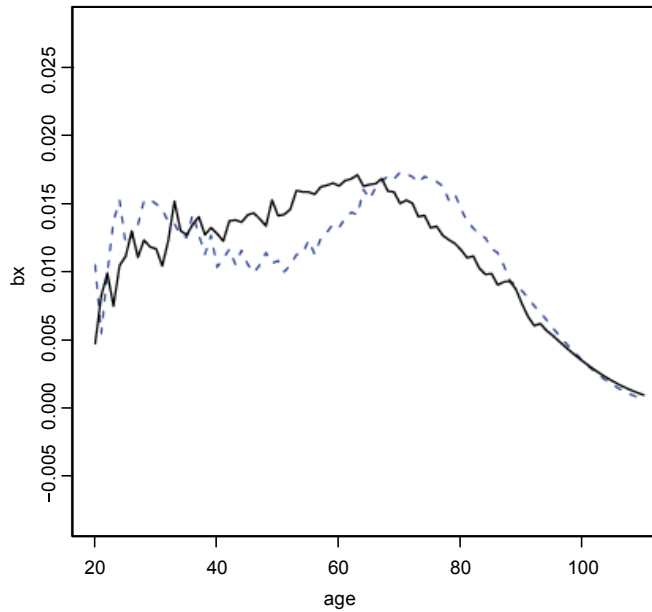


Figure 4.10 Lee-Carter weighing factors b_x for males (solid) and females in Finland in 1955–2008

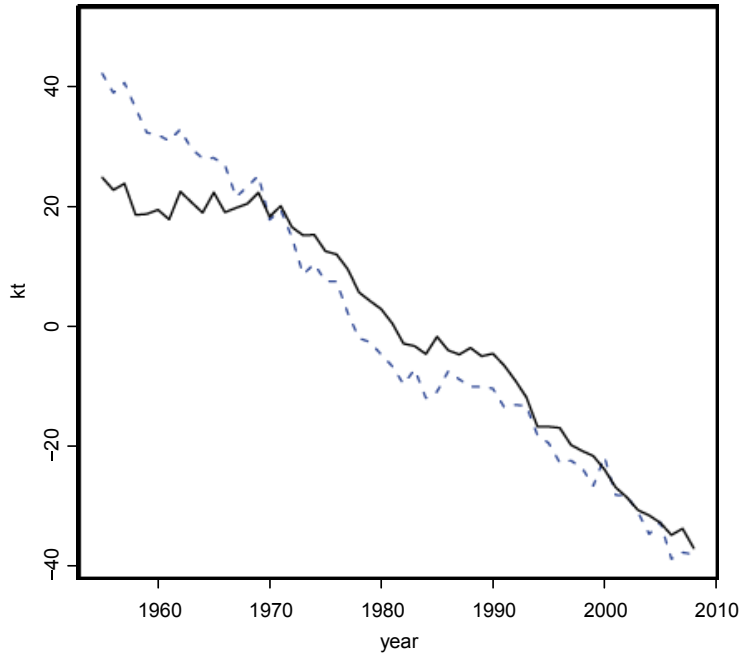


Figure 4.11 Lee-Carter mortality index k_t for males (solid) and females in Finland 1955–2008.

Posterior ARMA model parameters for the differenced k_t series were calculated by running the MCMC estimation separately for males and females for 50 000 iteration rounds. The results are shown in Figures 4.12 and 4.13. We note that both parameters are distributed differently for the genders, males having generally higher mean and lower standard deviation than females.

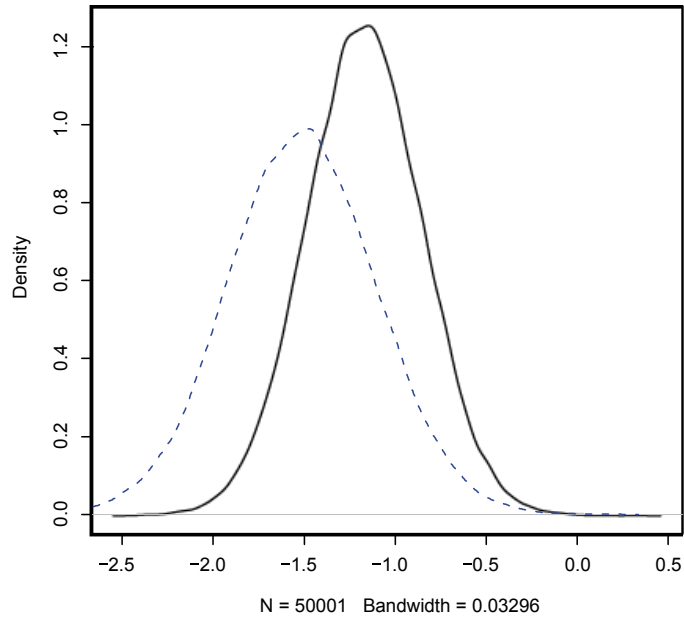


Figure 4.12 Posteriors of μ_m for males (solid) and females

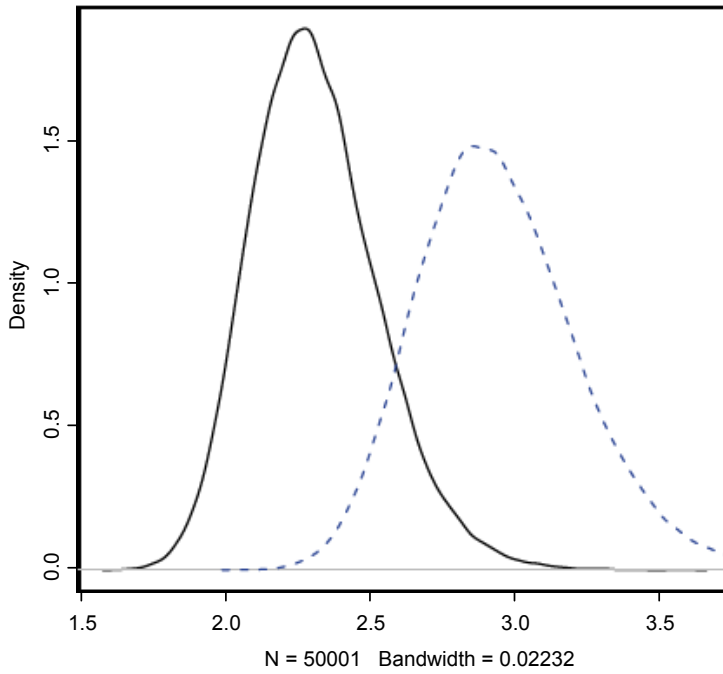


Figure 4.13 Posteriors of σ_m for males (solid) and females.

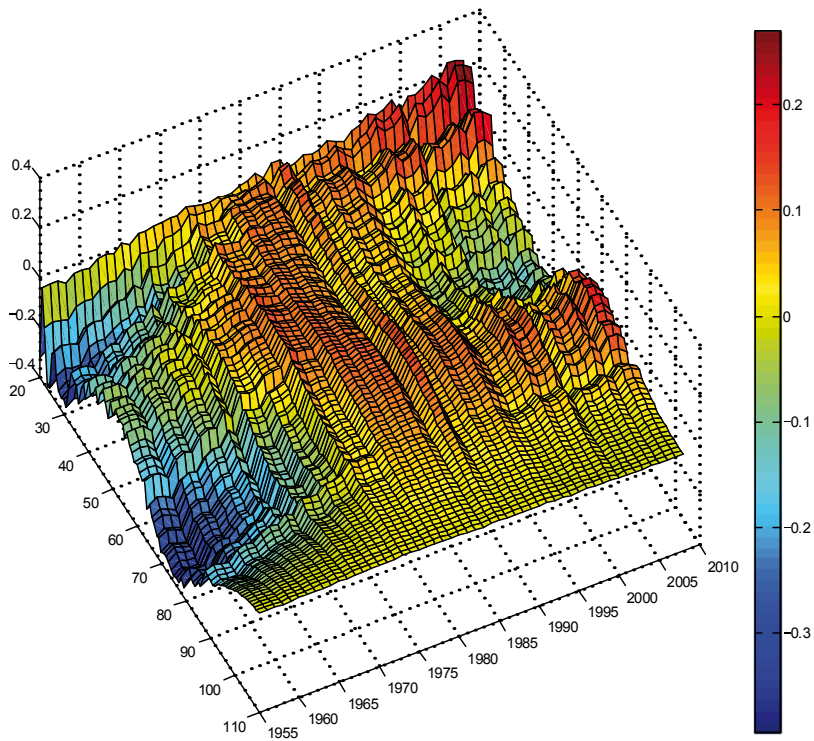


Figure 4.14 Difference of $k_t b_x$ between males and females for ages 20-110+ in 1955-2008

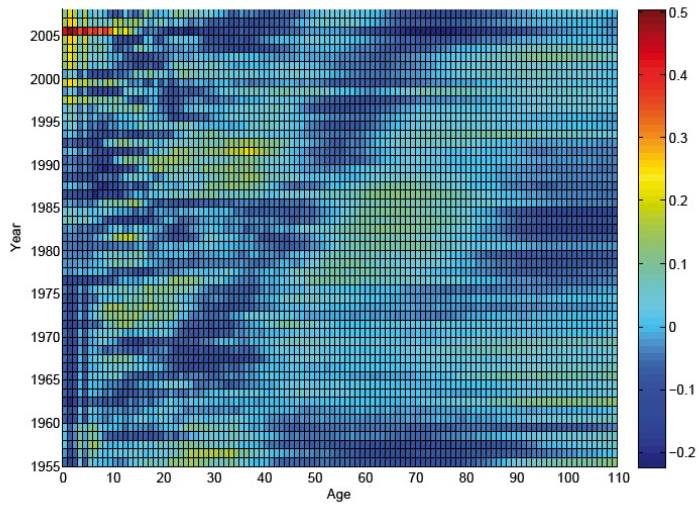


Figure 4.15 Fitted LBL minus L-C for ages $x=0,1,\dots,110+$ (x-axis) and years 1955-2008 (y-axis)

4.6 The local bilinear model

In the literature there have been numerous attempts to improve the Lee-Carter model, as for instance the review in Booth and Tickle (2008) illustrates. Our goal is to generalize the Lee-Carter (L-C) model in order to gain more flexibility in modeling different ages. This generalization, a local bilinear model (LBL), is developed in three steps. Firstly we estimate the Lee-Carter model using the standard SVD method explained above, but now we do it locally for each age x . We use several data windows $[x-h, x+h]$, $h=1, \dots, 10$, from which we choose the optimal bandwidth by cross-validation. Secondly, we estimate and validate the model parameters. Thirdly, we fit ARIMA(0,1,0) models for each age and generate forecasts of the model, which we then compare to the Lee-Carter forecasts. Finally, we choose the preferred model for our simulations.

4.6.1 Specification and bandwidth selection

In this section we specify the local bilinear model and determine the optimal size of the age interval to be included in the estimation step. The data cover the Finnish population for $t=1955, \dots, 2008$. We specify the Lee-Carter model locally for each age $x=10, 11, \dots, 90$, using different bandwidths, $W_{x,h} = [x-h, x+h]$, $h=1, \dots, 10$. When $y \in W_{x,h}$, the model for the death rate in year t or $m_{y,t}$ is

$$\log[m_{y,t}] = a_y + b_{x,y}k_{x,t} + \epsilon_{x,y,t}, \quad (4.9)$$

where a_y and $b_{x,y}$ are age-specific constant vectors, $k_{x,t}$ is a time-varying index of mortality, and $\epsilon_{x,y,t} \sim i.i.d.N(0, \sigma_l^2)$ is the error term. When the model is extended to cover the first and last h values of the age range, we define the middle term as $b_{10,y}k_{10,t} = b_{10,10}k_{10,t}$ for $y \in [0, 9]$, and as $b_{90,y}k_{90,t} = b_{90,90}k_{90,t}$ for $y \in [91, 100]$.

The age-profile vector a_y is estimated as the row mean of the data matrix, as in the Lee-Carter case above and it is first subtracted from the data. In the data analysis we have assumed the error term equal to zero and have not set further parameter restrictions on the model, as we use it only for comparative purposes below.

The optimal bandwidth selection is based on widely used least squares cross-validation (see e.g. Park and Marron, 1990, and Patil, 1993) as follows. We estimate the model separately for each age x using the punctured interval $W_{x,h} \setminus \{x\}$ in the SVD-estimation. We leave out age x in order to use it in the leave-one-out cross-validation. More specifically, we calculate the term $b_x k_t$ of the Lee-Carter formula by estimating $b_{x,y}$ locally as $(b_{x,y-1} + b_{x,y+1})/2$. We then compare the estimated result to the actual data, and choose the value of h which minimizes the Sum of Squared Errors,

$$\sum_{x,t} [0.5(b_{x,y-1} + b_{x,y+1})k_{x,t} - m'_{x,t}]^2, \quad (4.10)$$

where $m'_{x,t}$ comes from the log-mortality data matrix where the mean of each age has been subtracted. We find that the optimal bandwidth is 5 (see Table 4.2).

Table 4.2 Sum of Squared Errors for different bandwidths

h	SSE
1	72.0
2	66.9
3	62.9
4	60.3
5	58.0
6	59.2
7	59.3
8	59.7
9	60.1
10	61.0

4.6.2 Estimation and validation

The local bilinear model is estimated in the same way as in the previous section, except that now $h=5$ is fixed and we include the whole interval $W_{x,h}$ when running the SVD algorithm for age x . The results of these computations are discussed in the context of forecasting results below.

We validate our model by comparing it to the Lee-Carter model. This is done by computing the $b_{x,y}k_{x,t}$ terms in the local bilinear

model and the $b_x k_t$ terms in the Lee-Carter model and taking their differences. For the first and last 5 values in the LBL we used the first and the last available values as specified above. The resulting surface is shown in Figure 4.15. We observe that there are both positive and negative differences, which indicates that the LBL model fits the data better. The differences LBL minus L-C for the most part seem to be rather random. However, a notable exception is the cohort born around year 1940, i.e. during the wartime, for which the LBL model gives lower mortality. This is shown as a darker area or valley in the figure, which is due to negative values.²⁴

4.6.3 Deterministic forecasting and model comparison

Forecasting is based on the Random Walk with Drift (RWD) model as in the Lee-Carter case, but now we have a different model for each age. We approach the problem by developing deterministic forecasts and comparing the result of these two models.

The multipliers in the RWD models for each age are the differences between the last and the first observation divided by the number of data points minus 1. We smoothed the data by the cubic spline smoother *smooth.spline* in R for graduation.

Mortality forecasts for years $t=1, \dots, 75$ and ages $x=0, 1, 110+$ in the deterministic case are given by the formula

$$\log(M_{x,t}) = m_{x,0} + a_x + t c_x, \quad (4.11)$$

where $m_{x,0}$ is the last data point (year 2008) in the log-mortality data matrix, a_x is the subtracted mean value in the data, and c_x is the RWD coefficient described above. Finally we de-transform the results by the *exp* function. In Figures 4.16 and 4.17 we plot a couple of LBL forecasts together with the observed values.

²⁴Namely, $b_{x,y} k_{x,t} - b_x k_t = a_x + b_{x,y} k_{x,t} - a_x - b_x k_t = \ln[m''_{x,t}] - \ln[m_{x,t}]$, which is negative when the LBL-mortality $m''_{x,t}$ is smaller than the Lee-Carter mortality $m_{x,t}$.

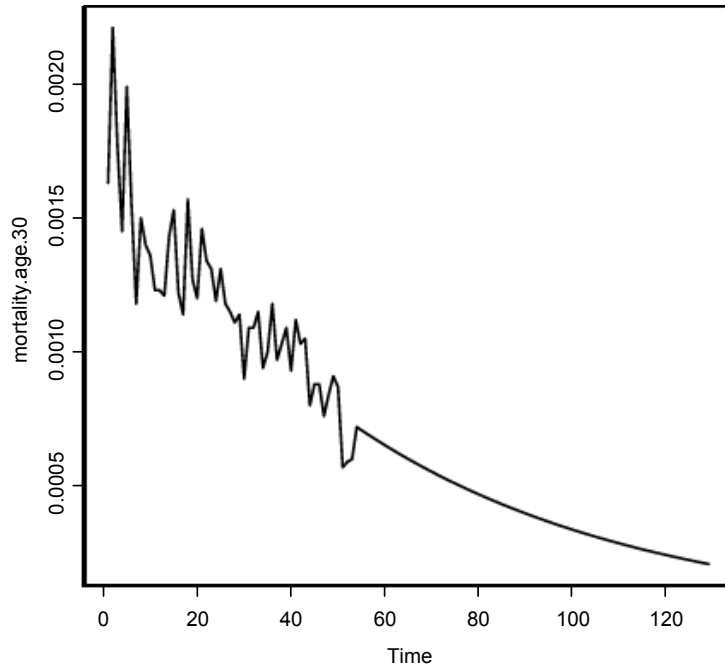


Figure 4.16 Observed and forecasted mortality for age $x=30$ in the LBL model

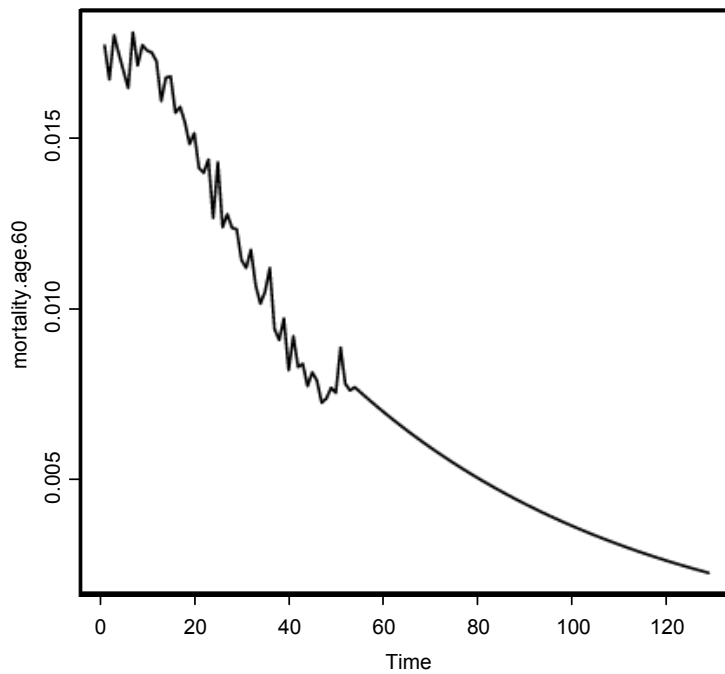


Figure 4.17 Observed and forecasted mortality for age $x=60$ in the LBL model

The final step in our deterministic calculation is to form a life table for the LBL and L-C models, which can be used to calculate life expectancies and compare forecasts. We use the actuarial estimator (cf. Alho and Spencer, 2005, p. 89) to get the 1-year-survival probabilities from the forecasted death rates for $t=1, \dots, 75$

$$p_{x,t} = (2 - M_{x,t}) / (2 + M_{x,t}), \quad (4.12)$$

where $x=0, 1, \dots, 109, 110+$. Using the familiar life table formulas with $l_{0,t} = 1$, we get the expected life time of infant for each t as

$$e_{0,t} = 1/2 + l_{1,t} + l_{2,t} + \dots, \quad (4.13)$$

$$l_{x,t} = p_{0,t} \times \dots \times p_{x-1,t}. \quad (4.14)$$

This standard method is explained in more detail in Gerber (1990), Appendix A, and in Alho and Spencer (2005) on page 80.

The results for the LBL and L-C models and their difference are shown in Tables 4.3, 4.4 and 4.5 respectively. We note that the LBL results are slightly greater for long forecasting horizons, i.e. the LBL model then gives slightly lower mortality forecasts than the L-C model does. The differences though are minor, which leads us to conclude that our LBL model supports the L-C model. The better fit of the LBL model does not significantly change the L-C forecasts despite the cohort effect we noted. Moreover, the L-C model is the far more simple of these two models. By the principle of parsimony we chose to base our mortality forecasting on the previously discussed MCMC simulation algorithm for the Lee-Carter model.

Table 4.3 Expected lifetime of infant during the 75-year forecasting period in the LBL model

year	e_0									
1	80.01	80.20	80.38	80.56	80.75	80.93	81.11	81.29	81.47	
10	81.65	81.82	82.00	82.17	82.35	82.52	82.69	82.86	83.03	83.20
20	83.36	83.53	83.70	83.86	84.02	84.19	84.35	84.51	84.67	84.83
30	84.98	85.14	85.29	85.44	85.60	85.75	85.90	86.05	86.20	86.35
40	86.49	86.64	86.78	86.93	87.07	87.21	87.35	87.49	87.63	87.77
50	87.90	88.04	88.17	88.31	88.44	88.57	88.70	88.83	88.96	89.09
60	89.22	89.35	89.47	89.60	89.72	89.84	89.96	90.08	90.20	90.32
70	90.44	90.56	90.68	90.79	90.91	91.02				

Table 4.4 Expected lifetime of infant during the 75-year forecasting period in the L-C model

year	e_0									
1	80.01	80.19	80.38	80.56	80.74	80.92	81.10	81.28	81.45	
10	81.63	81.80	81.97	82.14	82.31	82.48	82.65	82.82	82.98	83.15
20	83.31	83.47	83.63	83.79	83.95	84.10	84.26	84.41	84.57	84.72
30	84.87	85.02	85.17	85.32	85.47	85.62	85.76	85.90	86.04	86.18
40	86.32	86.46	86.60	86.74	86.88	87.01	87.14	87.28	87.41	87.54
50	87.67	87.80	87.93	88.05	88.18	88.31	88.43	88.55	88.67	88.80
60	88.92	89.04	89.15	89.27	89.39	89.50	89.62	89.73	89.84	89.96
70	90.07	90.18	90.29	90.39	90.50	90.61				

Table 4.5 Difference of expected lifetimes of infant in LBL and L-C forecasts

year	Δe_0
1	0.00
10	0.02
20	0.05
30	0.11
40	0.17
50	0.23
60	0.30
70	0.37
75	0.41

5 Dependence modeling

5.1 Introduction

The final building block in our model is the dependence structure. In addressing this question we must consider how the equity and bond total return indexes and mortality experience might be interrelated in different situations.

Asset managers typically base their investment strategies on the assumption that correlations between various asset classes are less

than perfect. Therefore diversification can provide benefits in terms of the risk-return profile when choosing the asset portfolio. However, dependence modeling is generally considered very challenging. This is true here also due to the comovements of equity and bond returns. The correlation between the log-returns on the SP500 and 5yB indexes in our whole data from 1926 to 2006 is small and positive, 0.059, but there are time periods of large positive and negative correlations. This is shown in Table 5.1, containing the correlations for 10-year periods.

Table 5.1 Correlation between the log-returns on the US equity and mid-term bond total return indexes SP500 and 5yB for various periods

years	correlation
1926–1935	0.12
1936–1945	0.38
1946–1955	−0.15
1956–1965	−0.67
1966–1975	0.13
1976–1985	0.37
1986–1995	0.70
1996–2006	−0.51

It is widely believed in the literature that the bond-stock relationship is time varying and includes nonlinearities. Andersson et al. (2008), after presenting a literature review, study the variation in daily data on US, UK and German stock and bond returns from January 1991 to August 2006. They conclude that stock and bond prices in general move in the same direction, but that there are periods of negative stock-bond return correlations which seem to coincide with the lowest levels of inflation expectations. Moreover, in line with the flight-to-quality strategy of asset managers, their results suggest that high stock market uncertainty (measured by the implied volatility of stock options) leads to a decoupling between stock and bond prices. Previously Gulko (2002) had shown that the periods of negative stock-bond correlation tend to coincide with stock market crashes. For a broad discussion on dynamic correlation modeling and forecasting we refer to Engle (2009), and for a discussion on asset prices during financial crisis we refer to Malz (2011), Ch. 14.

Kroner and Ng (1998) argue, in their study of asymmetric comovements of asset prices in the context of multivariate GARCH

models, that if the expected return on one asset class changes due to an asymmetric volatility effect, the correlation between the returns on that asset and on other assets which have not had a change in their expected returns should also change.

In addition to multivariate GARCH, there are other widely used multivariate models in economic and financial applications. A vector autoregressive (VAR) model is an important example. Engle and Granger (1987) introduced the notion of cointegrated processes and the Error Correction Model. For the application of a Vector Equilibrium Correction model to a pension insurance company, we refer to Koivu et al. (2005).

Non-linear dependence can be modeled by the so-called copula method. This method constructs the joint multivariate probability density as a product of the marginal densities and a copula function; see e.g. Denuit et al. (2005) or Nelsen (1999). However, one serious limitation of the copula approach is its static nature, i.e. it does not take into account the time varying aspect of the dependence.

When it comes to dependence between mortality and economic factors, there is nowadays little evidence to support the idea that economic factors and mortality would normally be correlated. However, historically the situation has been different due to famines etc; cf. the Finnish study in Turpeinen (1977). On the other hand, during economic shocks mortality might be higher due to more stressful living conditions etc, so we allow for this possibility in our dependence model.

Thus our next goal is to build a flexible model for the dependence structure that enables generation of stochastic scenarios. In this chapter we first discuss the structure of the simulation model. Next we formulate it mathematically, and then discuss the simulation algorithm for which more information is included in the appendix.

5.2 Model structure

The flow chart of the simulation model is as follows:

1. From the three independent parallel processes with posteriors Q_x , Q_r , Q_m we sample a parameter triple (θ, Ψ, Θ) , where $\theta = (\mu, \sigma, q, \alpha, \beta)$ correspond to the process for equity returns x_t , $\Psi = (\phi, \theta_y, \beta_y, \sigma_y)$ are the parameters for the interest rate returns r_t , and $\Theta = (\mu_m, \tau_m)$ and a_x, b_x are the parameters for

the mortality model $m_{x,t}$. The underlying data and estimation procedures were described in Chapters 2, 3 and 4.

2. Generate the innovations (ϵ_t, u_t, e_t) for the processes $(x_t, y_t, m_{x,t})$ based on the jump process J_t and the chosen correlation structure (Table 5.2 and 5.3).
3. Generate the processes x_t, r_t, k_t with a chosen dependence structure, where x_t is the equity return process (2.3), r_t is the bond return process (3.3), and k_t is the process driving the mortality rate $m_{x,t}$ in (4.3) for ages $x=0,1,\dots,110+$. From these processes we calculate the price indexes I_t^{eq} and I_t^B , mortality forecasts $m_{x,t}$ and other random variables needed in the case studies for the forecasting period $t=1,\dots,T$. For the mortality rate we assume the error term $\epsilon_{x,t} = 0$, for simplicity.

The dependence model, which is discussed below in more detail, has the following structure:

1. When $J_t = 0$ (no jump in the simulated yearly equity returns), we use multivariate normal innovations (ϵ_t, u_t, e_t) for $(x_t, y_t, m_{x,t})$ with the given correlation matrix of Table 5.2. The correlations can also be assumed to be zero.
2. When $J_t = 1$ (during the jump years) we can use modified innovations for interest rates and mortality based on parameters (s_u, s_e) given in Table 5.3. It is also possible to cancel this modification. In this case we use the original innovations and correlation $\rho_{u,e}$ between interest rates and mortality.

Table 5.2 Correlation matrix between equity, interest rate and mortality innovations ϵ_t, u_t and e_t

innov.	ϵ_t	u_t	e_t
ϵ_t	1	$\rho_{\epsilon,u}$	$\rho_{\epsilon,e}$
u_t		1	$\rho_{u,e}$
e_t			1

Table 5.3 Innovations during equity jump years

Innovations	Interest rates	Mortality
modified	$s_u u_t^* $	$s_e e_t^* $
unmodified	u_t	e_t

In Table 5.3 $u_t^*, u_t \sim i.i.d.N(0, \sigma_y^2)$, s_u is a chosen constant, $e_t^*, e_t \sim i.i.d.N(0, \tau_m^{-1})$, and s_e is a chosen constant.

5.3 Model specification

Although it is possible to assume independence for equity and bond total returns and mortality in our long-term forecasting model, there are periods when stock-bond correlation can be significantly positive or negative, and economic shocks may change its sign rapidly due to the flight-to-quality phenomenon. These facts, given with our risk management point of view, make it preferable to have a flexible dependence model that can be used for generating stochastic scenarios. We also want to be able to handle differently normal times and stressed times. A readily available indication of the latter regime in our model is the equity shock. Thus we build a hierarchical model based on the simulated stochastic process J_t that drives the equity shocks or jumps. Dependence modeling is then addressed in two parts, depending whether or not there is a shock in the equity index model. Because the equity shock is negative, we automatically get asymmetry in the dependence structure.

During normal years ($J_t = 0$) we assume that the innovations (ϵ_t, u_t, e_t) are distributed according to the 3-variate Normal distribution with given correlations $(\rho_{\epsilon,u}, \rho_{\epsilon,e}, \rho_{u,e})$. During the shock years ($J_t = 1$) we assume that the innovations u_t and e_t have latent dependence only via the J_t process.

We need to adjust the formulas for bond returns and for mortality to take into account the dependence structure. For convenience we review here the necessary model equations of Chapters 2 - 4. Our data are the year-end total return index values of SP500 and 5-year US bonds from 1925 to 2006, from which we calculated the yearly log-returns x_t and r_t respectively (i.e. x_1 is the log-return of 1926 for equities etc). We then specified the equity model in Chapter 2 as:

$$x_t = (1 - J_t)(\mu + \sigma\epsilon_t) - J_t Y_t, \quad (5.1)$$

where $t = 1, \dots, 81$, μ is the mean and σ is the standard deviation of no-jump years, and $\epsilon_t \sim i.i.d.N(0, 1)$. For the jump process we assumed that $J_t \sim i.i.d.Ber(q)$, $0 < q < 1$, and $Y_t \sim i.i.d.Gam(\alpha, \beta)$, $\alpha, \beta > 0$. Moreover, we assumed independence of these random variables.

In Chapter 3 the following ARMA(1,1) model was specified for $y_t = \ln(r_t + a)$, where $a > 0$ is a constant estimated by the method of moments:

$$y_t - \beta_y = \phi(y_{t-1} - \beta_y) + u_t - \theta_y u_{t-1}, \quad (5.2)$$

where β_y is the mean, and the ARMA parameters are restricted to $0 < \theta_y < \phi < 1$ based on stationarity and decaying autocorrelation requirements which we set during the data analysis. Now we adjust the model so that the innovations are defined differently for normal and equity jump years, when the shock modification is used. When $J_t = 0$, $u_t \sim i.i.d.N(0, \sigma_y^2)$. When $J_t = 1$, $u_t = s_u |u_t^*|$, where $u_t^* \sim i.i.d.N(0, \sigma_y^2)$ and s_u is a chosen constant.

In Chapter 4 the model developed by Lee and Carter (1992) for the death rate for age x in year t was given as

$$\log[m_{x,t}] = a_x + b_x k_t + \epsilon_{x,t}, \quad (5.3)$$

where a_x and b_x are age-specific constants, k_t is a time-varying index of mortality, and $\epsilon_{x,t} \sim i.i.d.N(0, \sigma_\epsilon^2)$ is the error term. The model for the yearly mortality index is

$$k_t = k_{t-1} + \mu_m + e_t, \quad (5.4)$$

where $e_t \sim i.i.d.N(0, \tau_m^{-1})$ when $J_t = 0$. When $J_t = 1$, $e_t = s_e |e_t^*|$, where $e_t^* \sim i.i.d.N(0, \tau_m^{-1})$ and s_e is a chosen constant, when the shock modification is used. Although for mortality it is not obvious to assume permanent influence of the shock, we chose not to subtract its effect from the next year's intensity in order to keep the model more simple.

It is important to note that the correlations we are using for the innovations are transformed through the model equations, which implies that they do not give the same correlations for the end-results, i.e. between the asset returns and mortality rates that we are forecasting. For instance we have two interest rate series, y_t and r_t , where the first concerns transformed data and thus cannot be used in forecasting. Therefore an iterative approach is required by the forecaster when setting the assumptions (cf. Case Study 3 below).

5.4 Simulation

As previously discussed, the simulation algorithm for the equity model generates the jump process J_t . Therefore the jump years are known and can be used in the hierarchical algorithm.

The algorithm to simulate from the multivariate Normal distribution, having the specified correlations, means and standard deviations, is typically based on a Choleski decomposition. This algorithm is discussed in standard books on numerical analysis, simulation, and statistics. We have instead used the R function *mvrnorm* in MASS library in our implementation.

We also add the option of using unaltered innovations for the shock years. In this case we generate all three innovations (ϵ_t, u_t, e_t) , as for the normal years, although we do not use the equity innovation term ϵ_t because the jump part of the formula is then activated.

The simulation steps, where the parameter uncertainty is taken into account by choosing new parameters from the MCMC and bootstrap samples for each forecast, were given in sec. 5.2. For more details on implementation see the appendix.

6 Pension insurance applications

6.1 Introduction

Our simulation model enables us to study longevity and market risk in pension insurance. We have discussed in the introductory chapter some fundamental aspects of pension schemes and solvency requirements. We move on to concrete situations via the following three case studies:

1. Single premium calculation and risk assessment of a whole life unit annuity for a cohort of 65-year old persons, where the cohort size varies, and comparison with an earlier study in Italy (Cruz, 2009, Ch. 14).
2. Extension of the previous case to the case of multiple large cohorts, and comparison with an earlier study in Japan (Fujisawa and Li, 2010).

3. The customer's view of an individual pension contract and a long-term analysis (for the broader view, see Alho et al., 2011).

6.2 Annuity premium and risk analysis for a cohort aged 65

In the first case study on annuities in a pension fund we analyse an index-linked unit whole life pension insurance contract without death benefit. We simulate the probability distribution of discounted annuity cash-flows and study the solvency of the pension fund. The same approach can be used to analyse more complex products. For instance, the pension rule O_t below could include interest rate guarantees or an equity return part (see e.g. Detering et al., 2011). Other interesting product features might include Variable Annuities (cf. Hardy, 2003), and profit sharing rules (cf. Cruz, 2009, Ch. 15).

6.2.1 Specifications of the simulation setting

We assume a cohort of pensioners exactly aged $x=65$ at the start of the base year $t=0$ (cf. Figure 6.1). Without regard to gender, we study four cohort sizes, $N_0 = \{1, 10, 100, 1000\}$. Life is assumed to end exactly at age $\omega=105$, at the latest.

We now specify the pension insurance contract for the cohort of individuals $i \in \{1, \dots, N_0\}$. The annual unit pension with indexation is defined for $t=1, 2, \dots, 39$ as

$$O_t = O_{t-1} \{1 + \max(r_t; 0)\}, \quad (6.1)$$

where $O_0 = 1 + \max(r_0; 0)$ and r_t is the simulated log-return on the bond index I_t^B , as defined in (3.1). In our calculations we assume that the return indexes are observed at the end of each year, and for convenience log-return is used in (6.1).

For each annuitant i we specify an indicator vector $I_i = (I_{i,0}, \dots, I_{i,39})$, where $I_{i,t}$ equals 1 if the person is alive at the start of year t and 0 otherwise. Let N_t denote the number of pensioners alive at the start of year t

$$N_t = \sum_i I_{i,t}. \quad (6.2)$$

The amount of pensions E_t paid out of the fund in year t is

$$E_t = O_t(N_t - wD_t), \quad (6.3)$$

where $D_t = N_t - N_{t+1}$ and, as a final pension a year's pension of $1 - w$, $w \in [0,1]$, is paid to each of the D_t persons. Having $w=0.5$, means that half a year's pension is paid. This is our assumption in the calculations below.

Pension liability is defined as

$$PV = \sum_t v_t E_t, \quad (6.4)$$

where $v_t = 1/I_t^B$. Note that PV is a random variable. Its randomness derives from the uncertainty of survival and stochastic interest rates. The expected pension liability is $EPV = E[PV]$, which can also be interpreted as the net single premium for the pension portfolio.

The assets of the pension fund, which consist of the premiums and solvency buffers for the yearly pension expenditures of unit pensions, are invested in the equity and bond indexes I_t^{eq} and I_t^B , $t=0,\dots,39$ as follows. Define

$$A_t = hI_t^{eq} + (1 - h)I_t^B, \quad (6.5)$$

where $h \in [0,1]$ is the fraction of equities, and the indexes are calculated from the simulated equity and bond returns x_t and r_t (cf. (2.3) and (3.3)) with log-returns as in (3.1). From the assets of the fund each year t the amount $(N_t - wD_t)A_t$ is sold to pay the pensions E_t . Their discounted difference is

$$\Delta = \sum_t v_t [(N_t - wD_t)A_t - E_t]. \quad (6.6)$$

6.2.2 Risk analysis

In this section we analyse the various sources of risk in our annuity model and define the required solvency buffers.

Uncertainty in our annuity product can be split into the following risk components:

1. The risk due to stochastic life times.
2. The risk due to the floor ($O_t \geq O_{t-1}$) of the pension index.
3. The risk due to equity investments.

In Solvency II the notion of underwriting risk includes for instance the longevity risk, and there is a specific solvency requirement for this purpose. In our case study, we address this risk, which is due to the difference between expected and observed life times, by calculating the distribution of EPV-PV and extracting the desired VaR or quantile point.

If we remove the floor of the pension index by defining $O'_t = I_t^B$ and assume $h=0$, the assets and the liabilities of the pension fund will depend on the same index I_t^B , and there will be no interest rate risk. When we include the floor or embedded option back in the product, i.e. use pension rule O_t , there is interest rate market risk even if $h=0$, because r_t can be negative. In Solvency II this kind of risk is split into 2 parts: the difference in the EPV is included in the best estimate part of technical provisions, and the tail risk is included partly in the SCR and partly in the risk margin part of technical provisions.²⁵ We illustrate these concepts below by calculating the present values with and without the floor. If the pension fund chooses to invest in equities ($h>0$), it always introduces market risk.

We address market risk based on the following question: how much of assets are needed at the start to ensure that the yearly pension payments can be covered with a given confidence level? We apply the approach of Alho and Spencer (2005), p. 84, to our simulation setting, and require that

$$Pr(\Delta \geq 0) = \alpha, \quad (6.7)$$

where $\alpha \in (0,1)$ and Δ was defined in the previous section. Alternative approaches to the above-mentioned question are discussed below.

²⁵SCR includes the tail risks for 1 year; the risk margin covers the cost of SCR for all future years.

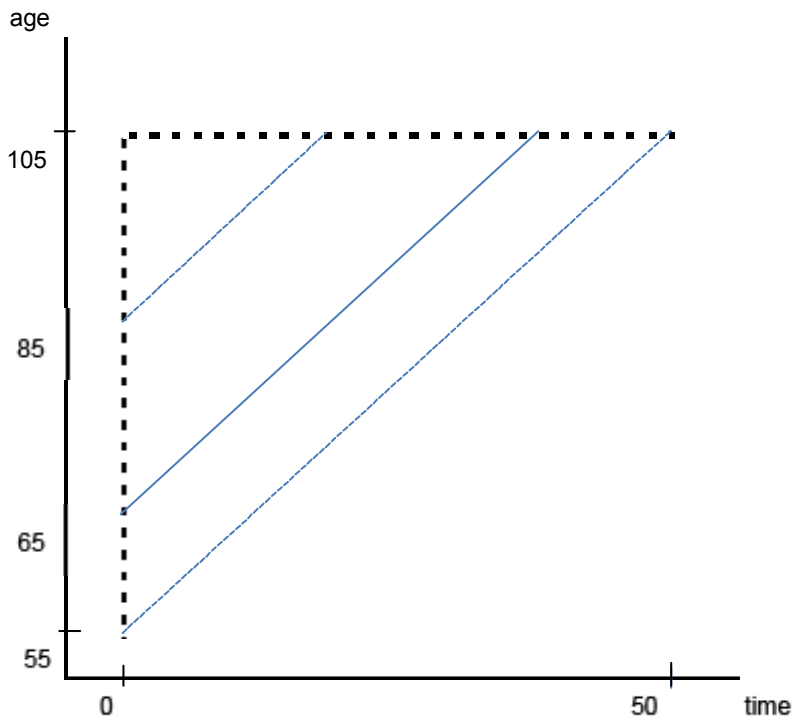


Figure 6.1 Lexis diagram for the case studies

6.2.3 Calculation method

We illustrate the solvency calculation for market and mortality risk using the following random vectors²⁶ resulting from N simulations carried out for cohort N_0 . For each simulation round, the profit or loss due to market risk is defined as $SB^{mkt} = \Delta$ and due to the mortality underwriting risk as $SB^{uw} = EPV - PV$. The solvency requirement for each risk is based on the distributions of SB^{mkt} and SB^{uw} , as in the last step in the algorithm below.

The total solvency requirement SB can be calculated from SB^{mkt} and SB^{uw} using the chosen aggregation method. The problem of summing up several random variables and VaR metrics is discussed in e.g. Embrechts and Puccetti (2010). The Solvency II aggregation methodology is based on the VaR approach, as discussed in Introduction. However, with our simulation model we can sum

²⁶All variables in this section are random variables except N , N_0 and t .

the generated random variables directly and obtain the empirical aggregate distribution, which is very useful.

The calculation proceeds from year t to year $t+1$, $t=0,\dots,38$, as follows. We already have available the necessary stochastic mortality and economic scenario files containing $T=80$ forecasting years and N simulation rounds for the modeled random variables x_t , r_t and $m_{x,t}$. Our algorithm for a single simulation round is:

1. Choose next equity and bond return series x_t and r_t and mortality forecast $m_{x,t}$. Calculate total return indices I_t^{eq} and I_t^B and discounting factors v_t from x_t and r_t according to the definitions in section 6.2.1.
2. Calculate the forecasted one-year survival probabilities $p_{x,t}$ from $m_{x,t}$ based on the actuarial estimator: $p_{x,t} = (2-m_{x,t})/(2+m_{x,t})$ (cf. (4.12)).
3. Calculate the survival probabilities $S_t = p_0 p_1 \dots p_{t-1}$, where the 1-year survival probabilities emerge cohort-wise for the chosen cohort from the previous step.
4. Sample a life time for each pensioner from the survival distribution by the inverse probability transform, i.e. sample a number from the uniform distribution to get a point on the y-axis of the survival distribution curve, and then take the corresponding inverse value or point on the x-axis to get the lifetime.
5. Calculate Δ and PV for the cohort N_0 .

After N simulation rounds:

6. Calculate EPV for the portfolio.
7. Next calculate the distributions of SB^{mkt} and SB^{uw} . Finally, choose the required solvency buffer SB so that it remains positive with the chosen risk metric, i.e. sort the simulation results and choose from the left tail of $-SB^{mkt}$ and $-SB^{uw}$ the desired VaR or TVaR, and aggregate the figures to get the total requirement SB . For the definition of TVaR we refer to Case Study 2 below.

Note that the cumulative definition of Δ allows buffering of profits and losses with respect to time, and that the asset management

strategy of the fund is determined at the start by h , i.e. we do not study more dynamic asset management strategies or hedging that might be used in practice. Furthermore, we do not differentiate between the roles and decision rules for the solvency buffer and pension liability. The terms and definitions above, which are deliberately different from Solvency I and Solvency II, and their alternative formulations are discussed in the final section of this case study.

6.2.4 Results

We simulated the above random variables for $N=3000$ simulation rounds of $T=80$ years under various assumptions and with the same simulation seed to make the results comparable.²⁷ Table 6.1 focuses on the premium or PV calculation for different cohort sizes. We note that for the larger cohorts there is less variation. The coefficient of variation decreases from 0.46 to 0.065 when the cohort increases from 1 to 1000. The assumptions for 3000 simulation rounds with cohort size 100 are the placeholder assumptions in our calculations unless otherwise stated. PV distribution with these assumptions is depicted in Figure 6.3.

Table 6.2 concentrates on the solvency calculation under the placeholder assumptions and different investment strategies, i.e. for different fractions of equities h . For instance when $h=0.5$, $VaR_{0.995}/EPV = 10.0/20.2 = 0.50$. We note that the market risk for equities is quite significant even under the above-mentioned cumulative approach.

Table 6.1 Simulated single premium (PV/N_0) results for different 65 year-old cohort sizes N_0

N_0	1%	Q1	Q2	mean	Q3	99%	sd/mean
1	1.5	13.5	20.7	20.0	26.9	38.8	0.46
10	12.8	18.1	20.2	20.3	22.3	28.3	0.16
100	17.2	19.2	20.1	20.2	21.1	24.7	0.076
1000	17.7	19.4	20.1	20.2	20.9	24.3	0.065

²⁷When $N=3000$, there is still notable random variation in the results. Repeating the simulation 10 times with different seeds and with the placeholder assumptions and $h = 0.5$, the results for $VaR_{0.995}$ were as follows. SB^{mkt} : min=9.9, mean=10.4, max=10.9, sd=0.4. SB^{uw} : min=5.0, mean=5.5, max=6.0, sd=0.3. The results of this section are thus at the lower end of the test sample.

Table 6.2 Simulated solvency buffer requirement for SB^{mkt} per annuitant for different equity weights h

h	$VarR_{0.99}$	$VarR_{0.995}$
0.00	4.2	5.6
0.25	5.4	6.5
0.50	9.1	10.0
0.75	11.4	12.5
1.00	13.1	14.2

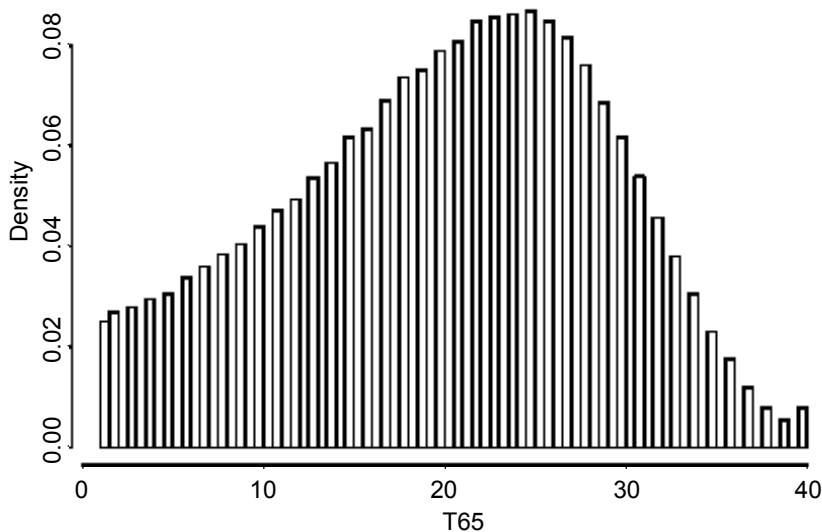


Figure 6.2 Simulated life times for a cohort aged 65 (total population). 3000 simulations, cohort size 100.

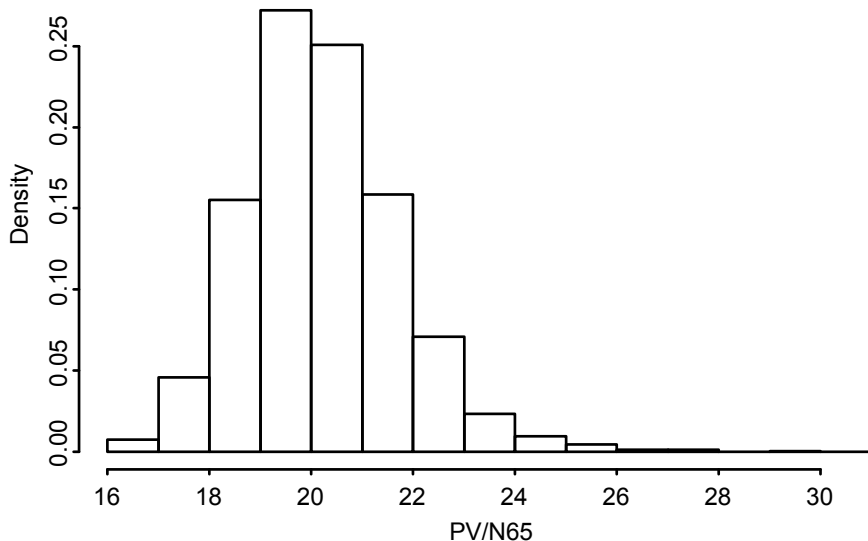


Figure 6.3 PV/N_0 of 3000 simulation rounds for a cohort aged 65 (total population). Cohort size 100.

In the calculations of this case study we have assumed independence between the equity and bond price indexes I_t^{eq} and I_t^B and between them and mortality. Aggregation and dependence options are analysed in Case Studies 2 and 3 below.

Two important features of traditional life and pension insurance contracts are the capital protection and the interest rate guarantee. In our case study the pension rule O_t includes the former. The price of this embedded option is $EPV - EPV' = 20.2 - 19.7 = 0.5$, where EPV' is calculated without the option. From the left tail of $(PV' - PV)$ we get VaR values of 4.2 and 5.6 for confidence levels 0.99 and 0.995 respectively. This is the same result as from SB^{mkt} with $h=0$; cf. the first row in Table 6.2.

We note that the pension rule with the floor introduces significant interest rate risk because r_t is negative once every 13 years on average. In practice insurers do not distribute all profits to the customer but only e.g. 80 percent, and they may also smoothen the result by summing up profits and losses over several years. We do not examine further the various forms of embedded options and profit sharing rules in this work.

Longevity risk charge for SB^{uw} is calculated in Table 6.3 for various risk measures. It derives from the difference between actual and expected lifetimes. This can be seen as follows. Assuming $O_t = A_t = v_t = 1$ for all t , and that a full pension is paid for the final year (i.e. $w=0$) yields $PV = \sum_t E_t = \sum_t N_t$. Now, $SB^{uw} = EPV - PV = \sum_t E[N_t] - \sum_t N_t$.

Table 6.3 Simulated solvency buffer requirement for SB^{uw} per annuitant

	$VarR_{0.99}$	$VarR_{0.995}$	$TVaR_{0.99}$	$TVaR_{0.995}$
SB^{uw}	4.5	5.0	6.0	7.2

The previous equation for SB^{uw} also illustrates the difference between the 1-year and multi-year approaches, in particular if the cohort size is small and yearly fluctuations are significant. In insurance company solvency assessment, e.g. in SII, it is commonly assumed that the cohort size is large enough so that only the undiversifiable or systemic risk needs to be taken into account. Alho concluded in his annuity study Alho (2008) that for a cohort size of around 30 the idiosyncratic uncertainty is smaller than the aggregate uncertainty. This is in line with our results (cf. Table 6.1).

In Cruz (2009), Chapter 14, Solvency II type calculations have been carried out by Olivieri and Pitacco using Italian mortality data and 7 differently parametrized mortality models of the Heligman-Pollard family. The expected lifetime of the 65 year-old cohort of males was 21.8 in their best estimate scenario, and for the other scenarios it was between 19.9 and 24.7. Moreover, the scenarios were differently weighted. The Italian data are not available to us, but for the Finnish total population data the median is 21, the first quartile is 14, and the third quartile is 27. Our model thus seems to produce more variation and uncertainty for the remaining lifetimes at the age of 65. Many other calculations were included in the Italian study. For instance, the net single premium for a unit annuity was 15.3 under the best estimate assumptions. The annuity did not include indexation or death benefit, and a constant discount interest rate of 3 percent was used. When we modified our calculation method by adding 0.03 to the yearly interest rates in the discount factors, the net single premium was 14.3 for the Finnish whole population data. SB^{uw} was now 2.7 for the 99.5 percent VaR, which gives the ratio $2.7/14.3=0.19$. In the Italian study the full run-off solvency requirement (termed [R3] in the paper) at $t=0$ for a cohort-size 100

plus their risk margin was $0.136+0.065=0.201$ or 20.1 percent of the best estimate technical provision. However, we need to keep in mind that both the data and the calculation methods differ, so that the results are not necessarily comparable.

6.2.5 Chosen methods and alternatives

In the EU the new Solvency II (SII) system will address mortality/longevity risk partly in the technical provisions and partly in the Solvency Capital Requirement (SCR). For the SCR there are two methods available. The standard approach uses simple stress tests, where the best estimate mortality is shifted or stressed upwards and downwards by a constant amount, and the resulting change in technical provisions is the basis for the required solvency buffer (cf. Cruz, 2009, Ch.14, or QIS5 specifications). We note that a parallel shift is not an accurate method, because the prediction interval for mortality is more parabolic than linear (cf. Chapter 4). The level of longevity risk stresses in QIS5 were $+0.15$ and -0.20 , which are within one standard deviation of our $m_{65,20}$ (cf. Table 4.1).

It is also interesting to note that the equity risk stresses in QIS5 were -0.39 for global equities and -0.49 for other equities but these numbers were subsequently adjusted by adding 0.09 to both. Our model, on the other hand, produces higher figures; cf. Chapter 2 and Case Study 3.

The internal modeling option in SII allows more freedom and realism in assessing longevity risk as well as other risks and their dependencies. On the other hand SII system does not cover pension funds but is limited to life and non-life and reinsurance companies.

We have analyzed above the solvency problem from a different point of view. For one thing, we have used a run-off approach (i.e. until the cohort vanishes) whereas SII uses the 1-year VaR approach for the solvency capital. In our case study we focused on PV and EPV whereas 'Technical provision' in SII is the best estimate plus a multi-year risk margin as explained in Introduction and in Cruz (2009), Ch. 14. The latter paper, and the earlier Olivieri and Pitacco (2009), address more generally the problem of 1-year versus multi-year risk measurement, in particular from the longevity risk point of view. They discuss and compare 4 alternative ways to calculate the SCR for annuities under the SII internal modeling option. A particular issue is whether a full run-off is analysed, and whether there is yearly checking

of the solvency position. Olivieri and Pitacco found these two full run-off alternatives not to differ significantly in terms of longevity risk. In our case studies future losses and profits are allowed to offset each other during the pension period.

It is a good idea to use several risk measures and measurement horizons in solvency analysis (cf. Wang and Koskinen, 2009). In Solvency II the so-called own risk and solvency assessment (ORSA) requires that the insurance companies also conduct longer term analyses. Olivieri and Pitacco recommended that both shorter and longer term measures be used in internal models. In our case study we addressed the market risk due to the embedded option or floor in the pension index. We found that it is important not to look only at the 1-year VaR; instead, one should examine the whole run off horizon when assessing this risk (cf. EIOPA, 2011).

In non-life insurance both 1-year and multi-year approaches have been used in solvency analysis, cf. Pentikäinen and Rantala (1982), sec. 3.1, and Daykin et al. (1994), Ch. 13. We applied the approach of the latter reference to our case study by multiplying the assets A_t in (6.6) by $K = 1 + SB^{mkt}/EPV$, and canceling the discounting by assuming $v_t = 1$ for all t . Then we classified each simulation path as a ruin if any yearly component of Δ was negative, and determined the 1-year ruin probability as follows. When $h = 0.5$, $K = 1.5$ at the confidence level of 0.995. The number of ruins divided by the number of simulations and the number of forecasting years is $600/3000/40=0.005$, which thus corresponds to the VaR level of 0.995. For $h = 0.25$ and for $h = 0.75$ the results were 0.004 and 0.006 respectively.

SII internal model users can choose any time horizon and risk measure, but the models must also produce the results according to the 1 year and 99.5 percent VaR definition of the SCR. The best estimate part in technical provisions should take into account the expected mortality improvements, and the risk margin part, together with the SCR longevity risk module, should allow for the remaining uncertainty. However, we believe that horizons longer than 1 year are more suitable for longevity risk analysis and that simplistic proxies should not be used in the SCR and risk margin calculations for practical SII implementations. Our modeling examples in this work and the reference literature such as Alho et al. (2011) highlight the significant long-term uncertainty in mortality forecasting. We note that the risk margin concept of Solvency II also has some theoretical weaknesses, as discussed in Cruz (2009), Ch. 17.

To conclude, there are different approaches available for the valuation and for the risk and solvency assessment of annuities. Some approaches focus more on the hedging of market and longevity risk through financial market instruments such as derivatives and longevity bonds; cf. Panjer and Boyle (1998), IAA (2010), Chapter IV, Hardy (2003), Chapter 7, and Cruz (2009), Ch. 15–16, and McWilliam (2011). An approach based on stochastic discounting factors or deflators and the valuation portfolio is presented in Wüthrich et al. (2008). Our methodology allows the main risks to be addressed from the statistical point of view. The approaches based on financial derivative methods are likely to suffer from the problems of long horizons (deep, liquid and transparent market is not available) and market incompleteness, as there are no market for some risks; cf. Kaliva et al. (2007). European Insurance and Occupational Pensions Authority concludes in EIOPA (2011) on page 17: 'Demographic risks and policyholder dynamic behaviour are very difficult to effectively hedge'. Hilli et al. (2011) study pension portfolio valuation and risk management in incomplete markets. Risk measurement is based on the distribution of wealth at the end of the forecasting horizon, and dynamic hedging and market consistent valuation is also discussed in the paper.

6.3 Annuity premium and risk analysis for multiple cohorts

We now extend the previous case study to allow for multiple cohorts of annuitants. Our additional reference in this section is the paper by Fujisawa and Li (2010), which we first review.

6.3.1 Literature review

Fujisawa and Li (2010) propose 3 longevity risk measures for a Defined Benefit plan using IFRS accounting: 1) longevity value-at-risk, 2) probability of longevity deficit, and 3) probabilistic corridor rule. They illustrate these concepts with a hypothetical pension plan in Japan, which includes 3000 pensioners with stationary age-structure based on the Japanese population. Mortality data are from the Human Mortality Database. Their stochastic mortality simulations,

including parameter uncertainty, are based on the model of Cairns et al. (2006) and the distribution of DB liability change: $Y(t) = PV(t) - PV(t_0)$, where $PV(t)$ is the aggregate figure when applying the single premium annuity formula for each pensioner in year t with a 3 percent discount rate (cf. \ddot{a}_x in Gerber, 1990, Appendix A), and t_0 is the current year. We note that in IFRS high quality corporate bond yields are used in the discounting. Another remark concerns the definition of $Y(t)$. Because there is no discount factor in the formula for $Y(t)$ to take into account the time value of money, t is the year when a new DB liability is calculated under new mortality assumptions but with the original population size and age structure. The pension fund population is therefore assumed stationary.

The various longevity risk indicators proposed for $Y(t)$ - namely Value-at-Risk (VaR), Tail-Value-at-Risk (TVaR), probability of longevity deficit (PLD) and its expectation, and a probabilistic corridor - are calculated for a 5-year window $t=2006,...,2010$ as follows:

$$VaR_p[Y(t)] = \inf\{y; P(Y(t) > y) \leq 1 - p\}, \quad (6.8)$$

$$TVaR_p[Y(t)] = E[Y(t)|Y(t) > VaR_p[Y(t)]], \quad (6.9)$$

$$PLD = P(Y(t) > 0), \quad (6.10)$$

$$E[PLD] = E[Y(t)|Y(t) > 0], \quad (6.11)$$

$$P(Z(t) > \tau) = P(Y(t)/PV(t_0) > \tau), \quad (6.12)$$

where $\tau > 0$ is the chosen threshold. We note that the threshold idea is methodologically similar to the viability region approach mentioned in the Introduction.

For further discussion of risk measures we refer to Wang and Koskinen (2009) and the references therein, where VaR and TVaR and some other risk measures are analysed and compared from the Solvency II internal model point of view.

6.3.2 Case study calculations

We now generalize the previous case study to allow for multiple cohorts of immediate whole life annuities without a death benefit. The definitions and the calculation methods are in general similar to Case Study 1. There are a few differences, however. Firstly, we calculate the cohort mortality $m'_{x,t}$ for those who are x years old at $t=0$ (cf. Figure 6.1) from the forecasted mortality rates $m_{x,t}$ as $m'_{x,t} = (m_{x,t} + m_{x+1,t})/2$.

Secondly, we assume that the cohort sizes are sufficiently large to render the diversifiable risk insignificant (cf. Case Study 1). Then we need not sample the individual lifetimes from the simulated survival function. Instead we can read directly from the survival function the remaining cohort size of yearly pension recipients. This implies also that we assume a full pension be paid for the final year.

Our first illustration is a generalisation of Case Study 1 for multiple cohorts. For the example, which is summarized in Table 6.4, we assumed the initial population to include 1000 persons of ages $x=55,60,\dots,85$ at $t=0$. We then calculated EPV, SB^{uw} and SB^{mkt} for the cohorts, with $h=0.4$. We note that both longevity risk and market risk are proportionally smaller at higher ages, which shows the effect of a shorter forecasting horizon in reducing uncertainty.²⁸ For instance, the required SB^{uw}/EPV is 0.23 and 0.11 at the ages 55 and 85, while the respective figures for the required SB^{mkt}/EPV are 0.43 and 0.25 at the 99 percent confidence level.

We illustrate the effect of multiple cohorts on the VaR measurement for the longevity risk as follows. Firstly, when we sum the VaR requirements for SB^{uw} for the cohorts in the table, the result is 22.4. This approach is similar to the SII SCR aggregation method for risk modules and their sub-modules, assuming perfect correlation (cf. QIS Technical Specification p. 95). Secondly, when we use our simulation model and sum the simulated (EPV-PV)'s of cohorts before taking the VaR from the aggregated distribution, the result is 22.7. The result that $22.7 > 22.4$ illustrates one weakness of the VaR metrics, that it is not sub-additive. When the dependence is high and the distributions are very skewed, this failure may show up (cf. Embrechts and Puccetti, 2010, or Denuit et al., 2005, and the references therein). Note that we have assumed large cohorts so that

²⁸This also applies to the VaR metric in sec. 1.4. In the basic VaR setting the time effect for different horizons is approximated by the square root of time multiplier.

diversifiable risk can be neglected. If this were taken into account, it would reduce aggregate variance and risk.

Table 6.4 Simulation of multiple cohorts of equal size 1000 at $t = 0$. Equity weight $h = 0.4$. $\text{VaR}_{0.99}$ for underwriting and market risk per annuitant, based on 3000 simulation rounds.

Age x	EPV	SB^{uw}	SB^{mkt}
55	29.9	7.0	12.8
60	24.8	5.2	10.1
65	20.0	4.0	7.7
70	15.5	2.7	5.6
75	11.5	1.8	3.7
80	8.1	1.1	2.4
85	5.5	0.6	1.4

For another illustration we consider a 1-year risk metric for longevity - $\text{VaR}_{0.95}/\text{EPV}$ - for a person aged 70 years. VaR for above-defined $Y(1)$ is first calculated based on the difference in annuity single premiums at $t=1$ and $t=0$ using the corresponding period life tables. The resulting risk metric with 3 percent discount rate is 0.029. Secondly, we approximate the effect of cohort approach by applying a multiplying factor $0.036/0.014$, which gives 0.075. The factor is the ratio of the coefficients of variation (standard deviation divided by mean) for cohort and period based annuity single premiums.

In the Japanese study $\text{VaR}_{0.95}/\text{EPV}$ for the pension fund was 0.041 for 1-year period. However, we need to keep in mind that the results are not necessarily comparable due to differences in methods and data, the full details of which are not easily analysed.

6.4 Annuities from the customer's point of view

In the final case study we concentrate on the customer's view and long-term aspects of annuities. First of all, it is important to distinguish between a whole life annuity with and without the death benefit. If the death benefit is included in the contract, the price will be higher, or to put it another way, the return on pension savings is lower. Without a death benefit, the remaining savings after the death are used for the benefit of the remaining pensioners in the fund.

Another fundamental property is the pooling effect of insurance. If the customer chooses not to participate in a pool, he faces a significant risk when choosing his monthly pension withdrawal: the annuity savings may not last long enough or the annuity may be smaller than necessary, depending on whether or not he lives longer than average. In other words the average lifetime assumption is not an appropriate financial planning tool for a single person. In Case Study 1 we found that the standard deviation divided by the mean of PV was 0.46 and 0.07 for the 65 year-old cohort consisting of a single person and of 1000 persons, respectively.

In our next example we sample a life time for a 25 year-old person for 300 000 times using our mortality simulation model. We assume that a constant premium is saved yearly until the retirement or death, if earlier. The death benefit is zero, or the accumulated savings are distributed to other policyholders. From age 65 a unit pension is taken until death. We disregard discounting (or we assume that indexing and discounting of cashflows cancel). We find that a yearly premium of 0.58 is sufficient on average. However, the variability of premium is high: a yearly premium of 0.88, 0.93 or 1 unit is needed to ascertain that the savings are sufficient for pensions at 90, 95 or 99 percent confidence level, respectively.

The third major consideration is the investment return and risk, which depend chiefly on the chosen asset mix for the pension savings. In traditional insurance contracts insurance company is in charge of asset management while in Unit Linked contracts it is on responsibility of customers. We can illustrate the risk and return profiles of different asset portfolios using the equity and bond log-returns x_t and r_t and their dependence as follows.

We simulated 3000 stochastic scenarios covering 80 years of equity and bond returns with selected asset mixes and dependence assumptions. The resulting summaries of the yearly log-returns are given in Table 6.5, where h is the proportion of equity investments in the asset portfolio, A_t , $\rho_{\epsilon,u}$ is the assumed correlation between the innovations, and s_u is the shock-innovation multiplier for bond returns.

Table 6.5 Simulated log-return statistics for asset portfolio A_t , when equity fraction (h) and dependence assumption $(\rho_{\varepsilon,u}, s_u)$ varies

h	$\rho_{\varepsilon,u}$	s_u	1%	Q1	Q2	mean	Q3	99%	sd	cor
0.5	0.0	-	-0.22	0.016	0.082	0.072	0.14	0.28	0.10	0
0.5	-0.5	-	-0.22	0.027	0.084	0.072	0.13	0.25	0.094	-0.22
0.5	0.5	-	-0.22	0.0067	0.079	0.072	0.14	0.31	0.11	0.30
0.5	0.0	2	-0.18	0.024	0.086	0.081	0.15	0.29	0.096	-0.27
0.5	0.0	-2	-0.25	0.012	0.079	0.067	0.14	0.28	0.11	0.20
0.0	0.0	-	-0.032	0.015	0.043	0.052	0.078	0.21	0.051	-
1.0	0.0	-	-0.48	-0.013	0.11	0.094	0.23	0.49	0.20	-

Using the bi-variate Normal assumption, we can calculate $VaR_{0.99}$ for the asset portfolio directly according to Section 1.4. Thus, for the assumed simulated estimates in the above table, we get $VaR_{0.99}$ as (0.17, 0.14, 0.20, 0.13, 0.19, 0.067, 0.37) for the rows respectively.

Shock-correlations have a significant effect in Table 6.5. When a positive shock-multiplier $s_u=2$ was used, equity shocks were compensated by positive interest rate shocks to the extent that the correlation of yearly log-returns was -0.27. The negative multiplier increased the correlation to 0.20.

Our next example considers investment risk and return in the long run for some typical asset portfolios in pension insurance. First, we simulate A_t 3000 times for the horizon of $t=(1,10,20,30,40)$ years when assuming $A_0=1$ and the fraction of equities $h=1/3$. The summary statistics of simulations are given in Table 6.6. Note that h is constant which implies yearly rebalancing. Next, Table 6.7 includes first percentiles and means for the same periods when $h=1/6$ and $h=2/3$. We note that a larger proportion of equities is likely to give higher returns in the long run but at the same time uncertainty increases also. Negative returns are possible for long periods of time as the columns of first percentiles show.

Table 6.6 Summary statistics of asset index A_t with $A_0 = 1$ and equity fraction $h = 1/3$. Forecasting horizon $t = 1, \dots, 40$.

t	1%	Q1	Q2	mean	Q3	99%
1	0.87	1.02	1.07	1.07	1.12	1.29
10	0.94	1.56	1.92	2.04	2.37	4.35
20	1.21	2.72	3.73	4.23	5.07	13.03
30	1.72	4.75	7.06	8.87	10.56	36.06
40	2.49	8.47	13.61	19.02	22.32	90.93

Table 6.7 Statistics of asset index A_t with $A_0 = 1$ and equity fraction $h_1 = 1/6$ and $h_2 = 2/3$. Forecasting horizon $t = 1, \dots, 40$.

t	1%(h_1)	mean(h_1)	1%(h_2)	mean(h_2)
1	0.94	1.07	0.74	1.09
10	0.97	1.91	0.70	2.46
20	1.18	3.68	0.84	6.22
30	1.56	7.22	1.23	15.98
40	2.22	14.39	1.76	42.85

Finally, we add three important considerations. The costs that the service provider(s) charge affect negatively the net returns. This should be considered carefully by the customer. We have not modeled this aspect in our work. The customer also has to keep in mind that it is the real return after inflation that is relevant for his or her standard of living. We have modeled only nominal returns because insurance companies typically do not give real return guarantees. Thus inflation adjustment of some sort is needed for the pension, e.g. based on the bond index as we have done, or on a long term observed average inflation of a chosen kind. In the U.S. the yearly inflation (calculated similarly as log-returns) during our data period from 1925 to 2006 in Morningstar (2007), was on average 0.030 with a standard deviation of 0.041. Figure 6.4 depicts the log-returns of equities, bonds and inflation during this period. The last consideration is the effect of taxes during both the saving and pension periods. This effect, however, is difficult to anticipate and to model. Therefore scenario testing of various alternatives may be the best starting point.

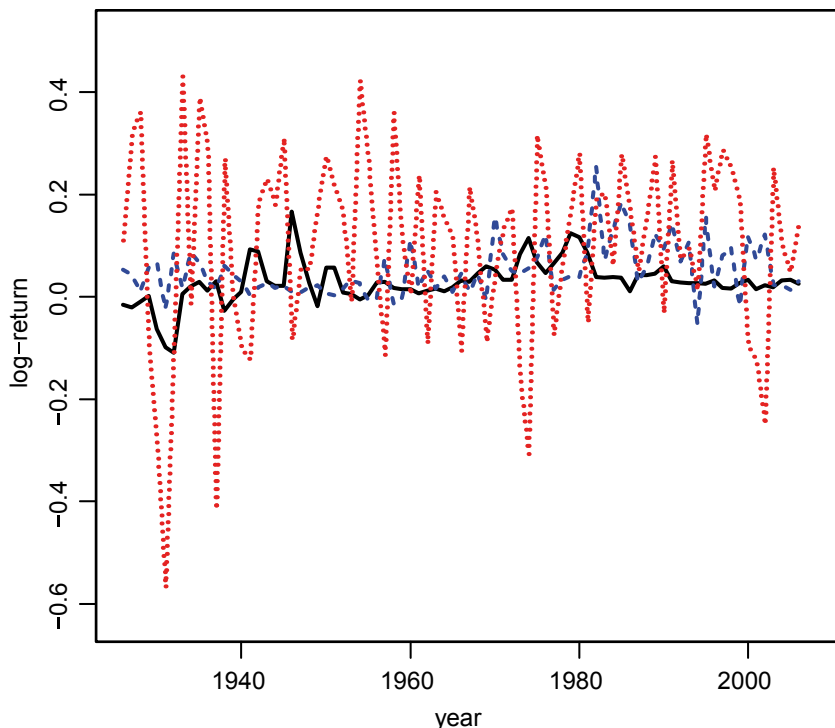


Figure 6.4 log-returns of the U.S. equities (dotted), bonds (dashed) and inflation (solid line) for 1926–2006

7 Discussion

We have studied the financing of longevity risk in pension insurance by developing a stylized and parsimonious, yet in our view realistic simulation model. The model takes into account the main sources of risk, the uncertainty of equity and bond returns and mortality, and their interrelations and the uncertainty relating to parameter estimation. We applied the model to topical and important long-term risk management problems in three case studies.

The model for equity returns is built on the assumption that the underlying stochastic process is a mixture of an uncorrelated process and a negative jump process. Bernoulli-mixture implies Geometric interarrival times with the memoryless property, which is desirable from the economic point of view. The Gamma-process for jump size allows fatter tails than the Normal distribution and is flexible enough to smooth the unwanted bimodality of year-end

SP500 returns. Our model includes some novel features, and it seems to give a good presentation of reality, as witnessed by the similarity of MLE and MCMC estimation results. VaR calculations were presented for several alternatives, which indicate that the jump model is more prudent than the Normal model. Moreover, the MLE and MCMC based results differ in the tail area, MCMC being more prudent.

Bond total return was modeled as an ARMA(1,1) process after a log-transform. This parsimonious model was chosen after testing with more complex alternatives. The uncertainty of the underlying process renders the data horizon of 81 observations rather short for specifying the model. In the data there are 2 regimes, the first having lower mean and volatility than the second, which started around 1970. Therefore we resorted to the 4 stylized facts for restricting the bootstrap parameters: 1) stationarity, 2) long cycles or a large AR parameter, 3) rapid corrections of large innovations or a large MA parameter, and 4) negative return approximately once every 10 years. The VaR results were different from the equity case because the log-returns of bonds are skewed to the right. Normal-approximation gave now larger results than the simulations.

The statistical mortality model of Lee and Carter (1992), which is trend-based but stochastic, is nowadays a standard tool for forecasting and risk management. Although the model is parsimonious, it is not easily outperformed in long-term forecasting applications although many variations of the model have been developed. We also developed a more complex version of the model, namely the local bilinear model, which gave a better fit to the data and revealed some interesting features of it. However, the model forecasts for expected life times were about the same for both models. Therefore we preferred the standard Lee-Carter model in our calculations, but used the MCMC approach to account for parameter uncertainty in the time index, which determines the long term improvement in longevity. We also analysed the difference that gender has on the Lee-Carter forecasts. This is an important area for the European insurance companies due to the recent EU court decision that requires unisex mortality tables for life insurance pricing. However, it is not clear to us how this decision is justified by the data.

We can use the mortality model to study the mortality forecasting uncertainty for different horizons. The increase in forecasting uncertainty for longer horizons is the key observation, which makes the pricing and risk management of annuities challenging. One solution to this problem is some kind of risk sharing arrangement

between the parties involved, e.g. the customer and the insurance company. For instance, in the Finnish statutory occupational pension scheme a longevity adjustment is applied to the pensions. It is calculated using the most recent data available before the cohort retires (cf. Alho and Spencer, 2005, Ch. 11.1.). On the other hand in voluntary individual pension insurance contracts provided by the life insurance companies this type of adjustment is not currently allowed under normal circumstances in Finland. Taking into account the major uncertainty for the long mortality forecasting horizons needed for annuities, this restriction may hinder the development of the annuity market in Finland. On the other hand, in some countries even more flexible rules for annuities are considered. In Sweden, for instance, there has been discussion of whether the annuity could and should be longevity-adjusted even after the person has retired, e.g at the age of 75. For further discussion, see Alho et al. (2011).

Dependence modeling was known to be challenging from the outset. Our model structure allows flexibility that can take into account correlations and shocks via the equity jump term. Our approach enables the modeling of many typical and atypical situations in a justified and novel way, by allowing asymmetric comovements of the variables.

Case studies for several annuity portfolios were included with risk analysis for the premiums and required solvency buffers. Particular attention was given to Solvency II (SII). The studies located some weaknesses and areas where SII internal models could be especially useful. Based on our studies we believe that pension insurance risk modeling is better done using a simulation model than by a formula-based approach. Our model takes a long-term risk management view of pension insurance, and can be used to supplement SII models in insurance companies' own risk and solvency assessments.

We applied the model to Value-at-Risk calculations, which showed that non-normal distributions and their aggregation is much better addressed by the simulation model than by the multi-Normal assumption. However, the analysis of extreme events, i.e. events that are rare but have severe impacts, is always difficult and subjective, due the difficulty of fully specifying the tail area of the distribution because of data limitations.

Further work could focus on yield curve modeling and market consistent valuation of liabilities. The simulation model could be used to study more complex products and more dynamic portfolios than in

the case studies. Another interesting question is why the cohort born in wartime seems to live longer. Is this due to the low-calory diet or something else?

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8 Appendix

8.1 Model implementation example

We have implemented the model with R language. The code has the following parts:

1. Preparations (data input, parameter assumptions, variable definitions and initializations)
2. Generation of parameters (equity model MCMC from sec. 2.3, mortality model MCMC from sec. 4.4, bond model bootstrap parameter input from sec. 3.4)
3. Generation of economic processes with given dependence structure and parameters (Ch. 5)
4. Case study 1 (generation of life times and insurance calculations, the algorithm and results are presented in sec. 6.2)
5. Case study 2 (generation of survival function and insurance calculations, the algorithm and results are presented in sec. 6.3)

A full run of the model includes running the first three blocks and the chosen case study in succession. Economic scenarios can be generated by running parts 1, 2 and 3. The R code and data are available from the author by e-mail.

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