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RELATIONS FOR MOMENTS OF PROGRESSIVELY TYPE-II RIGHT CENSORED ORDER STATISTICS FROM ERLANG-TRUNCATED EXPONENTIAL DISTRIBUTION

Mansoor Rashid Malik¹, Devendra Kumar²

ABSTRACT

In this paper, we establish some new recurrence relations for the single and product moments of progressively Type-II right censored order statistics from the Erlang-truncated exponential distribution. These relations generalize those established by Aggarwala and Balakrishnan (1996) for standard exponential distribution. These recurrence relations enable computation of mean, variances and covariances of all progressively Type-II right censored order statistics for all sample sizes in a simple and efficient manner. Further an algorithm is discussed which enable us to compute all the means, variances and covariances of Erlang-truncated exponential progressive Type-II right censored order statistics for all sample sizes n and all censoring schemes (R_1, R_2, \dots, R_m) , $m < n$. By using these relations, we tabulate the means and variances of progressively Type-II right censored order statistics of the Erlang-truncated exponential distribution.

Key words: Censoring, progressive Type-II right censored order statistics, single moments, product moments, recurrence relations, Erlang-truncated exponential distribution.

1. Introduction

Practitioners and statisticians are often faced with incomplete or censored data. In life testing, censored samples are present whenever the experimenter does not observe the failure times of all units placed on the life test. This may happen intentionally or unintentionally and may be caused, e.g. by time constraints on the test duration like in Type-I censoring, by requirements on the minimum number of observed failures, or by the structure of a technical system. Naturally, the probabilistic structure of the resulting incomplete data depends heavily on the censoring mechanism and so suitable inferential procedures become necessary. Progressive censoring can be described as a censoring method where units under test are removed from the life

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test at some prefixed or random inspection times. It allows for both failure and time censoring. Many modifications of the standard model have been developed, but the basic idea can be easily described by progressive Type-II censoring, which can also be considered as the most popular model. Under this scheme of censoring, from a total of n units placed simultaneously on a life test, only m are completely observed until failure. Then, given a censoring plan (R_1, R_2, \dots, R_m) :

- i) At the time $x_{1,m:n}$ of the first failure, R_1 of the $n - 1$ surviving units are randomly withdrawn (or censored) from the life-testing experiment.
- ii) At the time $x_{2,m:n}$ of the next failure, R_2 of the $n - 2 - R_1$ surviving units are censored, and so on.
- iii) Finally, at the time $x_{m,m:n}$ of the m th failure, all the remaining $R_m = n - m - R_1 - R_2 - \dots - R_{m-1}$ surviving units are censored.

Note that censoring takes place here progressively in m stages. Clearly, this scheme includes the complete sample situation and the conventional Type-II right censoring scenario as special cases. The ordered failure times $X_{1,m:n}^{(R_1, R_2, \dots, R_m)} \leq X_{2,m:n}^{(R_1, R_2, \dots, R_m)} \leq \dots \leq X_{m,m:n}^{(R_1, R_2, \dots, R_m)}$ arising from such a progressively Type-II right censored sample are called progressively Type-II censored order statistics. These are natural generalizations of the usual order statistics that were studied quite extensively during the past century. For more details one can refer to Balakrishnan and Cramer (2014). The following notations are used throughout this paper. In progressive censoring, the following notations are used:

- (i) $n, m, R_1, R_2, \dots, R_m$ are all integers.
- (ii) m is the sample size (which may be random in some models).
- (iii) n is the total number of units in the experiment.
- (iv) R_j is the number of (effectively employed) removals at the j -th censoring time.
- (v) (R_1, R_2, \dots, R_m) denotes the censoring scheme.

Recurrence relations for single and product moments for any continuous distribution can be used to compute all means, variances and covariance of such a distribution. Several authors obtained the recurrence relation for progressively type-II right censored order statistics for different distributions such as Cohin (1963), Mann [(1971), Thomas and Wilson (1972), Arnold (1992), Balakrishnan et al. (2012), Nikulin and Haghighi (2006), Nadarajah and Haghighi (2011), Joshi (1978), Balakrishnan and Malik (1986), Arnold et al. (1992), Viveros and Balakrishnan (1994), Balakrishnan and Sandhu (1995), Aggarwala and Balakrishnan (1996), Balakrishnan and Aggarwala (1998), Balakrishnan and Sultan (1998), Abd El-Baset and Mohammed (2003), David and Nagaraja (2003), Fernandez (2004), Balakrishnan et al. (2004), Sultan et al. (2006), Mahmoud et al. (2006), Balakrishnan (2007), Bal-

akrishnan et al. (2011), Balakrishnan and Saleh (2013). The moments of order statistics have been recursively derived, see Salah et al. (2008), Kumar and Sanku (2017) and Kumar et al. (2017), for a complete sample.

If the failure times are based on an absolutely continuous distribution function $F(x)$ with probability density function $f(x)$, the joint probability density function of the progressively censored failure times $X_{1:m:n}, X_{2:m:n}, \dots, X_{m:m:n}$, is given by [see Balakrishnan and Aggarwala (2000)].

$$f_{X_{1:m:n}, X_{2:m:n}, \dots, X_{m:m:n}}(x_1, x_2, \dots, x_m) \\ = A(n, m-1) \prod_{i=0}^m f(x_i) [1 - F(x_i)]^{R_i}, \quad -\infty < x_1 < x_2 < \dots < x_m < \infty, \quad (1)$$

where $f(x)$ and $F(x)$ are respectively the pdf and the cdf of the random sample and

$$A(n, m-1) = n(n - R_1 - 1) \cdots (n - R_1 - R_2 - \dots - R_{m-1} - m + 1). \quad (2)$$

The exponential distribution is the simplest distribution in terms of expression and analytical tractability. It is also widely used in reliability engineering. There is no doubt that the wide applicability of the exponential distribution even in inappropriate scenarios is motivated by its simplicity. However, the exponential distribution has a major problem of constant failure/hazard rate property, which makes it inappropriate for modelling data-sets from various complex life phenomena that may exhibit increasing, decreasing or bathtub hazard rate characteristics. El-Alosey (2007) proposed a two parameter Erlang-truncated exponential distribution. The probability density function (pdf) is given by

$$f(x; \beta, \lambda) = \beta(1 - e^{-\lambda})e^{-\beta x(1 - e^{-\lambda})}, \quad x \geq 0, \beta, \lambda > 0, \quad (3)$$

and the corresponding cumulative density function (cdf) is

$$F(x; \beta, \lambda) = 1 - e^{-\beta x(1 - e^{-\lambda})}, \quad x \geq 0, \beta, \lambda > 0. \quad (4)$$

Now in the view of (3) and (4), we have

$$f(x) = \beta(1 - e^{-\lambda})[1 - F(x)]. \quad (5)$$

where β is the shape parameter and λ is the scale parameter. It is important to note that the Erlang-truncated exponential distribution has a constant hazard rate function. The standard exponential distribution is the special case of Erlang-truncated

distribution for $\beta(1 - e^{-\lambda}) = 1$. We shall use the (5) in the following sections to derive some recurrence relations for single and product moments of progressively Type-II right censored order statistics arising from Erlang-truncated exponential distributions.

The inability of the Erlang-truncated exponential distribution to adequately model a variety of complex real data-sets, particularly lifetime ones, has stirred huge concern amongst distribution users and researchers alike and has summoned enormous research attention over the last two decades. Khan et al. (2010) obtained the recurrence relations for single and product moments of generalized order statistics of this distribution. Kulshrestha et al. (2013) obtained the marginal and joint moment generating functions of generalized order statistics and Kumar (2014) obtained the explicit expression for generalized order statistics.

Let X_1, X_2, \dots, X_n be a random sample from the Erlang-truncated exponential distribution with pdf and cdf given in (3) and (4) respectively. The corresponding progressive Type-II right censored order statistics with censoring scheme (R_1, R_2, \dots, R_m) , $m \leq n$ will be

$$X_{1:m:n}^{(R_1, R_2, \dots, R_m)}, X_{2:m:n}^{(R_1, R_2, \dots, R_m)}, \dots, X_{m:m:n}^{(R_1, R_2, \dots, R_m)}.$$

The single moments of the progressive Type-II right censored order statistics from the Erlang-truncated exponential distribution can be written as follows:

$$\begin{aligned} \mu_{i:m:n}^{(R_1, R_2, \dots, R_m)^{(k)}} &= E \left[x_{i:m:n}^{(R_1, R_2, \dots, R_m)^{(k)}} \right] \\ &= A(n, m-1) \int \int \dots \int_{0 < x_1 < x_2 < \dots < x_m < \infty} x_i^k f(x_1) \\ &\quad \times [1 - F(x_1)]^{R_1} f(x_2) [1 - F(x_2)]^{R_2} f(x_3) [1 - F(x_3)]^{R_3} \dots f(x_m) \\ &\quad \times [1 - F(x_m)]^{R_m} dx_1 dx_2 dx_3 \dots dx_m, \end{aligned} \quad (6)$$

where $f(\cdot)$ and $F(\cdot)$ are given respectively in (3), (4), and $A(n, m-1)$ as defined in (2). When $k = 1$, the superscript in the notation of the mean of the progressive Type-II right censored order statistics may be omitted without any confusion.

The outline of this note is as follows. Recurrence relations for single moments of progressive Type-II right censored order statistics from Erlang-truncated exponential distribution are given in section 2. Section 3 describes the recurrence relations for product moments of progressive Type-II right censored order statistics from Erlang-truncated exponential distribution. The recurrence algorithm is carried out in section 4 for Erlang-truncated exponential distribution. Further computations of means and variances from Erlang-truncated exponential progressive Type-II right

censored order statistics for some sample sizes n and some censoring schemes (R_1, R_2, \dots, R_m) , $m < n$ are tabulated in section 5.

2. Single moments of progressively Type-II censored order statistics

In this section, we establish several new recurrence relations satisfied by the single moments of progressive Type-II right censored order statistics from the Erlang-truncated exponential distribution. These recurrence relations may be used to compute the means, variances and covariances of Erlang-truncated exponential progressive Type-II right censored order statistics for all sample sizes n and all censoring schemes (R_1, R_2, \dots, R_m) , $m \leq n$.

Theorem 2.1: For $2 \leq m \leq n$ and $k \geq 0$,

$$\begin{aligned} \mu_{1:m:n}^{(R_1, R_2, \dots, R_m)^{(k+1)}} &= \frac{k+1}{(1+R_1)\beta(1-e^{-\lambda})} \mu_{1:m:n}^{(R_1, R_2, \dots, R_m)^{(k)}} \\ &- \frac{(n-R_1-1)}{(1+R_1)} \mu_{1:m-1:n}^{(R_1+1+R_2, \dots, R_m)^{(k+1)}}. \end{aligned} \quad (7)$$

Proof: From equations (5) and (6), we have

$$\begin{aligned} \mu_{1:m:n}^{(R_1, R_2, \dots, R_m)^{(k)}} &= A(n, m-1) \int \int \cdots \int_{0 < x_1 < x_2 < \cdots < x_m < \infty} \\ &\times L(x_2) f(x_2) [1-F(x_2)]^{R_2} f(x_3) [1-F(x_3)]^{R_3} \cdots f(x_m) \\ &\times [1-F(x_m)]^{R_m} dx_2 dx_3 \cdots dx_m, \end{aligned} \quad (8)$$

where

$$L(x_2) = \int_0^{x_2} x_1^k f(x_1) [1-F(x_1)]^{R_1} dx_1. \quad (9)$$

Using (5) in (9), we get

$$\begin{aligned} L(x_2) &= \int_0^{x_2} x_1^k \left\{ \beta(1-e^{-\lambda}) [1-F(x_1)] \right\} [1-F(x_1)]^{R_1} dx_1 \\ &= \beta(1-e^{-\lambda}) \int_0^{x_2} x_1^k [1-F(x_1)]^{R_1+1} dx_1. \end{aligned} \quad (10)$$

Integrating (10) by parts, we get after simplification

$$\begin{aligned} &= \frac{\beta(1-e^{-\lambda})}{k+1} \left[[1-F(x_2)]^{R_1+1} x_2^{k+1} + (R_1+1) \int_0^{x_2} x_1^{k+1} \right. \\ &\times \left. [1-F(x_1)]^{R_1} f(x_1) dx_1 \right]. \end{aligned} \quad (11)$$

Substituting the value of $L(x_2)$ from (11) in (8) and using (6), we simply have

$$\begin{aligned} \mu_{1:m:n}^{(R_1, R_2, \dots, R_m)^{(k)}} &= \frac{\beta(1 - e^{-\lambda})}{k+1} \left[(n - R_1 - 1) \mu_{1:m-1:n}^{(R_1+1+R_2, \dots, R_m)^{(k+1)}} \right. \\ &\quad \left. + (1 + R_1) \mu_{1:m:n}^{(R_1, R_2, \dots, R_m)^{(k+1)}} \right], \end{aligned}$$

rearranging the above equation gives the result in (7).

Theorem 2.2: For $m = 1, n = 1, 2, \dots$ and $k \geq 0$,

$$\mu_{1:1:n}^{(n-1)^{(k+1)}} = \frac{k+1}{n\beta(1 - e^{-\lambda})} \mu_{1:1:n}^{(n-1)^{(k)}}. \quad (12)$$

Proof: Theorem 2.2 may be proved by following exactly the same steps as those used in proving Theorem 2.1, which is presented above.

Remark 1. We may use the fact that the first progressive Type-II right censored order statistics is the same as the first usual order statistic from a sample of size n , regardless of the censoring scheme employed.

Theorem 2.3: For $2 \leq i \leq m-1, m \leq n$ and $k \geq 0$,

$$\begin{aligned} \mu_{i:m:n}^{(R_1, R_2, \dots, R_m)^{(k+1)}} &= \frac{1}{1 + R_i} \left[\frac{k+1}{\beta(1 - e^{-\lambda})} \mu_{i:m:n}^{(R_1, R_2, \dots, R_m)^{(k)}} \right. \\ &\quad - (n - R_1 - R_2 - \dots - R_i - i) \\ &\quad \times \mu_{i:m-1:n}^{(R_1, R_2, \dots, R_{i-1}, R_i+R_{i+1}+1, R_{i+2}, \dots, R_m)^{(k+1)}} \\ &\quad + (n - R_1 - R_2 - \dots - R_{i-1} - i + 1) \\ &\quad \left. \times \mu_{i-1:m-1:n}^{(R_1, R_2, \dots, R_{i-2}, R_{i-1}+R_i+1, R_{i+1}, \dots, R_m)^{(k+1)}} \right]. \quad (13) \end{aligned}$$

Proof: Theorem 2.3 may be proved by following exactly the same steps as those used in proving Theorem 2.1, which is presented above.

Theorem 2.4: For $2 \leq m \leq n$, and $k \geq 0$,

$$\begin{aligned} \mu_{m:m:n}^{(R_1, R_2, \dots, R_m)^{(k+1)}} &= \frac{k+1}{(1 + R_m)\beta(1 - e^{-\lambda})} \mu_{m:m:n}^{(R_1, R_2, \dots, R_m)^{(k)}} \\ &\quad + \mu_{m-1:m-1:n}^{(R_1, R_2, \dots, R_{m-2}, R_{m-1}+R_m+1)^{(k+1)}}. \quad (14) \end{aligned}$$

Proof: Theorem 2.4 may be proved by following exactly the same steps as those used in proving Theorem 2.1, which is presented above.

Remark 2. Using these recurrence relations, we can obtain all the single moments of all progressive Type-II right censored order statistics for all sample sizes and

censoring schemes (R_1, R_2, \dots, R_m) in a sample recursive manner. The recursive algorithm will be described in detail in section 4.

Corollary 2.1: For $\beta(1 - e^{-\lambda}) = 1$ in (7), we get the recurrence relation for single moments of progressively Type-II censored order statistics from the standard exponential distribution.

$$\begin{aligned} \mu_{1:m:n}^{(R_1, R_2, \dots, R_m)^{(k+1)}} &= \frac{1}{(1 + R_1)} \left[(k + 1) \mu_{1:m:n}^{(R_1, R_2, \dots, R_m)^{(k)}} \right. \\ &\quad \left. - (n - R_1 - 1) \mu_{1:m-1:n}^{(R_1+1, R_2, \dots, R_m)^{(k+1)}} \right], \end{aligned} \quad (15)$$

as obtained by Aggarwala and Balakrishnan [2].

Corollary 2.2: For $\beta(1 - e^{-\lambda}) = 1$ in (12), we get

$$\mu_{1:1:n}^{(n-1)^{(k+1)}} = \frac{k+1}{(n)} \mu_{1:1:n}^{(n-1)^{(k)}}, \quad (16)$$

as obtained by Aggarwala and Balakrishnan [2].

Corollary 2.3: For $\beta(1 - e^{-\lambda}) = 1$ in (13), we get

$$\begin{aligned} \mu_{i:m:n}^{(R_1, R_2, \dots, R_m)^{(k+1)}} &= \frac{1}{1 + R_i} \left[(k + 1) \mu_{i:m:n}^{(R_1, R_2, \dots, R_m)^{(k)}} \right. \\ &\quad - (n - R_1 - R_2 - \dots - R_i - i) \\ &\quad \times \mu_{i:m-1:n}^{(R_1, R_2, \dots, R_{i-1}, R_i + R_{i+1} + 1, R_{i+2}, \dots, R_m)^{(k+1)}} \\ &\quad + (n - R_1 - R_2 - \dots - R_{i-1} - i + 1) \\ &\quad \times \mu_{i-1:m-1:n}^{(R_1, R_2, \dots, R_{i-2}, R_{i-1} + R_i + 1, R_{i+1}, \dots, R_m)^{(k+1)}} \left. \right], \end{aligned} \quad (17)$$

as obtained by Aggarwala and Balakrishnan [2].

Corollary 2.4: For $\beta(1 - e^{-\lambda}) = 1$ in (14), we get

$$\begin{aligned} \mu_{m:m:n}^{(R_1, R_2, \dots, R_m)^{(k+1)}} &= \frac{k+1}{1 + R_m} \mu_{m:m:n}^{(R_1, R_2, \dots, R_m)^{(k)}} \\ &\quad + \mu_{m-1:m-1:n}^{(R_1, R_2, \dots, R_{m-2}, R_{m-1} + R_m + 1)^{(k+1)}}, \end{aligned} \quad (18)$$

as obtained by Aggarwala and Balakrishnan [2].

Deductions: For the special case $R_1 = R_2 = \dots = R_m = 0$ so that $m = n$ in which the progressive censored order statistics become the usual order statistics

$X_{1:n}, X_{2:n}, \dots, X_{n:n}$, then

(i) From Eq. (7): For $k \geq 0$, we get

$$\begin{aligned}\mu_{1:n}^{(k+1)} &= \frac{k+1}{\beta(1-e^{-\lambda})} \mu_{1:n}^{(k)} \\ &- (n-1) \mu_{1:n-1:n}^{(1,0,0,\dots,0)^{(k+1)}}\end{aligned}\quad (19)$$

(ii) From Eq. (13): For $k \geq 0$, we get

$$\begin{aligned}\mu_{i:n}^{(k+1)} &= \frac{k+1}{\beta(1-e^{-\lambda})} \mu_{i:n}^{(k)} \\ &- (n-i) \mu_{i:n}^{(k+1)} + (n-i+1) \mu_{i-1:n}^{(k+1)}\end{aligned}\quad (20)$$

3. Product moments of progressively Type-II censored order statistics

In this section, we establish some recurrence relations for product moments of the progressive Type-II right censored order statistics from the Erlang-truncated exponential distribution. The $(r, s)^{th}$ product moment of the progressive type-II right censored order statistics can be written as

$$\begin{aligned}\mu_{r,s;m:n}^{(R_1, R_2, \dots, R_m)} &= E \left[x_{r:m:n}^{(R_1, R_2, \dots, R_m)} x_{s:m:n}^{(R_1, R_2, \dots, R_m)} \right] \\ &= A(n, m-1) \int \int \dots \int_{0 < x_1 < x_2 < \dots < x_m < \infty} x_r \\ &\times x_s f(x_1) [1 - F(x_1)]^{R_1} f(x_2) \\ &\times [1 - F(x_2)]^{R_2} \dots f(x_m) [1 - F(x_m)]^{R_m} dx_1 dx_2 dx_3 \dots dx_m, \quad (21)\end{aligned}$$

where $f(\cdot)$ and $F(\cdot)$ are given respectively in (3) and (4) and $A(n, m-1)$ is defined in (2).

Theorem 3.1: For $1 \leq i < j \leq m-1$ and $m \leq n$,

$$\begin{aligned}\mu_{i,j;m:n}^{(R_1, R_2, \dots, R_m)} &= \frac{1}{R_j+1} \left[\frac{1}{\beta(1-e^{-\lambda})} \mu_{i,j;m:n}^{(R_1, R_2, \dots, R_m)} \right. \\ &- (n-R_1-1-\dots-R_j-j) \mu_{i,j;m-1:n}^{(R_1, R_2, \dots, R_{j-1}, R_j+R_{j+1}+1, \dots, R_m)} \\ &+ (n-R_1-1-\dots-R_{j-1}-j+1) \\ &\times \left. \mu_{i,j-1;m-1:n}^{(R_1, R_2, \dots, R_{j-1}+R_j+1, \dots, R_m)} \right].\end{aligned}\quad (22)$$

Proof: Using (5) and (6), we have

$$\begin{aligned}
 \mu_{i:m:n}^{(R_1, R_2, \dots, R_m)} &= A(n, m-1) \int \int \cdots \int_{0 < x_1 < \cdots < x_{j-1} < x_{j+1} < \cdots < x_m < \infty} \\
 &\times \left\{ \int_{x_{j-1}}^{x_{j+1}} \beta(1 - e^{-\lambda}) [1 - F(x_j)]^{R_j+1} dx_j \right\} \\
 &\times x_i f(x_1) [1 - F(x_1)]^{R_1} \cdots f(x_{j-1}) \\
 &\times [1 - F(x_{j-1})]^{R_{j-1}} f(x_{j+1}) [1 - F(x_{j+1})]^{R_{j+1}} \cdots f(x_m) \\
 &\times [1 - F(x_m)]^{R_m} dx_1 dx_2 \cdots dx_{j-1} dx_{j+1} \cdots dx_m. \quad (23)
 \end{aligned}$$

Integrating the innermost integral by parts, we obtain

$$\begin{aligned}
 \beta(1 - e^{-\lambda}) \int_{x_{j-1}}^{x_{j+1}} [1 - F(x_j)]^{R_j+1} dx_j &= \beta(1 - e^{-\lambda}) \left[x_{j+1} [1 - F(x_{j+1})]^{1+R_j} \right. \\
 &\quad \left. - x_{j-1} [1 - F(x_{j-1})]^{1+R_j} + (1 + R_j) \right. \\
 &\quad \left. \times \int_{x_{j-1}}^{x_{j+1}} [1 - F(x_j)]^{R_j} f(x_j) x_j dx_j \right],
 \end{aligned}$$

which, when substituted into equation (23) and using (21), we have

$$\begin{aligned}
 \mu_{i:m:n}^{(R_1, R_2, \dots, R_m)} &= \beta(1 - e^{-\lambda}) \left[(n - R_1 - 1 - \cdots - R_j - j) \right. \\
 &\times \mu_{i,j:m-1:n}^{(R_1, R_2, \dots, R_{j-1}, R_j+R_{j+1}+1, \dots, R_m)} - (n - R_1 - 1 - \cdots - R_{j-1} - j + 1) \\
 &\times \mu_{i,j-1:m-1:n}^{(R_1, R_2, \dots, R_{j-1}+R_j+1, \dots, R_m)} + (R_j + 1) \mu_{i,j:m:n}^{(R_1, R_2, \dots, R_m)} \left. \right].
 \end{aligned}$$

Upon rearrangement of this equation, we obtain the relation in (22).

Theorem 3.2: For $1 \leq i \leq m-1$ and $m \leq n$,

$$\begin{aligned}
 \mu_{i,m:n}^{(R_1, R_2, \dots, R_m)} &= \frac{1}{R_m + 1} \left[\frac{1}{\beta(1 - e^{-\lambda})} \mu_{i:m:n}^{(R_1, R_2, \dots, R_m)} \right. \\
 &\quad + (n - R_1 - 1 - \cdots - R_{m-1} - m + 1) \\
 &\quad \times \mu_{i,m-1:m-1:n}^{(R_1, R_2, \dots, R_{m-1}+R_m+1, \dots, R_m)} \left. \right]. \quad (24)
 \end{aligned}$$

Proof: The theorem 3.2 may be proved by following exactly the same steps as those used earlier in proving Theorem 3.1.

Remark 3. Using these recurrence relations, we can obtain all the product moments of progressive Type-II right censored order statistics for all sample sizes and censoring schemes (R_1, R_1, \dots, R_m) . The detailed recursive algorithm will be described in section 4.

Corollary 3.1: For $\beta(1 - e^{-\lambda}) = 1$ in (22), we get the recurrence relation for product moments of progressively Type-II consored order statistics from the standard exponential distribution.

$$\begin{aligned}\mu_{i,j:m:n}^{(R_1, R_2, \dots, R_m)} &= \frac{1}{R_j + 1} \left[\mu_{i:m:n}^{(R_1, R_2, \dots, R_m)} - (n - R_1 - 1 - \dots - R_j - j) \right. \\ &\times \mu_{i,j:m-1:n}^{(R_1, R_2, \dots, R_{j-1}, R_j + R_{j+1} + 1, \dots, R_m)} \\ &+ (n - R_1 - 1 - \dots - R_{j-1} - j + 1) \\ &\times \left. \mu_{i,j-1:m-1:n}^{(R_1, R_2, \dots, R_{j-1} + R_j + 1, \dots, R_m)} \right],\end{aligned}$$

as obtained by Aggarwala and Balakrishnan [2].

Corollary 3.2: For $\beta(1 - e^{-\lambda}) = 1$ in (24), we get

$$\begin{aligned}\mu_{i,m:m:n}^{(R_1, R_2, \dots, R_m)} &= \frac{1}{R_m + 1} \left[\mu_{i:m:n}^{(R_1, R_2, \dots, R_m)} \right. \\ &+ (n - R_1 - 1 - \dots - R_{m-1} - m + 1) \\ &\times \left. \mu_{i,m-1:m-1:n}^{(R_1, R_2, \dots, R_{m-1} + R_m + 1, \dots, R_m)} \right],\end{aligned}$$

as obtained by Aggarwala and Balakrishnan [2].

4. Illustration of the recursive computational algorithm

In this section, we describe the recursive computational algorithm that will produce all the means, variances and covariances of all progressively Type-II right censored order statistics for all sample sizes n and all choices of m and (R_1, R_2, \dots, R_m) from the Erlang-truncated exponential distribution.

4.1. Single moments

All the frist and second order moments with $m = 1$ for all sample sizes n can be obtained by setting $k = 0$ in equation (12) and then again setting $k = 1$ in the same equation. Next using equation (7), we can determine all the moments of the form $\mu_{1,2:n}^{(R_1, R_2)}$, $n = 2, 3, \dots$, which can in turn be used again with (7), to determine all moments of the form $\mu_{1,2:n}^{(R_1, R_2)^2}$, $n = 2, 3, \dots$. (14) can then be used to obtain $\mu_{2,2:n}^{(R_1, R_2)}$ for all R_1, R_2 and $n \geq 2$ and these values can be used to obtain all moments of the form $\mu_{1,2:n}^{(R_1, R_2)^2}$ by using equation (14) again. (7) can now be used again to obtain $\mu_{1,3:n}^{(R_1, R_2, R_3)}$, $\mu_{1,3:n}^{(R_1, R_2, R_3)^2}$ for all n, R_1, R_2 and R_3 and equation (13) can be used next to

obtain all moments of the form $\mu_{2,3:n}^{(R_1, R_2, R_3)}$, $\mu_{2,3:n}^{(R_1, R_2, R_3)^2}$. Finally, equation (14) can be used to obtain all moments of the form $\mu_{3,3:n}^{(R_1, R_2, R_3)}$, $\mu_{3,3:n}^{(R_1, R_2, R_3)^2}$. This process can be continued until all desired first and second order moments and hence all variances are obtained.

4.2. Product moments

From (24), all moments of the form $\mu_{m-1,m:m:n}^{(R_1, R_2, \dots, R_m)}$, $m = 2, 3, \dots, n$, can be determined, since only single moments, which have already been obtained, are needed to calculate them. Then, using (22), all moments of the form $\mu_{i-1,i:m:n}^{(R_1, R_2, \dots, R_m)}$, $2 \leq i < m$, can be obtained. From this point, using (24), we can obtain all moments of the form $\mu_{m-2,m:m:n}^{(R_1, R_2, \dots, R_m)}$, $m = 3, 4, \dots, n$, and subsequently, using (22), all the moments of the form $\mu_{i-2,i:m:n}^{(R_1, R_2, \dots, R_m)}$, $3 \leq i < m$. Continuing this way, all the desired product moments and hence all covariances can be obtained.

5. Numerical results

The recurrence relations obtained in the preceding sections allow us to evaluate the means, variances and covariances of Erlang-truncated exponential progressive Type-II right censored order statistics for all sample sizes n and all censoring schemes (R_1, R_2, \dots, R_m) , $m < n$. These quantities can be used for various inferential purposes; for example, they are useful in determining BLUEs of location/scale parameters and BLUPs of censored failure times. In this section we compute the means and variances of Erlang-truncated exponential progressive Type-II right censored order statistics for sample sizes up to 20 and for different choices of m and progressive schemes (R_1, R_2, \dots, R_m) , $m < n$.

In Table 1-2, we have computed the values of means of the progressive Type-II right censored order statistics for $\lambda = 2, 3$, $\beta = 3, 5$ and different values of m and n . One can see that the means are increasing with respect to m and n but decreasing with respect to β and λ . In Table 3-4, we have computed the variances of the progressive Type-II right censored order statistics for $\lambda = 2, 3$, $\beta = 3, 5$ and different values of m and n . We can see that variances are decreasing with respect to m , n and β , λ .

Tables for the skewness, kurtosis, product moments and covariances of the progressive Type-II right censored order statistics are not presented here but are available from the author on request. All computations here were performed using Mathematica. Mathematica like other algebraic manipulation packages allows for arbitrary precision, so the accuracy of the given values is not an issue.

Table 1. Means of progressively Type-II right censored order statistics for $\lambda = 2$ $\beta = 3$.

$m \downarrow$	$n \downarrow$	Scheme	Mean				
2	5	(0,3)	0.077101	0.173477			
2	5	(3,0)	0.077101	0.462607			
2	8	(6,0)	0.048188	0.433694			
2	8	(0,6)	0.048188	0.103260			
2	10	(8,0)	0.038550	0.424056			
2	10	(0,8)	0.038550	0.081384			
2	12	(10,0)	0.032125	0.417631			
2	12	(0,10)	0.032125	0.067171			
2	15	(13,0)	0.025700	0.411206			
2	15	(0,13)	0.025700	0.053236			
2	18	(16,0)	0.021416	0.406922			
2	18	(0,16)	0.021416	0.044093			
2	20	(18,0)	0.019275	0.404781			
2	20	(0,18)	0.019275	0.039565			
3	5	(2,0,0)	0.077101	0.269854	0.655359		
3	5	(0,0,2)	0.077101	0.173477	0.301979		
3	8	(5,0,0)	0.048188	0.240941	0.626447		
3	8	(0,0,5)	0.048188	0.103260	0.167511		
3	10	(7,0,0)	0.038550	0.231303	0.616809		
3	10	(0,0,7)	0.038550	0.081384	0.129572		
3	12	(9,0,0)	0.032125	0.224878	0.610384		
3	12	(0,0,9)	0.032125	0.067171	0.105722		
3	15	(12,0,0)	0.025700	0.218453	0.603959		
3	15	(0,0,12)	0.025700	0.053236	0.082890		
3	18	(15,0,0)	0.021416	0.214169	0.599675		
3	18	(0,0,15)	0.021416	0.044093	0.068187		
3	20	(17,0,0)	0.019275	0.212028	0.597534		
3	20	(0,0,17)	0.019275	0.039565	0.060982		
4	5	(1,0,0,0)	0.077101	0.205603	0.398356	0.783861	
4	5	(0,0,0,1)	0.077101	0.173477	0.301979	0.494732	
4	8	(4,0,0,0)	0.048188	0.176690	0.369443	0.754949	
4	8	(0,0,0,4)	0.048188	0.103260	0.167511	0.244612	
4	10	(6,0,0,0)	0.038550	0.167052	0.359805	0.745311	
4	10	(0,0,0,6)	0.038550	0.081384	0.129572	0.184645	
4	12	(8,0,0,0)	0.032125	0.160627	0.353380	0.738886	
4	12	(0,0,0,8)	0.032125	0.067171	0.105722	0.148556	
4	15	(11,0,0,0)	0.025700	0.154202	0.346955	0.732461	
4	15	(0,0,0,11)	0.025700	0.053236	0.082890	0.115016	
4	18	(14,0,0,0)	0.021416	0.149918	0.342671	0.728177	
4	18	(0,0,0,14)	0.021416	0.044093	0.068187	0.093888	
4	20	(16,0,0,0)	0.019275	0.147777	0.340530	0.726036	
4	20	(0,0,0,16)	0.019275	0.039565	0.060982	0.083658	
5	5	(0,0,0,0,0)	0.077101	0.173477	0.301979	0.494732	0.880238
5	8	(3,0,0,0,0)	0.048188	0.144564	0.273066	0.465819	0.851325
5	8	(0,0,0,0,3)	0.048188	0.103260	0.167511	0.244612	0.340989
5	10	(5,0,0,0,0)	0.038550	0.134927	0.263429	0.456181	0.841687
5	10	(0,0,0,0,5)	0.038550	0.081384	0.129572	0.184645	0.248896
5	12	(7,0,0,0,0)	0.032125	0.128501	0.257003	0.449756	0.835262
5	12	(0,0,0,0,7)	0.032125	0.067171	0.105722	0.148556	0.196744
5	15	(10,0,0,0,0)	0.025700	0.122076	0.250578	0.443331	0.828837
5	15	(0,0,0,0,10)	0.025700	0.053236	0.082890	0.115016	0.150062
5	18	(13,0,0,0,0)	0.021416	0.117793	0.246295	0.439048	0.824554
5	18	(0,0,0,0,13)	0.021416	0.044093	0.068187	0.093888	0.121424
5	20	(15,0,0,0,0)	0.019275	0.115651	0.244153	0.436906	0.822412
5	20	(0,0,0,0,15)	0.019275	0.039565	0.060982	0.083658	0.107753

Table 2. Means of progressively Type-II right censored order statistics for $\lambda = 3 \ \beta = 5$.

$m \downarrow$	$n \downarrow$	Scheme	Mean				
5	5(0,3)	(0,3)	0.042095	0.094715			
5	5(3,0)	(3,0)	0.042095	0.252574			
8	8(6,0)	(6,0)	0.026309	0.236789			
8	8(0,6)	(0,6)	0.026309	0.056378			
10	10(8,0)	(8,0)	0.021047	0.231527			
10	10(0,8)	(0,8)	0.021047	0.044434			
12	12(10,0)	(10,0)	0.017539	0.228019			
12	12(0,10)	(0,10)	0.017539	0.036674			
15	15(13,0)	(13,0)	0.014031	0.224511			
15	15(0,13)	(0,13)	0.014031	0.029066			
18	18(16,0)	(16,0)	0.011693	0.222172			
18	18(0,16)	(0,16)	0.011693	0.024074			
20	20(18,0)	(18,0)	0.010523	0.221003			
20	20(0,18)	(0,18)	0.010523	0.021601			
5	5(2,0,0)	(2,0,0)	0.042095	0.147335	0.357814		
5	5(0,0,2)	(0,0,2)	0.042095	0.094715	0.164875		
8	8(5,0,0)	(5,0,0)	0.026309	0.131549	0.342028		
8	8(0,0,5)	(0,0,5)	0.026309	0.056378	0.091458		
10	10(7,0,0)	(7,0,0)	0.021047	0.126287	0.336766		
10	10(0,0,7)	(0,0,7)	0.021047	0.044434	0.070744		
12	12(9,0,0)	(9,0,0)	0.017539	0.122779	0.333258		
12	12(0,0,9)	(0,0,9)	0.017539	0.036674	0.057722		
15	15(12,0,0)	(12,0,0)	0.014031	0.119271	0.329750		
15	15(0,0,12)	(0,0,12)	0.014031	0.029066	0.045256		
18	18(15,0,0)	(15,0,0)	0.011693	0.116932	0.327411		
18	18(0,0,15)	(0,0,15)	0.011693	0.024074	0.037229		
20	20(17,0,0)	(17,0,0)	0.010523	0.115763	0.326242		
20	20(0,0,17)	(0,0,17)	0.010523	0.021601	0.033295		
5	5(1,0,0,0)	(1,0,0,0)	0.042095	0.112255	0.217495	0.427974	
5	5(0,0,0,1)	(0,0,0,1)	0.042095	0.094715	0.164875	0.270114	
8	8(4,0,0,0)	(4,0,0,0)	0.026309	0.096469	0.201709	0.412188	
8	8(0,0,0,4)	(0,0,0,4)	0.026309	0.056378	0.091458	0.133554	
10	10(6,0,0,0)	(6,0,0,0)	0.021047	0.091207	0.196447	0.406926	
10	10(0,0,0,6)	(0,0,0,6)	0.021047	0.044434	0.070744	0.100812	
12	12(8,0,0,0)	(8,0,0,0)	0.017539	0.087699	0.192939	0.403418	
12	12(0,0,0,8)	(0,0,0,8)	0.017539	0.036674	0.057722	0.081108	
15	15(11,0,0,0)	(11,0,0,0)	0.014031	0.084191	0.189431	0.399910	
15	15(0,0,0,11)	(0,0,0,11)	0.014031	0.029066	0.045256	0.062796	
18	18(14,0,0,0)	(14,0,0,0)	0.011693	0.081852	0.187092	0.397571	
18	18(0,0,0,14)	(0,0,0,14)	0.011693	0.024074	0.037229	0.051261	
20	20(16,0,0,0)	(16,0,0,0)	0.010523	0.080683	0.185923	0.396402	
20	20(0,0,0,16)	(0,0,0,16)	0.010523	0.021601	0.033295	0.045676	
5	5(0,0,0,0,0)	(0,0,0,0,0)	0.042095	0.094715	0.164875	0.270114	0.480594
8	8(3,0,0,0,0)	(3,0,0,0,0)	0.026309	0.078929	0.149089	0.254328	0.464808
8	8(0,0,0,0,3)	(0,0,0,0,3)	0.026309	0.056378	0.091458	0.133554	0.186173
10	10(5,0,0,0,0)	(5,0,0,0,0)	0.021047	0.073667	0.143827	0.249066	0.459546
10	10(0,0,0,0,5)	(0,0,0,0,5)	0.021047	0.044434	0.070744	0.100812	0.135892
12	12(7,0,0,0,0)	(7,0,0,0,0)	0.017539	0.070159	0.140319	0.245558	0.456038
12	12(0,0,0,0,7)	(0,0,0,0,7)	0.017539	0.036674	0.057722	0.081108	0.107418
15	15(10,0,0,0,0)	(10,0,0,0,0)	0.014031	0.066651	0.136811	0.242051	0.452530
15	15(0,0,0,0,10)	(0,0,0,0,10)	0.014031	0.029066	0.045256	0.062796	0.081931
18	18(13,0,0,0,0)	(13,0,0,0,0)	0.011693	0.064313	0.134472	0.239712	0.450191
18	18(0,0,0,0,13)	(0,0,0,0,13)	0.011693	0.024074	0.037229	0.051261	0.066295
20	20(15,0,0,0,0)	(15,0,0,0,0)	0.010523	0.063143	0.133303	0.238543	0.449022
20	20(0,0,0,0,15)	(0,0,0,0,15)	0.010523	0.021601	0.033295	0.045676	0.058831

Table 3. Variances of progressively Type-II right censored order statistics for $\lambda = 2$ $\beta = 3$.

$m \downarrow$	$n \downarrow$	Scheme	Variance			
2	5	(0,3)	0.005944	0.015233		
2	5	(3,0)	0.005944	0.154559		
2	8	(6,0)	0.002322	0.150936		
2	8	(0,6)	0.002322	0.005355		
2	10	(8,0)	0.001486	0.150100		
2	10	(0,8)	0.001486	0.003320		
2	12	(10,0)	0.001032	0.149646		
2	12	(0,10)	0.001032	0.002260		
2	15	(13,0)	0.000660	0.149275		
2	15	(0,13)	0.000660	0.001418		
2	18	(16,0)	0.000458	0.149073		
2	18	(0,16)	0.000458	0.000972		
2	20	(18,0)	0.000371	0.148986		
2	20	(0,18)	0.000371	0.000783		
3	5	(2,0,0)	0.005944	0.043098	0.191713	
3	5	(0,0,2)	0.005944	0.015233	0.031745	
3	8	(5,0,0)	0.002322	0.039475	0.188090	
3	8	(0,0,5)	0.002322	0.005355	0.009483	
3	10	(7,0,0)	0.001486	0.038639	0.187254	
3	10	(0,0,7)	0.001486	0.003320	0.005643	
3	12	(9,0,0)	0.001032	0.038185	0.186800	
3	12	(0,0,9)	0.001032	0.002260	0.003746	
3	15	(12,0,0)	0.000660	0.037814	0.186428	
3	15	(0,0,12)	0.000660	0.001418	0.002298	
3	18	(15,0,0)	0.000458	0.037612	0.186227	
3	18	(0,0,15)	0.000458	0.000972	0.001553	
3	20	(17,0,0)	0.000371	0.037525	0.186140	
3	20	(0,0,17)	0.000371	0.000783	0.001241	
4	5	(1,0,0,0)	0.005944	0.022457	0.059611	0.208225
4	5	(0,0,0,1)	0.005944	0.015233	0.031745	0.068899
4	8	(4,0,0,0)	0.002322	0.018834	0.055988	0.204603
4	8	(0,0,0,4)	0.002322	0.005355	0.009483	0.015427
4	10	(6,0,0,0)	0.001486	0.017998	0.055152	0.203767
4	10	(0,0,0,6)	0.001486	0.003320	0.005643	0.008675
4	12	(8,0,0,0)	0.001032	0.017544	0.054698	0.203313
4	12	(0,0,0,8)	0.001032	0.002260	0.003746	0.005581
4	15	(11,0,0,0)	0.000660	0.017173	0.054326	0.202941
4	15	(0,0,0,11)	0.000660	0.001418	0.002298	0.003330
4	18	(14,0,0,0)	0.000458	0.016971	0.054125	0.202739
4	18	(0,0,0,14)	0.000458	0.000972	0.001553	0.002213
4	20	(16,0,0,0)	0.000371	0.016884	0.054037	0.202652
4	20	(0,0,0,16)	0.000371	0.000783	0.001241	0.001756
5	5	(0,0,0,0,0)	0.005944	0.015233	0.031745	0.068899
5	8	(3,0,0,0,0)	0.002322	0.011610	0.028123	0.065276
5	8	(0,0,0,0,3)	0.002322	0.005355	0.009483	0.015427
5	10	(5,0,0,0,0)	0.001486	0.010774	0.027287	0.064441
5	10	(0,0,0,0,5)	0.001486	0.003320	0.005643	0.008675
5	12	(7,0,0,0,0)	0.001032	0.010320	0.026833	0.063986
5	12	(0,0,0,0,7)	0.001032	0.002260	0.003746	0.005581
5	15	(10,0,0,0,0)	0.000660	0.009948	0.026461	0.063615
5	15	(0,0,0,0,10)	0.000660	0.001418	0.002298	0.003330
5	18	(13,0,0,0,0)	0.000458	0.009747	0.026259	0.063413
5	18	(0,0,0,0,13)	0.000458	0.000972	0.001553	0.002213
5	20	(15,0,0,0,0)	0.000371	0.009659	0.026172	0.063326
5	20	(0,0,0,0,15)	0.000371	0.000783	0.001241	0.001756

Table 4. Variances of progressively Type-II right censored order statistics for $\lambda = 3$ $\beta = 5$.

$m \downarrow$	$n \downarrow$	Scheme	Variance			
5	5(0,3)	(0,3)	0.001772	0.004540		
5	5(3,0)	(3,0)	0.001772	0.046073		
8	8(6,0)	(6,0)	0.000692	0.044993		
8	8(0,6)	(0,6)	0.000692	0.001596		
10	10(8,0)	(8,0)	0.000443	0.044744		
10	10(0,8)	(0,8)	0.000443	0.000989		
12	12(10,0)	(10,0)	0.000307	0.044609		
12	12(0,10)	(0,10)	0.000307	0.000673		
15	15(13,0)	(13,0)	0.000196	0.044498		
15	15(0,13)	(0,13)	0.000196	0.000422		
18	18(16,0)	(16,0)	0.000136	0.044438		
18	18(0,16)	(0,16)	0.000136	0.000290		
20	20(18,0)	(18,0)	0.000110	0.044412		
20	20(0,18)	(0,18)	0.000110	0.000233		
5	5(2,0,0)	(2,0,0)	0.001772	0.012847	0.057148	
5	5(0,0,2)	(0,0,2)	0.001772	0.004540	0.009463	
8	8(5,0,0)	(5,0,0)	0.000692	0.011767	0.056069	
8	8(0,0,5)	(0,0,5)	0.000692	0.001596	0.002826	
10	10(7,0,0)	(7,0,0)	0.000443	0.011518	0.055819	
10	10(0,0,7)	(0,0,7)	0.000443	0.000989	0.001682	
12	12(9,0,0)	(9,0,0)	0.000307	0.011383	0.055684	
12	12(0,0,9)	(0,0,9)	0.000307	0.000673	0.001116	
15	15(12,0,0)	(12,0,0)	0.000196	0.011272	0.055573	
15	15(0,0,12)	(0,0,12)	0.000196	0.000422	0.000685	
18	18(15,0,0)	(15,0,0)	0.000136	0.011212	0.055513	
18	18(0,0,15)	(0,0,15)	0.000136	0.000290	0.000463	
20	20(17,0,0)	(17,0,0)	0.000110	0.011186	0.055487	
20	20(0,0,17)	(0,0,17)	0.000110	0.000233	0.000370	
5	5(1,0,0,0)	(1,0,0,0)	0.001772	0.006694	0.017769	0.062071
5	5(0,0,0,1)	(0,0,0,1)	0.001772	0.004540	0.009463	0.020538
8	8(4,0,0,0)	(4,0,0,0)	0.000692	0.005614	0.016689	0.060991
8	8(0,0,0,4)	(0,0,0,4)	0.000692	0.001596	0.002826	0.004598
10	10(6,0,0,0)	(6,0,0,0)	0.000443	0.005365	0.016440	0.060742
10	10(0,0,0,6)	(0,0,0,6)	0.000443	0.000989	0.001682	0.002586
12	12(8,0,0,0)	(8,0,0,0)	0.000307	0.005230	0.016305	0.060606
12	12(0,0,0,8)	(0,0,0,8)	0.000307	0.000673	0.001116	0.001663
15	15(11,0,0,0)	(11,0,0,0)	0.000196	0.005119	0.016194	0.060496
15	15(0,0,0,11)	(0,0,0,11)	0.000196	0.000422	0.000685	0.000992
18	18(14,0,0,0)	(14,0,0,0)	0.000136	0.005059	0.016134	0.060435
18	18(0,0,0,14)	(0,0,0,14)	0.000136	0.000290	0.000463	0.000659
20	20(16,0,0,0)	(16,0,0,0)	0.000110	0.005033	0.016108	0.060409
20	20(0,0,0,16)	(0,0,0,16)	0.000110	0.000233	0.000370	0.000523
5	5(0,0,0,0,0)	(0,0,0,0,0)	0.001772	0.004540	0.009463	0.020538
8	8(3,0,0,0,0)	(3,0,0,0,0)	0.000692	0.003461	0.008383	0.019458
8	8(0,0,0,0,3)	(0,0,0,0,3)	0.000692	0.001596	0.002826	0.004598
10	10(5,0,0,0,0)	(5,0,0,0,0)	0.000443	0.003211	0.008134	0.019209
10	10(0,0,0,0,5)	(0,0,0,0,5)	0.000443	0.000989	0.001682	0.002586
12	12(7,0,0,0,0)	(7,0,0,0,0)	0.000307	0.003076	0.007998	0.019074
12	12(0,0,0,0,7)	(0,0,0,0,7)	0.000307	0.000673	0.001116	0.001663
15	15(10,0,0,0,0)	(10,0,0,0,0)	0.000196	0.002965	0.007888	0.018963
15	15(0,0,0,0,10)	(0,0,0,0,10)	0.000196	0.000422	0.000685	0.000992
18	18(13,0,0,0,0)	(13,0,0,0,0)	0.000136	0.002905	0.007827	0.018903
18	18(0,0,0,0,13)	(0,0,0,0,13)	0.000136	0.000290	0.000463	0.000659
20	20(15,0,0,0,0)	(15,0,0,0,0)	0.000110	0.002879	0.007801	0.018877
20	20(0,0,0,0,15)	(0,0,0,0,15)	0.000110	0.000233	0.000370	0.000523

6. Concluding remarks

In this paper, we have established several recurrence relations for the single and product moments of progressively Type-II right censored order statistics from the Erlang-truncated exponential distribution. These relations produce in a simple systematic manner all the means, variances and covariances of progressively Type-II right censored order statistics for all sample sizes and all progressive censoring schemes. we have computed the means and variances of Erlang-truncated exponential progressive Type-II right censored order statistics for sample sizes up to 20 and for different choices of m and progressive schemes (R_1, R_2, \dots, R_m) , $m < n$.

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