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Working Paper

Mixed Frequency Structural VARs

Working Paper, No. 2014/01

Provided in Cooperation with:

Norges Bank, Oslo

Suggested Citation: Foroni, Claudia; Marcellino, Massimiliano (2014) : Mixed Frequency Structural VARs, Working Paper, No. 2014/01, ISBN 978-82-7553-784-1, Norges Bank, Oslo, <https://hdl.handle.net/11250/2495985>

This Version is available at:

<https://hdl.handle.net/10419/210050>

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2014 | 01

Working Paper

Norges Bank Research

Mixed frequency structural VARs

Claudia Foroni and Massimiliano Marcellino

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Mixed Frequency Structural VARs*

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10 January 2014

Abstract

A mismatch between the time scale of a structural VAR (SVAR) model and that of the time series data used for its estimation can have serious consequences for identification, estimation and interpretation of the impulse response functions. However, the use of mixed frequency data, combined with a proper estimation approach, can alleviate the temporal aggregation bias, mitigate the identification issues, and yield more reliable responses to shocks. The problems and possible remedy are illustrated analytically and with both simulated and actual data.

JEL Classification Codes: C32, C43, E32

Keywords: Structural VAR, temporal aggregation, mixed frequency data, identification, estimation, impulse response function

*This Working Paper should not be reported as representing the views of Norges Bank. The views expressed are those of the authors and do not necessarily reflect those of Norges Bank. We would like to thank Pierre Guerin, Peter Hansen, Andrew Harvey, Helmut Lütkepohl, Frank Schorfheide for useful comments on a previous draft.

1 Introduction

Vector Autoregressive (VAR) models have acquired a prominent role in the toolbox for macroeconomic analysis, starting with the seminal work of Sims (1980). They can be used for forecasting and to characterize the comovements in macroeconomic series, but also to identify the key structural shocks driving the economy and their propagation mechanism, see e.g. Kilian (2011) for a recent survey on structural VARs.

A recent strand of research has focused on the use of mixed frequency data in standard VARs, in order to exploit all the available information and to avoid the problems associated with temporal aggregation, see e.g. Foroni, Ghysels and Marcellino (2013) for a survey on mixed frequency VARs (MF-VARs) and Marcellino (1999) for the effects of temporal aggregation in VAR models. Broadly speaking there are two main approaches to MF-VAR models. Either the model is based on high frequency data and some variables are treated as temporally aggregated, or the VAR model is directly based on mixed frequency data. The former approach, started by Zadrozny (1988), is more common and fits very well in a linear Gaussian state space framework, even though it inevitably involves latent shocks. Other examples of this approach include Mariano and Murasawa (2010), Chiu et al. (2011), Kuzin, Marcellino and Schumacher (2011, 2013), Schorfheide and Song (2011), Foroni and Marcellino (2013a), among others.

The second approach to MF-VAR is due to Ghysels (2011), who introduces a class of models driven by observable shocks, similar to the standard VAR approach. He decomposes each high frequency variable into a set of low frequency variables (for example, a monthly variable is decomposed into three quarterly variables) and jointly models the resulting variables with those originally available in low frequency. In this sense, the model is in the line of the U-MIDAS approach of Foroni, Marcellino and Schumacher (2013). The advantage of the setup in Ghysels (2011) is that standard VAR tools - that is OLS based estimation, Choleski factorizations, structural impulse response analysis, variance decompositions, etc. - can be readily applied to a mixed frequency setting. The cost of the approach is that there is a proliferation of parameters, which can be partly addressed by using Bayesian rather than classical estimation methods, but also that it can be difficult to recover the original monthly structural shocks from those in the stacked VAR.

In this paper we follow the first, more common, approach to MF-VAR modelling and show how it can be applied to specify and estimate mixed frequency structural VARs (MF-SVARs), which can then be used to conduct structural analysis. We proceed as in Foroni and Marcellino (2013b), who conducted a similar analysis for mixed frequency Dynamic Stochastic General Equilibrium (DSGE) models.

First, we discuss an analytical example that illustrates the danger of SVAR analyses based on temporally aggregated data, the common practice in the literature, and how mixed frequency data can alleviate the problems (Section 2). Then, we discuss estimation of the MF-SVAR (Section 3). Next, we provide Monte Carlo evidence on the empirical relevance of the temporal aggregation issues and on the good finite sample performance of the proposed estimation procedure for mixed frequency SVARs (Section 4). Finally, we present an empirical application based on US data to illustrate the theoretical results, analyze the importance of the temporal aggregation issues in practice, and show the usefulness of our proposed MF-SVAR (Section 5).

2 An analytical example in the SVAR context

To contextualize the analysis, we investigate the propagation mechanism of an exogenous monetary policy shock, a strongly debated issue in macroeconomics. A vast strand of literature has attempted to explain the effects of monetary policy using VAR models. Sims (1986), Strongin (1995), Christiano, Eichenbaum and Evans (1996), Bernanke and Mihov (1998), Sims and Zha (2006), among others have analyzed the US monetary policy using VAR models at quarterly frequency, following different identification strategies¹.

We start with a simple example, to provide analytical evidence. We consider a SVAR(1) model that includes three variables: the GDP growth, y_t^* , the inflation rate, π_t , and the policy rate, r_t . While two of these variables are available at monthly frequency, one of them, y_t^* , is released only quarterly. If the time frequency in which the agents take decisions is monthly, the best option for the econometrician would be to estimate the model at monthly frequency. Therefore, the ideal model for estimation is the following:

$$\begin{bmatrix} y_t^* \\ \pi_t \\ r_t \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} y_{t-1}^* \\ \pi_{t-1} \\ r_{t-1} \end{bmatrix} + \begin{bmatrix} u_{yt} \\ u_{\pi t} \\ u_{rt} \end{bmatrix}, \quad (1)$$

where the covariance matrix of $u_t = \begin{bmatrix} u_{yt} & u_{\pi t} & u_{rt} \end{bmatrix}'$ is a 3×3 non diagonal matrix Σ_u , and y_t^* is the (unobservable) monthly GDP growth.

To recover the unobserved structural monetary policy shock ε_{rt} , a researcher needs to impose some restrictions, and a very popular choice is to define B as a lower triangular matrix with positive elements on the main diagonal, based on the Choleski decomposition of the covariance matrix of u_t , $\Sigma_u = BB'$.

We therefore rewrite eq. (1) as:

$$\begin{bmatrix} y_t^* \\ \pi_t \\ r_t \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} y_{t-1}^* \\ \pi_{t-1} \\ r_{t-1} \end{bmatrix} + \begin{bmatrix} b_{11} & 0 & 0 \\ b_{21} & b_{22} & 0 \\ b_{31} & b_{32} & b_{33} \end{bmatrix} \begin{bmatrix} \varepsilon_{yt} \\ \varepsilon_{\pi t} \\ \varepsilon_{rt} \end{bmatrix}, \quad (2)$$

where

$$\varepsilon_t \sim (\mathbf{0}, I_3), \quad (3)$$

where $\varepsilon_t = (\varepsilon_{yt}, \varepsilon_{\pi t}, \varepsilon_{rt})'$.

The choice of a Choleski decomposition solves the identification problem by relying on the assumption that the recursive structure is justified by the delayed reaction of real and nominal variables to the monetary policy shock. There are of course other restrictions that solve the identification problem. The triangular form is just an example, and in practice it is the most common case (see, e.g., Eichenbaum & Evans (1995), Christiano, Eichenbaum & Evans (1996)). Therefore, for the purposes of our analysis we choose the Choleski identification scheme, but note

¹For an extensive review of the different strategies proposed in the literature to identify the effects of an exogenous shock to monetary policy see Christiano, Eichenbaum and Evans (1999). Other empirical studies are conducted at monthly level, see, among others, Sims (1992) for evidence on the effects of monetary policy in different countries, and Leeper, Sims and Zha (1996) for the US economy.

that the same analysis we conduct in the next sections can be repeated with a different ordering of variables or alternative identification approaches, without any major qualitative changes in our conclusions.

2.1 The common approach: aggregation at quarterly frequency

The VAR in eq. (2) cannot be directly estimated because one of the variables, y_t^* , is non observable. The common adopted solution is to estimate the model at a frequency at which all the variables are available, at a quarterly frequency in our example.

As a first step, we need to derive the correct representation of the data generating process at quarterly frequency. To do that, we need to aggregate the model described by eq. (2) at quarterly frequency. Time aggregation can be essentially seen as a two-step filter. In the first step, the variable is aggregated following an aggregation scheme $\omega(L)$, which can be seen as a one-sided filter. In the second step, the aggregated series $\omega(L)y_t$ is skip-sampled, so that we observe the variable only every k periods.

Let us rewrite the VAR as:

$$Y_t^* = AY_{t-1}^* + B\varepsilon_t, \quad \varepsilon_t \sim (\mathbf{0}, I_3), \quad (4)$$

where $Y_t^* = (y_t^*, \pi_t, r_t)'$, or, using the lag operator L :

$$(I - AL)Y_t^* = B\varepsilon_t, \quad \varepsilon_t \sim (\mathbf{0}, I_3). \quad (5)$$

For the sake of simplicity, we choose point-in-time sampling for all variables, i.e. $\omega(L) = 1$, meaning that the aggregate measure of GDP growth we observe corresponds to the monthly y_t^* every third period.

Then, we introduce a polynomial $B(L)$, such that $B(L)(I - AL)$ contains only powers of L^3 . In our case, we choose:

$$B(L) = (I + AL + A^2L^2). \quad (6)$$

Multiplying both sides of eq. (5) by the polynomial in (6) and $\omega(L)$, we obtain:

$$(I - A^3L^3)Y_t^* = (I + AL + A^2L^2)B\varepsilon_t, \quad (7)$$

or, equivalently,

$$Y_\tau = A^3Y_{\tau-1} + \xi_\tau, \quad (8)$$

where $\tau = 3t$ indicates quarters, with

$$\xi_\tau \sim (0, \Omega), \quad \Omega = BB' + ABB'A' + A^2BB'A'^2. \quad (9)$$

Since $\tau = 3, 6, \dots, 3t, \dots$, all the variables in Y_τ are observable. Hence, the econometrician estimates the aggregated process in eq. (8) as the following quarterly model:

$$y_\tau = Cy_{\tau-1} + \xi_\tau \quad (10)$$

with

$$\xi_\tau \sim (0, \Omega). \quad (11)$$

With this simple aggregation scheme, the true and estimated aggregated models coincide and are still a $VAR(1)$, whose coefficients and error variance-covariance matrix are functions of the parameters driving the monthly process. At this point the econometrician faces an identification issue: using quarterly data she obtains \hat{C} and $\hat{\Omega}$, but from these matrices she cannot uniquely identify A and B . In fact, abstracting from small sample estimation issues, knowledge of $C = A^3$, which in our example is a 3×3 matrix, does not allow to identify uniquely the parameters of A (since the multiplication, when operated across matrices, creates non-linear combinations of the original parameters, additional details are available in Appendix A.1).

The lack of identification of the matrix A translates into the lack of identification of the matrix B , since B should be recovered from Ω , which in turn depends on A and, in addition, it is a quadratic form in A . Therefore, in general, we cannot recover the underlying monthly parameters if we estimate the quarterly model.

In this specific example, starting with a $VAR(1)$ at monthly level we still have a $VAR(1)$ at quarterly level for the skip sampled variables. However, this is not necessarily true either with higher order VARs or with different aggregation schemes (see Marcellino (1999) for more details). It is likely that in the aggregated process there is an MA component, which is generally disregarded during the estimation of the aggregated process, or at best approximated with additional AR lags. Hence, there can also be a mismatch between the true and estimated aggregate models, which exacerbates the mentioned identification problems.

2.2 Exploiting data at different frequencies

Potentially useful information is discarded in aggregating series that are available at monthly frequency. Actually, we now show that, for this simple example, using the series at the frequency they are available, that is one series at quarterly frequency and the other two at the monthly frequency, allows us to recover all the parameters of the original monthly model, solving the identification issue.

For convenience, let us briefly recall the monthly SVAR described by eq. (2) and (3):

$$\begin{bmatrix} y_t^* \\ \pi_t \\ r_t \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} y_{t-1}^* \\ \pi_{t-1} \\ r_{t-1} \end{bmatrix} + \begin{bmatrix} b_{11} & 0 & 0 \\ b_{21} & b_{22} & 0 \\ b_{31} & b_{32} & b_{33} \end{bmatrix} \begin{bmatrix} \varepsilon_{yt} \\ \varepsilon_{\pi t} \\ \varepsilon_{rt} \end{bmatrix}, \quad (12)$$

where

$$\varepsilon_t \sim (\mathbf{0}, I_3), \quad (13)$$

with $\varepsilon_t = (\varepsilon_{yt}, \varepsilon_{\pi t}, \varepsilon_{rt})'$.

The monthly dynamics of GDP is:

$$y_t^* = a_{11}y_{t-1}^* + a_{12}\pi_{t-1} + a_{13}r_{t-1} + b_{11}\varepsilon_{yt}, \quad (14)$$

which aggregated at quarterly frequency becomes:²

$$\begin{aligned} y_t^* = & a_{11}^3 y_{t-3}^* + a_{12} \pi_{t-1} + a_{12} a_{11} \pi_{t-2} + a_{12} a_{11}^2 \pi_{t-3} + \\ & + a_{13} r_{t-1} + a_{13} a_{11} r_{t-2} + a_{13} a_{11}^2 r_{t-3} + \\ & + b_{11} \varepsilon_{yt} + a_{11} b_{11} \varepsilon_{yt-1} + a_{11}^2 b_{11} \varepsilon_{yt-2}, \end{aligned} \quad (15)$$

for $t = 3, 6, \dots, T - 3, T$.

Now, let us look at the dynamics of the inflation rate:

$$\pi_t = a_{21} y_{t-1}^* + a_{22} \pi_{t-1} + a_{23} r_{t-1} + b_{21} \varepsilon_{yt} + \varepsilon_{\pi t}. \quad (16)$$

We see that the inflation rate at time t is influenced by the GDP at time $t - 1$, which of course we cannot observe. But after some algebraic manipulations, simply recursively substituting y_{t-1}^* with its expression in eq. (14), we obtain:

$$\begin{aligned} \pi_t = & a_{21} a_{11}^2 y_{t-3}^* + a_{22} \pi_{t-1} + a_{21} a_{12} \pi_{t-2} + a_{21} a_{11} a_{12} \pi_{t-3} + \\ & + a_{23} r_{t-1} + a_{21} a_{13} r_{t-2} + a_{21} a_{11} a_{13} r_{t-3} + b_{22} \varepsilon_{\pi t} + \\ & + b_{21} \varepsilon_{yt} + a_{21} b_{11} \varepsilon_{yt-1} + a_{21} a_{11} b_{11} \varepsilon_{yt-2}, \end{aligned} \quad (17)$$

for $t = 3, 6, \dots, T - 3, T$, which depends only on observable values of y_t .

Repeating the same steps for the third variable, the interest rate, we have:

$$r_t = a_{31} y_{t-1}^* + a_{32} \pi_{t-1} + a_{33} r_{t-1} + b_{31} \varepsilon_{yt} + b_{32} \varepsilon_{\pi t} + b_{33} \varepsilon_{rt}, \quad (18)$$

which we can rewrite as:

$$\begin{aligned} r_t = & a_{31} a_{11}^2 y_{t-3}^* + a_{32} \pi_{t-1} + a_{31} a_{12} \pi_{t-2} + a_{31} a_{11} a_{12} \pi_{t-3} + \\ & + a_{33} r_{t-1} + a_{31} a_{13} r_{t-2} + a_{31} a_{11} a_{13} r_{t-3} + b_{33} \varepsilon_{rt} + b_{32} \varepsilon_{\pi t} + \\ & + b_{31} \varepsilon_{yt} + a_{31} b_{11} \varepsilon_{yt-1} + a_{31} a_{11} b_{11} \varepsilon_{yt-2}, \end{aligned} \quad (19)$$

for $t = 3, 6, \dots, T - 3, T$.

Since, following our aggregation scheme, y_t^* is observed for $t = 3, 6, \dots, T - 3, T$, also y_{t-3}^* is observed at $t = 3, 6, \dots, T - 3, T$. Hence, estimating eq. (15), (17) and (19) is possible because all the required data are available.

Eq. (15), (17) and (19) together uniquely identify all the parameters of the monthly SVAR. In particular, from eq. (15) the parameters a_{11} , a_{12} , a_{13} can be identified. Then, from eq. (17) we recover the parameters a_{21} , a_{22} , a_{23} and from eq. (19) a_{31} , a_{32} , a_{33} . From the covariance matrix, we can finally obtain b_{11} , b_{21} , b_{22} , b_{31} , b_{32} and b_{33} .³ Therefore, exploiting more information coming from data at different frequencies allows us to overcome the identification issues and recover the parameters that drive the model at the monthly frequency.

For more general models and aggregation schemes, the use of a mixed frequency approach will not totally eliminate the identification issues. However, it will still improve with respect to

²For details about the calculations in this subsection see Appendix A.2.

³More details on the identification of the parameters are available upon request.

the use of aggregate data only.

3 Estimation of a mixed frequency SVAR model

In order to estimate a mixed frequency structural VAR model we follow and extend to a general aggregation scheme the analysis of Mariano and Murasawa (2010), providing the state-space representation of the model to be estimated in a maximum-likelihood framework where the low frequency series are considered as high frequency series with missing observations.

We define $\{y_{1t}\}$ as the N_1 -variate low frequency series observable every m^{th} period, and $\{y_{2t}\}$ as the N_2 -variate high frequency series observable every period. $\{y_{1t}^*\}$ represents the latent unobservable high frequency series underlying $\{y_{1t}\}$, such that $y_{1t} = \omega(L) y_{1t}^*$ for each t , where l is the lag order of the polynomial $\omega(L)$ and $\omega(L) = \omega_0 + \omega_1 L + \dots + \omega_l L^l$ is the aggregation scheme (one-sided filter). Finally, we define the $N \times 1$ vectors y_t and y_t^* respectively as $\begin{pmatrix} y_{1t} \\ y_{2t} \end{pmatrix}$ and $\begin{pmatrix} y_{1t}^* \\ y_{2t} \end{pmatrix}$ for all t , where $N = N_1 + N_2$.

To simplify the notation, let us assume that $\mu = E(y_t) = 0$ and $\mu^* = E(y_t^*) = 0$.

The VAR model we want to estimate is therefore the following:

$$\Phi(L) y_t^* = u_t, \quad (20)$$

where $\Phi(L)$ is a polynomial in the lag operator of order p , and $u_t \sim N(0, \Sigma)$. Moreover, for all t the following relation must hold:

$$y_t = H(L) y_t^*, \quad (21)$$

where

$$H(L)_{N \times N} = \begin{pmatrix} \omega(L) I_{N_1 \times N_1} & 0_{N_1 \times N_2} \\ 0_{N_2 \times N_1} & I_{N_2 \times N_2} \end{pmatrix}.$$

The model in eq. (20) and (21) can be cast in a state-space form, and then estimated making use of the Kalman filter.

If $p \leq l + 1$, the state-space representation is the following:

$$s_t = A s_{t-1} + B \varepsilon_t, \quad (22)$$

$$y_t = C s_t, \quad (23)$$

where $\varepsilon_t \sim N(0, I_N)$, the state vector is defined as

$$s_t = \begin{pmatrix} y_t^{*'} & \dots & y_{t-l}^{*'} \end{pmatrix}'$$

(l+1)N × 1

and the matrices defined as

$$\begin{aligned} A_{(l+1)N \times (l+1)N} &= \begin{bmatrix} \Phi_1 & \dots & \Phi_p & 0_{N \times (l+1-p)N} \\ & I_{lN} & & 0_{lN \times N} \end{bmatrix}, \\ B_{(l+1)N \times N} &= \begin{bmatrix} \Sigma^{1/2} \\ 0_{lN \times N} \end{bmatrix}, \\ C_{N \times (l+1)N} &= \begin{bmatrix} H(0) & \dots & H(l) \end{bmatrix}. \end{aligned}$$

Since y_t is observable only every m^{th} period, it has periodically missing observations.

If $p > l + 1$, the state-space form is still as in eq. (22) and (23), but the state vector is now defined as

$$s_t = \begin{pmatrix} y_t^* & \dots & y_{t-p+1}^* \end{pmatrix}'$$

and the matrices are the following:

$$\begin{aligned} A_{Np \times Np} &= \begin{bmatrix} \Phi_1 & \dots & \Phi_{p-1} & \Phi_p \\ & I_{(p-1)N} & & 0_{(p-1)N \times N} \end{bmatrix}, \\ B_{Np \times N} &= \begin{bmatrix} \Sigma^{1/2} \\ 0_{(p-1)N \times N} \end{bmatrix}, \\ C_{N \times Np} &= \begin{bmatrix} H(0) & \dots & H(l) & 0_{N \times (p-(l+1)N)} \end{bmatrix}. \end{aligned}$$

Once the model is written in state-space form, we can estimate it by replacing the missing observations in y_t with zeros and applying the Kalman filter (see Mariano and Murasawa (2010) for details).⁴

4 A Monte Carlo exercise

We now want to assess whether the temporal aggregation issues are empirically relevant and evaluate the finite sample performance of the proposed estimation procedure for mixed frequency SVARs, using simulated data in a controlled setup.

Consistently with the literature on VAR models, we look at the impulse response functions, which summarize the information contained in the VAR coefficients and in the covariance matrix of the residuals. We choose to orthogonalize the errors with a Choleski decomposition.

Briefly recalling the theory behind the impulse response functions to apply it to our specific case, we know that for a VAR(1) as written in eq. (4), the MA coefficient matrices contain the impulse responses of the system, and the $i - th$ coefficient of the MA representation, ϕ_i , is equal to A^i . Moreover, since the covariance matrix of u_t , Σ_u can be decomposed as $\Sigma_u = BB'$, the matrix that describes the orthogonalized impulse response i periods after the shock is $\Theta_i = \phi_i B$.

In our case, the monthly impulse responses are therefore:

$$\Theta_0 = B \quad \Theta_1 = AB \quad \Theta_2 = A^2B \quad \Theta_3 = A^3B \quad \dots \quad \Theta_6 = A^6B \quad \dots \quad (24)$$

⁴Other approaches to deal with the missing observations are proposed by Durbin and Koopman (2001) and Schorfheide and Song (2012).

The impulse responses for the quarterly model defined in eq. (10) and (11) are:

$$\Psi_0 = Q \quad \Psi_1 = CQ \quad \Psi_2 = C^2Q \quad \dots \quad (25)$$

where Q is the matrix obtained from the decomposition of Ω . It is worth to stress that in the case of the impulse responses obtained from a quarterly model, the time horizon is in quarters and not in months. Therefore, if we want to compare the responses from quarterly and mixed frequency models we should compare the results on the same time scale. The impact at the time of the shock is $\Theta_0 = B$ in the monthly case and $\Psi_0 = Q$ in the quarterly case, the impact one quarter from now is equivalent to the impact three months from now, so it is $\Theta_3 = A^3B$ in one case and $\Psi_1 = CQ = A^3Q$ in the other, and generally the impact j quarters from now is respectively $\Theta_{3j} = A^{3j}B$ and $\Psi_j = C^jQ = A^{3j}Q$. It is therefore clear that in the VAR(1) case the differences in the impulse response functions are mainly driven by the differences between the covariance matrix estimated with the two approaches and the restrictions imposed (while dynamics is also relevant for higher order high frequency VARs). The use of mixed frequency data allows to trace the dynamics of the process also intra-quarterly. This is obviously not possible when quarterly data only are used.

Our aim is to run a small exercise to compare the different impulse responses when we use only quarterly data and when we exploit mixed frequency data. A similar exercise has been carried out by Chiu et al. (2011). Their analysis is conducted in a Bayesian framework and on a set of different monthly and quarterly variables, which include GDP, industrial production, inflation and unemployment rate. Their findings also support the importance of taking mixed frequency information into account in the estimation of structural VAR models.

4.1 Simulation design and results

The simulation design is closely related to those adopted in the small literature on MF-VARs in a structural context (see Ghysels (2011) and Chiu et al. (2011)) and to the design used in Foroni, Marcellino and Schumacher (2013) for another study on mixed-frequency data. It also allows to evaluate the effects of features such as the degree of persistence and extent of cross correlations of the variables under analysis. We look at bivariate systems, with one low- and one high-frequency series, which is the simplest framework to study impulse responses. The DGP given by the high frequency VAR

$$\begin{pmatrix} y_t \\ x_t \end{pmatrix} = \begin{pmatrix} \rho & \delta_l \\ \delta_h & \rho \end{pmatrix} \begin{pmatrix} y_{t-1} \\ x_{t-1} \end{pmatrix} + \begin{pmatrix} e_{y,t} \\ e_{x,t} \end{pmatrix}, \quad (26)$$

where y_t is the low frequency variable and x_t is the high frequency variable. With t we denote the high frequency time index with $t = 1, \dots, T \times m$. T defines the size of the estimation sample expressed in the low frequency unit. m denotes the sampling frequency of the low frequency variable y_t . To be consistent with the analytical example in Section 2, we assume that $\omega(L) = 1$. Thus, the low frequency variable y_t is available only for $t = m, 2m, \dots, T \times m$.

We focus our analysis on the sampling frequency $m = 3$, which represents the case of monthly and quarterly data.

In generating the variables we consider different combinations of parameters, in such a way

to ensure a non-explosive solution, and therefore stationarity of both y and x . In particular, we consider the following specifications of $\{\rho, \delta_l, \delta_h\}$:

$$\begin{aligned} &\{0.5, 0.4, 0.4\}, \{0.5, 0.8, 0.4\}, \\ &\{0.9, 0.08, 0.08\}, \{0.9, 0.1, 0.08\}. \end{aligned} \tag{27}$$

The shocks $e_{y,t}$ and $e_{x,t}$ are sampled independently from the normal distribution $N \sim (0, I_2)$.⁵

The number of observations in the sample is fixed to $T = 100$ for the low frequency variable, and therefore to 300 for the high frequency one.

All in all, the different parameter combinations cover a broad range of DGPs with different degrees of persistence and correlation between the high frequency and the low frequency variable.

In our Monte Carlo analysis we look at the impulse responses obtained when we estimate (26) with high frequency data, with low frequency data only (obtained skip-sampling the series), and with mixed frequency data. More in detail, the simulation of the data at monthly frequency allows us also to estimate the monthly VAR(1) process with standard techniques, and use it as a benchmark. Then, once the data are skip-sampled to quarterly frequency, we estimate the corresponding quarterly VAR(1) process, again by OLS.⁶ Finally, we consider the mixed frequency case in which only y is skip-sampled to mimic the availability of GDP at quarterly level, while x is available every month. In order to estimate the mixed frequency model, we follow the Kalman filter based procedure outlined in Section 3.

In each of the three cases, we apply the Choleski decomposition to make the shocks orthogonal. We compute the impulse response functions to trace out what happens to the system up to 8 quarters ahead in the quarterly model, and equivalently up to 24 months in the monthly and mixed frequency case.

In order to assess the effects of aggregation and the benefits from the use of mixed frequency data, we generate the data R times, and for each bivariate dataset we compute the impulse responses obtained from the process estimated at monthly, quarterly and mixed frequency. Then, we report the median impulse responses, and the 10th and 90th percentiles computed across replications and for the different parameter specifications. We fix the number of replications equal to $R = 1000$.

In Figures 1 to 4, we report the values of the impulse responses (median) and the confidence intervals, computed as the 10th and 90th percentiles. In the figures, we call period 1 the period of impact.

The mixed-frequency model allows to obtain monthly responses, as in the benchmark model. However, to make the results comparable with those obtained from the quarterly model, we compute the corresponding quarterly aggregates. The aggregation of the impulse responses is a delicate issue, and it is not straightforward how to do it, since it depends on the nature of the series. In our case, since we skip-sample the series to go from monthly to quarterly frequency, we follow the skip-sample scheme also for the impulse responses.

⁵We considered also many other specifications, in particular $\{0.1, 0.1, 0.1\}$, $\{0.1, 0.4, 0.4\}$, $\{0.1, 0.8, 0.8\}$, $\{0.5, 0.1, 0.1\}$, $\{0.5, 0.2, 0.2\}$, $\{0.5, 0.4, 0.2\}$, $\{0.9, 0.01, 0.01\}$, $\{0.9, 0.04, 0.04\}$ and $\{0.9, 0.08, 0.04\}$. The results are consistent with those described later on in the section.

⁶We consider a process with one lag, because we know from theory that the quarterly skip sampled aggregated process corresponding to a monthly VAR(1) is still a VAR(1). Therefore, we avoid any mis-specification issues.

We can summarize the main results as follows. First, if we estimate the process at the low frequency, the size of the impulse response is bigger than the one obtained at the monthly frequency, and generally outside the confidence bands of the monthly process, represented by the 10th and 90th percentiles of the distribution of impulse responses in our Monte Carlo experiment. This means that if we estimate a process at a lower frequency than the true frequency of the process, we may draw wrong conclusions on the size of the impact of the shock.

Second, the mixed frequency approach works quite well in capturing the salient features of the monthly process. If we look at the figures, we see how well the mixed frequency approach captures the dynamics of the monthly process. In most of the cases, the median response computed with mixed frequency data is very similar to the benchmark obtained when all the data are available at the higher frequency, and always inside the confidence bands of the monthly benchmark. Moreover, the same confidence bands are typically fairly similar to the monthly ones.

Third, a special consideration is due to Figure 1, in which we report the response of the low frequency variable to shocks in the same low frequency variable. This is the case in which both the quarterly and the mixed frequency approaches lose most of the information (the low frequency variable is the one which we do not observe at high frequency level). However, while the low frequency model overestimates once again the size of the response, this does not happen with the mixed frequency approach. With the latter, the median impulse response remains very similar in size to the monthly benchmark.

Finally, the results summarized above are valid across specifications, despite different correlations between low and high frequency variables.

4.2 Robustness analysis

In what follows, we assess the robustness of the results we have obtained so far to generalizations of the simulation design. Specifically, we first consider the weekly-quarterly (or monthly-annual) case, with $m = 12$. We then assess what happens when the true generating frequency is higher than what assumed in the mixed frequency model. Specifically, we look at the case in which the true DGP is a weekly process and we compare the structural estimates obtained with quarterly data to those resulting from mixed frequency monthly-quarterly data.

4.2.1 The case of weekly and quarterly data

We start comparing the impulse responses from SVAR models based on quarterly data and on mixed frequency weekly-quarterly data, in the case of a weekly VAR as DGP, so that the sampling frequency is $m = 12$.

In generating the data we consider the same combinations of parameters as in Section 4.1, which ensure stable solutions of our VAR(1) high frequency process. The analysis is conducted in the same way, comparing the impulse responses obtained with high, low and mixed frequency data. For computational reasons (the number of missing values is high and therefore the computational time increases substantially), we fix the number of replications to $R = 500$.

In Figures 5 to 8, we report the median value across replications of the SVAR based impulse responses and the confidence intervals, computed as the 10th and 90th percentiles.

We can summarize the main results as follows. First, we confirm the results obtained in the

case of sampling frequency $m = 3$. In particular, the size of the impulse response is bigger when we estimate the process at the low frequency. Second, the mixed frequency approach captures well the dynamic of the high frequency process: the median response computed with mixed frequency data is very similar to the one obtained when all the data are available at the higher frequency. The same consideration holds for the confidence bands, which are fairly similar in the two cases.

4.2.2 The mixed frequency process is also misspecified

We now address the case where the assumed mixed frequency is incorrect. We consider a weekly DGP while the model is estimated with monthly-quarterly or quarterly only data. Our goal is to check whether the mixed frequency approach still mitigates the problems arising with time aggregation.

Within the SVAR framework, we generate the weekly data in the same way as in Section 4.2.1, but in this case we consider our y_t available every quarter, and x_t available every month. Hence, we obtain the quarterly series by skip-sampling the high frequency equivalent every 12th period, and the monthly series by skip-sampling every fourth observation.

Figures 9 to 12 can be read as in the previous Monte Carlo experiments. We report the median value across replications of the impulse responses, and the confidence intervals computed as the 10th and 90th percentiles.⁷

The main results we obtain are the following. First, and in line with the previous findings, if we estimate the process at a frequency lower than the true one, the size of the impulse response remains bigger than the one obtained at the correct frequency. Now also the mixed frequency model is based on a lower frequency than the correct one. However if we compare the responses obtained with mixed frequency data to those obtained with quarterly data only, we see that the former are closer to the true ones than the latter.

5 An empirical example with US data

We now provide an empirical application to further illustrate the theoretical analysis conducted so far, and analyze the importance of the temporal aggregation issues in practice. We estimate a trivariate SVAR with data for the US economy, comparing the impulse response functions of a mixed frequency monthly-quarterly model to those obtained with a standard quarterly SVAR.

We consider output growth, inflation and interest rate, as three variables. The output growth is represented by the real GDP growth rate, and is available only quarterly. As inflation rate, we consider the growth rate of the consumer price index. The interest rate is represented by the Fed Fund rate (FFR). These last two variables are available also at monthly frequency. Therefore, we can conduct our analysis with one quarterly series (the GDP growth rate) and two monthly series: the inflation rate (as monthly change in the CPI index) and the FFR. When moving to a pure quarterly model, we aggregate the monthly series to a quarterly frequency by taking the sum of the three monthly observations over the quarter. When estimating the mixed-frequency model, the quarterly observable GDP growth rate is considered as the sum of

⁷For the benchmark high-frequency model, we obtain also intra-monthly responses. We do not analyze them, since they do not matter for the purposes of our analysis.

the three unobserved monthly GDP growth rates, but similar results are obtained when using a point in time sampling scheme.

The model is estimated using the Kalman filter, as outlined in Section 3, see also Mariano and Murasawa (2010). The sample we consider spans the period 1985-2007. We intentionally exclude the crisis period, to work on a stable sample period, since the treatment of the crisis is beyond the scope of this analysis.

In our exercise, we order the variables as GDP growth, inflation and interest rate, with a Choleski identification scheme. The BIC criterion for lag length selection indicates one lag in the quarterly model, and for theoretical coherence we assume one lag also for the (unobservable) high frequency model.

We compute the quarterly impulse responses to trace out the effects of the shocks up to 8 quarters ahead. For the mixed-frequency model, we can compute monthly responses and therefore we trace out the impulse responses up to 24 months ahead. However, as in the Monte Carlo experiments, in order to compare the results to those obtained from the quarterly model, we aggregate the monthly impulse responses to a quarterly frequency, and we focus on the quarterly aggregates, despite the possibility to analyze also the intra-quarterly dynamics in the mixed-frequency case. We sum the monthly impulse responses over the quarter, since the variables involved represent rates.

In Figure 13 we report the impulse response functions obtained with the two different methods. The red line indicates the response obtained with mixed frequency data, and the dashed red lines are the confidence bands. The blue line indicates the impulse response function estimated with quarterly data only, and the blue dashed lines are the corresponding bands. The bands are computed with a Monte Carlo method, with 1000 replications. The error bands represent the 5th and 95th percentile of the replications.

Figure 13 suggests that time aggregation plays a role in shaping the results, consistently with the findings of the Monte Carlo experiment. In the case of the GDP responses to shocks, we can see how including more information allows to reduce the uncertainty, which is reflected in tighter error bands when monthly information is included in the estimation. In the case of the response functions of the monthly variables to a shock to a monthly variable, we notice differences especially in the size of the reaction. It is interesting to see that for many of the periods considered, the two responses are not included in the confidence bands of the other approach, and sometimes even the standard errors do not intersect. The differences are particularly evident in the response of the interest rate. Using only quarterly data, we find a stronger and more persistent dynamic of the interest rate.

6 Conclusions

Summing up, the analytical, simulation based and empirical results we have obtained confirm that choosing the temporal frequency matters for structural analysis based on SVAR models.

The common approach in the empirical literature, just aggregating the data to the lowest available frequency, is inefficient and can distort the identification of the structural shocks and of their propagation mechanism.

We have shown that these problems can be solved or at least alleviated by using all the

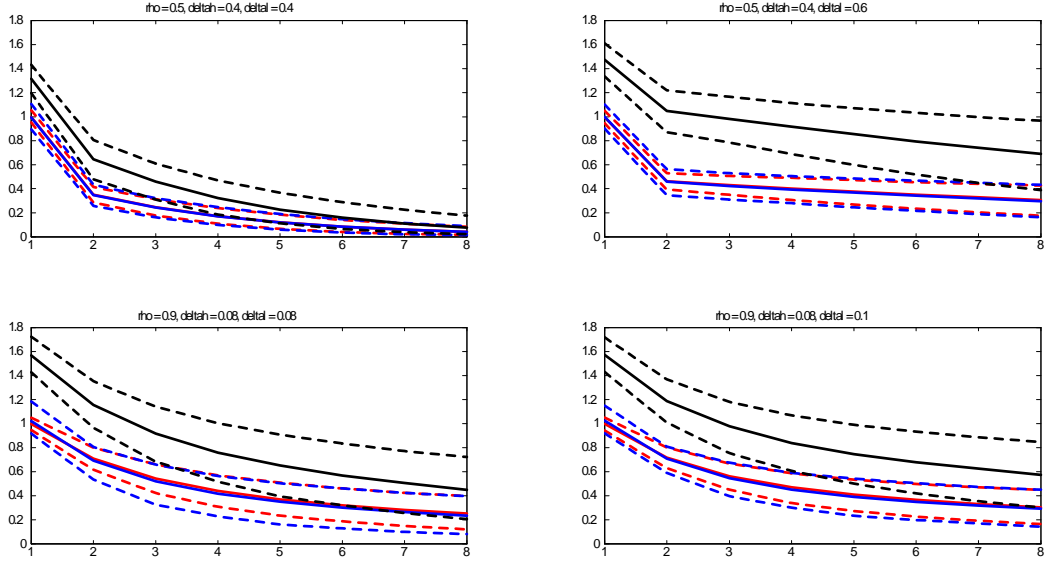
available mixed frequency information. This is a simple and sensible choice also in a structural VAR context.

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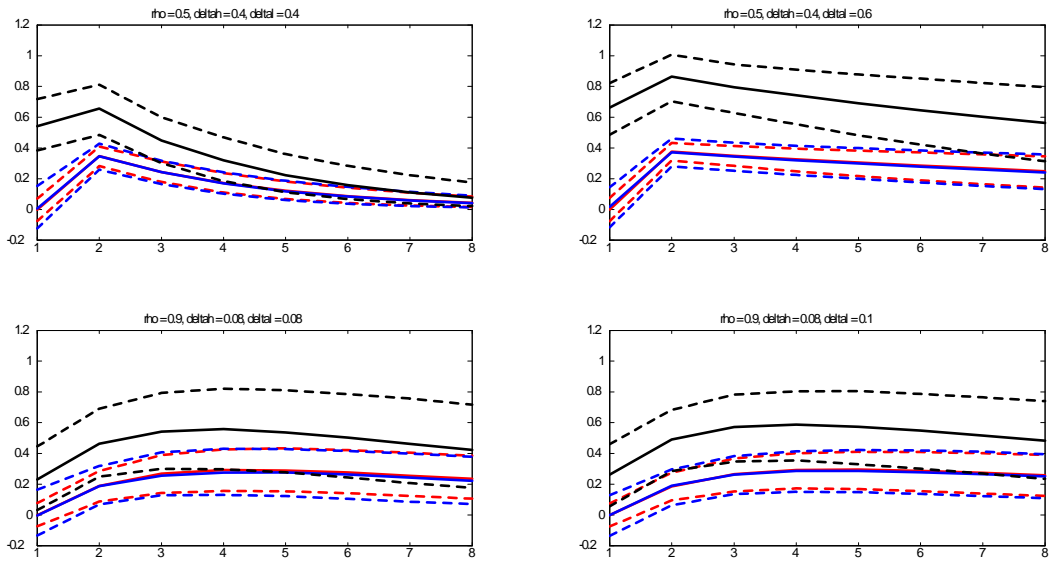
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Figure 1: Response of the low frequency variable y to a shock in y



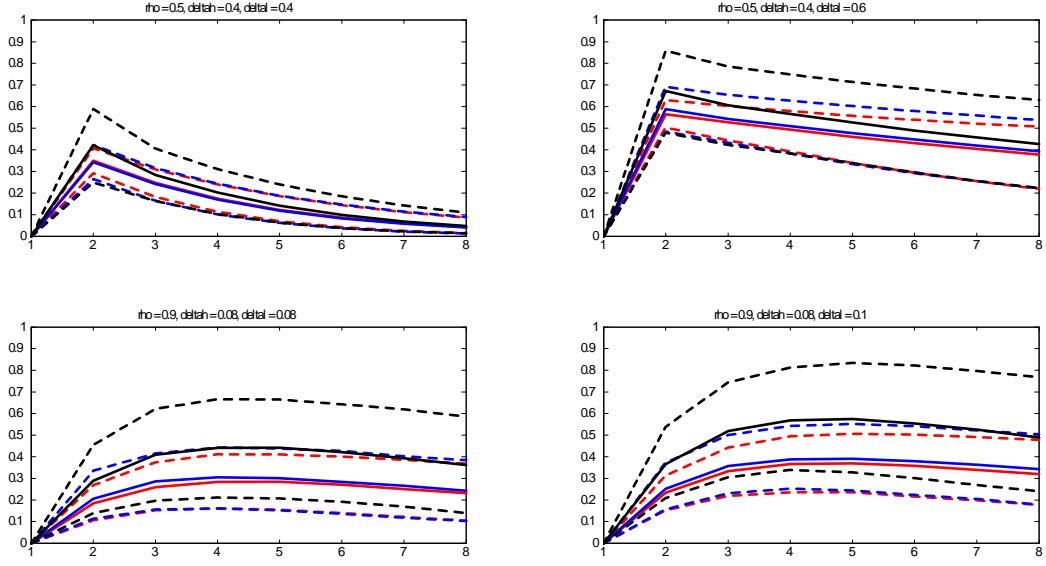
Notes: The figure reports the value of the impulse responses obtained simulating data from the DGP in eq. (26), for different parameter specifications and for different periods (indicated in quarters in the header). For each parameter specification, we report the value of the impulse responses obtained using the data at the high frequency (red), the data skip-sampled at the low-frequency (black), and the data at mixed-frequency (blue). We consider the median value, the 10th and 90th percentile across replications. The number of replication is 1000 for each parameter specification.

Figure 2: Response of the high frequency variable x to a shock in y



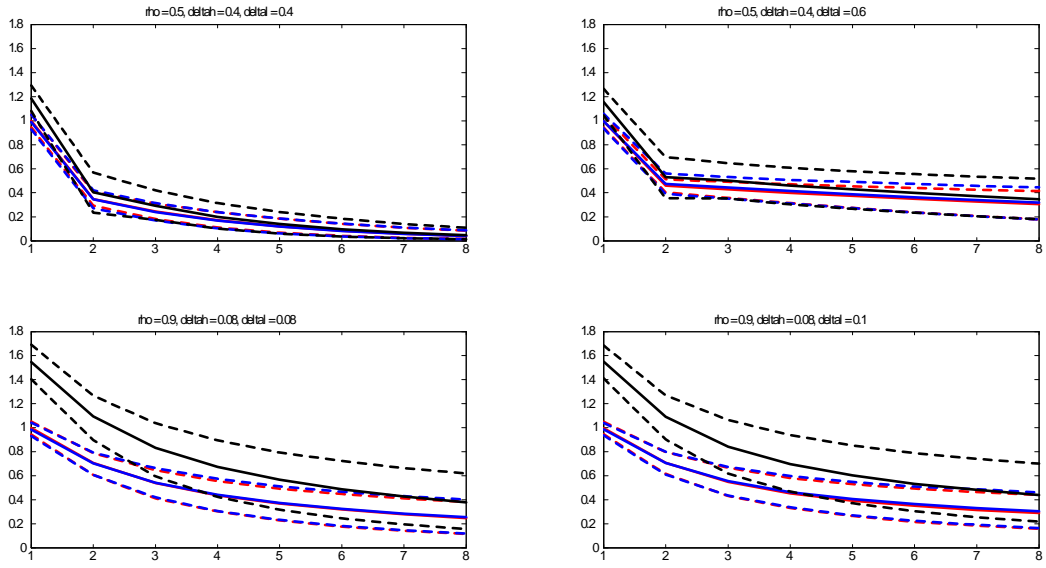
Notes: See notes at Figure 1.

Figure 3: Response of the low frequency variable y to a shock in x



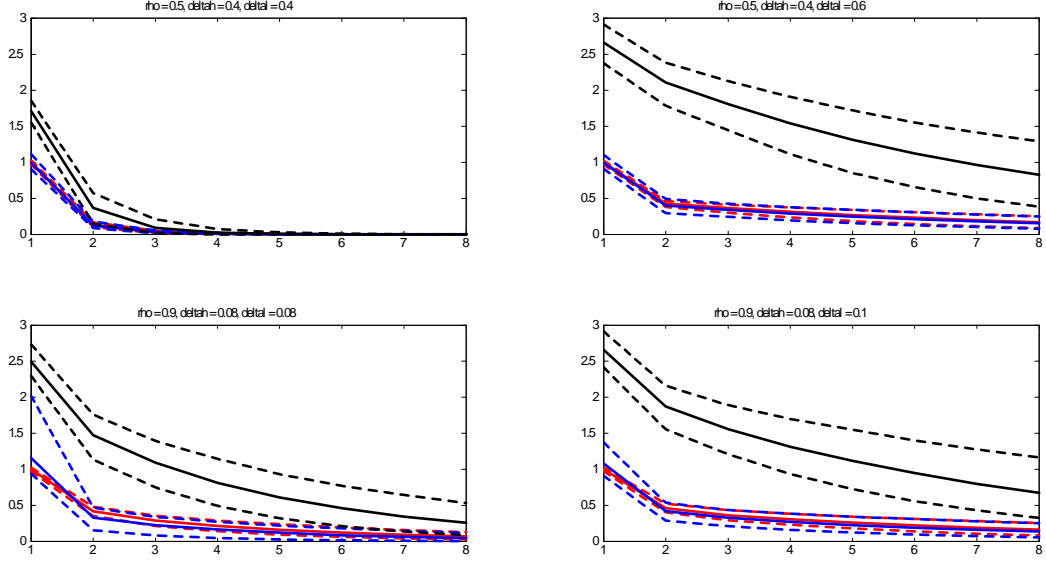
Notes: See notes at Figure 1.

Figure 4: Response of the high frequency variable x to a shock in x



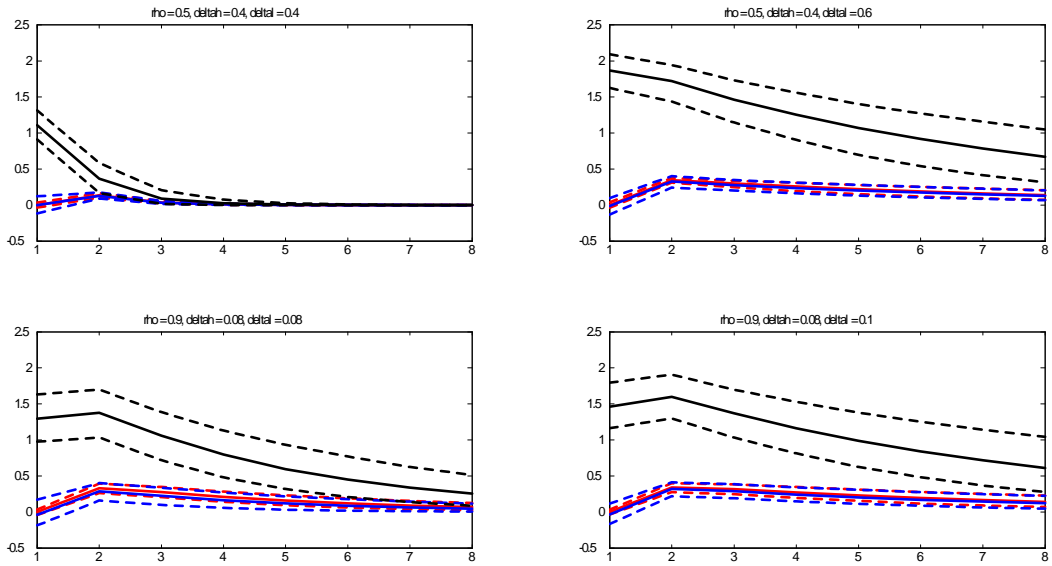
Notes: See notes at Figure 1.

Figure 5: Response of the low frequency variable y to a shock in y (DGP: weekly, data availability: weekly and quarterly)



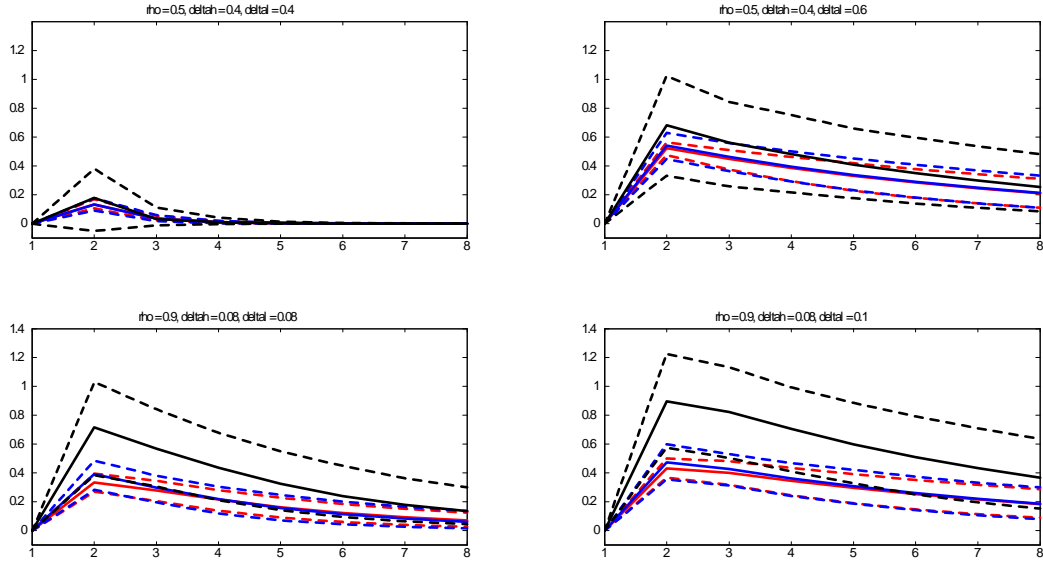
Notes: The figure reports the value of the impulse responses obtained simulating data from the DGP in eq. (26), for different parameter specifications and for different periods (indicated in quarters in the header). For each parameter specification, we report the value of the impulse responses obtained using the data at the weekly frequency (red), the data skip-sampled at the quarterly frequency (black), and the data at mixed-frequency with weekly and quarterly variables (blue). We consider the median value, the 10th and 90th percentile across replications. The number of replication is 500 for each parameter specification.

Figure 6: Response of the high frequency variable x to a shock in y (DGP: weekly, data availability: weekly and quarterly)



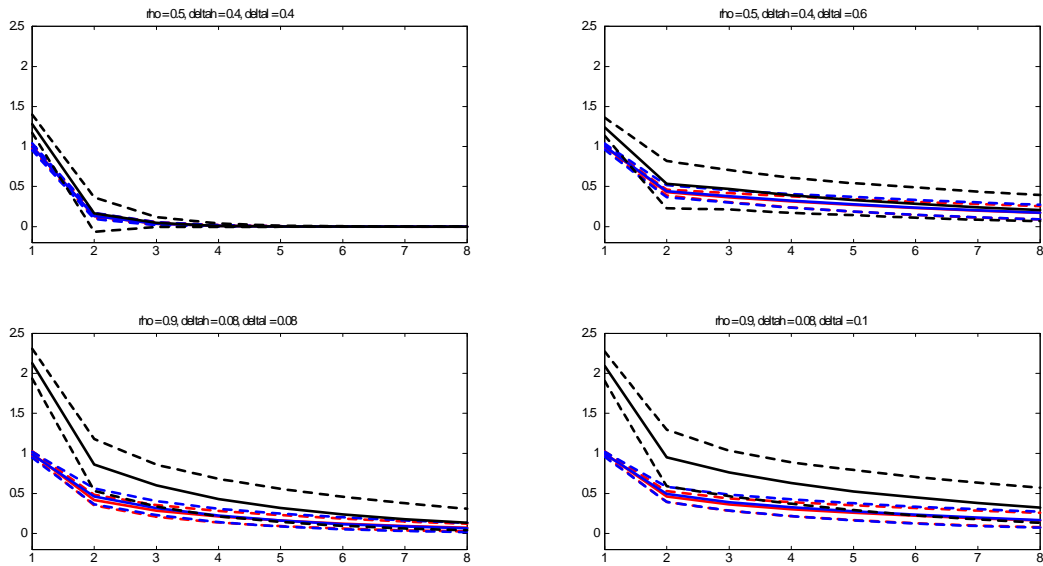
Notes: See notes at Figure 5.

Figure 7: Response of the low frequency variable y to a shock in x (DGP: weekly, data availability: weekly and quarterly)



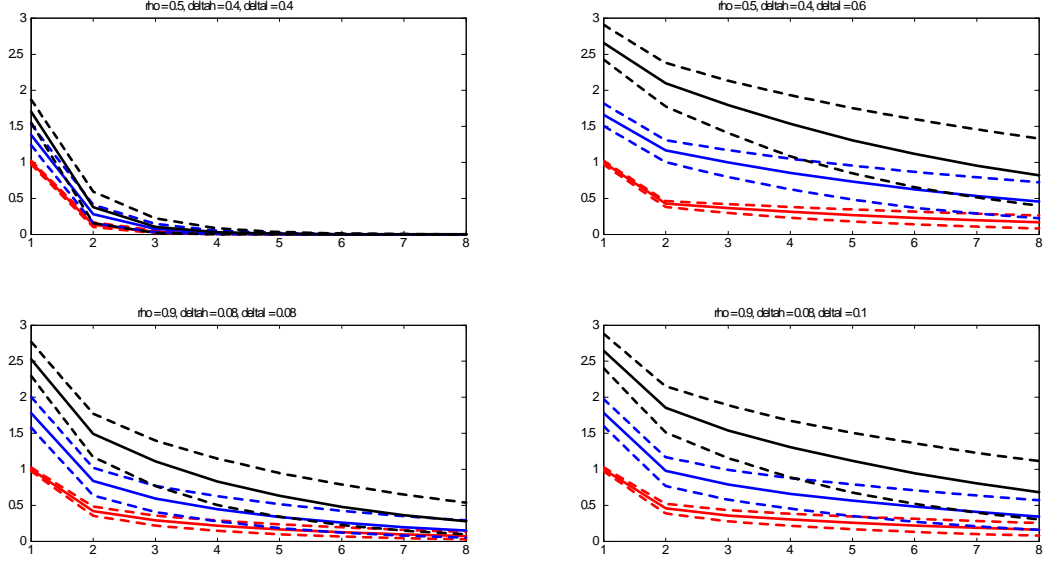
Notes: See notes at Figure 5.

Figure 8: Response of the high frequency variable x to a shock in x (DGP: weekly, data availability: weekly and quarterly)



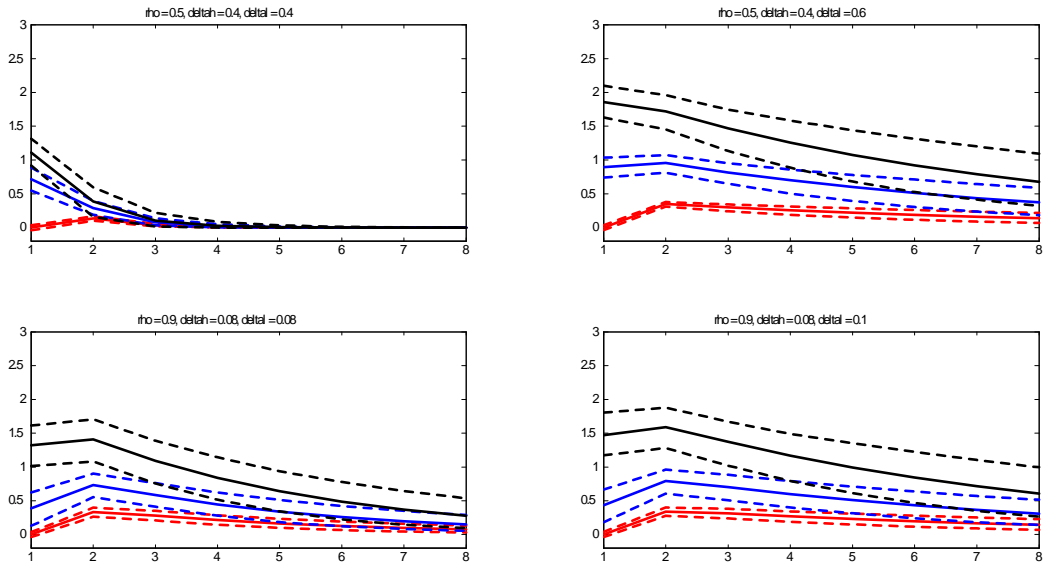
Notes: See notes at Figure 5.

Figure 9: Response of the low frequency variable y to a shock in y (DGP: weekly, data availability: monthly and quarterly)



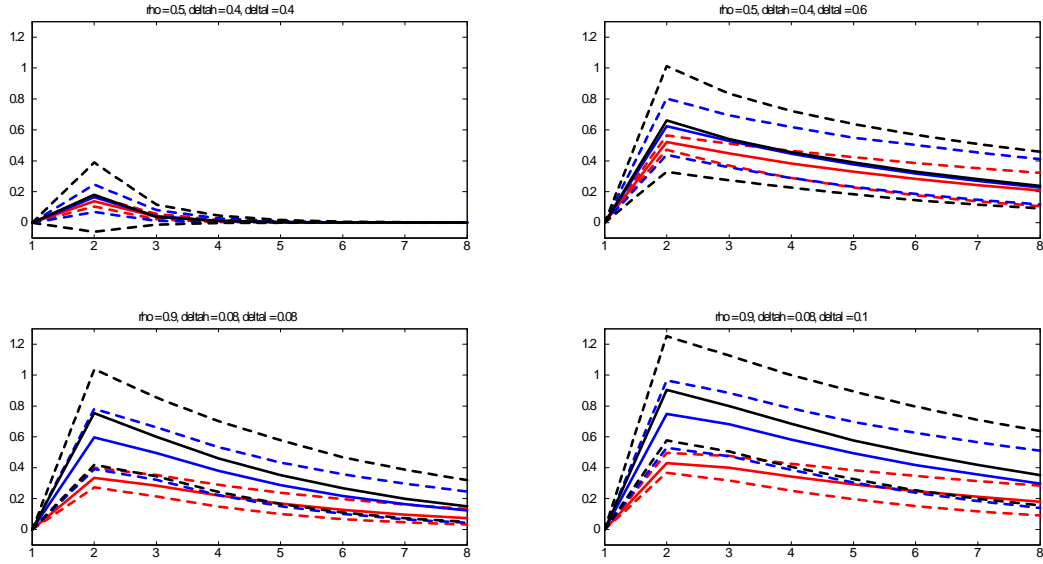
Notes: The figure reports the value of the impulse responses obtained simulating data from the DGP in eq. (26), for different parameter specifications and for different periods (indicated in months in the header). For each parameter specification, we report the value of the impulse responses obtained using the data at the weekly frequency (red), the data skip-sampled at the quarterly frequency (green), and the data at mixed-frequency with monthly and quarterly variables (blue). We consider the median value, the 10th and 90th percentile across replications. The number of replication is 1000 for each parameter specification.

Figure 10: Response of the high frequency variable x to a shock in y (DGP: weekly, data availability: monthly and quarterly)



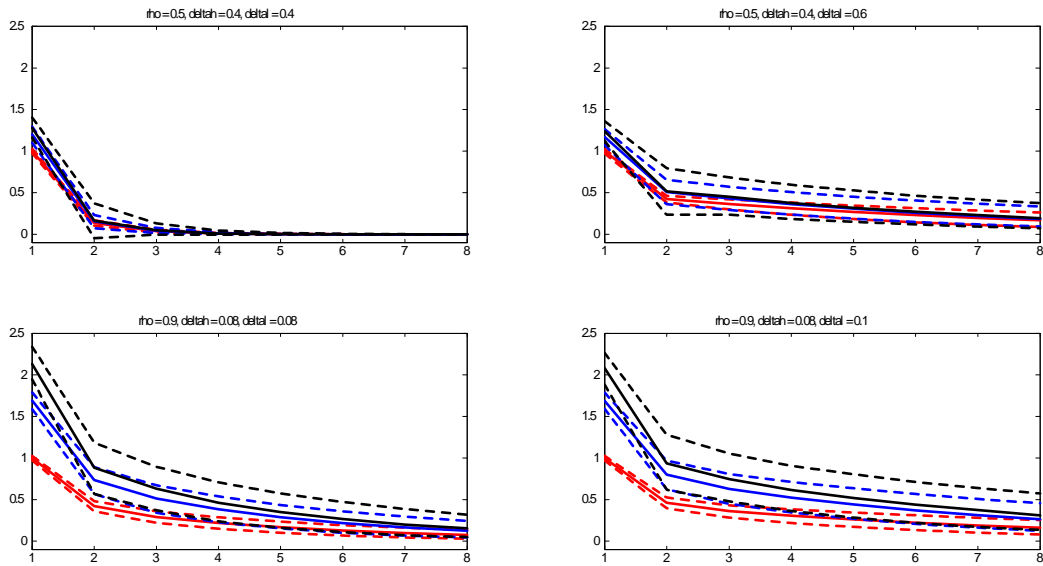
Notes: See notes at Figure 9.

Figure 11: Response of the low frequency variable y to a shock in x (DGP: weekly, data availability: monthly and quarterly)



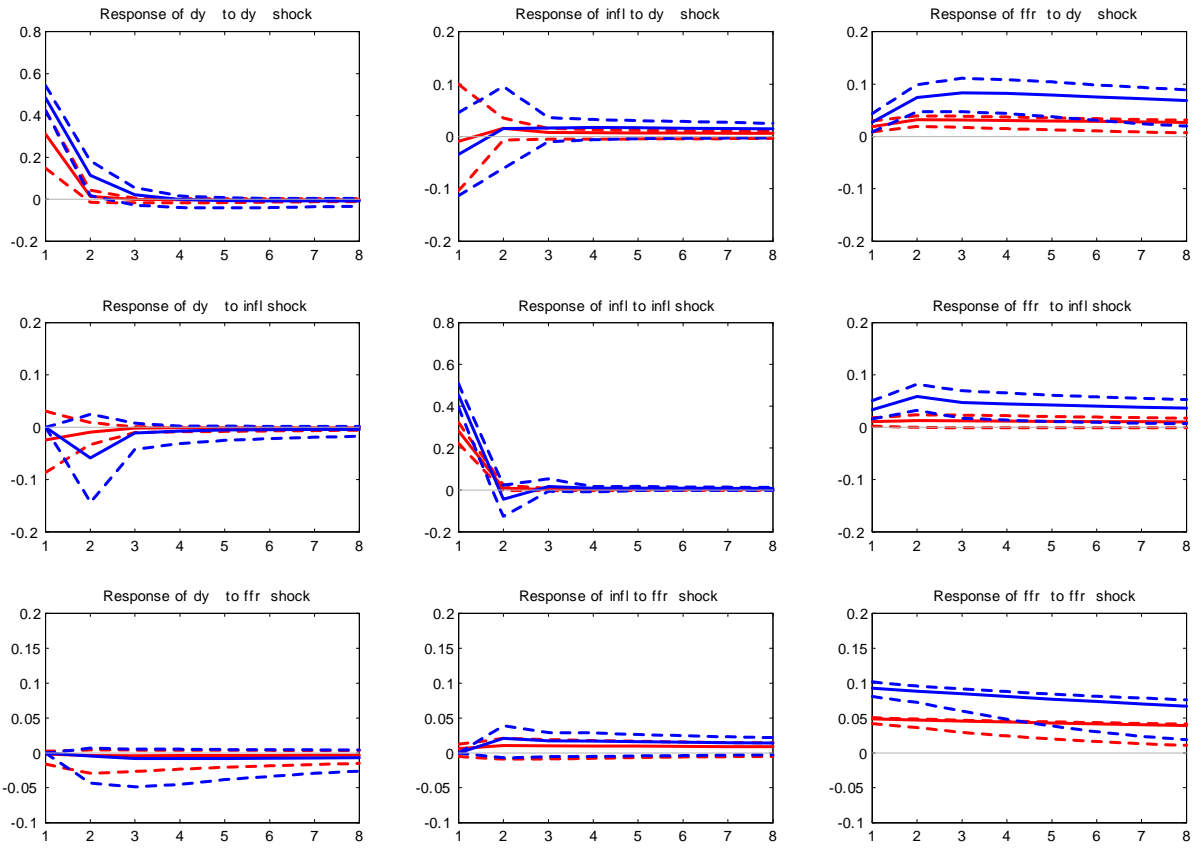
Notes: See notes at Figure 9.

Figure 12: Response of the high frequency variable x to a shock in x (DGP: weekly, data availability: monthly and quarterly)



Notes: See notes at Figure 9.

Figure 13: Impulse responses obtained with mixed-frequency and quarterly US data



Notes: The red lines represent the impulse responses obtained with mixed-frequency data. The red dashed lines are the confidence bands corresponding to the 5th and 95th percentile. The blue lines are the quarterly IRF. The blue dashed lines are the confidence bands corresponding to the 5th and 95th percentile. The data used cover the sample 1985 -2007.

A Appendix

A.1 Quarterly SVAR: identification issues

In Section 2.1 we aggregate the monthly SVAR at quarterly level and obtain:

$$y_\tau = A^3 y_{\tau-1} + \xi_\tau, \quad (28)$$

with

$$\xi_\tau \sim (0, \Omega), \quad \Omega = BB' + ABB'A' + A^2BB'A'^2. \quad (29)$$

The matrix A^3 has the following form:

$$A^3 = \begin{bmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \end{bmatrix}, \quad (30)$$

$$\begin{aligned} \alpha_{11} &= a_{11}^3 + (2a_{12}a_{21} + 2a_{13}a_{31})a_{11} + a_{12}(a_{21}a_{22} + a_{23}a_{31}) + a_{13}(a_{21}a_{32} + a_{31}a_{33}) \\ \alpha_{12} &= a_{12}(a_{22}^2 + a_{12}a_{21} + a_{23}a_{32}) + a_{11}(a_{11}a_{12} + a_{12}a_{22} + a_{13}a_{32}) + a_{13}(a_{12}a_{31} + a_{22}a_{32} + a_{32}a_{33}) \\ \alpha_{13} &= a_{13}(a_{33}^2 + a_{13}a_{31} + a_{23}a_{32}) + a_{11}(a_{11}a_{13} + a_{12}a_{23} + a_{13}a_{33}) + a_{12}(a_{13}a_{21} + a_{22}a_{23} + a_{23}a_{33}) \\ \alpha_{21} &= a_{21}(a_{11}^2 + a_{12}a_{21} + a_{13}a_{31}) + a_{22}(a_{11}a_{21} + a_{21}a_{22} + a_{23}a_{31}) + a_{23}(a_{11}a_{31} + a_{21}a_{32} + a_{31}a_{33}) \\ \alpha_{22} &= a_{22}^3 + a_{11}a_{12}a_{21} + 2a_{12}a_{21}a_{22} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} + 2a_{22}a_{23}a_{32} + a_{23}a_{32}a_{33} \\ \alpha_{23} &= a_{23}(a_{33}^2 + a_{13}a_{31} + a_{23}a_{32}) + a_{21}(a_{11}a_{13} + a_{12}a_{23} + a_{13}a_{33}) + a_{22}(a_{13}a_{21} + a_{22}a_{23} + a_{23}a_{33}) \\ \alpha_{31} &= a_{31}(a_{11}^2 + a_{12}a_{21} + a_{13}a_{31}) + a_{32}(a_{11}a_{21} + a_{21}a_{22} + a_{23}a_{31}) + a_{33}(a_{11}a_{31} + a_{21}a_{32} + a_{31}a_{33}) \\ \alpha_{32} &= a_{32}(a_{22}^2 + a_{12}a_{21} + a_{23}a_{32}) + a_{31}(a_{11}a_{12} + a_{12}a_{22} + a_{13}a_{32}) + a_{33}(a_{12}a_{31} + a_{22}a_{32} + a_{32}a_{33}) \\ \alpha_{33} &= a_{33}^3 + a_{11}a_{13}a_{31} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} + 2a_{13}a_{31}a_{33} + a_{22}a_{23}a_{32} + 2a_{23}a_{32}a_{33}. \end{aligned}$$

When we work with quarterly data, we obtain the estimate of a 3×3 matrix $C = A^3$ in eq. (28). In order to recover the parameters which drive the monthly process we should solve the following system for the parameters $a_{11}, a_{12}, \dots, a_{33}$:

$$\begin{cases} c_{11} = \alpha_{11}(a_{11}, a_{12}, \dots, a_{33}) \\ c_{12} = \alpha_{12}(a_{11}, a_{12}, \dots, a_{33}) \\ \vdots \\ c_{33} = \alpha_{33}(a_{11}, a_{12}, \dots, a_{33}) \end{cases}, \quad (31)$$

which is a highly non linear system. This system not always admits a unique solution, and many times has to be solved numerically.

In other words, solving the system (31), we want to find the matrix A , such that $A^3 = C$, and therefore we want to find the cube roots of a matrix. We can show in the easiest case possible that this is not necessarily unique. Let us consider for simplicity C as a 2×2 identity matrix⁸. It is obvious to see that $A = I_2$ is a cube root of C . However, it is possible to check

⁸The dimension of the matrix is set at $n = 2$ for simplicity, but it can be extended to any other n .

that any matrix

$$A = \begin{bmatrix} -d-1 & -\frac{1}{f}(d^2+d+1) \\ f & d \end{bmatrix}, \quad d, f \in \mathbf{R}, \quad f \neq 0, \quad (32)$$

is also a cube root. This is therefore a very easy example which shows that from C we cannot always uniquely identify A .

Moreover, since the error term is defined as

$$\xi_\tau \sim (0, \Omega), \quad \Omega = BB' + ABB'A' + A^2BB'A'^2. \quad (33)$$

the difficulties in identifying A translate into the problem of identifying B .

A.2 SVAR with mixed frequency data: identification issues

Let us consider the monthly process for GDP:

$$y_t^* = a_{11}y_{t-1}^* + a_{12}\pi_{t-1} + a_{13}r_{t-1} + b_{11}\varepsilon_{yt}, \quad (34)$$

and rewrite it as:

$$(1 - a_{11}L)y_t^* = a_{12}\pi_{t-1} + a_{13}r_{t-1} + \varepsilon_{yt}. \quad (35)$$

To aggregate the process at a quarterly frequency we need to pre-multiply both sides of eq. (35) by $b(L) = (1 + a_{11}L + a_{11}^2L^2)$ to obtain

$$\begin{aligned} y_t^* &= a_{11}^3y_{t-3}^* + a_{12}\pi_{t-1} + a_{12}a_{11}\pi_{t-2} + a_{12}a_{11}^2\pi_{t-3} + \\ &\quad + a_{13}r_{t-1} + a_{13}a_{11}r_{t-2} + a_{13}a_{11}^2r_{t-3} + \\ &\quad + b_{11}\varepsilon_{yt} + a_{11}b_{11}\varepsilon_{yt-1} + a_{11}^2b_{11}\varepsilon_{yt-2}. \end{aligned} \quad (36)$$

Now, let us move to the monthly process for inflation:

$$\pi_t = a_{21}y_{t-1}^* + a_{22}\pi_{t-1} + a_{23}r_{t-1} + b_{21}\varepsilon_{yt} + b_{22}\varepsilon_{\pi t}. \quad (37)$$

Since we do not observe y_{t-1}^* , we substitute it with its dynamics in eq. (34):

$$\begin{aligned} \pi_t &= a_{21}(a_{11}y_{t-2}^* + a_{12}\pi_{t-2} + a_{13}r_{t-2} + b_{11}\varepsilon_{yt-1}) + \\ &\quad + a_{22}\pi_{t-1} + a_{23}r_{t-1} + b_{21}\varepsilon_{yt} + b_{22}\varepsilon_{\pi t}, \end{aligned} \quad (38)$$

and then again, we substitute y_{t-2}^* :

$$\begin{aligned} \pi_t &= a_{21} \left(a_{11}(a_{11}y_{t-3}^* + a_{12}\pi_{t-3} + a_{13}r_{t-3} + b_{11}\varepsilon_{yt-2}) + \right. \\ &\quad \left. + a_{12}\pi_{t-2} + a_{13}r_{t-2} + \varepsilon_{yt-1} \right) + \\ &\quad + a_{22}\pi_{t-1} + a_{23}r_{t-1} + b_{21}\varepsilon_{yt} + b_{22}\varepsilon_{\pi t}, \end{aligned} \quad (39)$$

obtaining

$$\begin{aligned}\pi_t = & a_{21}a_{11}^2y_{t-3}^* + a_{22}\pi_{t-1} + a_{21}a_{12}\pi_{t-2} + a_{21}a_{11}a_{12}\pi_{t-3} + \\ & + a_{23}r_{t-1} + a_{21}a_{13}r_{t-2} + a_{21}a_{11}a_{13}r_{t-3} + b_{22}\varepsilon_{\pi t} + \\ & + b_{21}\varepsilon_{yt} + a_{21}b_{11}\varepsilon_{yt-1} + a_{21}a_{11}b_{11}\varepsilon_{yt-2},\end{aligned}\quad (40)$$

which depends only on observable values of y_t^* .

The same can be done for r_t , obtaining:

$$\begin{aligned}r_t = & a_{31}a_{11}^2y_{t-3}^* + a_{32}\pi_{t-1} + a_{31}a_{12}\pi_{t-2} + a_{31}a_{11}a_{12}\pi_{t-3} + \\ & + a_{33}r_{t-1} + a_{31}a_{13}r_{t-2} + a_{31}a_{11}a_{13}r_{t-3} + b_{33}\varepsilon_{rt} + b_{32}\varepsilon_{\pi t} + \\ & + b_{31}\varepsilon_{yt} + a_{31}b_{11}\varepsilon_{yt-1} + a_{31}a_{11}b_{11}\varepsilon_{yt-2}.\end{aligned}\quad (41)$$

From eq. (36) the parameters a_{11} , a_{12} , a_{13} can be identified. From eq. (40) we recover the parameters a_{21} , a_{22} , a_{23} and from eq. (41) a_{31} , a_{32} , a_{33} .

From equations (36), (40) and (41) we obtain three series of residuals: $\xi_{yt} = b_{11}\varepsilon_{yt} + a_{11}b_{11}\varepsilon_{yt-1} + a_{11}^2b_{11}\varepsilon_{yt-2}$, $\xi_{\pi t} = b_{22}\varepsilon_{\pi t} + b_{21}\varepsilon_{yt} + a_{11}b_{11}\varepsilon_{yt-1} + a_{21}a_{11}b_{11}\varepsilon_{yt-2}$ and $\xi_{rt} = b_{33}\varepsilon_{rt} + b_{32}\varepsilon_{\pi t} + b_{31}\varepsilon_{yt} + a_{31}b_{11}\varepsilon_{yt-1} + a_{31}a_{11}b_{11}\varepsilon_{yt-2}$ and with covariance matrix

$$\Sigma_{\xi} = \begin{bmatrix} s_1^2 & s_{21} & s_{31} \\ s_{21} & s_2^2 & s_{32} \\ s_{31} & s_{32} & s_3^3 \end{bmatrix}, \quad (42)$$

where the elements are defined as:

$$\begin{aligned}s_1^2 &= var(b_{11}\varepsilon_{yt} + a_{11}b_{11}\varepsilon_{yt-1} + a_{11}^2b_{11}\varepsilon_{yt-2}) \\ s_{21} &= cov\left(\begin{array}{c} b_{22}\varepsilon_{\pi t} + b_{21}\varepsilon_{yt} + a_{11}b_{11}\varepsilon_{yt-1} + a_{21}a_{11}b_{11}\varepsilon_{yt-2}, \\ b_{11}\varepsilon_{yt} + a_{11}b_{11}\varepsilon_{yt-1} + a_{11}^2b_{11}\varepsilon_{yt-2} \end{array}\right) \\ s_2^2 &= var(b_{22}\varepsilon_{\pi t} + b_{21}\varepsilon_{yt} + a_{11}b_{11}\varepsilon_{yt-1} + a_{21}a_{11}b_{11}\varepsilon_{yt-2}) \\ s_{31} &= cov\left(\begin{array}{c} b_{11}\varepsilon_{yt} + a_{11}b_{11}\varepsilon_{yt-1} + a_{11}^2b_{11}\varepsilon_{yt-2}, \\ b_{33}\varepsilon_{rt} + b_{32}\varepsilon_{\pi t} + b_{31}\varepsilon_{yt} + a_{31}b_{11}\varepsilon_{yt-1} + a_{31}a_{11}b_{11}\varepsilon_{yt-2} \end{array}\right) \\ s_{32} &= cov\left(\begin{array}{c} b_{22}\varepsilon_{\pi t} + b_{21}\varepsilon_{yt} + a_{11}b_{11}\varepsilon_{yt-1} + a_{21}a_{11}b_{11}\varepsilon_{yt-2}, \\ b_{33}\varepsilon_{rt} + b_{32}\varepsilon_{\pi t} + b_{31}\varepsilon_{yt} + a_{31}b_{11}\varepsilon_{yt-1} + a_{31}a_{11}b_{11}\varepsilon_{yt-2} \end{array}\right) \\ s_3^3 &= var(b_{33}\varepsilon_{rt} + b_{32}\varepsilon_{\pi t} + b_{31}\varepsilon_{yt} + a_{31}b_{11}\varepsilon_{yt-1} + a_{31}a_{11}b_{11}\varepsilon_{yt-2})\end{aligned}\quad (43)$$

Solving this system, we are able to recover the parameters b_{21} , b_{31} , b_{32} , σ_y^2 , σ_{π}^2 and σ_r^2 .