

Feinstein, Zachary et al.

Working Paper

Sensitivity of the Eisenberg-Noe Clearing Vector to Individual Interbank Liabilities

Working Paper, No. 13/2017

Provided in Cooperation with:

Norges Bank, Oslo

Suggested Citation: Feinstein, Zachary et al. (2017) : Sensitivity of the Eisenberg-Noe Clearing Vector to Individual Interbank Liabilities, Working Paper, No. 13/2017, ISBN 978-82-7553-991-3, Norges Bank, Oslo,
<https://hdl.handle.net/11250/2495570>

This Version is available at:

<https://hdl.handle.net/10419/210123>

Standard-Nutzungsbedingungen:

Die Dokumente auf EconStor dürfen zu eigenen wissenschaftlichen Zwecken und zum Privatgebrauch gespeichert und kopiert werden.

Sie dürfen die Dokumente nicht für öffentliche oder kommerzielle Zwecke vervielfältigen, öffentlich ausstellen, öffentlich zugänglich machen, vertreiben oder anderweitig nutzen.

Sofern die Verfasser die Dokumente unter Open-Content-Lizenzen (insbesondere CC-Lizenzen) zur Verfügung gestellt haben sollten, gelten abweichend von diesen Nutzungsbedingungen die in der dort genannten Lizenz gewährten Nutzungsrechte.

Terms of use:

Documents in EconStor may be saved and copied for your personal and scholarly purposes.

You are not to copy documents for public or commercial purposes, to exhibit the documents publicly, to make them publicly available on the internet, or to distribute or otherwise use the documents in public.

If the documents have been made available under an Open Content Licence (especially Creative Commons Licences), you may exercise further usage rights as specified in the indicated licence.



<https://creativecommons.org/licenses/by-nc-nd/4.0/deed.no>

WORKING PAPER

Sensitivity of the Eisenberg-Noe clearing vector to individual interbank liabilities

NORGES BANK
RESEARCH

13 | 2017

ZACHARY FEINSTEIN,
WEIJIE PANG,
BIRGIT RUDLOFF,
ERIC SCHAANNING,
STEPHAN STURM AND
MACKENZIE WILDMAN



NORGES BANK

Working papers fra Norges Bank, fra 1992/1 til 2009/2 kan bestilles over e-post:

FacilityServices@norges-bank.no

Fra 1999 og senere er publikasjonene tilgjengelige på www.norges-bank.no

Working papers inneholder forskningsarbeider og utredninger som vanligvis ikke har fått sin endelige form. Hensikten er blant annet at forfatteren kan motta kommentarer fra kolleger og andre interesserte. Synspunkter og konklusjoner i arbeidene står for forfatterens regning.

Working papers from Norges Bank, from 1992/1 to 2009/2 can be ordered by e-mail

FacilityServices@norges-bank.no

Working papers from 1999 onwards are available on www.norges-bank.no

Norges Bank's working papers present research projects and reports (not usually in their final form) and are intended inter alia to enable the author to benefit from the comments of colleagues and other interested parties. Views and conclusions expressed in working papers are the responsibility of the authors alone.

ISSN 1502-819-0 (online)

ISBN 978-82-7553-991-3 (online)

Sensitivity of the Eisenberg–Noe clearing vector to individual interbank liabilities.*

Zachary Feinstein[†] Weijie Pang[‡] Birgit Rudloff[§]
Eric Schaanning[¶] Stephan Sturm[‡] Mackenzie Wildman^{||}

Abstract

We quantify the sensitivity of the Eisenberg–Noe clearing vector to estimation errors in the bilateral liabilities of a financial system. The interbank liabilities matrix is a crucial input to the computation of the clearing vector. However, in practice central bankers and regulators must often estimate this matrix because complete information on bilateral liabilities is rarely available. As a result, the clearing vector may suffer from estimation errors in the liabilities matrix. We quantify the clearing vector’s sensitivity to such estimation errors and show that its directional derivatives are, like the clearing vector itself, solutions of fixed point equations. We describe estimation errors utilizing a basis for the space of matrices representing permissible perturbations and derive analytical solutions to the maximal deviations of the Eisenberg–Noe clearing vector. This allows us to compute upper bounds for the worst case perturbations of the clearing vector. Moreover, we quantify the probability of observing clearing vector deviations of a certain magnitude, for uniformly or normally distributed errors in the relative liability matrix.

Applying our methodology to a dataset of European banks, we find that perturbations to the relative liabilities can result in economically sizeable differences that could lead to an underestimation of the risk of contagion. Importantly, our results allow regulators to bound the error of their simulations.

Keywords: Systemic risk, model risk, Eisenberg–Noe clearing vector, sensitivity analysis, interbank networks, contagion.

*The views expressed in this work are those of the authors and do not necessarily reflect the views of Norges Bank. This material is based upon work supported by the National Science Foundation under Grant No. 1321794.

[†]Washington University in St. Louis, Department of Electrical & Systems Engineering, 1 Brookings Drive, Green Hall, Room 2160B, St. Louis, MO 63130, USA, zfeinstein@ese.wustl.edu.

[‡]Worcester Polytechnic Institute, Department of Mathematical Sciences, 100 Institute Road, Worcester, MA 01609-2280 USA, wpang@wpi.edu and ssurm@wpi.edu.

[§]Vienna University of Economics and Business, Welthandelsplatz 1, Building D4, 1020 Vienna, Austria, brudloff@wu.ac.at.

[¶]Norges Bank Research, Bankplassen 2, 0107 Oslo, Norway. Part of this work was undertaken while Eric Schaanning was at Imperial College London, eric.schaanning@norges-bank.no.

^{||}University of California Santa Barbara, Department of Statistics & Applied Probability, South Hall, Santa Barbara, CA 93106-3080. Part of this work was undertaken while Mackenzie Wildman was at Lehigh University, wildman@pstat.ucsb.edu.

1 Introduction

An important stream of the literature on contagion in networks has focused on interbank contagion, building on the network model of Eisenberg and Noe (2001). Central banks and regulators have applied the model to study default cascades in the banking systems in their jurisdictions (Anand et al. (2014), Hałaj and Kok (2015), Boss et al. (2004), Elsinger et al. (2013), Upper (2011), Gai et al. (2011)). Hüser (2015) offers a comprehensive review of the interbank contagion literature. Recently, the Bank of England has extended this model to analyse solvency contagion in the UK financial system (Bardoscia et al. (2017)). Moreover, extensions of this model have been developed to include effects such as fire sales (e.g. Cifuentes et al. (2005), Nier et al. (2007), Gai and Kapadia (2010), Chen et al. (2016), Amini et al. (2016a,b), Weber and Weske (2017), Feinstein (2017a), Feinstein and El-Masri (2017), Feinstein (2017b), Di Gangi et al. (2015)), cross-ownership (e.g. Elsinger (2009), Elliott et al. (2014), Weber and Weske (2017)), bankruptcy costs (e.g. Elsinger (2009), Rogers and Veraart (2013), Elliott et al. (2014), Glasserman and Young (2015), Weber and Weske (2017)), and multiple maturities (e.g. Capponi and Chen (2015), Kusnetsov and Veraart (2016), Feinstein (2017b)). Regulators have identified the inclusion of such contagion mechanisms in stress tests as a key priority (Basel Committee on Banking Supervision (2015), Anderson (2016)). Furthermore, recent research illustrates that accounting for feedback effects and contagion can change the pass/fail result in stress tests for individual institutions (Cont and Schaanning (2017)).

A key ingredient required to estimate contagion in these models is the so-called liabilities matrix L , where L_{ij} is the nominal liability of bank i to bank j . Often, the exact bilateral exposures are not known and thus need to be estimated (Hałaj and Kok (2013), Anand et al. (2015), Elsinger et al. (2013), Hałaj and Kok (2015)). Despite considerable efforts after the crisis to improve data collection, data gaps have not been closed yet. Beyond logistical issues like the standardization of reporting formats and the creation of unique and universal institution identifiers, further hurdles remain, such as legal restrictions that limit regulators’ access only to data pertinent to their respective jurisdictions. Therefore, the estimation of specific bilateral exposures remains an important issue (Langfield et al. (2014), Anand et al. (2015, 2017), Financial Stability Board and International Monetary Fund (2015)). The early literature often used entropy maximizing techniques to “fill in the blanks” in the liabilities matrix given the total assets and liabilities of banks (viz. the row and column sums of L). However, a growing empirical literature has shown that real-world interbank networks look quite different from the homogeneous networks that are obtained with such techniques (Bech and Atalay (2010), Mistrulli (2011), Cont et al. (2013), Soramäki et al. (2007)). A recent Bayesian method to estimate the bilateral liabilities, given the total liabilities and potential other prior information, is proposed in Gandy and Veraart (2016) and applied to reconstruct CDS markets in Gandy and Veraart (2017). In particular, Mistrulli (2011), Gandy and Veraart (2016) show how wide estimates of systemic risk may fluctuate when estimating contagion on real-world and heterogeneous networks versus uniform networks. This highlights the pivotal role that the matrix of bilateral exposures plays in quantifying the extent of contagion when computing default cascades. Capponi et al. (2016) studies the effects of the network topology on systemic risk through the use of majorization-based tools. To the best of our knowledge, Liu and Staum (2010) is so far the only paper that performs a sensitivity analysis of the Eisenberg–Noe model. Their analysis is, however, limited to the sensitivity of the clearing vector with respect to the initial net worth of each bank.

The main contribution of this paper is to perform a detailed sensitivity analysis of the clearing vector with respect to the interbank liabilities in the standard Eisenberg–Noe framework. To this end, we define directional derivatives of order k of the clearing vector with respect to “perturbation

matrices,” which quantify the estimation errors in the relative liability matrix. This allows us to derive an exact Taylor series for the clearing vector. Moreover, we introduce a set of “basis matrices,” which specify a notion of fundamental directions for the directional derivative. We demonstrate that the directional derivative of the clearing vector can be written as a linear combination of these basis matrices. We proceed to use this result to study two optimisation problems that quantify the maximal deviation of the clearing vector from its “true” value, and obtain explicit solutions for both problems. These analytical results additionally provide an upper bound to the (first-order) worst case perturbation error. We extend these results by computing the probability of observing deviations of a given magnitude when the estimation errors are either uniformly or normally distributed.

Finally, we illustrate our results both in a small four-bank network and using a dataset of European banks. Our results suggest that, though the set of defaulting banks may remain stable across different bilateral interbank networks (calibrated to the same data set), the deviation of the clearing vector from perturbations in the relative liabilities can be large.

In this paper, we occasionally consider external liabilities along with the interbank liabilities. We aggregate all external liabilities into a single external “societal firm.” This additional “bank” is a stand-in for the entirety of the economy that is not included in the financial network. This is discussed in more details in, e.g., Glasserman and Young (2015). In particular, as utilized in Feinstein et al. (2017), the impact on the wealth of the societal firm can be used as an aggregate measure for the health of the financial network as a whole. We will make use of the societal firm in a similar way in order to study the effects of estimation errors in the interbank liabilities on external stakeholders.

The organization of the paper is as follows. In Section 2 we present the Eisenberg–Noe framework and prove initial continuity results of that model. We then study directional derivatives and the Taylor series of the Eisenberg–Noe clearing payments with respect to the relative liabilities matrix. These results allow us to consider the sensitivity of the clearing payments. In Section 3 we use the directional derivatives in order to determine the perturbations to the relative liabilities matrix that present the “worst” errors in terms of misspecification of the clearing payments and impact to society. These results are extended to also consider the probability of the various estimation errors. In Section 4 we implement our sensitivity analysis on data calibrated to a network of European banks. Section 5 concludes. An appendix provides technical details on the orthogonal basis of perturbation matrices.

2 Sensitivity analysis of Eisenberg–Noe clearing vector

We consider a financial system consisting of n banks, $\mathcal{N} = \{1, \dots, n\}$. For $i, j \in \mathcal{N}$, $L_{ij} \geq 0$ is the **nominal liability** of bank i to bank j . (External liabilities can be considered as well through the introduction of an “external” bank 0. This is discussed in more detail in Section 3.2.) Equivalently, L_{ij} is the exposure of bank j to bank i . $L \in \mathbb{R}^{n \times n}$ is called the **liabilities matrix** of the financial network, and we assume that no bank has an exposure to itself, i.e., $L_{ii} = 0$ for all $i \in \mathcal{N}$. The **total liability** of bank i is given by $\bar{p}_i = \sum_{j=1}^n L_{ij}$. The **relative liability** of bank i to bank j is denoted $\pi_{ij} \in [0, 1]$, where $\pi_{ij} = \frac{L_{ij}}{\bar{p}_i}$ when $\bar{p}_i > 0$. We allow $\pi_{ij} \in [0, 1]$ to be arbitrary when $\bar{p}_i = 0$ and only require $\sum_{j=1}^n \pi_{ij} = 1$. (Note that the arbitrary choice of π_{ij} in the case $\bar{p}_i = 0$ has no impact on the outcome of the Eisenberg–Noe model since the transpose of the relative liability matrix Π is multiplied by the incoming payment vector $p(\Pi)$, whose j^{th} entry is 0 when $\bar{p}_j = 0$ (cf. (2)).) We denote the **relative liability matrix** $\Pi \in \mathbb{R}^{n \times n}$. Any relative liability matrix Π must belong to the set of admissible matrices $\mathbf{\Pi}^n$, defined as the set of all right stochastic matrices with

Assets	Representation	Liabilities	Representation
Interbank	$a_i^{IB} = \sum_{j=1}^n L_{ji}$	Interbank	$l_i^{IB} = \sum_{j=1}^n L_{ij}$
External	x_i	External	L_{i0}
		Capital	c_i

Table 1: Stylized bank balance sheet

entries in $[0, 1]$ and all diagonal entries 0:

$$\mathbf{\Pi}^n := \left\{ \Pi \in [0, 1]^{n \times n} \mid \forall i : \pi_{ii} = 0, \sum_{j=1}^n \pi_{ij} = 1 \right\}. \quad (1)$$

Finally, denote the **external assets** of bank i from outside the banking system by $x_i \geq 0$. A bank balance sheet then takes the simplified form of Table 1, and a financial system is given by the triplet $(\Pi, x, \bar{p}) \in \mathbf{\Pi}^n \times \mathbb{R}_+^n \times \mathbb{R}_+^n$.

A bank is solvent when the sum of its net external assets and performing interbank assets exceeds its total liabilities. In this case, the bank honours all of its obligations. However, if the value of its obligations is greater than the bank's net assets plus performing interbank assets, then the bank will default and repay its obligations pro-rata. (This corresponds to the assumption that all interbank and external claims can be aggregated to a single figure per bank and that all creditors of a defaulting bank are paid *pari passu*.) These rules yield a **clearing vector** as the solution of the fixed point problem

$$p(\Pi) = \bar{p} \wedge (x + \Pi^\top p(\Pi)). \quad (2)$$

Let $p : \mathbf{\Pi}^n \rightarrow \mathbb{R}_+^n; \Pi \mapsto p(\Pi)$ be the fixed point function with parameters (x, \bar{p}) . As proved in (Eisenberg and Noe 2001, Theorem 2), the clearing vector is unique if a system of banks is regular. Regularity is defined as follows: A **surplus set** $S \subseteq \mathcal{N}$ is a set of banks in which no bank in the set has any obligations to a bank outside of the set and the sum over all banks' external net asset values in the set is positive, i.e., $\forall (i, j) \in S \times S^c : \pi_{ij} = 0$ and $\sum_{i \in S} x_i > 0$. Next, consider the financial system as a directed graph in which there is a directed link from bank i to bank j if $L_{ij} > 0$. Denote the **risk orbit** of bank i as $o(i) = \{j \in \mathcal{N} \mid \text{there exists a directed path from } i \text{ to } j\}$. This means that the risk orbit of bank i is the set of all banks which may be affected by the default of bank i . A system is **regular** if every risk orbit is a surplus set. Uniqueness of the clearing vector has important consequences in terms of the continuity of the function p , which in turn is important for our sensitivity analysis. For this reason we will proceed under the assumption that our financial system is regular.

Proposition 2.1. *Consider a regular financial system (Π, x, \bar{p}) in which x and \bar{p} are fixed. The function p , defined via (2), is continuous with respect to $\Pi \in \mathbf{\Pi}^n$.*

Proof. This proof follows the logic of (Feinstein et al. 2017, Lemma 5.2) and (Ren et al. 2014, Theorem 4). Fix the net assets x and total obligation \bar{p} . Let $\phi : [0, \bar{p}] \times \mathbf{\Pi}^n \rightarrow [0, \bar{p}]$ be the function defined by $\phi(\hat{p}, \Pi) := (\phi_1(\hat{p}, \Pi), \dots, \phi_n(\hat{p}, \Pi))^\top$, where

$$\phi_i(\hat{p}, \Pi) = \bar{p}_i \wedge \left(x_i + \sum_{j=1}^n \pi_{ji} \hat{p}_j \right), \quad i \in \mathcal{N}.$$

The function ϕ is jointly continuous with respect to the payment vector \hat{p} and the relative liabilities π_{ij} for $i, j \in \mathcal{N}$. Because the system is regular and thus has a unique fixed point, it follows from

(Feinstein et al. 2017, Proposition A.2) that the graph

$$\text{graph}(p) = \{(\Pi, \hat{p}) \in \mathbf{\Pi}^n \times [0, \bar{p}] \mid \phi(\hat{p}, \Pi) = \hat{p}\}$$

is closed. Define the projection $\Psi : \mathbf{\Pi}^n \times [0, \bar{p}] \rightarrow \mathbf{\Pi}^n$ as $\Psi(\Pi, p) = \Pi$. By (Feinstein et al. 2017, Proposition A.3), Ψ is a closed mapping in the product topology. Then, in order to show that p is continuous, take $U \subset [0, \bar{p}]$ closed. Then

$$p^{-1}[U] = \{\Pi \in \mathbf{\Pi}^n \mid p(\Pi) \in U\} = \Psi(\text{graph}(p) \cap (\mathbf{\Pi}^n \times U)).$$

The graph of p is closed and $\mathbf{\Pi}^n$ is closed by definition. Hence $p^{-1}[U]$ is closed and the function p is continuous with respect to Π . \square

We finish these preliminary notes by considering a simple example of the Eisenberg–Noe clearing payments under a system of $n = 4$ banks. We will return to this example throughout as a simple illustrative case study.

Example 2.2. *Consider the following example of a network consisting of four banks in which the bank’s nominal interbank liabilities are given by*

$$L = \begin{pmatrix} 0 & 7 & 1 & 1 \\ 3 & 0 & 3 & 3 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix},$$

as shown in Figure 1(a). Assume the banks’ external assets are given by the vector $x = (0, 2, 2, 2)^\top$. With 0 net worth and positive liabilities, Bank 1 defaults initially. The Eisenberg–Noe clearing vector (2) can be easily computed to be $p = (4.5, 7.5, 3, 3)^\top$, showing that Bank 2 also defaults through contagion. The realized interbank payments are shown in Figure 1(b). Banks who are in default are coloured red and payments that are repaid less than whole are also coloured red. The edge widths are proportional to the payment size.

2.1 Quantifying estimation errors from the (relative) liabilities matrices

We assume that some estimation error is attached to the entries of the relative liability matrix, leading to a deviation of the clearing vector from the “true” clearing vector $p(\Pi)$. Denote the true relative liabilities matrix by Π and let $\Pi + h\Delta$ denote the liabilities matrix that includes some estimation error, for a perturbation matrix Δ and size $h \in \mathbb{R}$. First we consider the class of perturbation matrices, $\mathbf{\Delta}^n(\Pi)$, under which we assume that the existence or non-existence of a link between two banks is known to the regulator and hence, the error is limited to a misspecification of the size of that link. Remark 2.9, Corollary 3.2 and Corollary 3.16 will utilize the results in this section to provide bounds for the perturbation error in general without predetermining existence or non-existence of links.

Definition 2.3. *For a fixed $\bar{p} \in \mathbb{R}_+^n$, define the set of relative liability perturbation matrices by*

$$\mathbf{\Delta}^n(\Pi) := \left\{ \Delta \in \mathbb{R}^{n \times n} \mid \forall i : \delta_{ii} = 0, \sum_{j=1}^n \delta_{ij} = 0, \sum_{j=1}^n \delta_{ji} \bar{p}_j = 0, \text{ and } (\pi_{ij} = 0) \Rightarrow (\delta_{ij} = 0) \forall j \right\}.$$

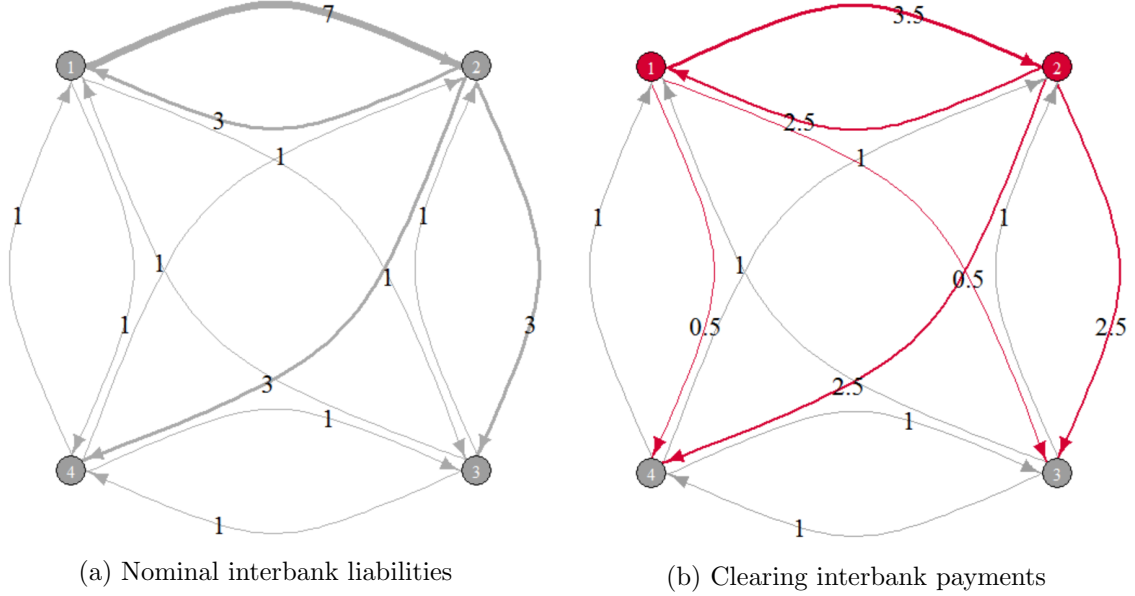


Figure 1: Initial network defined in Example 2.2

The summation conditions ensure that the total liabilities and total assets, respectively, of each bank are left unchanged by the perturbation. Of course it is not possible to have $\Pi + h\Delta \in \mathbf{\Pi}^n$ for any $h \in \mathbb{R}$. Throughout this work we consider perturbation magnitudes in a bounded interval, $h \in (-h^*, h^*)$, where

$$h^* := \min \left\{ \min_{\delta_{ij} < 0, \bar{p}_i > 0} \frac{-\pi_{ij}}{\delta_{ij}}, \min_{\delta_{ij} > 0, \bar{p}_i > 0} \frac{1 - \pi_{ij}}{\delta_{ij}} \right\} > 0,$$

for any $\Delta \in \mathbf{\Delta}^n(\Pi)$ to assure $\Pi + h\Delta \in \mathbf{\Pi}^n$. We exclude from this calculation of h^* any bank i where $\bar{p}_i = 0$ since this has no impact on the results. We will consider in particular a bounded set of perturbations

$$\mathbf{\Delta}_F^n(\Pi) := \mathbf{\Delta}^n(\Pi) \cap \{ \Delta \in \mathbb{R}^{n \times n} \mid \|\Delta\|_F \leq 1 \},$$

where $\|\cdot\|_F$ is the Frobenius norm, i.e., $\|\Delta\|_F = \sqrt{\sum_{i=1}^n \sum_{j=1}^n |\delta_{ij}|^2}$. This bounded set of perturbations will be of importance in Section 3.

Remark 2.4. A more general case can be considered in which one allows for errors that create links where there were none. This set is defined as follows: For a fixed $\bar{p} \in \mathbb{R}_+^n$,

$$\overline{\mathbf{\Delta}}^n(\Pi) := \left\{ \Delta \in \mathbb{R}^{n \times n} \mid \forall i : \delta_{ii} = 0, \sum_{j=1}^n \delta_{ij} = 0, \sum_{j=1}^n \delta_{ji} \bar{p}_j = 0, \text{ and } (\pi_{ij} = 0) \Rightarrow (\delta_{ij} \geq 0) \forall j \right\}.$$

We will consider in particular the bounded set of perturbations

$$\overline{\mathbf{\Delta}}_F^n(\Pi) := \overline{\mathbf{\Delta}}^n(\Pi) \cap \{ \Delta \in \mathbb{R}^{n \times n} \mid \|\Delta\|_F \leq 1 \}.$$

2.2 Directional derivatives of the Eisenberg–Noe clearing vector

Next, we analyse the error when using the clearing vector of a perturbed liability matrix, $p(\Pi + h\Delta)$, instead of the clearing vector of the original liability matrix, $p(\Pi)$, for small perturbations $h\Delta$, with $\Delta \in \mathbf{\Delta}^n(\Pi)$.

Definition 2.5. Let $\Delta \in \mathbf{\Delta}^n(\Pi)$. In the case that the following limit exists, we define the directional derivative of the clearing vector $p(\Pi)$ in the direction of a perturbation matrix Δ as

$$\mathcal{D}_\Delta p(\Pi) := \lim_{h \rightarrow 0} \frac{p(\Pi + h\Delta) - p(\Pi)}{h}.$$

The first order Taylor expansion of p about Π gives

$$p(\Pi + h\Delta) - p(\Pi) = h\mathcal{D}_\Delta p(\Pi) + O(h^2).$$

The following theorem provides an explicit formula for the directional derivative of the clearing vector for a fixed financial network.

Theorem 2.6. Let (Π, x, \bar{p}) be a regular financial system. The directional derivative of the clearing vector $p(\Pi)$ in the direction of a perturbation matrix $\Delta \in \mathbf{\Delta}^n(\Pi)$ exists almost everywhere and is given by

$$\mathcal{D}_\Delta p(\Pi) = \left(I - \text{diag}(d)\Pi^\top \right)^{-1} \text{diag}(d)\Delta^\top p(\Pi), \quad (3)$$

where $\text{diag}(d)$ is the diagonal matrix defined as $\text{diag}(d_1, \dots, d_n)$, where

$$d_i := \mathbb{1}_{\{x_i + \sum_{j=1}^n \pi_{ji} p_j(\Pi) < \bar{p}_i\}}.$$

Here, (3) holds outside of the measure-zero set $\{x \in \mathbb{R}_+^n \mid \exists i \in \mathcal{N} \text{ s.t. } x_i + \sum_{j=1}^n \pi_{ji} p_j(\Pi) = \bar{p}_i\}$ in which some bank is exactly at the brink of default.

Proof. We assume that the net external assets lie in the set

$$\left\{ x \in \mathbb{R}_+^n \mid \nexists i \in \mathcal{N} \text{ s.t. } x_i + \sum_{j=1}^n \pi_{ji} p_j(\Pi) = \bar{p}_i \right\}.$$

Denote $\alpha^{(1)} = x + \Pi^\top p(\Pi) = (\alpha_1^{(1)}, \dots, \alpha_n^{(1)})^\top$ and $\alpha^{(2)} = x + (\Pi + h\Delta)^\top p(\Pi + h\Delta) = (\alpha_1^{(2)}, \dots, \alpha_n^{(2)})^\top$. By continuity of p with respect to Π (Proposition 2.1) we have for all $i \in \mathcal{N}$, $\alpha_i^{(2)} \rightarrow \alpha_i^{(1)}$ as $h \rightarrow 0$ and thus $\mathbb{1}_{\{\{\alpha_i^{(1)} < \bar{p}_i\} \cap \{\alpha_i^{(2)} > \bar{p}_i\}\}} \rightarrow 0$, $\mathbb{1}_{\{\{\alpha_i^{(1)} > \bar{p}_i\} \cap \{\alpha_i^{(2)} < \bar{p}_i\}\}} \rightarrow 0$ and $\mathbb{1}_{\{\{\alpha_i^{(1)} < \bar{p}_i\} \cap \{\alpha_i^{(2)} < \bar{p}_i\}\}} \rightarrow \mathbb{1}_{\{\alpha_i^{(1)} < \bar{p}_i\}}$. To prove the existence of $\mathcal{D}_\Delta p(\Pi)$, we will show that the following two limits,

$$\begin{aligned} \overline{\mathcal{D}_\Delta p(\Pi)}_i &= \limsup_{h \rightarrow 0} \frac{p_i(\Pi + h\Delta) - p_i(\Pi)}{h} & \text{and} \\ \underline{\mathcal{D}_\Delta p(\Pi)}_i &= \liminf_{h \rightarrow 0} \frac{p_i(\Pi + h\Delta) - p_i(\Pi)}{h} \end{aligned}$$

are equal for each component. Consider the upper limit

$$\begin{aligned}
\overline{\mathcal{D}_{\Delta p}(\Pi)}_i &= \limsup_{h \rightarrow 0} \frac{p_i(\Pi + h\Delta) - p_i(\Pi)}{h} \\
&= \limsup_{h \rightarrow 0} \frac{1}{h} \left(\left(\bar{p}_i \wedge \left(x_i + \sum_{j=1}^n (\pi_{ji} + h\delta_{ji}) p_j(\Pi + h\Delta) \right) \right) - \left(\bar{p}_i \wedge \left(x_i + \sum_{j=1}^n \pi_{ji} p_j(\Pi) \right) \right) \right) \\
&= \limsup_{h \rightarrow 0} \left(0 \times \mathbb{1}_{\{\{\alpha_i^{(1)} > \bar{p}_i\} \cap \{\alpha_i^{(2)} > \bar{p}_i\}\}} + \frac{\bar{p}_i - (x_i + \sum_{j=1}^n \pi_{ji} p_j(\Pi))}{h} \mathbb{1}_{\{\{\alpha_i^{(1)} < \bar{p}_i\} \cap \{\alpha_i^{(2)} > \bar{p}_i\}\}} \right. \\
&\quad + \frac{x_i + \sum_{j=1}^n \pi_{ji} p_j(\Pi + h\Delta) + h \sum_{j=1}^n \delta_{ji} p_j(\Pi + h\Delta) - \bar{p}_i}{h} \mathbb{1}_{\{\{\alpha_i^{(1)} > \bar{p}_i\} \cap \{\alpha_i^{(2)} < \bar{p}_i\}\}} \\
&\quad \left. + \frac{\sum_{j=1}^n \pi_{ji} (p_j(\Pi + h\Delta) - p_j(\Pi)) + h \sum_{j=1}^n \delta_{ji} p_j(\Pi + h\Delta)}{h} \mathbb{1}_{\{\{\alpha_i^{(1)} < \bar{p}_i\} \cap \{\alpha_i^{(2)} < \bar{p}_i\}\}} \right) \\
&= \left(\sum_{j=1}^n \pi_{ji} \overline{\mathcal{D}_{\Delta p}(\Pi)}_j + \sum_{j=1}^n \delta_{ji} p_j(\Pi) \right) \mathbb{1}_{\{\alpha_i^{(1)} < \bar{p}_i\}} \\
&= d_i \sum_{j=1}^n \pi_{ji} \overline{\mathcal{D}_{\Delta p}(\Pi)}_j + d_i \sum_{j=1}^n \delta_{ji} p_j(\Pi) =: \Psi_i(\overline{\mathcal{D}_{\Delta p}(\Pi)})
\end{aligned}$$

for some function $\Psi : \mathbb{R}^n \rightarrow \mathbb{R}^n$.

Similarly, we get

$$\mathcal{D}_{\Delta p}(\Pi)_i = d_i \sum_{j=1}^n \pi_{ji} \mathcal{D}_{\Delta p}(\Pi)_j + d_i \sum_{j=1}^n \delta_{ji} p_j(\Pi) = \Psi_i(\mathcal{D}_{\Delta p}(\Pi)).$$

Hence, both $\overline{\mathcal{D}_{\Delta p}(\Pi)}$ and $\mathcal{D}_{\Delta p}(\Pi)$ are fixed points of the same mapping Ψ . Assuming that this fixed point problem has a unique solution it follows

$$\overline{\mathcal{D}_{\Delta p}(\Pi)}_i = \mathcal{D}_{\Delta p}(\Pi)_i,$$

for all $i \in \mathcal{N}$. Therefore, under this assumption, $\mathcal{D}_{\Delta p}(\Pi)$ is well defined and it is the solution to the fixed point equation

$$\mathcal{D}_{\Delta p}(\Pi) = \Psi(\mathcal{D}_{\Delta p}(\Pi)) = \text{diag}(d)\Pi^\top \mathcal{D}_{\Delta p}(\Pi) + \text{diag}(d)\Delta^\top p(\Pi).$$

Next, we proceed to show that $(I - \text{diag}(d)\Pi^\top)$ is invertible, which establishes uniqueness of the fixed point and the directional derivative (3) to conclude the proof.

First, assume that $\text{diag}(d)\Pi^\top$ is irreducible, i.e., the graph with adjacency matrix $\text{diag}(d)\Pi^\top$ has directed paths in both directions between any two vertices $i \neq j$. Then by the Perron–Frobenius Theorem (see, e.g., (Gentle 2007, Section 8.7.2)), $\text{diag}(d)\Pi^\top$ has an eigenvector $v > \mathbf{0}$ corresponding to eigenvalue $\rho(\text{diag}(d)\Pi^\top)$, where $\rho(\cdot)$ is the spectral radius of a matrix. As eigenvectors are only unique up to a multiplicative constant, we may assume $\|v\|_1 = 1$. Under the assumption of a regular system, at least one bank must be solvent, i.e., there exists some i such that $\text{diag}(d)_{ii} = 0$. This implies that there exists a column such that the column sum of $\text{diag}(d)\Pi^\top$ is strictly less than 1. In fact, any insolvent institution j with obligations to bank i will have column sum of $\text{diag}(d)\Pi^\top$ strictly less than 1. If all banks are solvent, $\text{diag}(d)$ is the zero matrix and the result is trivial. Thus there is some matrix $M \geq 0$, $M \neq 0$ so that each column sum of $\text{diag}(d)\Pi^\top + M$ is 1, i.e.

$$\mathbf{1}^\top (\text{diag}(d)\Pi^\top + M) = \mathbf{1}^\top.$$

Note that the column sums of $\text{diag}(d)\Pi^\top$ are at most 1 since each row sum of Π is 1. Therefore the spectral radius of $\text{diag}(d)\Pi^\top$ must be less than or equal to 1. Moreover, we must have $\rho(\text{diag}(d)\Pi^\top) < 1$. Otherwise, $\rho(\text{diag}(d)\Pi^\top) = 1$, which along with the scaling of the eigenvector so that $\|v\|_1 = 1$ implies

$$1 = \mathbf{1}^\top v = \mathbf{1}^\top (\text{diag}(d)\Pi^\top + M)v = \mathbf{1}^\top (v + Mv) = 1 + \mathbf{1}^\top Mv > 1,$$

as $\Pi^\top v = 1v$ by the definition of eigenvalues. Therefore, we can conclude that, in the case $\text{diag}(d)\Pi^\top$ is irreducible, $\rho(\text{diag}(d)\Pi^\top) < 1$.

Now suppose that $\text{diag}(d)\Pi^\top$ is reducible, i.e., $\text{diag}(d)\Pi^\top$ is similar to a block upper triangular matrix D , with irreducible diagonal blocks D_i , $i = 1, \dots, m$ for some $m < n$. Under the assumption of a regular system, each D_i has at least one column whose sum is strictly less than 1. As in the preceding case, this implies that $\rho(D_i) < 1$ for each i and therefore

$$\rho(\text{diag}(d)\Pi^\top) = \rho(D) < 1.$$

Since the maximal eigenvalue of $\text{diag}(d)\Pi^\top$ is strictly less than 1, 0 cannot be an eigenvalue of $I - \text{diag}(d)\Pi^\top$. This suffices to show that $I - \text{diag}(d)\Pi^\top$ is invertible. \square

2.3 A Taylor series for the Eisenberg–Noe clearing payments

In the same manner, we can define higher order directional derivatives.

Definition 2.7. For $k \geq 1$, we define the k^{th} order directional derivative of the clearing vector with respect to a perturbation matrix Δ as

$$\mathcal{D}_\Delta^{(k)} p(\Pi) := \lim_{h \rightarrow 0} \frac{\mathcal{D}_\Delta^{(k-1)} p(\Pi + h\Delta) - \mathcal{D}_\Delta^{(k-1)} p(\Pi)}{h}, \quad (4)$$

when the limit exists, and

$$\mathcal{D}_\Delta^{(0)} p(\Pi) = p(\Pi).$$

Remarkably, as Theorem 2.8 shows, all higher order derivatives also have an explicit formula, which allows us to obtain an exact Taylor series for the clearing vector. We impose an additional assumption on allowable perturbations $h\Delta$ so that the matrix $\text{diag}(d)$ (as defined in Theorem 2.6) as a function of $\Pi + h\Delta$ is fixed with respect to h , i.e., we require h sufficiently small so that the same subset of banks is in default when the liability matrix is $\Pi + h\Delta$ as when the liability matrix is Π . Let

$$\begin{aligned} \bar{h}^{**} &:= \sup \left\{ h \leq h^* \mid \begin{array}{l} x_i + \sum_{j=1}^n \pi_{ji} p_j(\Pi) < \bar{p}_i \\ \Leftrightarrow x_i + \sum_{j=1}^n (\pi_{ji} + h\delta_{ji}) p_j(\Pi + h\Delta) < \bar{p}_i \quad \forall i \in \mathcal{N} \end{array} \right\}, \\ \underline{h}^{**} &:= \inf \left\{ h \geq -h^* \mid \begin{array}{l} x_i + \sum_{j=1}^n \pi_{ji} p_j(\Pi) < \bar{p}_i \\ \Leftrightarrow x_i + \sum_{j=1}^n (\pi_{ji} + h\delta_{ji}) p_j(\Pi + h\Delta) < \bar{p}_i \quad \forall i \in \mathcal{N} \end{array} \right\}, \\ h^{**} &:= \min\{-\underline{h}^{**}, \bar{h}^{**}\}. \end{aligned} \quad (5)$$

We necessarily have $h^{**} > 0$ because we exclude the measure-zero set $\{x \in \mathbb{R}_+^n \mid \exists i \in \mathcal{N} \text{ s.t. } x_i + \sum_{j=1}^n \pi_{ji} p_j(\Pi) = \bar{p}_i\}$ in which a bank is exactly at the brink of default.

Theorem 2.8. Let (Π, x, \bar{p}) be a regular financial system. Then for $\Delta \in \mathbf{\Delta}^n(\Pi)$, and for all $k \geq 1$:

$$\begin{aligned} \mathcal{D}_\Delta^{(k)} p(\Pi) &= k(I - \text{diag}(d)\Pi^\top)^{-1} \text{diag}(d)\Delta^\top \mathcal{D}_\Delta^{(k-1)} p(\Pi) \\ &= k! \left((I - \text{diag}(d)\Pi^\top)^{-1} \text{diag}(d)\Delta^\top \right)^k p(\Pi), \end{aligned} \quad (6)$$

where $\mathcal{D}_\Delta^{(0)}p(\Pi) = p(\Pi)$. Moreover, for $h \in (-h^{**}, h^{**})$, the Taylor series

$$p(\Pi + h\Delta) = \sum_{k=0}^{\infty} \frac{h^k}{k!} \mathcal{D}_\Delta^{(k)}p(\Pi)$$

converges and has the following representation

$$p(\Pi + h\Delta) = \left(I - h(I - \text{diag}(d)\Pi^\top)^{-1} \text{diag}(d)\Delta^\top \right)^{-1} p(\Pi) \quad (7)$$

outside of the measure-zero set $\{x \in \mathbb{R}_+^n \mid \exists i \in \mathcal{N} \text{ s.t. } x_i + \sum_{j=1}^n \pi_{ji} p_j(\Pi) = \bar{p}_i\}$.

Proof. We prove the result by induction. Theorem 2.6 shows the result for $k = 1$. We now assume that equation (6) holds for k and we proceed to show that it holds for $k + 1$. As in Theorem 2.6, we show the existence of (4) by computing the two limits:

$$\begin{aligned} \overline{\mathcal{D}_\Delta^{(k+1)}p(\Pi)}_i &= \limsup_{h \rightarrow 0} \frac{\mathcal{D}_\Delta^{(k)}p(\Pi + h\Delta)_i - \mathcal{D}_\Delta^{(k)}p(\Pi)_i}{h} \quad \text{and} \\ \underline{\mathcal{D}_\Delta^{(k+1)}p(\Pi)}_i &= \liminf_{h \rightarrow 0} \frac{\mathcal{D}_\Delta^{(k)}p(\Pi + h\Delta)_i - \mathcal{D}_\Delta^{(k)}p(\Pi)_i}{h}. \end{aligned}$$

The first order Taylor approximation for matrix inverses gives by the differentiation rules for the matrix inverse (cf. (Gentle 2007, p. 152)) for $X, Y \in \mathbb{R}^{n \times n}$ and h small enough: $(X + hY)^{-1} \approx X^{-1} - hX^{-1}YX^{-1}$. Applying this fact with $X = I - \text{diag}(d)\Pi^\top$ and $Y = -\text{diag}(d)\Delta^\top$, we have

$$(I - \text{diag}(d)(\Pi + h\Delta)^\top)^{-1} \approx (I - \text{diag}(d)\Pi^\top)^{-1} + h(I - \text{diag}(d)\Pi^\top)^{-1} \text{diag}(d)\Delta^\top (I - \text{diag}(d)\Pi^\top)^{-1}.$$

Additionally, we note that the k^{th} order derivative, similarly to all lower order derivatives, is continuous with respect to the relative liabilities matrix Π (by assumption of the induction) $\mathcal{D}_\Delta^{(k)}p(\Pi) = k!((I - \text{diag}(d)\Pi^\top)^{-1} \text{diag}(d)\Delta^\top)^k p(\Pi)$, where $p(\Pi)$ and $(I - \text{diag}(d)\Pi^\top)^{-1}$ are both continuous with respect to Π (see Proposition 2.1 and the continuity of the matrix inverse). Consider now the upper limit

$$\begin{aligned} \overline{\mathcal{D}_\Delta^{(k+1)}p(\Pi)} &= \limsup_{h \rightarrow 0} \frac{\mathcal{D}_\Delta^{(k)}p(\Pi + h\Delta) - \mathcal{D}_\Delta^{(k)}p(\Pi)}{h} \\ &= \limsup_{h \rightarrow 0} \frac{k}{h} \left((I - \text{diag}(d)(\Pi + h\Delta)^\top)^{-1} \text{diag}(d)\Delta^\top \mathcal{D}_\Delta^{(k-1)}p(\Pi + h\Delta) \right. \\ &\quad \left. - (I - \text{diag}(d)\Pi^\top)^{-1} \text{diag}(d)\Delta^\top \mathcal{D}_\Delta^{(k-1)}p(\Pi) \right) \\ &= \limsup_{h \rightarrow 0} k \frac{(I - \text{diag}(d)\Pi^\top)^{-1} \text{diag}(d)\Delta^\top \mathcal{D}_\Delta^{(k-1)}p(\Pi + h\Delta) - \mathcal{D}_\Delta^{(k-1)}p(\Pi)}{h} \\ &\quad + \limsup_{h \rightarrow 0} \frac{k}{h} (I - \text{diag}(d)\Pi^\top)^{-1} \text{diag}(d)\Delta^\top (I - \text{diag}(d)\Pi^\top)^{-1} \text{diag}(d)\Delta^\top \mathcal{D}_\Delta^{(k-1)}p(\Pi + h\Delta) \\ &= k(I - \text{diag}(d)\Pi^\top)^{-1} \text{diag}(d)\Delta^\top \mathcal{D}_\Delta^{(k)}p(\Pi) \\ &\quad + k(I - \text{diag}(d)\Pi^\top)^{-1} \text{diag}(d)\Delta^\top (I - \text{diag}(d)\Pi^\top)^{-1} \text{diag}(d)\Delta^\top \mathcal{D}_\Delta^{(k-1)}p(\Pi) \\ &= k(I - \text{diag}(d)\Pi^\top)^{-1} \text{diag}(d)\Delta^\top \mathcal{D}_\Delta^{(k)}p(\Pi) + (I - \text{diag}(d)\Pi^\top)^{-1} \text{diag}(d)\Delta^\top \mathcal{D}_\Delta^{(k)}p(\Pi) \\ &= (k + 1)(I - \text{diag}(d)\Pi^\top)^{-1} \text{diag}(d)\Delta^\top \mathcal{D}_\Delta^{(k)}p(\Pi). \end{aligned}$$

Similarly, we obtain $\underline{\mathcal{D}}_{\Delta}^{(k+1)} p(\Pi) = (k+1)(I - \text{diag}(d)\Pi^{\top})^{-1} \text{diag}(d)\Delta^{\top} \underline{\mathcal{D}}_{\Delta}^{(k)} p(\Pi)$. The existence of the limit and the result (6) follow for all $k \geq 1$.

With the above results on all k^{th} order directional derivatives, we now consider the full Taylor expansion. First, by the definition of h^{**} given in (5), $\text{diag}(d)$ is fixed for $h \in (-h^{**}, h^{**})$. By the definition of the clearing payments p (given in (2)) and defaulting firms $\text{diag}(d)$ (defined in Theorem 2.6), along with the fact that $I - \text{diag}(d)(\Pi + h\Delta)^{\top}$ is invertible (as shown in the proof of Theorem 2.6 since $(\Pi + h\Delta, x, \bar{p})$ remains a regular system by $h \in (-h^{**}, h^{**}) \subseteq (-h^*, h^*)$), we have

$$\begin{aligned} p(\Pi + h\Delta) &= \text{diag}(d)(x + (\Pi + h\Delta)^{\top} p(\Pi + h\Delta)) + (I - \text{diag}(d))\bar{p} \\ &= (I - \text{diag}(d)(\Pi + h\Delta)^{\top})^{-1} \left(\text{diag}(d)x + (I - \text{diag}(d))\bar{p} \right). \end{aligned} \quad (8)$$

Similarly we find that

$$p(\Pi) = (I - \text{diag}(d)\Pi^{\top})^{-1} \left(\text{diag}(d)x + (I - \text{diag}(d))\bar{p} \right). \quad (9)$$

By combining (8) and (9), we immediately find

$$p(\Pi + h\Delta) = (I - \text{diag}(d)(\Pi + h\Delta)^{\top})^{-1} (I - \text{diag}(d)\Pi^{\top}) p(\Pi).$$

Additionally, we can show that

$$(I - \text{diag}(d)(\Pi + h\Delta)^{\top})^{-1} (I - \text{diag}(d)\Pi^{\top}) = \left(I - h(I - \text{diag}(d)\Pi^{\top})^{-1} \text{diag}(d)\Delta^{\top} \right)^{-1}$$

directly by

$$\begin{aligned} &(I - \text{diag}(d)(\Pi + h\Delta)^{\top})^{-1} (I - \text{diag}(d)\Pi^{\top}) \left(I - h(I - \text{diag}(d)\Pi^{\top})^{-1} \text{diag}(d)\Delta^{\top} \right) \\ &= (I - \text{diag}(d)(\Pi + h\Delta)^{\top})^{-1} (I - \text{diag}(d)\Pi^{\top} - h \text{diag}(d)\Delta^{\top}) \\ &= (I - \text{diag}(d)(\Pi + h\Delta)^{\top})^{-1} (I - \text{diag}(d)(\Pi + h\Delta)^{\top}) = I. \end{aligned}$$

Therefore, for any $h \in (-h^{**}, h^{**})$, we find

$$p(\Pi + h\Delta) = \left(I - h(I - \text{diag}(d)\Pi^{\top})^{-1} \text{diag}(d)\Delta^{\top} \right)^{-1} p(\Pi),$$

i.e., (7).

Now let us consider the perturbations of size h within the neighbourhood

$$\mathcal{H} := \left\{ h \in \mathbb{R} \mid |h| < \min \left\{ h^{**}, \frac{1}{\rho\left((I - \text{diag}(d)\Pi^{\top})^{-1} \text{diag}(d)\Delta^{\top}\right)} \right\} \right\}.$$

We will employ the following property of matrix inverses (see (Meyer 2000, p. 126)): If $X, Y \in \mathbb{R}^{n \times n}$ so that X^{-1} exists and $\lim_{k \rightarrow \infty} (X^{-1}Y)^k = 0$, then

$$(X + Y)^{-1} = \sum_{k=0}^{\infty} (-X^{-1}Y)^k X^{-1}.$$

We take $X = I - \text{diag}(d)\Pi^{\top}$ and $Y = -h \text{diag}(d)\Delta^{\top}$. Since $\rho(h(I - \text{diag}(d)\Pi^{\top})^{-1} \text{diag}(d)\Delta^{\top}) = |h| \rho\left((I - \text{diag}(d)\Pi^{\top})^{-1} \text{diag}(d)\Delta^{\top}\right) < 1$ by the assumption that $|h| < \frac{1}{\rho\left((I - \text{diag}(d)\Pi^{\top})^{-1} \text{diag}(d)\Delta^{\top}\right)}$,

we have

$$\lim_{k \rightarrow \infty} \left[h(I - \text{diag}(d)\Pi^{\top})^{-1} \text{diag}(d)\Delta^{\top} \right]^k = 0,$$

using a property of the spectral radius (see (Meyer 2000, p. 617)). Thus, by combining this result with (8), we have

$$\begin{aligned}
p(\Pi + h\Delta) &= (I - \text{diag}(d)(\Pi + h\Delta)^\top)^{-1} \left(\text{diag}(d)x + (I - \text{diag}(d))\bar{p} \right) \\
&= \sum_{k=0}^{\infty} \left(h(I - \text{diag}(d)\Pi^\top)^{-1} \text{diag}(d)\Delta^\top \right)^k (I - \text{diag}(d)\Pi^\top)^{-1} \left(\text{diag}(d)x + (I - \text{diag}(d))\bar{p} \right) \\
&= \sum_{k=0}^{\infty} \left(h(I - \text{diag}(d)\Pi^\top)^{-1} \text{diag}(d)\Delta^\top \right)^k p(\Pi) \\
&= \sum_{k=0}^{\infty} \frac{h^k}{k!} \mathcal{D}_\Delta^{(k)} p(\Pi).
\end{aligned}$$

The penultimate equality above follows directly from (9). The last equality follows directly from the definition of the k^{th} order directional derivatives proven above. Thus we have shown the full Taylor expansion is exact on $\mathcal{H} \subseteq (-h^{**}, h^{**})$.

Finally, since we have already shown that (7) is exact for any $h \in (-h^{**}, h^{**})$ and

$$\left(-h(I - \text{diag}(d)\Pi^\top)^{-1} \text{diag}(d)\Delta^\top \right)$$

is singular for at least one of the elements $h \in \left\{ -\frac{1}{\rho((I - \text{diag}(d)\Pi^\top)^{-1} \text{diag}(d)\Delta^\top)}, \frac{1}{\rho((I - \text{diag}(d)\Pi^\top)^{-1} \text{diag}(d)\Delta^\top)} \right\}$ by construction, it must follow that $h^{**} \leq \frac{1}{\rho((I - \text{diag}(d)\Pi^\top)^{-1} \text{diag}(d)\Delta^\top)}$. That is, $\mathcal{H} = (-h^{**}, h^{**})$. \square

Remark 2.9. *We can extend the Taylor series expansion results to the more general space of perturbation matrices $\bar{\Delta}^n(\Pi)$ rather than $\Delta^n(\Pi)$. Over such a domain the Taylor series (7) is only guaranteed to converge for*

$$h \in \left[0, \min \left\{ \bar{h}^{**}, \frac{1}{\rho((I - \text{diag}(d)\Pi^\top)^{-1} \text{diag}(d)\Delta^\top)} \right\} \right),$$

as negative perturbations are not feasible.

3 Perturbation errors

In this section we study in detail estimation errors in an Eisenberg–Noe framework, relying on the directional derivatives discussed in the previous section. Specifically we calculate both maximal errors as well as the error distribution assuming a specific distribution of the mis-estimation of the interbank liabilities, notably uniform and Gaussian. We do this first in the original Eisenberg–Noe model, considering the Euclidean norm of the clearing vector as objective. Then we turn to an enhanced model that includes an additional node representing society and study the effect of estimation errors on the payout to society.

3.1 Deviations of the clearing vector

We concentrate first on the L^2 -deviation of the actual clearing vector from the estimated one.

3.1.1 Largest shift of the clearing vector

We return to the first order directional derivative to quantify the largest shift of the clearing vector for estimation errors in the relative liability matrix given by perturbations in $\mathbf{\Delta}^n(\Pi)$. Let $\Delta \in \mathbf{\Delta}^n(\Pi)$ and assume that for a given $h \in \mathbb{R} : \Pi + h\Delta \in \mathbf{\Pi}^n$. Then, the worst case estimation error under $\mathbf{\Delta}^n(\Pi)$ is given as

$$\max_{\Delta \in \mathbf{\Delta}^n(\Pi)} \|p(\Pi + h\Delta) - p(\Pi)\|_2^2.$$

In order to remove the dependence on h and the magnitude of Δ , we consider instead the bounded set of directions $\mathbf{\Delta}_F^n(\Pi)$ and infinitesimal perturbations,

$$\max_{\Delta \in \mathbf{\Delta}_F^n(\Pi)} \lim_{h \rightarrow 0} \frac{\|p(\Pi + h\Delta) - p(\Pi)\|_2^2}{h} = \max_{\Delta \in \mathbf{\Delta}_F^n(\Pi)} \|\mathcal{D}_\Delta p(\Pi)\|_2^2.$$

In this section, we call $\|\mathcal{D}_\Delta p(\Pi)\|_2^2$ the estimation error and $\max_{\Delta \in \mathbf{\Delta}_F^n(\Pi)} \|\mathcal{D}_\Delta p(\Pi)\|_2^2$ the maximal deviation in the clearing vector under $\mathbf{\Delta}_F^n(\Pi)$. Because Δ appears via a linear term in (3), this allows us to use a basis of perturbation matrices in an elegant way to quantify the deviation of the Eisenberg–Noe clearing vector under the space of perturbations $\mathbf{\Delta}_F^n(\Pi)$.

Throughout the following results we will take advantage of an orthonormal basis $\vec{E}(\Pi) = (E_1, \dots, E_d)$ of the space $\mathbf{\Delta}^n(\Pi)$. More details of this space are given in Appendix A.

Proposition 3.1. *Let (Π, x, \bar{p}) be a regular financial system. The worst case first order estimation error under $\mathbf{\Delta}_F^n(\Pi)$ is given by*

$$\max_{\Delta \in \mathbf{\Delta}_F^n(\Pi)} \|\mathcal{D}_\Delta p(\Pi)\|_2^2 = \left(\|\mathcal{D}_{\vec{E}(\Pi)} p(\Pi)\|_2^o \right)^2 \quad (10)$$

for any choice of basis $\vec{E}(\Pi)$ where $\|\cdot\|_2^o$ denotes the spectral norm of a matrix. Furthermore, the largest shift of the clearing vector is achieved by

$$\Delta^*(\Pi) := \pm \sum_{k=1}^d z_k E_k,$$

where z_k are the components of the (normalised) eigenvector corresponding to the maximum eigenvalue of $(\mathcal{D}_{\vec{E}(\Pi)} p(\Pi))^\top \mathcal{D}_{\vec{E}(\Pi)} p(\Pi)$.

Proof. Note first that any perturbation matrix $\Delta \in \mathbf{\Delta}^n(\Pi)$ can be written as a linear combination of basic perturbation matrices, i.e., $\Delta = \sum_{k=1}^d z_k E_k$. Thus,

$$\|\mathcal{D}_\Delta p(\Pi)\|_2^2 = \left\| \sum_{k=1}^d z_k \mathcal{D}_{E_k} p(\Pi) \right\|_2^2 = z^\top (\mathcal{D}_{\vec{E}(\Pi)} p(\Pi))^\top \mathcal{D}_{\vec{E}(\Pi)} p(\Pi) z.$$

Immediately this implies, denoting the largest eigenvalue of a matrix A by $\lambda_{\max}(A)$,

$$\begin{aligned} \max_{\Delta \in \mathbf{\Delta}_F^n(\Pi)} \|\mathcal{D}_\Delta p(\Pi)\|_2^2 &= \max_{\|z\|_2 \leq 1} z^\top (\mathcal{D}_{\vec{E}(\Pi)} p(\Pi))^\top \mathcal{D}_{\vec{E}(\Pi)} p(\Pi) z \\ &= \lambda_{\max} \left((\mathcal{D}_{\vec{E}(\Pi)} p(\Pi))^\top \mathcal{D}_{\vec{E}(\Pi)} p(\Pi) \right) \\ &= \left(\|\mathcal{D}_{\vec{E}(\Pi)} p(\Pi)\|_2^o \right)^2. \end{aligned}$$

Finally, the independence of the solution from the choice of basis $\vec{E}(\Pi)$ is a direct result of Proposition A.2. \square

We can use this result on the maximum deviations of the clearing vector under $\Delta_F^n(\Pi)$ in order to provide an upper bound of the worst case perturbation error without predetermining the existence or non-existence of links.

Corollary 3.2. *Let (Π, x, \bar{p}) be a regular financial system. The worst case first order estimation error under all perturbations is bounded by*

$$\max_{\Delta \in \overline{\Delta}_F^n(\Pi)} \|\mathcal{D}_{\Delta} p(\Pi)\|_2^2 \leq (\|\mathcal{D}_{\vec{E}(\Pi_C)} p(\Pi)\|_2^o)^2 \quad (11)$$

for any choice of orthonormal basis $\vec{E}(\Pi_C)$ of any completely connected network Π_C . In the case that Π itself is a completely connected network then this upper bound is attained.

Proof. For all Π and all completely connected networks Π_C , we have $\overline{\Delta}_F^n(\Pi) \subseteq \Delta_F^n(\Pi_C)$. Hence, using (3), one obtains

$$\begin{aligned} \max_{\Delta \in \overline{\Delta}_F^n(\Pi)} \|\mathcal{D}_{\Delta} p(\Pi)\|_2^2 &\leq \max_{\Delta \in \Delta_F^n(\Pi_C)} \|\mathcal{D}_{\Delta} p(\Pi)\|_2^2 \\ &= \max_{\|z\|_2 \leq 1} \left\| (I - \text{diag}(d)\Pi^\top)^{-1} \text{diag}(d) \left[\sum_{k=1}^d z_k E_k \right]^\top p(\Pi) \right\|_2^2 \\ &= (\|\mathcal{D}_{\vec{E}(\Pi_C)} p(\Pi)\|_2^o)^2, \end{aligned}$$

where $\vec{E}(\Pi_C) := (E_1, \dots, E_d)$ is an orthonormal basis of the space $\Delta^n(\Pi_C)$. As in Proposition 3.1, the independence of the solution from the choice of basis $\vec{E}(\Pi_C)$ is a direct result of Proposition A.2. \square

Example 3.3. *We return to Example 2.2 and consider the same toy network consisting of four banks in which each bank's nominal liabilities are shown in Figure 1(a). The largest shift of the clearing vector (10) under $\Delta_F^4(\Pi)$, as described in Proposition 3.1, is given by the matrix*

$$\Delta^*(\Pi) = \begin{pmatrix} 0 & 0.3230 & -0.1615 & -0.1615 \\ -0.0381 & 0 & 0.0190 & 0.0190 \\ 0.0571 & -0.4845 & 0 & 0.4274 \\ 0.0571 & -0.4845 & 0.4274 & 0 \end{pmatrix}.$$

As this network is complete, this is furthermore a solution to both optimization problems (10) and (11) for the worst case perturbation. Additionally, the upper bound in Corollary 3.2 is attained. This perturbation is depicted in Figure 2. As before, banks who are in default are coloured red. The edges are labeled with the perturbation of the respective link between banks that achieves this greatest estimation error. The edge linking one node to another is red if the greatest estimation error under the set of perturbations $\Delta_F^n(\Pi)$ occurs when we have overestimated the value of this link and green if we have underestimated it. Note that due to the symmetry of the optimal estimation error problem, $-\Delta^*(\Pi)$ is also optimal and thus the interpretation of red and green links in Figure 2 can be reversed. Indeed, when studying the deviation of the clearing vector, the solutions $\Delta^*(\Pi)$ and $-\Delta^*(\Pi)$ are equivalent. When analysing the shortfall of payments to society in Section 3.2, this will be no longer the case. Edge widths are proportional to the absolute value of the entries in $\Delta^*(\Pi)$. Though our Taylor expansion results (Theorem 2.8) are provided for $h \in (-h^{**}, h^{**})$

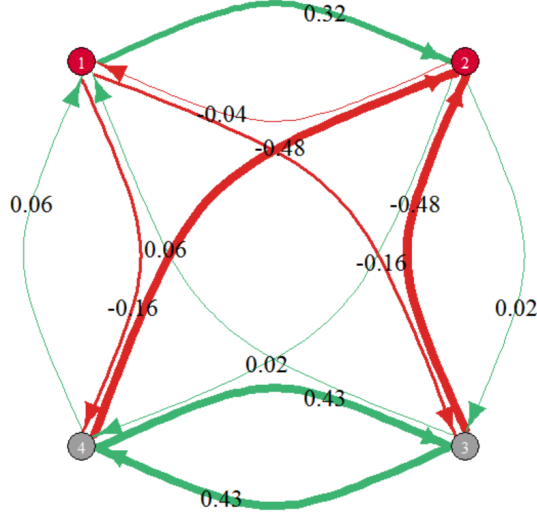


Figure 2: Worst case network perturbation under $\Delta_F^n(\Pi)$ defined in Example 3.3

only, the strict inequality is only necessary if h^{**} denotes the perturbation size at which a new bank defaults, not when a connection is removed. So when $h = h^{**} \approx 0.688$, we obtain

$$L^* = \begin{pmatrix} 0 & 9 & 0 & 0 \\ 2.76 & 0 & 3.12 & 3.12 \\ 1.12 & 0 & 0 & 1.88 \\ 1.12 & 0 & 1.88 & 0 \end{pmatrix},$$

which has the clearing vector

$$\hat{p} \approx (4.11, 6.11, 3, 3)^\top.$$

One can immediately verify that L^* has indeed the same total interbank assets and liabilities for each bank, but they are distributed in a different manner. Hence, in this example, there can be a deviation of up to 15% in the relative norm of the clearing vector for a network that is still consistent with the total assets and total liabilities.

Remark 3.4. It may be desirable to normalize the first order estimation errors by, e.g., the clearing payments or total nominal liabilities, rather than considering the absolute error. In a general form, let $A \in \mathbb{R}^{n \times n}$ denote a normalization matrix (e.g., $A = \text{diag}(p(\Pi))^{-1}$ or $A = \text{diag}(\bar{p})^{-1}$). Then we can extend the results of Proposition 3.1 and Corollary 3.2 by

$$\begin{aligned} \max_{\Delta \in \Delta_F^n(\Pi)} \|AD_{\Delta} p(\Pi)\|_2^2 &= (\|AD_{\bar{E}(\Pi)} p(\Pi)\|_2^0)^2 \\ \max_{\Delta \in \Delta_F^n(\Pi)} \|AD_{\Delta} p(\Pi)\|_2^2 &\leq (\|AD_{\bar{E}(\Pi_C)} p(\Pi)\|_2^0)^2 \end{aligned}$$

for any completely connected network Π_C . Similarly the distribution results presented below can be generalized by considering $AD_{\bar{E}(\Pi)} p(\Pi)$ in place of $\mathcal{D}_{\bar{E}(\Pi)} p(\Pi)$.

3.1.2 Clearing vector deviation for uniformly distributed estimation errors

In this section, we will extend the above analysis to the case when estimation errors are uniformly distributed. This is done by considering the linear coefficients z for the basis of perturbation

matrices to be chosen uniformly on the d -dimensional Euclidean unit ball. Then $\Delta = \sum_{k=1}^d z_k E_k$ is a perturbation matrix.

Proposition 3.5. *Let (Π, x, \bar{p}) be a regular financial system. The distribution of the estimation error when the perturbations are uniformly distributed in the L^2 -unit ball is given by*

$$\mathbb{P}(\|\mathcal{D}_\Delta p(\Pi)\|_2^2 \leq \alpha) = \frac{\text{vol}(\{w \in \mathbb{R}^d \mid w^\top w \leq 1, w^\top \Lambda w \leq \alpha\}) \Gamma(\frac{d}{2} + 1)}{\pi^{d/2}}, \quad \alpha \geq 0,$$

where Λ is the diagonal matrix with elements given by the eigenvalues of $(\mathcal{D}_{\vec{E}(\Pi)} p(\Pi))^\top \mathcal{D}_{\vec{E}(\Pi)} p(\Pi)$ for any choice of orthonormal basis $\vec{E}(\Pi)$, vol denotes the volume operator, and Γ is the gamma function.

Proof. Let z be uniform on the d -dimensional unit ball. Then $\Delta = \sum_{k=1}^d z_k E_k$ is a perturbation matrix. One obtains

$$\begin{aligned} \mathbb{P}(\|\mathcal{D}_\Delta p(\Pi)\|_2^2 \leq \alpha) &= \mathbb{P}\left((\mathcal{D}_\Delta p(\Pi))^\top \mathcal{D}_\Delta p(\Pi) \leq \alpha\right) \\ &= \mathbb{P}\left(z^\top (\mathcal{D}_{\vec{E}(\Pi)} p(\Pi))^\top \mathcal{D}_{\vec{E}(\Pi)} p(\Pi) z \leq \alpha\right). \end{aligned}$$

The matrix $(\mathcal{D}_{\vec{E}(\Pi)} p(\Pi))^\top \mathcal{D}_{\vec{E}(\Pi)} p(\Pi)$ is diagonalizable because it is real and symmetric. Therefore we can write

$$(\mathcal{D}_{\vec{E}(\Pi)} p(\Pi))^\top \mathcal{D}_{\vec{E}(\Pi)} p(\Pi) = V^\top \Lambda V,$$

where Λ is a diagonal matrix of the eigenvalues and V is orthonormal. Combining the above equations, we have

$$\mathbb{P}\left(z^\top (\mathcal{D}_{\vec{E}(\Pi)} p(\Pi))^\top \mathcal{D}_{\vec{E}(\Pi)} p(\Pi) z \leq \alpha\right) = \mathbb{P}\left(z^\top V^\top \Lambda V z \leq \alpha\right).$$

Then since z is uniform on the unit ball and $VV^\top = I$, $w = Vz$ is also uniform on the unit ball and thus we have

$$\begin{aligned} \mathbb{P}(z^\top V^\top \Lambda V z \leq \alpha) &= \mathbb{P}(w^\top \Lambda w \leq \alpha) \\ &= \frac{\text{vol}(\{w \mid w^\top w \leq 1, w^\top \Lambda w \leq \alpha\})}{\text{vol}(\{w \mid w^\top w \leq 1\})} \\ &= \frac{\text{vol}(\{w \mid w^\top w \leq 1, w^\top \Lambda w \leq \alpha\}) \Gamma(\frac{d}{2} + 1)}{\pi^{d/2}}. \end{aligned}$$

As in Proposition 3.1, the independence of the distribution from the choice of basis $\vec{E}(\Pi_C)$ is a direct result of Proposition A.2. \square

Remark 3.6. *In the case where $\alpha \leq \min_k \lambda_k$ or $\alpha \geq \max_k \lambda_k$ then $\mathbb{P}(\|\mathcal{D}_\Delta p(\Pi)\|_2^2 \leq \alpha)$ can explicitly be given by $\alpha^d \prod_{k=1}^d \frac{1}{\sqrt{\lambda_k}}$ and 1 respectively where $\{\lambda_k \mid k = 1, \dots, d\}$ is the collection of eigenvalues of $(\mathcal{D}_{\vec{E}(\Pi)} p(\Pi))^\top \mathcal{D}_{\vec{E}(\Pi)} p(\Pi)$. In the case that $\min_k \lambda_k < \alpha < \max_k \lambda_k$, the probability $\mathbb{P}(\|\mathcal{D}_\Delta p(\Pi)\|_2^2 \leq \alpha)$ can be given via the volume formula provided in Proposition 3.5 as d nested integrals,*

$$\frac{\Gamma(\frac{d}{2} + 1)}{\pi^{d/2}} \int_{-1}^1 \int_{-\sqrt{1-x_1^2}}^{\sqrt{1-x_1^2}} \cdots \int_{-\sqrt{1-\sum_{k=1}^{m-1} x_k^2}}^{\sqrt{1-\sum_{k=1}^{m-1} x_k^2}} \int_{-\sqrt{\frac{\alpha - \sum_{k=1}^m \lambda_{[k]} x_k^2}{\lambda_{[m+1]}}}^{\sqrt{\frac{\alpha - \sum_{k=1}^m \lambda_{[k]} x_k^2}{\lambda_{[m+1]}}}} \cdots \int_{-\sqrt{\frac{\alpha - \sum_{k=1}^{d-1} \lambda_{[k]} x_k^2}{\lambda_{[d]}}}^{\sqrt{\frac{\alpha - \sum_{k=1}^{d-1} \lambda_{[k]} x_k^2}{\lambda_{[d]}}}} dx_d \cdots dx_1,$$

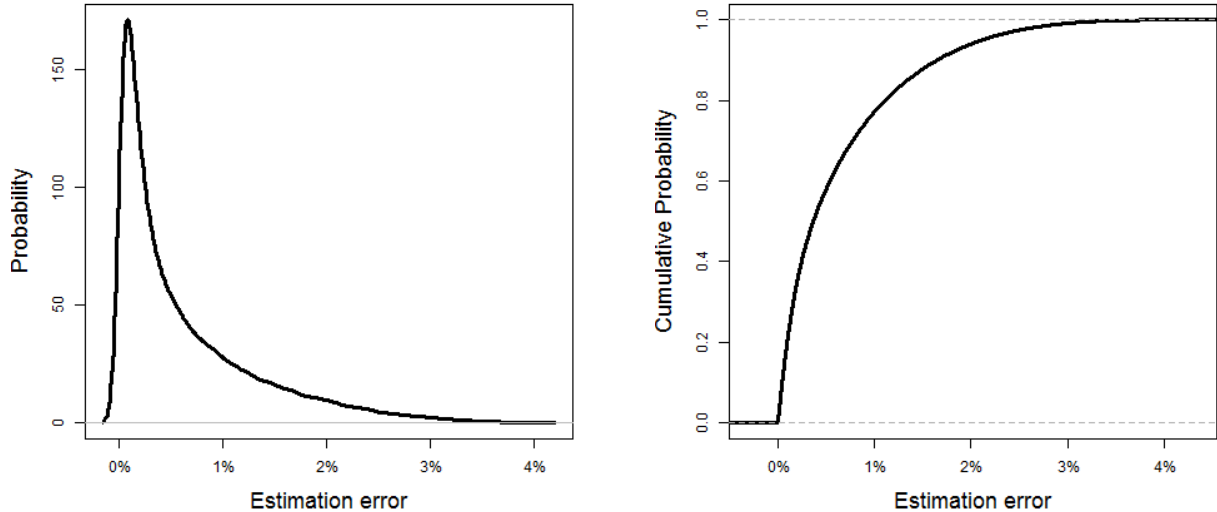


Figure 3: The Probability density (left) and the CDF (right) of the relative estimation error, $\|\mathcal{D}_\Delta p(\Pi)\|_2^2 / \|p(\Pi)\|_2^2$, under uniform perturbations Δ as described in Example 3.7

where $\lambda_{[m]} \leq \alpha \leq \lambda_{[m+1]}$ and $\lambda_{[m]}$ is a reordering of the eigenvalues such that $0 \leq \lambda_{[1]} \leq \lambda_{[2]} \leq \dots \leq \lambda_{[d]}$.

Example 3.7. We return again to Example 2.2 to consider perturbations Δ sampled from the uniform distribution. Figure 3 shows the density and CDF estimation for the relative estimation error, $\|\mathcal{D}_\Delta p(\Pi)\|_2^2 / \|p(\Pi)\|_2^2$, corresponding to our stylized four-bank network. The probabilities are estimated from 100,000 simulated uniform perturbations.

3.1.3 Clearing vector deviation for normally distributed estimation errors

We extend our analysis from the previous subsection by considering normally distributed perturbations. To do so, we consider the linear coefficients z for the basis of perturbation matrices to be chosen distributed according to the standard d -dimensional multivariate standard Gaussian distribution. Then $\sum_{k=1}^d z_k E_k$ is a perturbation matrix Δ . Though our prior results on the deviations of the clearing payments have been within the unit ball $\Delta_F^n(\Pi)$, under a Gaussian distribution the magnitude of the perturbation matrices are no longer bounded by 1 and thus the estimation errors can surpass the worst case errors determined in Proposition 3.1 and Corollary 3.2.

Proposition 3.8. Let (Π, x, \bar{p}) be a regular financial system. The distribution of estimation errors where the perturbations are distributed with respect to the standard normal is given by the moment generating function

$$M(t) := \det(I - 2\Lambda t)^{-1/2},$$

where Λ is the diagonal matrix with elements given by the eigenvalues of $(\mathcal{D}_{\vec{E}(\Pi)} p(\Pi))^\top \mathcal{D}_{\vec{E}(\Pi)} p(\Pi)$ for any orthonormal basis $\vec{E}(\Pi)$.

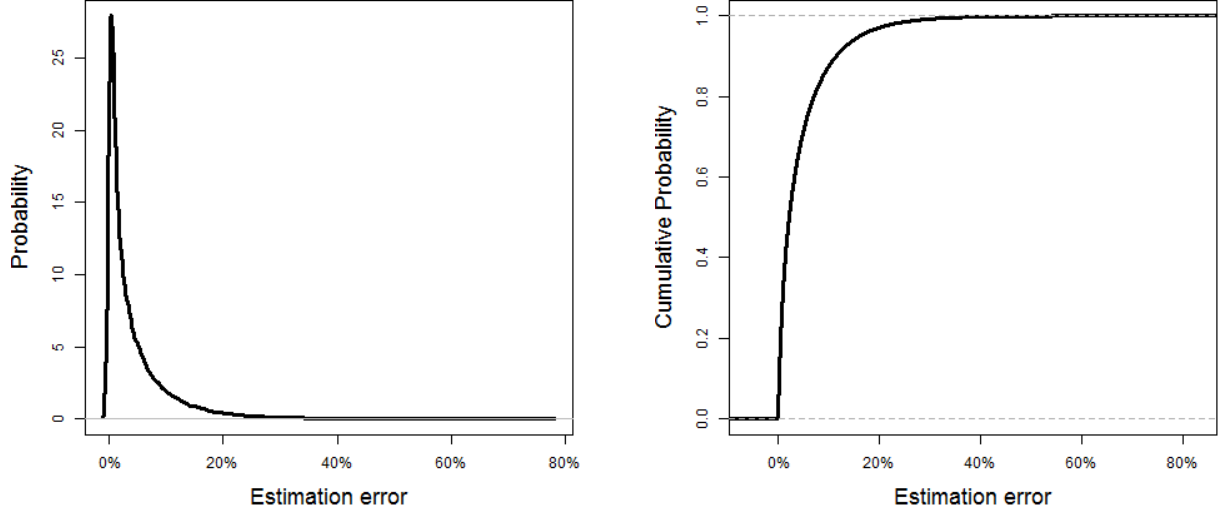


Figure 4: The estimated probability density (left) and the CDF (right) of the estimation error, $\|\mathcal{D}_\Delta p(\Pi)\|_2^2 / \|p(\Pi)\|_2^2$, under standard Gaussian perturbations Δ as described in Example 3.10.

Proof. Let z be a d -dimensional standard normal Gaussian random variable. Then $\Delta = \sum_{k=1}^d z_k E_k$ is a perturbation matrix. As in Proposition 3.5, we can write

$$z^\top (\mathcal{D}_{\vec{E}(\Pi)} p(\Pi))^\top \mathcal{D}_{\vec{E}(\Pi)} p(\Pi) z = z^\top V^\top \Lambda V z,$$

where Λ is the diagonal matrix of eigenvalues of $(\mathcal{D}_{\vec{E}(\Pi)} p(\Pi))^\top \mathcal{D}_{\vec{E}(\Pi)} p(\Pi)$ and V is orthonormal. Since $z \sim N(0, I)$ and $VV^\top = I$, we have $w = Vz \sim N(0, VV^\top = I)$. Therefore,

$$z^\top V^\top \Lambda V z = w^\top \Lambda w = w^\top \Lambda^{1/2} \Lambda^{1/2} w.$$

Then $y = \Lambda^{1/2} w \sim N(0, \Lambda)$ and so each component $y_k \sim N(0, \lambda_k)$ and the y_k 's are independent. Therefore

$$w^\top \Lambda^{1/2} \Lambda^{1/2} w = y^\top y = \sum_{k=1}^d y_k^2.$$

The distribution of y_k^2 is $\Gamma(1/2, 2\lambda_k)$, and thus the sum $\sum_{k=1}^d y_k^2$ has the moment generating function

$$M(t) = \prod_{k=1}^d (1 - 2\lambda_k t)^{-1/2},$$

where λ_k are the eigenvalues of $(\mathcal{D}_{\vec{E}(\Pi)} p(\Pi))^\top \mathcal{D}_{\vec{E}(\Pi)} p(\Pi)$. As in Proposition 3.1, the independence of the distribution from the choice of basis $\vec{E}(\Pi_C)$ is a direct result of Proposition A.2. \square

Remark 3.9. A closed form for the density of the distribution found in Proposition 3.8 is given in equation (7) of Mathai (1982).

Example 3.10. We return again to Example 2.2 to consider perturbations Δ sampled from the standard normal distribution. Figure 4 shows the density and CDF estimation for the relative estimation error, $\|\mathcal{D}_{\Delta}p(\Pi)\|_2^2/\|p(\Pi)\|_2^2$, corresponding to our stylized four-bank network. The probabilities are estimated from 100,000 simulated Gaussian perturbations.

3.2 Impact to the payout to society

In this section, we assume that in addition to their interbank liabilities, banks also have a liability to society. Here, society is used as totum pro parte, encompassing all non-financial counterparties, corporate, individual or governmental. Hence, the set of institutions becomes $\mathcal{N}_0 = \{0\} \cup \mathcal{N}$. Without loss of generality, we assume that all banks $i \in \mathcal{N}$ owe money to at least one counterparty $j \in \mathcal{N}_0$ within the system. Otherwise, a bank who owes no money can be absorbed by the society node as it plays the same role within the model structure. The question of interest is then how the payout to society may be mis-estimated (and in particular overestimated) given estimation errors in the relative liabilities matrix. (Often the societal payments would be prioritized over interbank debt (rather than paid pro-rata as we study herein). Such a modification in the direction of Elsinger (2009) for inclusion of prioritized payments and seniority structure is possible, though we restrict ourselves to the original Eisenberg–Noe framework.)

The interbank liability matrix L of the previous section is expanded to $L_0 \in \mathbb{R}^{(n+1) \times (n+1)}$ given by

$$L_0 = \left[\begin{array}{ccc|c} 0 & \cdots & L_{1n} & L_{10} \\ \vdots & \ddots & \vdots & \vdots \\ L_{n1} & \cdots & 0 & L_{n0} \\ \hline 0 & \cdots & 0 & 0 \end{array} \right] = \left[\begin{array}{c|c} L & l_0 \\ \hline 0 & 0 \end{array} \right],$$

where $l_0 = (L_{10}, \dots, L_{n0})^\top$ is the society liability vector. We require that at least one bank has an obligation to society, i.e., $L_{i0} > 0$ for some $1 \leq i \leq n$. The total liability of bank i is now given by $\bar{p}_i = \sum_{j=0}^n L_{ij}$. As stated above, we also require that each bank owes to at least one counterparty within the system (possibly society), i.e., $\bar{p}_i > 0$ for all $i \in \mathcal{N}$. The relative liability matrix Π_0 is transformed accordingly, i.e., $\pi_{ij} \in [0, 1]$ and $\pi_{ij} = \frac{L_{ij}}{\bar{p}_i}$. An admissible relative liability matrix Π_0 thus belongs to the set of all right stochastic matrices with entries in $[0, 1]$, all diagonal entries 0, and at least one $\pi_{i0} > 0$:

$$\mathbf{\Pi}_0^n := \left\{ \Pi_0 \in [0, 1]^{(n+1) \times (n+1)} \mid \forall i : \pi_{ii} = 0, \sum_{j=0}^n \pi_{ij} = 1 \text{ and } \exists i \text{ s.t. } \pi_{i0} > 0 \right\}.$$

An admissible *interbank* relative liability matrix Π thus belongs to the set

$$\mathbf{\Pi}_I^n := \left\{ \Pi \in [0, 1]^{n \times n} \mid \forall i : \pi_{ii} = 0, \sum_{j=1}^n \pi_{ij} \leq 1 \text{ and } \exists i \text{ s.t. } \sum_{j=1}^n \pi_{ij} < 1 \right\},$$

which has the same properties as the original interbank relative liability matrix $\mathbf{\Pi}^n$ defined in (1), except that row sums are smaller or equal to 1, with at least one strictly smaller than 1.

The following result is implicitly used in the subsequent sections. This provides us with the ability to, e.g., consider the directional derivative with respect to the payments made by the n financial firms without considering the societal node (which is equal to 0 by assumption).

Proposition 3.11. *If (Π_0, x, \bar{p}) is a regular network then $I - \text{diag}(d)\Pi$ is invertible.*

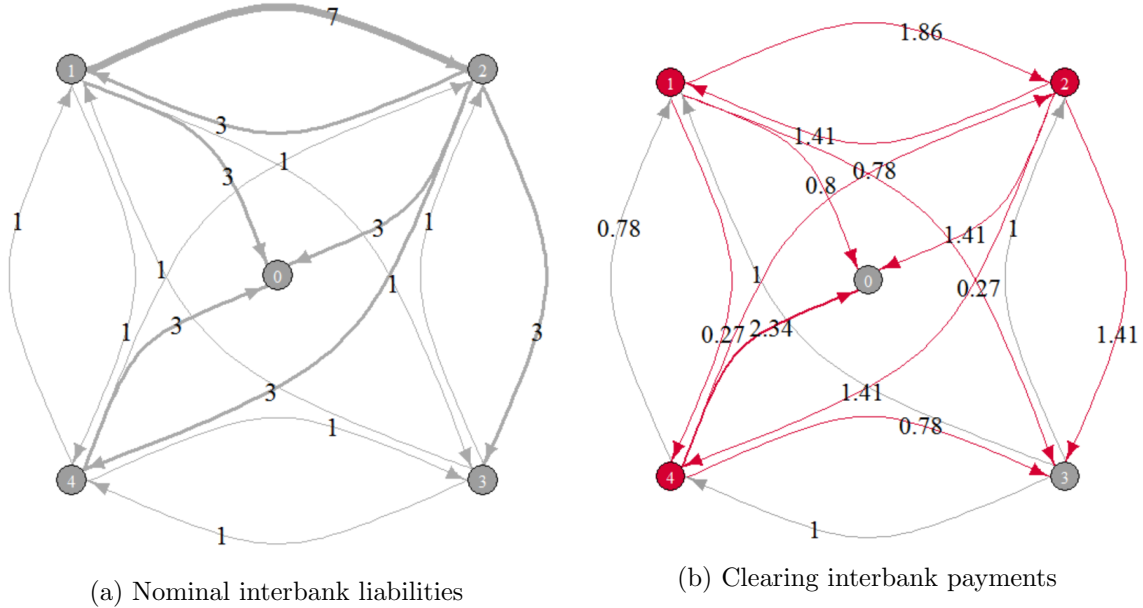


Figure 5: Initial network defined in Example 3.12

Proof. This follows immediately from

$$I - \text{diag}(d_0)\Pi_0^\top = \begin{pmatrix} I - \text{diag}(d)\Pi^\top & -\text{diag}(d)\pi_0 \\ \mathbf{0}^\top & 1 \end{pmatrix},$$

where $\pi_0 = (\pi_{10}, \dots, \pi_{n0})^\top$ and d_0 is the vector of default indicators (of length $n + 1$ to include the societal node). In particular, since $\det(I - \text{diag}(d_0)\Pi_0^\top) \neq 0$ (as shown in the proof of Theorem 2.6), we can conclude that $\det(I - \text{diag}(d)\Pi^\top) \neq 0$. \square

Example 3.12. We include now a society node into our example from Section 2. The nominal interbank liabilities and liabilities from each bank to society are shown in Figure 5(a). Note that at least one bank has an obligation to society and the society does not owe to any bank. As above, the banks' external assets are given by the vector $x = (0, 2, 2, 2)^\top$. The clearing payments, or the amount of its obligations that each bank is able to repay, is given in Figure 5(b). Banks who are in default are coloured red, as are the liabilities that are not repaid in full.

3.2.1 Largest reduction in the payout to society

Next, we use the directional derivative in order to quantify how estimation errors, under $\Delta_F^n(\Pi)$ in the interbank relative liability matrix, could lead to an overestimation of the payout to society. As it turns out, this problem also has an elegant solution using the basis of perturbation matrices discussed in Appendix A. We assume that (Π_0, x, \bar{p}) is a regular financial system and additionally that both the relative liabilities to society $\pi_0 = (\pi_{10}, \dots, \pi_{n0})^\top$ and the total liabilities \bar{p} are exactly known.

Definition 3.13. Let (Π_0, x, \bar{p}) be a regular financial system. The payout to society is defined as the quantity $\pi_0^\top p(\Pi)$ where $p(\Pi)$ is the clearing vector of the n firms.

Herein we consider the relative liabilities matrix Π_0 to be an estimation of the true relative liabilities. We thus consider the perturbations of the estimated clearing vectors to determine the maximum

amount that the payout to society may be overestimated. To study the optimisation problem of minimizing the payout to society, we assume that at least one bank, but not all banks, default. The following proposition shows that this assumption excludes only trivial cases.

Proposition 3.14. *Let (Π_0, x, \bar{p}) be a regular system with the interbank relative liability matrix $\Pi \in \mathbf{\Pi}_I^n$ and $\Delta \in \mathbf{\Delta}^n(\Pi)$. If all banks default, or if no bank defaults, then the payout to society remains unchanged for an arbitrary admissible perturbation Δ .*

Proof. Let Δ be an arbitrary perturbation matrix. We show that in both cases $\pi_0^\top \mathcal{D}_\Delta p(\Pi) = 0$.

1. Assume that no bank defaults. Then $\text{diag}(d) = 0$, and the result holds as $\mathcal{D}_\Delta p(\Pi) = 0$.
2. Assume all banks default. Then $\text{diag}(d) = I$. Hence, $\pi_0^\top \mathcal{D}_\Delta p(\Pi) = \pi_0^\top (I - \Pi^\top)^{-1} \Delta^\top p(\Pi)$. Note that $\pi_0^\top (I - \Pi^\top)^{-1} = \mathbf{1}^\top$, because by definition $\pi_0^\top = \mathbf{1}^\top (I - \Pi^\top)$. Using this and the definitions of $\mathcal{D}_\Delta p(\Pi)$ and Δ , it follows $\pi_0^\top \mathcal{D}_\Delta p(\Pi) = \sum_{i=1}^n \sum_{j=1}^n \delta_{ji} p_j(\Pi) = 0$.

□

Let $\Delta \in \mathbf{\Delta}^n(\Pi)$ and assume that for a given $h \in \mathbb{R} : \Pi + h\Delta \in \mathbf{\Pi}_I^n$. Then, the minimum payout to society is

$$\min_{\Delta \in \mathbf{\Delta}^n(\Pi)} \pi_0^\top p(\Pi + h\Delta).$$

In order to remove the dependence on h and the magnitude of Δ , we subtract the constant term $\pi_0^\top p(\Pi)$ and consider instead

$$\min_{\Delta \in \mathbf{\Delta}_F^n(\Pi)} \lim_{h \rightarrow 0} \pi_0^\top \frac{p(\Pi + h\Delta) - p(\Pi)}{h} = \min_{\Delta \in \mathbf{\Delta}_F^n(\Pi)} \pi_0^\top \mathcal{D}_\Delta p(\Pi).$$

As in Section 3.1.1, using the basis of perturbation matrices $\vec{E}(\Pi)$ of $\mathbf{\Delta}^n(\Pi)$ (see Appendix A), we can compute the shortfall to society due to perturbations in the relative liability matrix in $\mathbf{\Delta}_F^n(\Pi)$.

Proposition 3.15. *Let (Π_0, x, \bar{p}) be a regular financial system. The largest shortfall in payments to society due to estimation errors in the liability matrix in $\mathbf{\Delta}_F^n(\Pi)$ is given by*

$$\min_{\Delta \in \mathbf{\Delta}_F^n(\Pi)} \pi_0^\top \mathcal{D}_\Delta p(\Pi) = -\|\pi_0^\top \mathcal{D}_{\vec{E}(\Pi)} p(\Pi)\|_2.$$

Furthermore, the largest shortfall to society is achieved by

$$\Delta_0^*(\Pi) := -\sum_{k=1}^d \frac{\pi_0^\top \mathcal{D}_{E_k} p(\Pi)}{\|\pi_0^\top \mathcal{D}_{\vec{E}(\Pi)} p(\Pi)\|_2} E_k.$$

Additionally, both the largest shortfall and the perturbation matrix that attains that shortfall are independent of the chosen basis $\vec{E}(\Pi)$.

Proof. Since the problem

$$\min \pi_0^\top \mathcal{D}_{\vec{E}(\Pi)} p(\Pi) z \quad \text{s.t.} \quad \|z\|_2 \leq 1,$$

has a linear objective, it is equivalent to

$$\min \pi_0^\top \mathcal{D}_{\vec{E}(\Pi)} p(\Pi) z \quad \text{s.t.} \quad z^\top z = 1.$$

By the necessary Karush–Kuhn–Tucker conditions, we know that any solution to this problem must satisfy

$$\begin{aligned} (\mathcal{D}_{\vec{E}(\Pi)} p(\Pi))^\top \pi_0 + 2\mu z &= 0, \\ z^\top z &= 1, \end{aligned}$$

for some $\mu \in \mathbb{R}$. The first condition implies $z^* = -\frac{(\mathcal{D}_{\vec{E}(\Pi)} p(\Pi))^\top \pi_0}{2\mu}$. Plugging this into the second implies that $\mu = \pm \frac{\|\pi_0^\top \mathcal{D}_{\vec{E}(\Pi)} p(\Pi)\|_2}{2}$. With two possible solutions we plug these back into the original objective to find that the minimum is attained at $\mu = \frac{\|\pi_0^\top \mathcal{D}_{\vec{E}(\Pi)} p(\Pi)\|_2}{2}$ for an optimal value of:

$$\pi_0^\top \mathcal{D}_{\vec{E}(\Pi)} p(\Pi) z^* = -\frac{(\pi_0^\top \mathcal{D}_{\vec{E}(\Pi)} p(\Pi)) (\pi_0^\top \mathcal{D}_{\vec{E}(\Pi)} p(\Pi))^\top}{\|\pi_0^\top \mathcal{D}_{\vec{E}(\Pi)} p(\Pi)\|_2} = -\|\pi_0^\top \mathcal{D}_{\vec{E}(\Pi)} p(\Pi)\|_2.$$

Therefore, the solution is

$$\Delta_0^*(\Pi) = \sum_{k=1}^d z_k^* E_k = -\sum_{k=1}^d \frac{\pi_0^\top \mathcal{D}_{E_k} p(\Pi)}{\|\pi_0^\top \mathcal{D}_{\vec{E}(\Pi)} p(\Pi)\|_2} E_k.$$

By Proposition A.3, this result is independent of the choice of basis matrices. \square

Corollary 3.16. *Let (Π_0, x, \bar{p}) be a regular financial system. The worst case shortfall to society is bounded from below by*

$$\min_{\Delta \in \overline{\Delta}_F^n(\Pi)} \pi_0^\top \mathcal{D}_\Delta p(\Pi) \geq -\|\pi_0^\top \mathcal{D}_{\vec{E}(\Pi_C)} p(\Pi)\|_2,$$

where $\vec{E}(\Pi_C)$ is any orthonormal basis of perturbation matrices of any completely connected network Π_C . In the case that Π itself is a completely connected network then this upper bound is attained.

Proof. This follows by the same logic as Corollary 3.2 through the inclusion $\overline{\Delta}_F^n(\Pi) \subseteq \Delta_F^n(\Pi_C)$ for any completely connected network Π_C . The independence of this result to the choice of orthonormal basis $\vec{E}(\Pi)$ follows as in Proposition 3.15. \square

Example 3.17. *We continue the discussion from Example 3.12: The perturbation resulting in the greatest shortfall for the society's payout, as described in Proposition 3.15, is given by the matrix*

$$\Delta_0^* = \begin{pmatrix} 0 & 0.16 & -0.46 & 0.30 \\ 0.11 & 0 & 0.16 & -0.27 \\ 0.06 & 0.04 & 0 & -0.10 \\ -0.26 & -0.34 & 0.60 & 0 \end{pmatrix}.$$

This perturbation is depicted in Figure 6. Each edge is labeled with the perturbation of the respective link between banks that achieves this greatest reduction in payout to society. As before, banks who are in default are coloured red. The edge linking one node to another is red if the greatest reduction in payout occurs when we have overestimated the value of this link and green if, in the worst case under $\Delta_F^n(\Pi)$, we have underestimated the value of this link. Edge widths are proportional to the absolute value of the entries in $\Delta_0^*(\Pi)$. In contrast to Example 3.3, note that $-\Delta_0^*(\Pi)$ is **not** a solution anymore. As this network is complete, this also equals the worst case shortfall of -1.4513 , which is nearly 32% of the entire estimated payment to society.

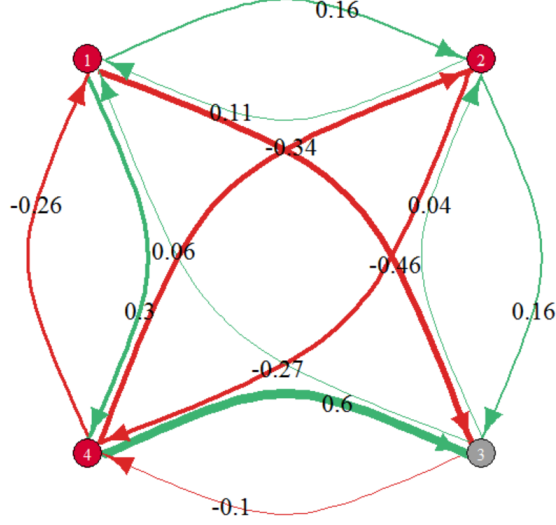


Figure 6: The perturbation in $\Delta_F^n(\Pi)$ which generates the largest shortfall for society defined in Example 3.17.

3.2.2 Shortfall to society for uniformly distributed estimation errors

In this section we compute the reduction in the payout to society when the perturbations are uniformly distributed. To do so, we consider the linear coefficients z for the basis of perturbation matrices to be chosen uniformly from the d -dimensional Euclidean unit ball. Then $\Delta = \sum_{k=1}^d z_k E_k$ is a perturbation matrix.

Proposition 3.18. *Let (Π_0, x, \bar{p}) be a regular financial system. The distribution of changes in payments to society where the perturbations are uniformly distributed on the unit ball is given by*

$$\mathbb{P}(\pi_0^\top \mathcal{D}_\Delta p(\Pi) \leq \alpha) = \frac{1}{2} + \frac{\alpha}{\|(\mathcal{D}_{\vec{E}(\Pi)} p(\Pi))^\top \pi_0\|_2} \frac{\Gamma(1 + \frac{d}{2})}{\sqrt{\pi} \Gamma(\frac{1+d}{2})} {}_2F_1\left(\frac{1}{2}, \frac{1-d}{2}; \frac{3}{2}; \frac{\alpha^2}{\|(\mathcal{D}_{\vec{E}(\Pi)} p(\Pi))^\top \pi_0\|_2^2}\right)$$

for $\alpha \in [-\|(\mathcal{D}_{\vec{E}(\Pi)} p(\Pi))^\top \pi_0\|_2, \|(\mathcal{D}_{\vec{E}(\Pi)} p(\Pi))^\top \pi_0\|_2]$ and 0 for $\alpha \leq -\|(\mathcal{D}_{\vec{E}(\Pi)} p(\Pi))^\top \pi_0\|_2$ and 1 for $\alpha \geq \|(\mathcal{D}_{\vec{E}(\Pi)} p(\Pi))^\top \pi_0\|_2$. In the above equation, ${}_2F_1$ is the standard hypergeometric function. Furthermore, this distribution holds for any choice of basis matrices $\vec{E}(\Pi)$.

Proof. Let z be a uniform random variable on the unit ball in \mathbb{R}^d centered at the origin. Then $\Delta = \sum_{k=1}^d z_k E_k$ is a perturbation matrix. Note that by linearity of the directional derivative, we have

$$\mathcal{D}_\Delta(\pi_0^\top p(\Pi)) = \pi_0^\top \mathcal{D}_{\vec{E}(\Pi)} p(\Pi) z,$$

where $\mathcal{D}_{\vec{E}(\Pi)}p(\Pi) = (\mathcal{D}_{E_1}(p(\Pi)), \dots, \mathcal{D}_{E_d}(p(\Pi)))$. Since z is uniform on the unit ball,

$$\begin{aligned} \mathbb{P}\left(\mathcal{D}_{\Delta}(\pi_0^\top p(\Pi)) \leq \alpha\right) &= \mathbb{P}\left(\pi_0^\top \mathcal{D}_{\vec{E}(\Pi)}p(\Pi)z \leq \alpha\right) \\ &= \frac{\text{vol}(\{z \in \mathbb{R}^d \mid \pi_0^\top \mathcal{D}_{\vec{E}(\Pi)}p(\Pi)z \leq \alpha, z^\top z \leq 1\})}{\text{vol}(\{z \in \mathbb{R}^d \mid z^\top z \leq 1\})} \end{aligned} \quad (12)$$

$$\begin{aligned} &= \frac{\text{vol}\left(\left\{z \in \mathbb{R}^d \mid \left((\mathcal{D}_{\vec{E}(\Pi)}p(\Pi))^\top \pi_0\right)^\top z \leq \alpha, z^\top z \leq 1\right\}\right)}{\text{vol}(\{z \in \mathbb{R}^d \mid z^\top z \leq 1\})} \\ &= \frac{\text{vol}\left(\left\{z \in \mathbb{R}^d \mid \left(\frac{(\mathcal{D}_{\vec{E}(\Pi)}p(\Pi))^\top \pi_0}{\|(\mathcal{D}_{\vec{E}(\Pi)}p(\Pi))^\top \pi_0\|_2}\right)^\top z \leq \frac{\alpha}{\|(\mathcal{D}_{\vec{E}(\Pi)}p(\Pi))^\top \pi_0\|_2}, z^\top z \leq 1\right\}\right)}{\text{vol}(\{z \in \mathbb{R}^d \mid z^\top z \leq 1\})} \\ &= \frac{\text{vol}\left(\left\{z \in \mathbb{R}^d \mid e_1^\top z \leq \frac{\alpha}{\|(\mathcal{D}_{\vec{E}(\Pi)}p(\Pi))^\top \pi_0\|_2}, z^\top z \leq 1\right\}\right)}{\text{vol}(\{z \in \mathbb{R}^d \mid z^\top z \leq 1\})} \end{aligned} \quad (13)$$

$$\begin{aligned} &= \begin{cases} 0 & \text{if } \alpha < -\|(\mathcal{D}_{\vec{E}(\Pi)}p(\Pi))^\top \pi_0\|_2 \\ \frac{1}{2}I_\theta\left(\frac{1+d}{2}, \frac{1}{2}\right) & \text{if } \alpha \in \|(\mathcal{D}_{\vec{E}(\Pi)}p(\Pi))^\top \pi_0\|_2 \times [-1, 0] \\ 1 - \frac{1}{2}I_\theta\left(\frac{1+d}{2}, \frac{1}{2}\right) & \text{if } \alpha \in \|(\mathcal{D}_{\vec{E}(\Pi)}p(\Pi))^\top \pi_0\|_2 \times [0, 1] \\ 1 & \text{if } \alpha > \|(\mathcal{D}_{\vec{E}(\Pi)}p(\Pi))^\top \pi_0\|_2 \end{cases}, \quad \theta = \frac{\|(\mathcal{D}_{\vec{E}(\Pi)}p(\Pi))^\top \pi_0\|_2^2 - \alpha^2}{\|(\mathcal{D}_{\vec{E}(\Pi)}p(\Pi))^\top \pi_0\|_2^2} \\ &= \begin{cases} 0 & \text{if } \alpha < -\|(\mathcal{D}_{\vec{E}(\Pi)}p(\Pi))^\top \pi_0\|_2 \\ \frac{1}{2} + \frac{\alpha}{\|(\mathcal{D}_{\vec{E}(\Pi)}p(\Pi))^\top \pi_0\|_2} \frac{\Gamma(1+\frac{d}{2})}{\sqrt{\pi}\Gamma(\frac{1+d}{2})} {}_2F_1\left(\frac{1}{2}, \frac{1-d}{2}; \frac{3}{2}; \frac{\alpha^2}{\|(\mathcal{D}_{\vec{E}(\Pi)}p(\Pi))^\top \pi_0\|_2^2}\right) & \text{if } \alpha \in \|(\mathcal{D}_{\vec{E}(\Pi)}p(\Pi))^\top \pi_0\|_2 \times [-1, 1] \\ 1 & \text{if } \alpha > \|(\mathcal{D}_{\vec{E}(\Pi)}p(\Pi))^\top \pi_0\|_2 \end{cases} \end{aligned}$$

where $I_\theta(a, b)$ is the regularized incomplete beta function (see, e.g., (DLMF, Chapter 8.17)) and ${}_2F_1$ is the standard hypergeometric function (see, e.g., (DLMF, Chapter 15)). Equation (12) follows from considering the probability by taking the ratio of the volume of the fraction of the unit ball satisfying the probability event to the full volume of the unit ball. Equation (13) follows by symmetry of the unit ball and since $(\mathcal{D}_{\vec{E}(\Pi)}p(\Pi))^\top \pi_0 / \|(\mathcal{D}_{\vec{E}(\Pi)}p(\Pi))^\top \pi_0\|_2$ has unit norm. The penultimate result follows directly from the volume of the spherical cap (see, e.g., (Li 2011, Equation (2))). The final result follows from properties of the regularized incomplete beta function (see, e.g., (DLMF, Chapter 8.17)), i.e.,

$$I_\theta\left(\frac{1+d}{2}, \frac{1}{2}\right) = 1 - 2\sqrt{1-\theta} \frac{\Gamma(1+\frac{d}{2})}{\sqrt{\pi}\Gamma(\frac{1+d}{2})} {}_2F_1\left(\frac{1}{2}, \frac{1-d}{2}; \frac{3}{2}; 1-\theta\right),$$

with $\theta = \frac{\|(\mathcal{D}_{\vec{E}(\Pi)}p(\Pi))^\top \pi_0\|_2^2 - \alpha^2}{\|(\mathcal{D}_{\vec{E}(\Pi)}p(\Pi))^\top \pi_0\|_2^2}$, and noting that the case for α positive and negative can be written under the same equation using the standard hypergeometric function. The independence of this result to the choice of orthonormal basis $\vec{E}(\Pi)$ follows as in Proposition 3.15 as the distribution only depends on the basis $\vec{E}(\Pi)$ through the norm $\|(\mathcal{D}_{\vec{E}(\Pi)}p(\Pi))^\top \pi_0\|_2$. \square

Example 3.19. We return to Example 3.12 and consider perturbations Δ sampled from the uniform distribution. The left and right panels of Figure 7 show the density and the CDF respectively for the relative reduction in society payout under uniformly distributed errors in our stylized four-bank

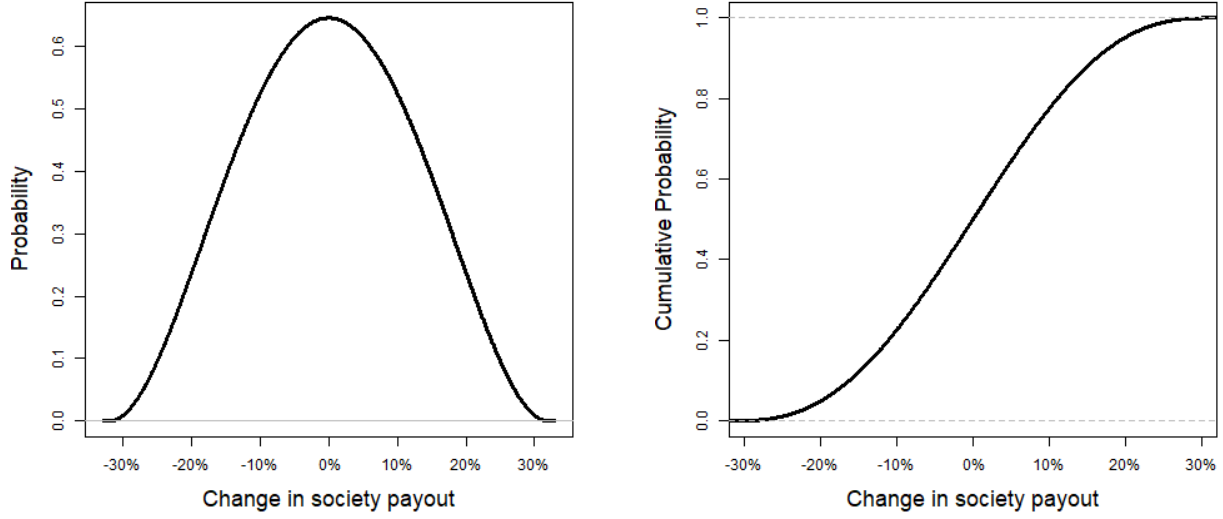


Figure 7: Estimated probability density (left) and CDF (right) of relative reduction in payout to society, $\frac{\pi_0^\top \mathcal{D}_\Delta p(\Pi)}{\pi_0^\top p(\Pi)}$, under uniform perturbations Δ as described in Example 3.19

network. Figure 8(a) shows both the largest reduction and increase in the payout to society as well as various confidence intervals for the change in the payout as a function of the perturbation size, h . As h^* and h^{**} depend on the choice of perturbation matrix Δ , we present the confidence intervals on an extrapolated interval for $h \in [0, 1]$.

3.2.3 Shortfall to society for normally distributed estimation errors

We will now consider the same problem as above under the assumptions that the errors follow a standard normal distribution. As in Section 3.1.3, we note that the magnitude of the perturbations is no longer bounded by 1 in this setting.

Proposition 3.20. *Let (Π_0, x, \bar{p}) be a regular financial system. The distribution of changes to payments to society where the perturbations follow a multivariate standard normal distribution is given by*

$$\mathcal{D}_\Delta(\pi_0^\top p(\Pi)) \sim N\left(0, \|(\mathcal{D}_{\vec{E}(\Pi)} p(\Pi))^\top \pi_0\|_2^2\right).$$

Furthermore, this distribution holds for any choice of basis matrices $\vec{E}(\Pi)$.

Proof. Let z be a d -dimensional standard normal Gaussian random variable. The result follows immediately by linearity and affine transformations of the multivariate Gaussian distribution. The independence of this result to the choice of orthonormal basis $\vec{E}(\Pi)$ follows as in Proposition 3.15 as the distribution only depends on the basis $\vec{E}(\Pi)$ through the norm $\|(\mathcal{D}_{\vec{E}(\Pi)} p(\Pi))^\top \pi_0\|_2$. \square

Example 3.21. *We return once more to Example 3.12 to consider perturbations Δ sampled from the standard normal distribution. Figure 8(b) shows various confidence intervals for the relative change in payout to society under normally distributed errors under $\Delta^4(\Pi)$, as a function of the*

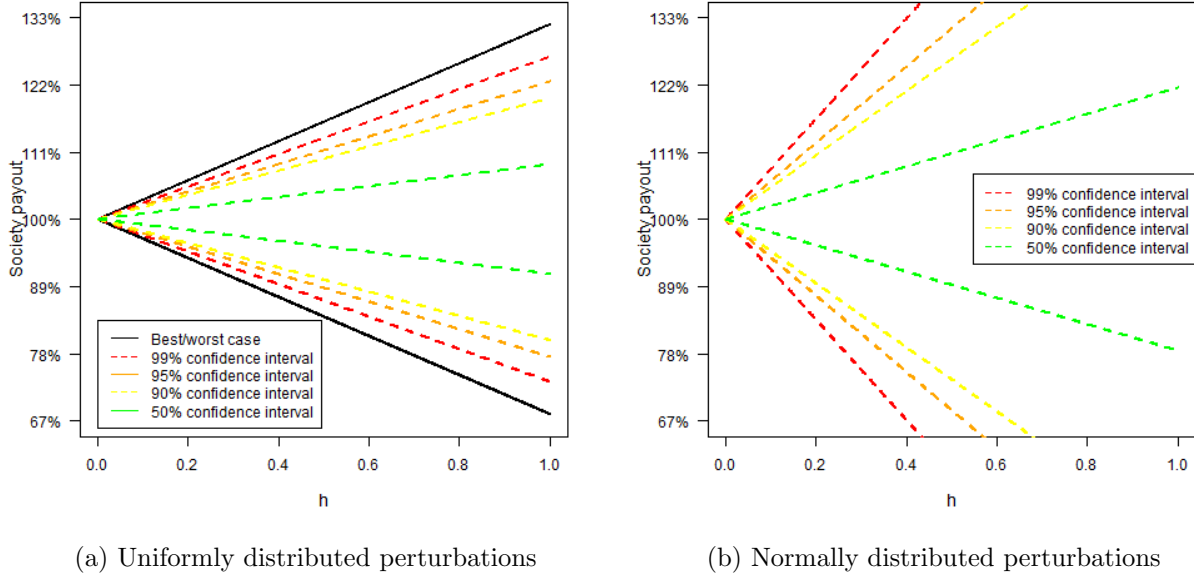


Figure 8: Largest increase and decrease of the payout to society and confidence intervals for the payout as a function of the size h of perturbations in $\Delta_F^n(\Pi)$, respectively $\Delta^n(\Pi)$, where perturbations Δ are sampled uniformly (left) and from the standard Gaussian distribution (right) for the stylized four-bank system with society as described in Example 3.12.

perturbation size, h . As h^* and h^{**} depend on the choice of perturbation matrix Δ we present the confidence intervals on an extrapolated interval for $h \in [0, 1]$.

4 Empirical application: assessing the robustness of systemic risk analyses

In this section, we study the robustness of conclusions that can be drawn from systemic risk studies that use the Eisenberg-Noe algorithm to model direct contagion. We use the same dataset from 2011 of European banks from the European Banking Authority that has been used in previous studies relying on the Eisenberg–Noe framework (Gandy and Veraart (2016), Chen et al. (2016)). As in these papers, given the heuristic approach to the dataset, our exercise should be considered to be an illustration of our results and methodology, rather than a realistic full-fledged empirical analysis.

With respect to the model’s data requirements, the EBA dataset only provides information on the total assets TA_i , the capital c_i and a proxy for interbank exposures, a_i^{IB} . To populate the remaining key variables of the Eisenberg–Noe model, we therefore first assume, as in Chen et al. (2016), that for each bank the interbank liabilities are equal to the interbank assets. Furthermore, we assume that all non-interbank assets are external assets, and the non-interbank liabilities are

liabilities to a society sink-node. Hence,

$$\begin{aligned} l_i^{IB} &:= a_i^{IB}, \\ L_{i0} &:= TA_i - l_i^{IB} - c_i, \\ a_i^0 &:= TA_i - a_i^{IB}. \end{aligned}$$

Consequently, the Eisenberg–Noe model variables are

$$\begin{aligned} \text{Total liabilities: } \bar{p}_i &= L_{i0} + l_i^{IB}, \\ \text{Total external assets: } x_i &= a_i^0. \end{aligned}$$

Note that each bank’s net worth hence exactly corresponds to the book value of equity, or the banks’ capitals: $TA_i - \bar{p}_i = a_i^0 + a_i^{IB} - l_i^{IB} - L_{i0} = c_i$.

The final key ingredient to the model is the (relative) liabilities matrix. This is usually highly confidential data, and is not provided in the EBA data set. In Gandy and Veraart (2016), Gandy and Veraart propose an elegant Bayesian sampling methodology to generate individual interbank liabilities, given information on the total interbank liabilities and total interbank assets of each bank. The authors have developed an *R*-package called “systemicrisk” that implements a Gibbs sampler to generate samples from this conditional distribution. As our analysis requires an initial liability matrix, we use the European Banking Authority (EBA) data as input to their code in order to generate such a liability matrix. As suggested by (Gandy and Veraart 2016, Section 5.3), we perturb the interbank liabilities l_i^{IB} slightly (such that they are not exactly equal to the interbank assets, while keeping the total sums equal) to fulfill the condition that L be connected along rows and columns. We then run their algorithm, with parameters $p = 0.5$, $\text{thinning} = 10^4$, $n_{\text{burn-in}} = 10^9$, $\lambda = \frac{pn(n-1)}{\sum_{i=1}^N a_i^{IB}} \approx 1.217810^{-3}$, to create one realisation of a 87×87 network of banks from the data. (We needed to exclude banks DE029, LU45 and SI058 because the mapping of the data to the model as described above created violations of the conditions for the algorithm and resulted in an error message.)

For simplicity and to consider an extreme event that would trigger a systemic crisis in the European banking system, we analyze what might have happened if Greece had defaulted on its debt and exited the Eurozone. We study this shock by decreasing the external assets of each bank by its individual Greek exposures, i.e. setting Greek bond values to zero. In our sensitivity analysis we resample the underlying liabilities matrix from the Gandy & Veraart algorithm Gandy and Veraart (2016) 1000 times.

In each of our 1000 simulated networks considered there were 9 specific institutions that default on their debts in the Eisenberg–Noe framework; in only 3 simulated networks (0.3% of all simulations) there were between 1 and 3 additional banks that fail. As such, the traditional analysis of sensitivity of the Eisenberg–Noe framework would conclude that this contagion model is robust to errors in the relative liabilities matrix. This is consistent with the work of, e.g., Glasserman and Young (2015).

However, we now consider the maximal deviation sensitivity analysis under $\Delta_F^n(\Pi)$ in both the estimation errors and the payments to society in each of our 1000 simulated networks. Figure 9(a) depicts the empirical density of the maximal deviation estimation errors $\frac{\|D_{\Delta}p(\Pi)\|_2^2}{\|p(\Pi)\|_2^2}$ for $\Delta \in \Delta_F^n(\Pi)$. Figure 9(b) depicts the empirical density of maximal fractional shortfalls to society $\frac{D_{\Delta}e_0(\Pi)}{e_0(\Pi)}$. We also depict the upper bound of the worst case perturbation errors for each of the 1000 simulated networks.

Notably in Figure 9(a) we see that the shape of the network, calibrated to the same EBA data set, can vastly change the impact that the worst case estimation error has under perturbations

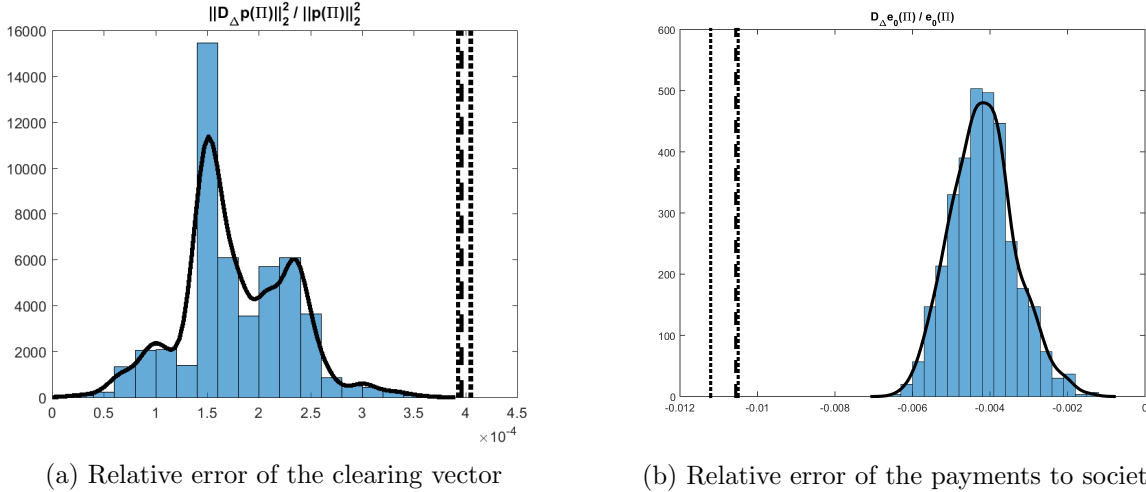


Figure 9: Empirical densities of the relative errors in the Eisenberg–Noe framework as a function of random networks calibrated to the same EBA dataset. The dotted vertical lines indicate the maximal and minimal empirical values of the upper bound of the worst case and the dashed line indicates the median upper bound

in $\Delta_F^n(\Pi)$. In this plot of the empirical densities, we see the range of normalized worst case first order estimation errors range from 0 to nearly 4×10^{-4} . That is a 0 to 2% normed deviation of the clearing payments (while the value of $\|p(\Pi)\|_2$ itself has only minor variations: a total range of under 27 M EUR compared to its norm of near 5 trillion EUR for the different simulated networks Π). The upper bound on these perturbation errors (for the norm rather than norm squared) is approximately 2%, and as can be seen in Figure 9(a), the range of obtained upper bounds is very small. This indicates that such a bound is rather insensitive to the initial relative liability matrix Π . Therefore any such computed upper bound is of value to a regulator, even if the initial estimate of the relative liabilities Π is incorrect.

When we consider instead Figure 9(b) we see that the density is more bell shaped, again with a large variation from the least change (roughly -0.001) to the most change (roughly -0.007) in the normalized impact to society; this proves as with Figure 9(a) that the underlying network can provide large differences in the apparent stability of a simulation to validation. While these values may appear small, the 10^{-3} arises from normalising the deviation of the clearing vector with the value of the societal node but still amounts to a variation on the order of 23.2 - 162.4 billion EUR. Thus this sensitivity is as if entire banks' assets vanished from the wealth of society. The upper bound of these perturbation errors is approximately twice as high as the obtained maximal deviations computed under $\Delta_F^n(\Pi)$. Notably, the median upper bound of the worst case error is nearly equal to the minimum possible value, though with a skinny tail reaching off to greater errors.

5 Conclusion

In this paper we analyse the sensitivity of the Eisenberg–Noe clearing payments Eisenberg and Noe (2001) to misspecification or estimation errors in the relative liabilities matrix. We accomplish this by determining the directional derivative of the clearing payments with respect to the relative liabilities matrix. We extend this result to consider the full Taylor expansion of the fixed points to

determine the clearing payments as a closed-form perturbation of an initial solution.

We further study worst case and probabilistic interpretations of our perturbation analysis. In particular, our results provide an upper bound on the largest shift for the clearing vector and a lower bound for the shortfall to society for general network perturbations. In a numerical case study of the European banking system, we demonstrate that, even when the set of defaulting firms remains constant, the clearing payments and wealth of society can be greatly impacted. This is true even in the case that the existence and non-existence of links is pre-specified. When the existence and non-existence of links is unknown, then the upper bound of the errors can be utilized which generally provides errors that are significantly less sensitive to the initial estimate of the relative liabilities and roughly twice as large as the errors under pre-specification of links.

A An orthonormal basis for perturbation matrices

We construct here an orthonormal basis for the matrices in $\Delta^n(\Pi)$. To fix ideas, consider the case $n = 4$, where the general form of a matrix $\Delta \in \Delta^4(\Pi_C)$ for a fully connected network Π_C can be written as

$$\Delta^4(\Pi_C) = \left\{ \text{diag}(\bar{p})^{-1} \begin{pmatrix} 0 & z_1 & z_2 & -z_1 - z_2 \\ z_3 & 0 & z_4 & -z_3 - z_4 \\ z_5 & -\sum_{k=1}^5 z_k & 0 & \sum_{k=1}^4 z_k \\ -z_3 - z_5 & \sum_{k=2}^5 z_k & -z_2 - z_4 & 0 \end{pmatrix} \mid z \in \mathbb{R}^5 \right\},$$

from which it is clear that there are 5 degrees of freedom. It is easy to see that in general one has $d = n^2 - 3n + 1$ degrees of freedom. In the case $n = 4$, two such basis elements \hat{E}_1 and \hat{E}_2 are given by

$$\hat{E}_1 = \begin{pmatrix} 0 & \frac{1}{\bar{p}_1} & 0 & \frac{-1}{\bar{p}_1} \\ 0 & 0 & 0 & 0 \\ 0 & \frac{-1}{\bar{p}_3} & 0 & \frac{1}{\bar{p}_3} \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad \hat{E}_2 = \begin{pmatrix} 0 & 0 & \frac{1}{\bar{p}_1} & \frac{-1}{\bar{p}_1} \\ 0 & 0 & 0 & 0 \\ 0 & \frac{-1}{\bar{p}_3} & 0 & \frac{1}{\bar{p}_3} \\ 0 & \frac{1}{\bar{p}_4} & \frac{-1}{\bar{p}_4} & 0 \end{pmatrix}.$$

In general we note that $\Delta^n(\Pi)$ is a closed, convex polyhedral set; we will take advantage of this fact in order to generate a general method for constructing basis matrices for $\Delta^n(\Pi)$, as follows:

1. Define

$$\vec{\Delta}^n(\Pi) := \left\{ \delta \in \mathbb{R}^{n^2} \mid \begin{aligned} &\delta_{i+n(i-1)} = 0, \quad \sum_{j=1}^n \delta_{i+n(j-1)} = 0, \\ &\sum_{j=1}^n \bar{p}_j \delta_{n(i-1)+j} = 0, \quad \mathbf{1}_{\{\pi_{ij}=0\}} \delta_{i+n(j-1)} = 0 \quad \forall i, j \end{aligned} \right\}$$

to be a vectorised version of $\Delta^n(\Pi)$.

2. Construct a matrix $A(\Pi) \in \mathbb{R}^{(n^2+2n) \times n^2}$ so that $\vec{\Delta}^n(\Pi) = \{\delta \in \mathbb{R}^{n^2} \mid A(\Pi)\delta = 0\}$. Note that the total degrees of freedom for $\vec{\Delta}^n(\Pi)$ (and therefore also for $\Delta^n(\Pi)$) is given by the rank of the matrix $A(\Pi)$. We include enough rows in the matrix $A(\Pi)$ in order to ensure that the n row sums and n (weighted) column sums are 0 and that components of δ are equal to zero based on $\pi_{ij} = 0$.
3. An orthonormal basis of $\vec{\Delta}^n(\Pi)$ can be found by generating the orthonormal basis $\{e_1, \dots, e_d\}$ of the null space of $A(\Pi)$.

4. Finally our basis matrices $\{E_1, \dots, E_d\}$ can be generated by reshaping the basis of the null space of $A(\Pi)$ by setting $E_{k;i,j} := e_{k;i+n(j-1)}$ for any $k = 1, \dots, d$ and $i, j \in \mathcal{N}$.

Definition A.1. *The set*

$$\vec{E}^n(\Pi) := \{E_1, \dots, E_d\}$$

is an orthonormal basis of perturbation matrices for the relative liability matrix Π . Additionally, the vector

$$\mathcal{D}_{\vec{E}(\Pi)}p(\Pi) := (\mathcal{D}_{E_1}p(\Pi), \dots, \mathcal{D}_{E_d}p(\Pi)) \in \mathbb{R}^{n \times d}$$

is a vector of basis directional derivatives for the relative liability matrix Π .

We define two matrices to be orthogonal when their vectorised forms are orthogonal in \mathbb{R}^{n^2} , and note that, by construction, any matrix in the basis of perturbation matrices $\vec{E}^n(\Pi)$ has unit Frobenius norm.

Proposition A.2. *Let $\Pi \in \Pi^n$. Then the set of eigenvalues of $(\mathcal{D}_{\vec{E}(\Pi)}p(\Pi))^\top \mathcal{D}_{\vec{E}(\Pi)}p(\Pi)$ is the same for any choice of orthonormal basis of perturbation matrices $\vec{E}(\Pi)$. Additionally, if $z(\lambda, \vec{E}(\Pi)) \in \mathbb{R}^d$ is the eigenvector corresponding to eigenvalue λ and basis $\vec{E}(\Pi)$, then $\sum_{k=1}^d z_k(\lambda, \vec{E}(\Pi)) E_k$ is independent of the choice of basis.*

Proof. Let E be the vectorised version of $\vec{E}(\Pi)$ and let $F \neq E$ be a different orthonormal basis. By linearity of the directional derivative (see Theorem 2.6) we can immediately state that $(\mathcal{D}_{\vec{E}(\Pi)}p(\Pi))^\top \mathcal{D}_{\vec{E}(\Pi)}p(\Pi) = E^\top C E$ for some matrix $C \in \mathbb{R}^{n^2 \times n^2}$. Let (λ, v) be an eigenvalue and eigenvector pair for the operator $E^\top C E$ and let $z \in \mathbb{R}^d$ such that $Ev = Fz$. We will show that (λ, z) is an eigenvalue and eigenvector pair for $F^\top C F$ and thus the proof is complete:

$$\lambda z = \lambda F^\top F z = \lambda F^\top E v = F^\top E(\lambda v) = F^\top E E^\top C E v = F^\top C F z.$$

The last equality follows from the fact that $EE^\top = FF^\top$ is the unique projection matrix onto $\vec{\Delta}^n(\Pi)$. \square

Proposition A.3. *Let $\Pi \in \Pi^n$. Then $\|(\mathcal{D}_{\vec{E}(\Pi)}p(\Pi))^\top c\|_2$ is independent of the choice of orthonormal basis of perturbation matrices $\vec{E}(\Pi)$ and for any fixed vector $c \in \mathbb{R}^n$.*

Proof. Let E and F be two distinct basis matrices for the vectorized perturbation space $\vec{\Pi}^n(\Pi)$ as in the proof of Proposition A.2. By linearity of the directional derivative (see Theorem 2.6) we can immediately state that $(\mathcal{D}_{\vec{E}(\Pi)}p(\Pi))^\top c = E^\top \tilde{c}$ for some vector $\tilde{c} \in \mathbb{R}^{n^2}$. Immediately we can see that $\|E^\top \tilde{c}\|_2 = \|F^\top \tilde{c}\|_2$ since $EE^\top = FF^\top$ is the unique projection matrix onto $\vec{\Delta}^n(\Pi)$. \square

B Acknowledgements

The collaboration leading to this article was initiated at the AMS Mathematics Research Community 2015 in Financial Mathematics, supported by the National Science Foundation, Division of Mathematical Sciences, under Grant No. 1321794.

References

- Amini H, Filipović D, Minca A (2016a) Systemic risk with central counterparty clearing. Swiss Finance Institute Research Paper No. 13-34, Swiss Finance Institute, URL https://papers.ssrn.com/sol3/papers.cfm?abstract_id=2275376.
- Amini H, Filipović D, Minca A (2016b) Uniqueness of equilibrium in a payment system with liquidation costs. *Operations Research Letters* 44(1):1–5, URL <https://doi.org/10.1016/j.orl.2015.10.005>.
- Anand K, Bédard-Pagé G, Tractlet V (2014) Stress testing the Canadian banking system: A system-wide approach. *Bank of Canada Financial Stability Review* URL <http://www.bankofcanada.ca/wp-content/uploads/2014/06/fsr-june2014-anand.pdf>.
- Anand K, Craig B, von Peter G (2015) Filling in the blanks: network structure and interbank contagion. *Quantitative Finance* 15(4):625–636, URL <https://doi.org/10.1080/14697688.2014.968195>.
- Anand K, van Lelyveld I, Ádám Banai, Friedrich S, Garratt R, Hałaj G, Figue J, Hansen I, Jaramillo SM, Lee H, Molina-Borboa JL, Nobili S, Rajan S, Salakhova D, Silva TC, Silvestri L, de Souza SRS (2017) The missing links: A global study on uncovering financial network structures from partial data. *Journal of Financial Stability* ISSN 1572-3089, URL <http://dx.doi.org/10.1016/j.jfs.2017.05.012>.
- Anderson RW (2016) Stress testing and macroprudential regulation: A transatlantic assessment. *Systemic Risk Center, Financial Markets Group & CEPR Press* URL http://voxeu.org/sites/default/files/Stress_testing_eBook.pdf.
- Bardoscia M, Barucca P, Brinley Codd A, Hill J (2017) The decline of solvency contagion risk. *Bank of England Staff Working Paper* 662, URL <http://www.bankofengland.co.uk/research/Documents/workingpapers/2017/swp662.pdf>.
- Basel Committee on Banking Supervision (2015) Making supervisory stress tests more macroprudential: Considering liquidity and solvency interactions and systemic risk. BIS Working Paper 29, URL <http://www.bis.org/bcbs/publ/wp29.pdf>.
- Bech ML, Atalay E (2010) The topology of the federal funds market. *Physica A: Statistical Mechanics and its Applications* 389(22):5223 – 5246, URL <https://doi.org/10.1016/j.physa.2010.05.058>.
- Boss M, Elsinger H, Summer M, Thurner S (2004) Network topology of the interbank market. *Quantitative Finance* 4(6):677–684, URL <https://doi.org/10.1080/14697680400020325>.
- Capponi A, Chen PC (2015) Systemic risk mitigation in financial networks. *Journal of Economic Dynamics & Control* 58:152–166, URL <https://doi.org/10.1016/j.jedc.2015.06.008>.
- Capponi A, Chen PC, Yao DD (2016) Liability concentration and systemic losses in financial networks. *Operations Research* 64(5):1121–1134, URL <https://doi.org/10.1287/opre.2015.1402>.
- Chen N, Liu X, Yao DD (2016) An optimization view of financial systemic risk modeling: Network effect and market liquidity effect. *Operations Research* 64(5):1089–1108, URL <https://doi.org/10.1287/opre.2016.1497>.
- Cifuentes R, Ferrucci G, Shin HS (2005) Liquidity risk and contagion. *Journal of the European Economic Association* 3(2-3):556–566, URL <https://doi.org/10.1162/jeea.2005.3.2-3.556>.
- Cont R, Moussa A, Santos EB (2013) Network structure and systemic risk in banking systems. *Handbook on Systemic Risk*, 327 – 368 (Cambridge University Press), URL <https://doi.org/10.1017/CB09781139151184.018>.
- Cont R, Schaanning E (2017) Fire sales, indirect contagion and systemic stress testing. Norges Bank Working Paper 02/2017, URL <https://ssrn.com/abstract=2541114>.
- Di Gangi D, Lillo F, Pirino D (2015) Assessing systemic risk due to fire sales spillover through maximum entropy network reconstruction. *Working Paper* URL http://papers.ssrn.com/sol3/papers.cfm?abstract_id=2639178.
- DLMF (2017) *NIST Digital Library of Mathematical Functions*. <http://dlmf.nist.gov/>, Release 1.0.15 of 2017-06-01, URL <http://dlmf.nist.gov/>, f. W. J. Olver, A. B. Olde Daalhuis, D. W. Lozier, B. I. Schneider, R. F. Boisvert, C. W. Clark, B. R. Miller and B. V. Saunders, eds.
- Eisenberg L, Noe TH (2001) Systemic risk in financial systems. *Management Science* 47(2):236–249, URL <https://doi.org/10.1287/mnsc.47.2.236.9835>.

- Elliott M, Golub B, Jackson MO (2014) Financial networks and contagion. *American Economic Review* 104(10):3115–53, URL <https://doi.org/10.1257/aer.104.10.3115>.
- Elsinger H (2009) Financial networks, cross holdings, and limited liability. Österreichische Nationalbank (Austrian Central Bank) Working Paper 156, URL https://www.oenb.at/dam/jcr:b2f1721c-7af9-4646-9568-d93faec67bac/wp156_tcm16-138413.pdf.
- Elsinger H, Lehar A, Summer M (2013) Network models and systemic risk assessment. *Handbook on Systemic Risk*, 287 – 305, URL <https://doi.org/10.1017/CB09781139151184.016>.
- Feinstein Z (2017a) Financial contagion and asset liquidation strategies. *Operations Research Letters* 45(2):109–114, URL <https://doi.org/10.1016/j.orl.2017.01.004>.
- Feinstein Z (2017b) Obligations with physical delivery in a multi-layered financial network. *Working Paper* URL <https://arxiv.org/abs/1702.07936>.
- Feinstein Z, El-Masri F (2017) The effects of leverage requirements and fire sales on financial contagion via asset liquidation strategies in financial networks. *Statistics & Risk Modeling* URL <http://dx.doi.org/10.1515/strm-2015-0030>.
- Feinstein Z, Rudloff B, Weber S (2017) Measures of systemic risk. *SIAM Journal on Financial Mathematics* to appear, URL <https://arxiv.org/pdf/1502.07961v5.pdf>.
- Financial Stability Board, International Monetary Fund (2015) The financial crisis and information gaps - sixth progress report on the implementation of the G-20 data gaps initiative URL <http://www.imf.org/external/np/g20/pdf/2015/6thprogressrep.pdf>.
- Gai P, Haldane A, Kapadia S (2011) Complexity, concentration and contagion. *Journal of Monetary Economics* 58(5):453–470, URL <https://doi.org/10.1016/j.jmoneco.2011.05.005>.
- Gai P, Kapadia S (2010) Contagion in financial networks. Bank of England Working Papers 383, Bank of England, URL https://papers.ssrn.com/sol3/papers.cfm?abstract_id=1577043.
- Gandy A, Veraart LAM (2016) A Bayesian methodology for systemic risk assessment in financial networks. *Management Science* URL <https://doi.org/10.1287/mnsc.2016.2546>.
- Gandy A, Veraart LAM (2017) Adjustable network reconstruction with applications to CDS exposures. *Working Paper* URL <https://ssrn.com/abstract=2895754>.
- Gentle JE (2007) *Matrix Algebra* (Springer-Verlag New York), ISBN 978-0-387-70872-0.
- Glasserman P, Young HP (2015) How likely is contagion in financial networks? *Journal of Banking and Finance* 50:383–399, URL <https://doi.org/10.1016/j.jbankfin.2014.02.006>.
- Halaj G, Kok C (2013) Assessing interbank contagion using simulated networks. *Computational Management Science* 10(2-3):157–186, URL <https://doi.org/10.1007/s10287-013-0168-4>.
- Halaj G, Kok C (2015) Modelling the emergence of the interbank networks. *Quantitative Finance* 15(4):653–671, URL <http://dx.doi.org/10.1080/14697688.2014.968357>.
- Hüser AC (2015) Too interconnected to fail: A survey of the interbank networks literature. *Journal of Network Theory in Finance* 1(3):1–50, URL <https://doi.org/10.21314/JNTF.2015.001>.
- Kusnetsov M, Veraart LAM (2016) Interbank clearing in financial networks with multiple maturities. *Working Paper* URL <https://ssrn.com/abstract=2854733>.
- Langfield S, Liu Z, Ota T (2014) Mapping the UK interbank system. *Journal of Banking and Finance* 45:288–303, URL <http://dx.doi.org/10.1016/j.jbankfin.2014.03.031>.
- Li S (2011) Concise formulas for the area and volume of a hyperspherical cap. *Asian Journal of Mathematics and Statistics* 4(1):66–70, URL <http://dx.doi.org/10.3923/ajms.2011.66.70>.
- Liu M, Staum J (2010) Sensitivity analysis of the Eisenberg and Noe model of contagion. *Operations Research Letters* 38(5):489–491, URL <https://doi.org/10.1016/j.orl.2010.07.007>.
- Mathai AM (1982) Storage capacity of a dam with gamma type inputs. *Annals of the Institute of Statistical Mathematics* 34(1):591–597, URL <https://doi.org/10.1007/BF02481056>.
- Meyer CD (2000) *Matrix Analysis and Applied Linear Algebra* (SIAM), ISBN 978-0-898714-54-8.
- Mistrulli PE (2011) Assessing financial contagion in the interbank market: Maximum entropy versus observed interbank lending patterns. *Journal of Banking and Finance* 35(5):1114 – 1127, URL <https://doi.org/10.1016/j.jbankfin.2010.09.018>.

- Nier E, Yang J, Yorulmazer T, Alentorn A (2007) Network models and financial stability. *Journal of Economic Dynamics and Control* 31(6):2033–2060, URL <https://doi.org/10.1016/j.jedc.2007.01.014>.
- Ren X, Yuan GX, Jiang L (2014) The framework of systemic risk related to contagion, recovery rate and capital requirement in an interbank network. *Journal of Financial Engineering* 1(1):1450004, URL <https://doi.org/10.1142/S2345768614500044>.
- Rogers LCG, Veraart LAM (2013) Failure and rescue in an interbank network. *Management Science* 59(4):882–898, URL <https://doi.org/10.1287/mnsc.1120.1569>.
- Soramäki K, Bech ML, Arnold J, Glass RJ, Beyeler WE (2007) The topology of interbank payment flows. *Physica A: Statistical Mechanics and its Applications* 379(1):317–333, URL <https://doi.org/10.1016/j.physa.2006.11.093>.
- Upper C (2011) Simulation methods to assess the danger of contagion in interbank markets. *Journal of Financial Stability* 7(3):111–125, URL <https://doi.org/10.1016/j.jfs.2010.12.001>.
- Weber S, Weske K (2017) The joint impact of bankruptcy costs, fire sales and cross-holdings on systemic risk in financial networks. *Probability, Uncertainty and Quantitative Risk* 2(1):9, URL <https://doi.org/10.1186/s41546-017-0020-9>.