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SAMPLE ALLOCATION IN ESTIMATION OF PROPORTION IN A FINITE POPULATION DIVIDED AMONG TWO STRATA

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ABSTRACT

The problem of estimating a proportion of objects with a particular attribute in a finite population is considered. The classical estimator is compared with the estimator, which uses the information that the population is divided among two strata. Theoretical results are illustrated with a numerical example.

Key words: survey sampling, sample allocation, stratification, estimation, proportion.

1. Introduction

Consider a population $U = \{u_1, u_2, ..., u_N\}$ which contains a finite number of *N* units. In this population we can observe objects which have a given characteristic (property), for example sex, defectiveness, support for a particular candidate in elections, etc. Let *M* denote an unknown number of units in the population with a given property. We would like to estimate *M*, or equivalently, a proportion (fraction) $\theta = \frac{M}{N}$. A sample of size *n* is drawn using simple random sampling without replacement scheme. In the sample the number of objects with a particular attribute is observed. This number is a random variable. To be formal, let ξ be a random variable describing number of units having a certain attribute in the sample. The random variable ξ has hypergeometric distribution (Zieliński 2010) and its statistical model is

$$(\{0,1,\ldots,n\},\{H(N,\theta N,n),\theta\in\langle 0,1\rangle\}),\tag{1}$$

with probability distribution function

$$P_{\theta,N,n}\left\{\xi=x\right\} = \frac{\binom{\theta N}{x}\binom{(1-\theta)N}{n-x}}{\binom{N}{n}},\tag{2}$$

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for integer *x* from interval $\langle \max\{0, n - (1 - \theta)N\}, \min\{n, \theta N\} \rangle$. Unbiased estimator with minimal variance of the parameter θ is $\hat{\theta}_c = \frac{\xi}{n}$ (Bracha 1998). Variance of that estimator equals

$$D_{\theta}^{2}\hat{\theta}_{c} = \frac{1}{n^{2}}D_{\theta}^{2}\xi = \frac{\theta(1-\theta)}{n}\frac{N-n}{N-1} \text{ for all } \theta.$$
(3)

It is easy to calculate that variance $D_{\theta}^2 \hat{\theta}_c$ takes on its maximal value at $\theta = \frac{1}{2}$.

2. Stratified estimator

Let contribution of the first strata be w_1 , i.e. $w_1 = N_1/N$. Hence, the overall proportion θ equals

$$\boldsymbol{\theta} = w_1 \boldsymbol{\theta}_1 + w_2 \boldsymbol{\theta}_2, \tag{4}$$

where $w_2 = 1 - w_1$. It seems intuitively obvious to take as our estimate of θ ,

$$\hat{\theta}_{w} = w_1 \frac{\xi_1}{n_1} + w_2 \frac{\xi_2}{n_2},\tag{5}$$

where n_1 and n_2 denote sample sizes from the first and the second strata, respectively. Now, we have two random variables describing the number of units with a particular attribute in samples drawn from each strata:

$$\xi_1 \sim H(N_1, \theta_1 N_1, n_1), \quad \xi_2 \sim H(N_2, \theta_2 N_2, n_2).$$
 (6)

The whole sample size equals $n = n_1 + n_2$. The question now arises: how shall we choose n_1 and n_2 to obtain the best estimate of θ ? This problem concerns sample allocation between strata. One of known approaches to this problem is proportional allocation (Armitage 1943, Cochran 1977). Sample sizes n_1 and n_2 are proportional to w_1 and w_2 ,

$$n_1 = w_1 n$$
 and $n_2 = w_2 n$. (7)

The second approach to sample allocation is Neyman Allocation (Neyman 1934). This method gives values of n_1 and n_2 , which minimize the variance of estimator $\hat{\theta}_w$ for given θ_1 and θ_2 . The values of n_1 and n_2 are as follows

$$n_i = \frac{w_i \sqrt{\theta_i (1 - \theta_i)}}{\sum_i w_i \sqrt{\theta_i (1 - \theta_i)}} n, \quad i = 1, 2.$$
(8)

Neyman Allocation requires knowledge of the parameters θ_1 and θ_2 . Those magnitudes would be known exactly when the population were subjected to exhaustive

sampling. Usually values θ_1 and θ_2 are estimated from a preliminary sample. In some cases fairly good estimates of θ_1 and θ_2 are available from past experience (Armitage 1943).

Since our aim is to estimate θ , hence the parameter θ_1 will be considered as a nuisance one. This parameter will be eliminated by appropriate averaging. Note that for a given $\theta \in [0, 1]$, parameter θ_1 is a fraction M_1/N_1 (it is treated as the number, not as the random variable) from the set

$$\mathscr{A} = \left\{ a_{\theta}, a_{\theta} + \frac{1}{N_1}, a_{\theta} + \frac{2}{N_1}, \dots, b_{\theta} \right\},\tag{9}$$

where

$$a_{\theta} = \max\left\{0, \frac{\theta - w_2}{w_1}\right\}$$
 and $b_{\theta} = \min\left\{1, \frac{\theta}{w_1}\right\}$ (10)

and let L_{θ} be cardinality of \mathscr{A} .

Theorem. Estimator $\hat{\theta}_w$ is an unbiased estimator of θ .

Proof. Note that for a given θ there are L_{θ} values of θ_1 and θ_2 giving θ . Hence, averaging with respect to θ_1 is made assuming the uniform distribution of θ_1 on the set $\{a_{\theta}, \ldots, b_{\theta}\}$. We have

$$E_{\theta}\hat{\theta}_{w} = E_{\theta}\left(w_{1}\frac{\xi_{1}}{n_{1}} + w_{2}\frac{\xi_{2}}{n_{2}}\right) = \frac{1}{L_{\theta}}\sum_{\theta_{1}\in\mathscr{A}}\left(\frac{w_{1}}{n_{1}}E_{\theta_{1}}\xi_{1} + \frac{w_{2}}{n_{2}}E_{\frac{\theta-w_{1}\theta_{1}}{w_{2}}}\xi_{2}\right)$$
$$= \frac{1}{L_{\theta}}\sum_{\theta_{1}\in\mathscr{A}}\left(\frac{w_{1}}{n_{1}}\frac{\theta_{1}N_{1}n_{1}}{N_{1}} + \frac{w_{2}}{n_{2}}\frac{\frac{\theta-w_{1}\theta_{1}}{w_{2}}N_{2}n_{2}}{N_{2}}\right) \quad (11)$$
$$= \theta$$

for all θ .

Averaged variance of estimator $\hat{\theta}_w$ equals:

$$D_{\theta}^{2} \hat{\theta}_{w} = D_{\theta}^{2} \left(w_{1} \frac{\xi_{1}}{n_{1}} + w_{2} \frac{\xi_{2}}{n_{2}} \right) =$$

$$= \frac{1}{L_{\theta}} \sum_{\theta_{1} \in \mathscr{A}} \left(\left(\frac{w_{1}}{n_{1}} \right)^{2} D_{\theta_{1}}^{2} \xi_{1} + \left(\frac{w_{2}}{n_{2}} \right)^{2} D_{\frac{\theta-w_{1}\theta_{1}}{w_{2}}}^{2} \xi_{2} \right) =$$

$$= \frac{1}{L_{\theta}} \sum_{\theta_{1} \in \mathscr{A}} \left[\frac{w_{1}^{2}}{n_{1}} \theta_{1} (1 - \theta_{1}) \frac{N_{1} - n_{1}}{N_{1} - 1} + \frac{w_{2}^{2}}{n_{2}} \frac{\theta - w_{1}\theta_{1}}{w_{2}} \left(1 - \frac{\theta - w_{1}\theta_{1}}{w_{2}} \right) \frac{N_{2} - n_{2}}{N_{2} - 1} \right].$$
(12)

Let $f = \frac{n_1}{n}$ denote the contribution of the first strata in the sample. For $0 < \theta < w_1$

variance of $\hat{\theta}_w$ equals $\left(a_{\theta} = 0 \text{ and } b_{\theta} = \frac{\theta}{w_1}\right)$:

$$\frac{h(f)}{-6(N_1-1)(N_2-1)Nf(1-f)n}\theta + \frac{(N_2-1)N_1 - (N(n+1) - 2(N_1+n))f + (N-2)nf^2}{3(N_1-1)(N_2-1)f(1-f)n}\theta^2,$$
(13)

where

$$h(f) = N_1(N_2 - 3N_1(N_2 - 1) - 1) + (3N_1^2(N_2 - 1) + 3N_2^2 + 2n + N_1(6N_2n - 3N_2^2 - 4n + 1) - N_2(4n + 1)) f + 2(N_1(2 - 3N_2) + 2N_2 - 1)nf^2$$
(14)

For $w_1 \leq \theta \leq 1 - w_1$ variance of $\hat{\theta}_w$ equals $(a_{\theta} = 0 \text{ and } b_{\theta} = 1)$:

$$\frac{\frac{(N_2 - (1 - f)n)}{(N_2 - 1)(1 - f)n}\theta(1 - \theta) + \frac{N_1\left(2(N+1)f^2 + (3NN_2 + N_2 - N_1 - 2n(N+1))f - N_1(N_2 - 1)\right)}{6N^2(N_2 - 1)nf(1 - f)}$$
(15)

To obtain explicit formula for variance of $\hat{\theta}_w$ for $1 - w_1 < \theta < 1$ it is sufficient to replace θ by $1 - \theta$ in (13). Observe that variance $D_{\theta}^2 \hat{\theta}_w$ depends on size *n* of the sample, size *N* of the population, contribution w_1 of the first strata in population and contribution *f* of the first strata in the sample. In Figure 1 variances of $\hat{\theta}_w$ and $\hat{\theta}_c$ are drawn against θ , for N = 100000, n = 100, $w_1 = 0.4$ and f = 0.3.





It is easy to note that $D_{\theta}^2 \hat{\theta}_w = D_{1-\theta}^2 \hat{\theta}_w$ and $D_0^2 \hat{\theta}_w = 0$.

Maximum of variance $D_{\theta}^2 \hat{\theta}_w$ determines for which value of unknown parameter θ estimation of θ is the worst one. After the analysis of variance of $\hat{\theta}_w$, it is seen that the maximal variance may be in the one of the intervals: $(0, w_1)$, $(w_1, 1 - w_1)$ or $(1 - w_1, 1)$. It depends on the values of w_1 and f. In Figures 2, 3, 4 and 5 variance of $\hat{\theta}_w$ as well as variance of $\hat{\theta}_c$ is drawn for N = 100000, n = 100, $w_1 = 0.4$ and f = 0.2, 0.4, 0.6, 0.9.





The point at which $D_{\theta}^2 \hat{\theta}_w$ takes on the maximal value may be located in interval $(0, w_1)$ or in interval $(w_1, 1 - w_1)$. Hence, to find the global maximum due to θ , we have to find local maximum in both intervals. Denote by θ^* a local maximum point in interval $(0, w_1)$ (local maximum point in interval $(1 - w_1, 1)$ is $1 - \theta^*$). In an interval $(w_1, 1 - w_1)$ local maximum is achieved at $\theta = 1/2$. Let $\tilde{\theta}$ denote a global

maximum point, i. e. $\tilde{\theta} = 1/2$ or $\tilde{\theta} = \theta^*$, hence

$$\max_{\theta \in \langle 0,1 \rangle} D_{\theta}^2 \hat{\theta}_w = \max\left\{ D_{0.5}^2 \hat{\theta}_w, D_{\theta^*}^2 \hat{\theta}_w \right\}.$$
(16)

Regardless of which point is the global maximum point $(1/2 \text{ or } \theta^*)$, the maximum of the variance $D_{\theta}^2 \hat{\theta}_w$ depends on size *n* of the sample, size *N* of the population, contribution w_1 of the first strata in the population and the contribution *f* of the first strata in the sample. Values N, n, w_1 are treated as given. It may be seen that for given w_1 , variance $D_{\theta}^2 \hat{\theta}_w$ may be smaller as well as greater than $D_{\theta}^2 \hat{\theta}_c$. We would like to find optimal *f*, which minimizes maximal variance $D_{\tilde{\theta}}^2 \hat{\theta}_w$.

3. Results

A general formula for the optimal f is unobtainable, because of complexity of symbolic computation. But for given N, w_1 and n numerical solution is easy to obtain. Table 1 shows some numerical results for N = 100000 and n = 100.

w_1	f ^{opt}	n_1^{opt}	$D^2_{\widetilde{ heta}} \hat{ heta}_w$	$D_{0.5}^2 \hat{ heta}_c$	$\left(1-rac{D_{\widetilde{ heta}}^2\hat{ heta}_w}{D_{0.5}^2\hat{ heta}_c} ight)\cdot 100\%$
0.05	0.018	2	0.0004645	0.0025	81%
0.10	0.041	4	0.0008404	0.0025	66%
0.15	0.071	7	0.0011328	0.0025	55%
0.20	0.111	11	0.0013493	0.0025	46%
0.25	0.166	17	0.0015004	0.0025	40%
0.30	0.250	25	0.0015984	0.0025	36%
0.35	0.350	35	0.0017045	0.0025	32%
0.40	0.400	40	0.0017982	0.0025	28%
0.45	0.450	45	0.0018544	0.0025	26%
0.50	0.500	50	0.0018731	0.0025	25%

Table 1. Maximal variances $D_{\tilde{\theta}}^2 \hat{\theta}_w$

Source: Own calculations.

In the first column of Table 1. the values of w_1 are given. In the second column, optimal contribution of the first strata in the sample is shown. It is a value f, which gives minimum of $D_{\tilde{\theta}}^2 \hat{\theta}_w$. Column n_1^{opt} shows optimal sample size from the first strata (called averaged sample allocation). The values of minimal (maximal) variances $D_{\tilde{\theta}}^2 \hat{\theta}_w$ are given in the fourth column. The next column contains maximal variance $D_{0.5}^2 \hat{\theta}_c$. The last column shows how much estimator $\hat{\theta}_w$ is better than $\hat{\theta}_c$.

4. Summary

In the paper a new approach to the sample allocation between strata was proposed. Two estimators of an unknown fraction θ in the finite population were considered: standard estimator $\hat{\theta}_c$ and stratified estimator $\hat{\theta}_w$. It was shown that both estimators are unbiased. Their variances were compared. It appears that for a given sample size there exists its optimal allocation between strata, i.e. the allocation for which variance of $\hat{\theta}_w$ is smaller than variance of $\hat{\theta}_c$. Since a theoretical comparison seems to be impossible, hence a numerical example was presented. In that example it was shown that variance of the stratified estimator may be smaller at least 25% with respect to variance of the classical estimator. For such an approach there is no need to estimate unknown θ_1 and θ_2 by preliminary sample. It will be interesting to generalize the above results to the case of more than two "subpopulations". Work on the subject is in progress.

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