

Bell, Clive; Gersbach, Hans; Komarov, Evgenij

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Untimely Destruction: Pestilence, War and Accumulation in the Long Run

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Clive Bell
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Clive Bell

University of Heidelberg

Hans Gersbach

CER-ETH, CEPR and IZA

Evgenij Komarov

CER-ETH

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IZA – Institute of Labor Economics

Schaumburg-Lippe-Straße 5–9
53113 Bonn, Germany

Phone: +49-228-3894-0
Email: publications@iza.org

www.iza.org

ABSTRACT

Untimely Destruction: Pestilence, War and Accumulation in the Long Run*

This paper analyses the effects of disease and war on the accumulation of human and physical capital. We employ an overlapping-generations framework in which young adults, confronted with such hazards and motivated by old-age provision and altruism, make decisions about investments in schooling and reproducible capital. A poverty trap exists for a wide range of stationary war losses and premature adult mortality. If parents are altruistic and their sub-utility function for own consumption is more concave than that for the children's human capital, the only possible steady-state growth path involves full education. Otherwise, steady-state paths with incompletely educated children may exist, some of them stationary ones. We also examine, analytically and with numerical examples, a growing economy's robustness in a stochastic environment. The initial boundary conditions have a strong influence on outcomes in response to a limited sequence of destructive shocks.

JEL Classification: D91, E13, I15, I25, O11, O41

Keywords: premature mortality, capital accumulation and destruction, steady states, poverty traps, overlapping generations

Corresponding author:

Hans Gersbach
EHT Zürich
Department of Management, Technology and Economics
Zürichbergstrasse 18
8092 Zürich
Switzerland
E-mail: hgersbach@ethz.ch

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1 Introduction

Dürer's woodcut, 'The Four Horsemen of the Apocalypse', is a terrifying vision of the great scourges of humanity from time immemorial. This paper deals with three of them – pestilence, war and death, with their accompanying destruction of human and physical capital. Its particular concern is how these calamities affect the accumulation of capital, with special reference to the existence of growth paths and poverty traps. The treatment is necessarily stylised, simple and, in contrast to Dürer's masterpiece, desiccated.

In such a setting, the distinction between human and physical capital is vital. Not only are they complementary in production, but they are also, in general, subject to different, albeit not fully independent, hazard rates. The attendant risks are not, moreover, equally insurable. These considerations weigh heavily in the decision of how much to invest and in what form, with all the ensuing consequences for material prosperity over the long run.

A few selected examples of such calamities will convey some flavour of the historical dimensions of what is involved. The Black Death carried off about one-third of the entire European population between 1347 and 1352. The so-called 'Spanish influenza' pandemic of 1918-1920 is estimated to have caused at least 50 million deaths globally, with exceptionally high mortality among young adults. In recent times, the AIDS pandemic, far slower in its course like the disease itself, still threatens to rival that figure, despite the improved availability of anti-retroviral therapies. Pestilence and war also ride together. Half a million died in an outbreak of smallpox in the Franco-Prussian War of 1870-71 (Morgan, 2002). For every British soldier killed in combat in the Crimean War (1854-56), another ten died of dysentery, and in the Boer War (1899-1902), the ratio was still one to five.

War losses in the 20th Century make for especially grim reading. Between 15 and 20 million people died in the First World War, the majority of them young men. Almost two million French soldiers fell, including nearly 30 per cent of the conscript classes of 1912-15. Joining this companionship of death were over 2 million Germans, including almost two of every five boys born between 1892 and 1895 (Keegan, 1998: 6-7), almost a million members of the British Empire's armed forces, and many millions more in those of Imperial Austria, Russia and Turkey. Its continuation, the Second World War, was conducted, in every respect, on a much vaster scale. Most estimates suggest that it resulted in at least 50 million deaths, directly and indirectly. Among them were 15 million or more Soviet

soldiers and civilians, 6 million Poles (20 per cent of that country's pre-war population) and at least 4 million Germans (Keegan, 1990: 590-1). With these staggering human losses went the razing of German and Japanese cities and massive destruction in the western part of the Soviet Union as well as the states of Eastern Europe. The catalogue of conflicts in the second half of the 20th Century is also unbearably long, with particularly appalling casualties in South-east Asia and Rwanda.

Great epidemics and wars capture the headlines and grip the imagination, but the majority of those adults who die prematurely fall victim to low-level, 'everyday' causes, especially in poor countries: notable killers are endemic communicable diseases, accidents, violence and childbirth. These are competing hazards – one dies only once –, but their combined effect is not wholly negligible even in contemporary O.E.C.D. countries. In many poorer ones, it is quite dismaying. According to the WHO (2007), those who had reached the age of 20 in the O.E.C.D. group could expect to live, on average, another 60 years or so, their counterparts in China and India another 50-55 years, and those in sub-Saharan Africa but 30-40. The odds that a 20-year old in the O.E.C.D. group would not live to see his or her 40th birthday were 1 or 2 in a 100, rising to 2.5-5 in a 100 for the 50th birthday. These odds were just a little worse for young Chinese, decidedly worse for young Indians, and for young Africans much less favourable than those of Russian roulette – in some countries where the AIDS epidemic was raging, indeed, scarcely better than the toss of a fair coin.

The human and material losses so inflicted, whether they are caused by great epidemics and wars, or endemic communicable diseases and low-level conflicts, have long-run as well as immediate economic consequences. Taking as given agents' preferences and the technologies for producing output and human capital in the presence of these hazards, we address the following questions.

1. Under what conditions are steady-state growth paths outcomes in equilibrium?
2. Are such paths possible when parents are moved by altruism; and if so, is stronger altruism conducive to faster steady-state growth?
3. If mortality and destruction rates do not vary over time, are both secular, low-level stagnation and steady-state growth possible equilibria, thus establishing the existence of a poverty trap?

4. If mortality and destruction rates are stochastic, under what conditions would the economy fall into such a trap when it would otherwise be growing?

The overlapping generations model (OLG) offers the natural framework within which to analyse the long-run consequences of economic behaviour in such environments. In the variant adopted here, there are children, young (working) adults and the old. Young adults decide how much schooling the children will receive and how much to put aside to yield a stock of physical capital in the next period. In doing so, they are bound by certain social norms, which govern the distribution of aggregate current consumption among the three generations. Untimely destruction can undo these plans, however carefully laid. The children may die prematurely at some point in young adulthood; and war can wreak havoc on the newly formed capital stock. These losses, if they occur, will reduce the resources available to satisfy claims on consumption in old age in the period that follows. Parents may also be motivated by altruism towards their children, so that their premature deaths will be felt as a distinct loss quite independently of the ensuing reduction in old-age consumption under the prevailing social norms – and arguably all the more keenly if the children have been well educated. The institutional form within which all this takes place is assumed to be a very large extended family, in which the surviving young adults raise all surviving children. Given such pooling, the law of large numbers makes the level of consumption in old age – for those who survive to enjoy it – virtually certain when mortality and war loss rates are forecast unerringly, but even then, the idiosyncratic risk of dying earlier remains. War losses are wholly uninsurable and operate much like cohort-specific mortality. When these rates are stochastic, as is wholly plausible, they constitute unavoidable systemic risks, with consequent effects on investment in both forms of capital.

Our main insights are as follows. Since balanced growth paths with endogenous physical and human capital may not exist – as Uzawa (1961) pointed out in his classic contribution – we first establish conditions for the existence of two extreme steady states, namely, permanent backwardness with no education and unbounded growth with a fully educated population, which we term ‘progress’. Without altruism the well-known poverty trap always exists under standard conditions. Moreover, both backwardness and progress may both exist as equilibria for a wide range of mortality and destruction rates. Parents’ altruism influences the set of balanced growth paths in two ways. First, if sufficiently

strong, it can rule out backwardness. Yet with a robust numerical example, we show that even under quite strong altruism, a poverty trap can exist. Second, in the presence of altruism, progress is the only steady-state path other than backwardness if the sub-utility function for own consumption is more strongly concave than that for parents' evaluation of their children's human capital. If this latter condition is reversed, other steady-state paths with incompletely educated children may exist, some of them stationary, even if altruism is strong.

We also establish conditions for the local stability of a poverty trap. In some settings, both extreme states may be locally stable equilibria, which contrasts with results from corresponding models in which only human capital accumulation matters. We also provide conditions for balanced growth paths with intermediate levels of schooling.

Finally, we explore whether a growing economy can withstand an outbreak of war, a severe epidemic, or a combination of both, as stochastic events; for such destructive events, even if temporary, may pitch a growing economy into backwardness. We establish that these risks depress investment in both physical and human capital; and only extreme destruction of physical capital could induce an increase in schooling. We also establish thresholds for human and physical capital above which an economy can withstand a particular configuration of shocks. We show with simulations that the duration of adverse events – wars or epidemics – is often decisive in determining whether an economy can regain growth.

There is a substantial literature on the relationship between the health of populations and aggregate economic activity. Notable is the general empirical observation that good health has a positive and statistically significant effect on aggregate output (Barro and Sala-I-Martin, 1995; Bloom and Canning, 2000; Bloom, Canning and Sevilla, 2001). What is especially relevant for present purposes, however, is a body of work on the macroeconomic effects of AIDS, in which there are varying points of emphasis. Corrigan, Glomm and Méndez (2005a, 2005b), for example, adopt a two-generation OLG framework in which the epidemic can affect schooling and the accumulation of physical capital, but expectations about future losses play no role. In two contrasting studies of South Africa, Young (2005) uses a Solovian model to estimate the epidemic's impact on living standards through its effects on schooling and fertility, with a constant savings rate; whereas Bell, Devarajan and Gersbach (2006) apply a two-generation OLG model with pooling through extended

families and a vital role for expectations, but no role for physical capital.

Closely related theoretical contributions include Chakraborty (2004), in whose OLG framework endogenous mortality is at centre-stage. Better health promotes growth by improving longevity, and investment in health emerges as a prerequisite for sustained growth. Individual investment in health is also the prime mechanism in Augier and Yaly (2013). Young adults, whose only income is wages, pay a fixed fraction thereof as taxes into a fund managed by the government. This fund provides all capital for the next period, with the gross returns going to the survivors. In Boucekkine and Laffargue's (2010) two-period framework with heterogeneous levels of human capital, a rise in mortality among adults in the first period reduces the proportion of young adults with low human capital in the second period because the mortality rate among children at the end of the first rises more sharply in poor families. The number of orphans in the first period increases, however, so that the proportion of young adults with low human capital in the second period will increase if orphans go poorly educated. Bell and Gersbach (2013) analyse growth paths and poverty traps when epidemics take the form of two-period shocks to mortality, paying particular attention to their effects on inequality in nuclear family systems, albeit without a place for physical capital.

A salient feature of these studies is the central importance, if only implicitly, of premature adult mortality. Physical capital, when it does appear, is not subject to similar hazards. Voigtländer and Voth (2009, 2013) take a Malthusian position in explaining the rise of growth in early modern Europe. Disease and war rode together, but '[war] destroyed human life quickly while not wreaking havoc on infrastructure on a scale comparable to modern wars.' (Voigtländer and Voth (2013: 175). In contrast, the possibility of destruction on such a scale is an essential element of the present paper, in which there are no fixed factors like land. Furthermore, the second part of the paper deals with the robustness of a growing economy to shocks: both destruction rates are stochastic. In this connection, exponential depreciation at a constant rate in Solovian models does not lend itself to the task of representing the shocks of war losses. To our knowledge, no other contribution addresses the possibilities of long-term growth and stagnation when both forms of premature destruction are salient features of the environment wherein agents make decisions about accumulation.

The paper's theme is also broadly related to the existence and relevance of 'balanced growth paths'. The classic problem examined by Uzawa (1961) is whether such paths exist in neoclassical growth models with capital accumulation, population growth and labour- or capital-augmenting technological progress. Grossman et al. (2016) establish that balanced growth requires either an absence of capital-augmenting technological change or a unitary elasticity of substitution between physical and human capital, in which case the forms of factor-augmenting technical change are all equivalent. In this connection, we explore a complementary balanced growth problem: does balanced growth exist in an OLG framework with endogenous physical and human capital accumulation, with or without altruism? We establish conditions on the utility functions with respect to altruism and own consumption that allow balanced growth without imposing very strong restrictions on the production technology.

The plan of the paper is as follows: Section 2 lays out the model and specifies the general problem to be solved. There follows an analysis of steady states, which necessarily involves unchanging mortality and destruction rates. Sections 3 and 4 establish not only the conditions for the existence of a stable, low-level equilibrium in which all generations go uneducated, but also that these conditions and those under which steady-state growth is also an equilibrium in the environment in question can be satisfied simultaneously, thus establishing the existence of a poverty trap. Settings in which the destruction rates are stochastic are treated in Sections 5 to 7. Section 8 briefly draws together the chief conclusions.

2 The Model

There are three overlapping generations: children, who split their time between schooling and work; young adults, who work full time; and the old, who are active neither economically nor in raising children. The timing of events within each period t relates to a generation born in period $t - 1$, thus becoming young adults at the start of period t . It is displayed in Figure 1. Those individuals who survive into full old age in the following period $t + 1$ therefore live for the three periods $t - 1$ to $t + 1$.

All individuals belong to numerous, identical and very large extended families. The number

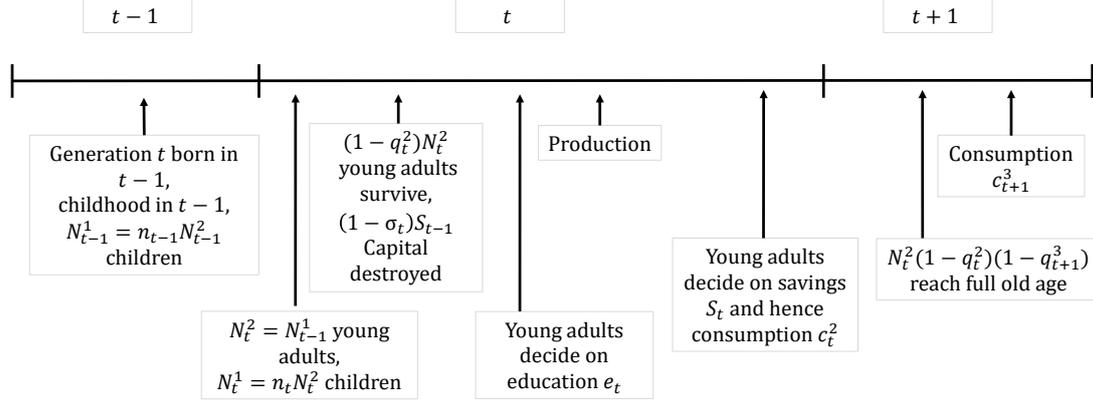


Figure 1: Sequence of Events for the Generation Born in Period $t - 1$.

of young adults in each family at the beginning of period t is N_t^2 . They marry and have children at once. Mortality among children occurs only in infancy, and any child who dies is replaced immediately. After such replacement fertility, each couple within the extended family has $2n_t$ children, all of whom survive into adulthood in the next period. Thus, n_t is the net reproduction rate (NNR). Death then claims some young adults and some of those who have just entered old age. The surviving young adults rear all children collectively and decide how to allocate the children's time between schooling and work, and the resulting aggregate output between consumption and savings, whereby certain social rules govern the claims of children and the old in relation to the consumption of young adults. The numbers of young adults and their offspring who reach maturity are, therefore,

$$N_t^2 = n_{t-1}N_{t-1}^2 \text{ and } N_t^1 = n_t N_t^2,$$

respectively. The numbers of young and old adults who make claims on output in period t are as follows:

$$(1 - q_t^2)N_t^2 \text{ young adults survive to raise all children, and}$$

$$(1 - q_t^3)N_t^3 \text{ old adults survive to full old age, where } N_t^3 = (1 - q_{t-1}^2)N_{t-1}^2$$

and q_t^a denotes the premature mortality rate among age group $a(= 2, 3)$. All adults who do reach full old age in period t die at the end of that period.

Two social rules govern consumption-sharing in the extended family:

- (i) When each surviving young adult consumes c_t^2 , each child consumes βc_t^2 ($\beta < 1$).
- (ii) All surviving old adults receive the share ρ of the family's current 'full income', \bar{Y}_t , which is the level of output that would result if all children were to work full time.¹ Since the extended family is very large, each surviving old adult will consume

$$c_t^3 = \frac{\rho \bar{Y}_t}{(1 - q_t^3) N_t^3}. \quad (1)$$

Output is produced under constant returns to scale by means of labour augmented by human capital (that is, labour is measured in efficiency units) and physical capital, which is made of the same stuff as output. All individuals are endowed with one unit of time. The time the child spends in school in period t is denoted by $e_t \in [0, 1]$. Each young adult possesses λ_t efficiency units of labour, each child γ units. Each fully educated child ($e_t = 1$) requires $w (< 1)$ young adults as teachers, so that the direct cost of providing each child with schooling in the amount e_t is $w \lambda_t e_t$, measured in units of human capital. The total endowment of the surviving young adults' human capital is $\Lambda_t \equiv (1 - q_t^2) N_t^2 \lambda_t$; $\bar{L}_t \equiv \Lambda_t + \gamma N_t^1$ is the household's endowment of labour (measured in efficiency units) at time t ; and the amount of labour supplied to the production of the aggregate good is

$$L_t \equiv [(1 - q_t^2 - w n_t e_t) \lambda_t + n_t \gamma (1 - e_t)] N_t^2.$$

The aggregate savings of the previous period, S_{t-1} , like the cohort of children entering adulthood, are also subject to losses early in the current one, and what does remain has a lifetime of one period. The capital stock available for current production is therefore $K_t = \sigma_t S_{t-1}$, where $\sigma_t \in (0, 1]$ is the survival rate in period t . The current levels of aggregate output and full income are, respectively,

$$Y_t = F(L_t, \sigma_t S_{t-1}) \quad (2)$$

¹The determination of β and ρ is discussed in Section 3.3. A variant of the rule governing old-age provision is discussed in Section 6.

and, putting $e_t = 0$,

$$\bar{Y}_t \equiv Y_t(e_t = 0) = F(\Lambda_t + \gamma N_t^1, \sigma_t S_{t-1}),$$

where the function F is assumed to be monotonically increasing in both arguments, continuously differentiable and homogeneous of degree 1, with both inputs necessary in production.

Full income is available to finance the consumption of all three generations in keeping with the social rules, savings to provide the capital stock in the next period, and investment in the children's education.

$$P_t c_t^2 + S_t + \rho \bar{Y}_t = Y_t, \quad (3)$$

where $P_t \equiv [(1 - q_t^2) + \beta n_t] N_t^2$ is effectively the price of one unit of a young adult's consumption in terms of output, the numéraire.

The formation of human capital involves the contributions of parents' human capital as well as formal education. The human capital attained by a child on reaching adulthood is assumed to be given by

$$\lambda_{t+1} = z_t h(e_t) \lambda_t + 1. \quad (4)$$

The multiplier $z_t (> 0)$ represents the strength with which capacity is transmitted across generations; and it may depend on the number of children each surviving young adult must raise. The function $h(\cdot)$ may be thought of as representing the educational technology, albeit with the fixed pupil-teacher ratio of $1/w$. Let $h(\cdot)$ be an increasing, differentiable function on $[0, 1]$, with $h(0) = 0$ and $\lim_{e \rightarrow 0^+} h'(e) < \infty$. The property $h(0) = 0$ implies that unschooled children attain, as adults, only some basic level of human capital, which has been normalised to unity.

2.1 Preferences And Choices

Young adults, who make all allocative decisions, have preferences over lotteries involving current consumption, consumption in old age and, if they are altruistic, the human capital attained by the children in their care. When deciding on an allocation (c_t^2, e_t, S_t) , young

adults must forecast mortality and destruction rates in the coming period. If these forecasts are unerring, as would be the case in a steady state, those who survive into old age will obtain c_{t+1}^3 , from (1), which the law of large numbers renders virtually non-stochastic. The stochastic element in the lotteries in question therefore arises only from the individual risks of failing to reach old age and, where altruism towards the children is concerned, that the latter will suffer the misfortune to die prematurely in young adulthood. In this connection, let there be full altruism towards adopted children. If, in contrast, the outbreaks of war and disease in the future are viewed as stochastic events, there will be systemic risks. The analysis of such environments is deferred to Sections 5-7.

The surviving young adults' preferences are assumed to be additively separable in $(c_t^2, c_{t+1}^3, \lambda_{t+1})$ and von Neumann-Morgenstern in form:

$$V_t = u(c_t^2) + \delta(1 - q_{t+1}^3)u(c_{t+1}^3) + \frac{b(1 - q_{t+1}^2)}{(1 - q_t^2)}n_tv(\lambda_{t+1}), \quad (5)$$

where δ is the pure impatience factor and b is a taste parameter for altruism. The term $\frac{1}{1-q_t^2}$ accounts for the children in the extended family whose parents have died.² The sub-utility functions u and v are assumed to be strictly concave, where u satisfies $\lim_{c \rightarrow 0} u'(\cdot) = \infty$. In view of the considerations they represent, there are strong reasons to suppose that these functions are not the same.

The surviving young adults' decision problem is as follows:

$$\max_{(c_t^2, e_t, S_t)} V_t \text{ s.t. (1) - (4), } c_t^2 \geq 0, e_t \in [0, 1], S_t \geq 0. \quad (6)$$

When solving it, they note the current state variables, $(n_t, z_t, N_t^1, N_t^2, N_t^3, q_t^2, q_t^3, \lambda_t, K_t)$, and form beliefs about all relevant future levels. Note that these decisions in period t are not influenced by their successors in subsequent periods. Let (c_t^{20}, e_t^0, S_t^0) solve (6).

The evolution of the economy is governed by the following difference equations:

$$\lambda_{t+1}^0 = z_t h(e_t^0) \lambda_t + 1 \text{ and } K_{t+1} = \sigma_{t+1} S_t^0.$$

In what follows, the superscript '0' will be dropped if no confusion would arise.

²If only natural children count, the 'adjustment' for adopted children $1/(1 - q_t^2)$ drops out.

A preliminary step is to normalise the system by the size of the cohort N_t , exploiting the assumption that F is homogeneous of degree one. Let $l_t \equiv L_t/N_t^2$ and $s_t \equiv S_t/N_t^2$, so that (1) and (3) can be written respectively as

$$c_{t+1}^3 = \frac{\rho n_t}{(1 - q_{t+1}^3)(1 - q_t^2)} \cdot F \left[(1 - q_{t+1}^2) \lambda_{t+1}(e_t) + n_{t+1} \gamma, \frac{\sigma_{t+1} s_t}{n_t} \right] \quad (7)$$

and

$$[(1 - q_t^2) + \beta n_t] c_t^2 + s_t + \rho F \left[(1 - q_t^2) \lambda_t + n_t \gamma, \frac{\sigma_t s_{t-1}}{n_{t-1}} \right] = F \left(l_t, \frac{\sigma_t s_{t-1}}{n_{t-1}} \right). \quad (8)$$

Normalised output is

$$y_t \equiv F \left((1 - q_t^2 - w n_t e_t) \lambda_t + n_t \gamma (1 - e_t), \sigma_t s_{t-1} / n_{t-1} \right).$$

The analogous definition of normalised full income is $\bar{y}_t \equiv F(\bar{l}_t, \sigma_t s_{t-1} / n_{t-1})$, where $\bar{l}_t \equiv \bar{L}_t / N_t^2$ denotes the normalised endowment of labour at time t . Closely associated with these normalisations is the ratio $\zeta_t \equiv \lambda_t / s_{t-1}$, which arises from investment decisions in the previous period.

Together with the constraints $c_t^2 \geq 0$, $e_t \in [0, 1]$ and $s_t \geq 0$, the budget identity (8) defines the set of all feasible allocations (c_t^2, e_t, s_t) . Upon substitution for c_{t+1}^3 from (7) into (5), it is seen that V_t is likewise defined in the same space.

3 Steady States

In a steady state, the levels of inputs, output and (dated) consumption all grow at a constant rate. Thus, the parameters $\mathbf{q}_t, n_t, \sigma_t$, and z_t are constant, as is the level of the children's education e_t .³ A special case is that wherein all per capita levels are constant, though population may be growing. In a slight abuse of terminology, this will be called a stationary state, even if the population is not constant. If all per capita levels are growing at the same, positive rate, the economy is said to be on a steady-state growth path. There are two notable steady states, which involve the extreme values of education.

³Since $\lambda_{t+1} = z_t h(e_t) \lambda_t + 1$, the latter property is implied by the former.

If a whole generation of children goes uneducated ($e_t = 0$), so that $\lambda_{t+1} = 1$, the state of backwardness is said to rule in period $t + 1$. If such a state, once reached, becomes permanent, the associated stationary equilibrium implies the existence of a poverty trap. If, at the other extreme, children born in period t enjoy a full education ($e_t = 1$), and all generations that follow them do likewise, this will be called the progressive state, or simply ‘progress’. It may, under certain conditions to be explored below, be an equilibrium which exhibits steady-state growth. A fundamental question to be answered is whether permanent backwardness and a progressive growth path are both possible equilibria of an economy of the kind treated here.

So much for the extremes, but are there also steady-state equilibria in which there is some constant level of education short of a full one? If both backwardness and progress are possible equilibria, considerations of continuity suggest that there exists at least one stationary state with $e_t = e^s \in (0, 1) \forall t$. If, moreover, steady-state growth with $e_t^0 = 1$ is not an equilibrium path, are there such growth paths with e_t^0 constant and sufficiently close to 1?

It will be helpful to rewrite V_t as a function of the decision variables:

$$V_t = u(c_t^2) + \chi_t u \left(\frac{\rho n_t \bar{y}_{t+1}}{(1 - q_{t+1}^3)(1 - q_t^2)} \right) + \nu_t v(z_t h(e_t) \lambda_t + 1), \quad (9)$$

where

$$\chi_t \equiv \delta(1 - q_{t+1}^3) \text{ and } \nu_t \equiv \frac{b(1 - q_{t+1}^2)n_t}{(1 - q_t^2)}.$$

The budget constraint (8) can be expressed as $y_t = [(1 - q_t^2) + \beta n_t]c_t^2 + s_t + \rho \bar{y}_t$. Hence, the associated Lagrangian is

$$\Phi_t = V_t + \mu_t [y_t - [(1 - q_t^2) + \beta n_t]c_t^2 - s_t - \rho \bar{y}_t], \quad (10)$$

whose multiplier is μ_t . Note that y_t depends on the amount of child labor $1 - e_t$. The assumptions on u (the Inada condition at $c_t^2 = 0$) ensure that, at the optimum, $c_t^2 > 0$. By assumption, physical capital is necessary in production. Hence, if some young adults are forecast to survive into full old age ($q_{t+1}^3 < 1$), so that $\chi_t > 0$, then $s_t^0 > 0$.

The associated f.o.c. are set out in Appendix 1. Those w.r.t. c_t^2 and s_t yield

$$\frac{u'(c_t^2)}{u'(c_{t+1}^3)} = \frac{\sigma_{t+1}\delta\rho[(1-q_t^2) + \beta n_t]}{(1-q_t^2)} \cdot F_2 \left[\bar{l}_{t+1}, \frac{\sigma_{t+1}s_t}{n_t} \right], \quad (11)$$

which holds for all $e_t \in [0, 1]$. Those with respect to c_t^2 and e_t yield:

$$\begin{aligned} \delta\rho u'(c_{t+1}^3) \cdot F_1 \left[\bar{l}_{t+1}, \frac{\sigma_{t+1}s_t}{n_t} \right] zh'(e_t) + bv'(\lambda_{t+1})zh'(e_t) \geq \\ \frac{(w\lambda_t + \gamma)u'(c_t^2)}{(1-q^2 + \beta n_t)\lambda_t} F_1 \left[l_t, \frac{\sigma_t s_{t-1}}{n_{t-1}} \right], \quad e \leq 1, \end{aligned} \quad (12)$$

where the inequality holds in the other direction for $e \geq 0$. Substituting from (11) in (12), we obtain, for all interior solutions $e_t \in (0, 1)$,

$$\frac{v'(\lambda_{t+1})}{u'(c_t^2)} = \frac{\left(w + \frac{\gamma}{\lambda_t}\right) F_1 \left[l_t, \frac{\sigma_t s_{t-1}}{n_{t-1}} \right] - \left((1-q^2)zh'(e_t)/\sigma_{t+1}\right) F_1 \left[\bar{l}_{t+1}, \frac{\sigma_{t+1}s_t}{n_t} \right] / F_2 \left[\bar{l}_{t+1}, \frac{\sigma_{t+1}s_t}{n_t} \right]}{b(1-q^2 + \beta n_t)zh'(e_t)}. \quad (13)$$

3.1 Conditions For Backwardness

Given stationary demographic conditions, output per head can increase only if there is some form of technical progress. If time t does not appear as an explicit argument of F , the only possible form of technical progress in the present framework is the labour-augmenting kind, which is expressed by an increase in the average level of human capital possessed by those supplying labour to production. The first question to be answered, therefore, is whether allocations in which no generation receives any schooling can be equilibria, with the result that $\lambda_t = 1 \forall t$. The second, related question is whether such a state is locally stable. If it is, then backwardness – should it once occur – will persist: there will be a poverty trap. The third question, which is of central importance, is whether, in a given stationary setting, both backwardness and progress can be equilibria.

We examine young adults' choice of e_t when they expect the next generation to choose $e_{t+1} = 0$. Given this expectation, $\lambda_t = 1 \forall t$ will be a steady state of the economy if, and only if, each and every generation's optimal choice is $e_t = 0$. We therefore seek to establish

conditions that yield a steady-state path $e_t^0 = 0 \forall t$. Along such a path,

$$\bar{y}_t = y_t(e_t^0 = 0) = F \left[(1 - q_t^2) + n_t \gamma, \frac{\sigma_t s_{t-1}}{n_{t-1}} \right] \forall t,$$

since $\lambda_t = z_t h(0) \lambda_{t-1} + 1 = 1 \forall t$.

Dropping the index t in equation (11), we have

$$\frac{\delta \rho \sigma [(1 - q^2) + \beta n]}{(1 - q^2)} F_2 \left[(1 - q^2) + n \gamma, \frac{\sigma s}{n} \right] u'(c^3) - u'(c^2) = 0.$$

The budget constraint (8) specialises to

$$[(1 - q^2) + \beta n] c^2 + s = (1 - \rho) F[(1 - q^2) + \gamma n, \sigma s/n],$$

and (7) to

$$c^3 = \frac{n \rho}{(1 - q^2)(1 - q^3)} \cdot F[(1 - q^2) + \gamma n, \sigma s/n].$$

Remark: $F[(1 - q^2) + \gamma n, \sigma s/n]$ is the output per young adult at the *start* of each period. Each of them has n children, but only the fraction $(1 - q^2)$ of these adults survive early adulthood. The deceased make no claims on full income in the following period.

Substituting for c^2 and c^3 in (11), we obtain an equation in s , given the constellation (n, q^2, q^3, σ) and the parameters $(\rho, \beta, \gamma, \delta)$. Denote the smallest positive value of s that satisfies this equation by $s^b = s^b(n, q^2, q^3, \sigma)$.

The final step is to examine the counterpart of (13) when $e_t = 0 \forall t$. Rearranging terms, we obtain

$$\left((\gamma + w) - \frac{1 - q^2}{\sigma} \cdot \frac{z h'(0)}{F_2(\bar{l}, \frac{\sigma s}{n})} \right) u'(c^2) F_1(\bar{l}, \frac{\sigma s}{n}) \geq [(1 - q^2) + \beta n] b v'(1) z h'(0), \quad (14)$$

where the derivatives are evaluated at the arguments $((1 - q^2) + \gamma n, \sigma s^b/n)$.

If parents are at all altruistic, the r.h.s. of (14) will be positive, so the said condition can

hold as a strict inequality at the hypothesised $e_t^0 = 0$ only if

$$(1 - q^2)zh'(0) < \sigma(\gamma + w) \cdot F_2[(1 - q^2) + \gamma n, \sigma s^b/n]. \quad (15)$$

A small investment in a child's education will yield $zh'(0)$ units of human capital, over and above the basic endowment of unity, in the next period, with the fraction $1 - q^2$ of all children surviving early adulthood, and so contributing to output.

The cost of this investment involves the sum of the opportunity and direct costs of education at the margin, measured in units of human capital. When $\lambda_t = 1$, this combined direct cost is $(\gamma + w)$ for each child, which is surely less than unity. For a child is much less productive than an uneducated adult and w is the teacher-pupil ratio, with some allowance for an administrative overhead. The alternative is to invest in physical capital. The marginal product thereof, F_2 , is a pure number, since capital is made of the same stuff as output. When adjusted by the survival rate σ , it measures the yield of investing a little more in physical capital, the proportional claim on future full income being ρ for both forms of investment. Hence, σF_2 is the opportunity cost of investing a little in education, considering only making provision for one's old age.

We make the following assumption, which will be relaxed in Section 4:

Assumption 1.

$$u(c_t) = \ln c_t \quad \text{and} \quad v(\lambda_{t+1}) = \ln \lambda_{t+1} .$$

Under Assumption 1, a sufficient condition for (15) to hold is derived as follows.

$$\frac{u'(c_t^2)}{u'(c_{t+1}^3)} = \frac{c_{t+1}^3}{c_t^2} = \frac{[(1 - q^2) + \beta n]n\rho}{(1 - q^2)(1 - q^3)[(1 - \rho) - s^b/F]},$$

where F is evaluated at the arguments $((1 - q^2) + \gamma n, \sigma s^b/n)$. Recalling (11) and noting that $s^b > 0$, it is seen that (15) will hold if

$$\frac{n}{\delta(1 - q^2)(1 - q^3)(1 - \rho)} > \frac{zh'(0)}{\gamma + w}. \quad (16)$$

The l.h.s. depends only on fertility and mortality rates, and the social norm and preference parameters ρ and δ ; the r.h.s. only on those representing the costs of education and the associated marginal yield of human capital at $e_t = 0$. This separation establishes the

existence of a measurable subset of all these parameters such that (15) will indeed hold. Since $n \geq 1$, $\delta < 1$ and both mortality rates and ρ are positive, this is not a very exacting condition, even though $\gamma + w < 1$. In particular, it creates some scope for z to exceed 1, and hence of fulfilling the growth requirement $zh(1) > 1$.

In the absence of altruism ($b = 0$), condition (15) is also sufficient to ensure the existence of a locally stable, steady-state equilibrium in which there is no investment in human capital, children work full time, and output per head is stationary. It does not, however, rule out $zh(1) > 1$, and hence the possible existence of a steady-state path along which output per head grows without limit. If condition (15) holds strongly, then by continuity, the same conclusions will also hold if the altruism motive is sufficiently weak, since the latter implies that the r.h.s of (14) will be small and hence that (14) will hold as a strict inequality. If, however, altruism is strong, such a low-level equilibrium may well not exist. We summarize our findings in Proposition 1.

Proposition 1.

Under Assumption 1, conditions (14) and $zh(1) > 1$ are compatible, especially if altruism is not too strong and the survival rates for investments in both forms of capital are similar. If (14) holds as a strict inequality, backwardness will be a locally stable state. If both conditions hold, an escape can be followed by an asymptotic approach to a steady-state growth path along which output per head increases without bound.

3.2 Conditions For Both a Poverty Trap and Progress

On any steady-state growth path, λ_t and s_{t-1} will increase without bound, and when they are sufficiently large, the contribution of γ in the relevant terms can be neglected. The (asymptotic) rate of growth of λ_t and s_t at any fixed e , denoted by $g(e)$, is given by (4): $1 + g(e) = zh(e)$. A growth path with $e_t = e$ is feasible, therefore, only if the education technology and intergenerational transmission of human capital satisfy the condition $zh(e) > 1$. Each path is effectively defined by the value of e . The state of progress is a steady state with $e = 1 \forall t$.

The f.o.c for positive investment in education may be written as

$$\begin{aligned} \delta \rho u'(c_{t+1}^3) F_1 \left[(1 - q^2) \zeta, \frac{\sigma}{n} \right] z h'(e) + b v'(\lambda_{t+1}) z h'(e) \geq \\ \frac{w u'(c_t^2)}{1 - q^2 + \beta n} F_1 \left[(1 - q^2 - w n e) \zeta, \frac{\sigma}{n} \right], \quad e \leq 1, \end{aligned} \quad (17)$$

so that (13) becomes

$$\frac{v'(\lambda_{t+1})}{u'(c_t^2)} \geq \frac{w F_1 \left[l_t, \frac{\sigma s_{t-1}}{n} \right] - ((1 - q^2) z h'(e) / \sigma) F_1 \left[\bar{l}_{t+1}, \frac{\sigma s_t}{n} \right] / F_2 \left[\bar{l}_{t+1}, \frac{\sigma s_t}{n} \right]}{b[(1 - q^2) + \beta n] z h'(e)}, \quad e \leq 1, \quad (18)$$

The following conditions must be satisfied if both states are to be equilibria.

- (i) Condition (15), which must hold for backwardness ($e_t^0 = 0 \forall t$) to be a locally stable equilibrium.
- (ii) $z h(1) > 1$, so that unbounded growth results when $e_t = 1 \forall t$.
- (iii) $e_t^0 = 1$ along the steady-state path $e = 1$.

In order to ensure that V_t is concave over the feasible set, we also impose

- (iv) $Z(e_t) \equiv v(z h(e_t) \lambda_t + 1)$ is concave $\forall e_t \in [0, 1]$.

This technical requirement is satisfied if $h(e)$ is concave.⁴ We now examine whether these conditions can be met simultaneously.

The social norms, as represented by the values of the parameters β and ρ , play an important role. In the state of backwardness, $\lambda_t = 1$, and although a child's endowment of human capital, γ , is smaller, his or her potential contribution to output will be relatively important. If $\beta < \gamma$, the (relative) claim on the common pot is, in a sense, less than the child's potential contribution, thus favouring child labour over education. This consideration argues for keeping β fairly close to γ , n being exogenous.

The old-age generation's claim to the fraction ρ of current full income can be regarded as stemming from its investments in the previous period. Under pure individualism, with no family considerations other than pooling for insurance purposes, this claim comprises

⁴This analytically convenient restriction on h is not easy to square with the fact that there is the need to lay secure foundations early on in schooling in order to enable the rapid development of wider abilities later on.

the imputed share of physical capital in current output and the return to investments in educating their children. In the state of backwardness, there are no investments in education, so that ρ would then be the said imputed share. In the state of progress, the direct cost of educating each child is $w\lambda_t$ and full income is larger than output, the actual input of human capital being $(1 - q^2 - wn)\lambda_t$. Thus, ρ is a weighted average of physical capital's imputed share in current output and the combined imputed share of physical capital and, neglecting the opportunity cost of the children's endowment, $wn\lambda_t$ units of human capital. Since wn is unlikely to be much greater than 0.1 and altruism enters through v , this way of regarding the norm expressed by ρ argues for keeping its value fairly close to physical capital's imputed share of output in the state of progress. With this preliminary settled, we turn to conditions (i)-(iii).

Condition (i). A sufficient condition for (15) to hold is (16), which is independent of η, b, σ and F and imposes only a mild restriction on $h(e)$, thereby leaving considerable scope to satisfy conditions (ii) and (iii).

Condition (ii). To illustrate, let $h = e$, so that $h'(e) = h(1) = 1$ and $z > 1$ yields $g(1) > 0$.

Condition (iii). Let us define $\zeta(1)$ as the constant ratio of λ_t and s_{t-1} along the state of progress with $e = 1$. Then in this state, $F[(1 - q^2 - wn)\zeta(1), \sigma/n]$ is obtained using (44), which is derived in the proof of Lemma 1 below (see Appendix 1). When F is Cobb-Douglas, $y_t = A \cdot l_t^{1-\alpha} k_t^\alpha$, and $u = \ln c_t$, this condition specialises to

$$A\zeta(1)^{1-\alpha}(\sigma/n)^\alpha[(1 - q^2 - wn)^{1-\alpha} - \rho(1 - q^2)^{1-\alpha}] = \left(1 + \frac{\sigma}{(1 - q^3)\delta}\right) zh(1),$$

that is,

$$\begin{aligned} F[(1 - q^2 - wn)\zeta(1), \sigma/n] &= \frac{(1 - q^2 - wn)^{1-\alpha}}{[(1 - q^2 - wn)^{1-\alpha} - \rho(1 - q^2)^{1-\alpha}]} \cdot \left(1 + \frac{\sigma}{(1 - q^3)\delta}\right) zh(1), \\ &\equiv F[.^p]. \end{aligned} \tag{19}$$

where it should be noted that the right-hand side is independent of the TFP-parameter A .

It is proved in Appendix 1, that the condition for progress to be an equilibrium is

$$zh'(1) \geq \frac{w(1 - \alpha)}{(1 - q^2 - wn)} \cdot F[.^p] \cdot \left(\frac{b\sigma zh(1)}{(1 - q^3)\delta} + \frac{1 - \alpha}{(1 - q^2)n\alpha} \right)^{-1}. \tag{20}$$

In the absence of altruism ($b = 0$), (15) is both necessary and sufficient to ensure the existence of backwardness as a stable equilibrium. Given $u = \ln c_t$ and $h(e) = e$, conditions (i), (ii) and (iii) will be satisfied if there exists a $z > 1$ such that

$$\frac{n(\gamma + w)}{\delta(1 - q^2)(1 - q^3)(1 - \rho)} > z \geq \frac{\alpha n w (1 - q^2)}{(1 - q^2 - wn)} \cdot F[(1 - q^2 - wn)\zeta(1), \sigma/n], \quad (21)$$

whereby the weak inequality is also a necessary condition. The left-hand inequality is readily satisfied by very large ranges of plausible parameter values, with $z > 1$. The same holds for that on the right. For it is seen that although $F[(1 - q^2 - wn)\zeta(1), \sigma/n] > zh(1) > z$, the term $\alpha n w$ takes values quite close to zero, plausibly in the range $[0.02, 0.07]$. We summarize our findings in proposition 2.

Proposition 2.

If $u = \ln c_t$ and F is Cobb-Douglas, both backwardness and progress will be equilibria if there exists an $h(e_t)$ and a z such that conditions (16) and (20) are satisfied, whereby altruism is sufficiently weak. In the absence of altruism ($b = 0$), condition (21) is both necessary and sufficient when $h(e) = e$.

The robust numerical examples that follow in Section 3.3 confirm that a poverty trap, coupled with steady-state growth as an alternative equilibrium, will exist for a wide range of functional forms and plausible parameter values. The function $h(e)$, for example, may be sufficiently weakly convex. If it is strictly convex for all e close to zero, but weakly concave thereafter, it will restrict $h'(0)$ without necessarily making $h'(1)$ too small. Technologies close to Cobb-Douglas will also serve, as will sub-utility functions close to $u = \ln c_t$.

3.3 Numerical Examples

Let $h(e) = d_1 \cdot e - d_2 \cdot e^{d_3}$. Table 1 sets out the whole constellation of parameter values: $h(e)$ is fairly weakly concave ($d_3 = 1.5$), with $h(1) = 0.8$, $h'(0) = 1$ and $h'(1) = 0.7$. Long-run growth at a steady rate is feasible: $zh(1) = 1 + g(1) = 1.2$.

In the first variant, there is no altruism ($b = 0$). Backwardness is an equilibrium; for

$$zh'(0) = 1.5 < \frac{n(\gamma + w)}{\delta(1 - q^2)(1 - q^3)(1 - \rho)} = \frac{1.2(0.6 + 0.075)}{0.85(1 - 0.1)(1 - 0.3)(1 - 0.35)} = 2.327,$$

Table 1: Poverty traps and progress: a constellation of parameter values

Parameter	Value	Variable
n	1.2	net reproduction rate
q^2	0.1	mortality rate at the start of young adulthood
q^3	0.3	mortality rate at the close of young adulthood
σ	0.75	survival rate of physical capital
γ	0.6	a child's endowment of human capital
d_1	1	a parameter of $h(e)$
d_2	0.2	a parameter of $h(e)$
d_3	1.5	a parameter of $h(e)$
z	1.5	transmission factor for human capital formation
w	0.075	teacher-pupil ratio
A	5	TFP parameter
α	1/3	elasticity of output w.r.t. physical capital
δ	0.85	pure impatience factor
b	(0, 0.1)	taste parameter for altruism
ρ	0.35	share of current full income accruing to the old
β	0.325	share parameter for a child's consumption

this inequality being itself a sufficient condition for (15) to hold. The progressive state will also be an equilibrium if, and only if,

$$zh'(1) \geq \frac{\alpha nw(1 - q^2)}{(1 - q^2 - wn)} \cdot F[(1 - q^2 - wn)\zeta(1), \sigma/n].$$

Using the chosen parameter values, we have $zh'(1) = 1.05$, which substantially exceeds

$$\begin{aligned} & \frac{\alpha nw(1 - q^2)}{(1 - q^2 - wn)} \cdot F[(1 - q^2 - wn)\zeta(1), \sigma/n] \\ = & \frac{(1.2 \cdot 0.075/3)(1 - 0.1)}{(1 - 0.1 - 0.09)} \frac{(1 - 0.1 - 0.09)^{2/3}}{(1 - 0.1 - 0.09)^{2/3} - 0.35(1 - 0.1)^{2/3}} \cdot \left(1 + \frac{0.75}{(1 - 0.3)0.85}\right) 1.2 \\ & = 0.1448. \end{aligned}$$

The scope for substantial changes to this constellation of values, while satisfying the conditions in question, is evidently large. Of particular interest is its robustness to altruism. When $b = 0.1$, the said condition is easily satisfied: $zh'(1) = 1.05 \geq 0.1742$, whereby 0.1742 is not much larger than the value 0.1448 when $b = 0$. We conclude that even quite strong altruism is compatible with a poverty trap.

3.4 The Choice Of Schooling

To establish whether there are also other, ‘intermediate’, equilibria, we now analyze how the choice of the level of schooling depends on the stocks of human and physical capital. The optimal level, e_t^0 , depends on the state of the world today, expectations about the state that will rule in the next period, and various state variables, including λ_t and ζ_t .

Let F be Cobb-Douglas. Substituting (7) and (8) into (11) and recalling assumption 1, we obtain

$$\begin{aligned} \frac{\zeta_{t+1}}{\zeta_t^\alpha} &= \left(\frac{zh(e_t) + \frac{1}{\lambda_t}}{\alpha\delta(1-q^3)} + 1 \right) \cdot \\ &\left(A \left(1 - q^2 - wne_t + \frac{n\gamma(1-e_t)}{\lambda_t} \right)^{1-\alpha} - \rho A \left(1 - q^2 + \frac{n\gamma}{\lambda_t} \right)^{1-\alpha} \right)^{-1} \left(\frac{\sigma}{n} \right)^{-\alpha} \equiv \psi(e_t; \lambda_t; \cdot). \end{aligned} \quad (22)$$

Likewise, from (13), we have

$$\begin{aligned} \frac{\zeta_{t+1}}{\zeta_t^\alpha} &= \left(\frac{\delta(1-q^3)(1-q^2)}{1-q^2 + \frac{n\gamma}{\lambda_{t+1}(e_t, \lambda_t)}} + \frac{bn}{1-\alpha} \right) \cdot \\ &\left(\frac{\alpha\delta \left(w + \frac{\gamma}{\lambda_t} \right)}{zh'(e_t)} (1-q^3) A \left(1 - q^2 - wne_t + \frac{n\gamma(1-e_t)}{\lambda_t} \right)^{-\alpha} \left(\frac{\sigma}{n} \right)^\alpha \right)^{-1} \equiv \phi(e_t; \lambda_t; \cdot). \end{aligned} \quad (23)$$

It is seen that this particular iso-elastic combination of preferences and technology yields an optimal choice that depends only on the state variable λ_t and the various parameters; for the optimum satisfies $\psi(e_t^0; \lambda_t; \cdot) = \phi(e_t^0; \lambda_t; \cdot)$, which is independent of the physical capital stock inherited from the decision s_{t-1} (> 0). The extreme values of e_t are covered by noting that if $\psi(e_t; \cdot) = \phi(e_t; \cdot)$ is satisfied by $e_t' \leq 0$ or $e_t' \geq 1$, then $e_t^0 = 0$ or $e_t^0 = 1$, respectively. With e_t^0 thus determined, $\psi(e_t^0; \lambda_t; \cdot) = \zeta_{t+1}/\zeta_t^\alpha$ yields s_t^0 .

To summarise these results precisely:

Proposition 3.

If $u = \ln(c_t)$ and F is Cobb-Douglas, the functions ψ and ϕ yield the optimum values of e_t

and ζ_{t+1} as follows:

- (i) If $e_t < 0$ satisfies $\psi(e_t; \cdot) = \phi(e_t; \cdot)$, then $e_t^0 = 0$ and $\psi(0; \cdot) = \zeta_{t+1}^1 / \zeta_t^\alpha$ yields the value of ζ_{t+1} ($= \lambda_{t+1}(e_t^0 = 0) / s_t^0 = 1 / s_t^0$).
- (ii) If $e_t \in (0, 1)$ satisfies $\psi(e_t; \cdot) = \phi(e_t; \cdot)$, then both (36) and (37) yield the value of ζ_{t+1} .
- (iii) If $e_t > 1$ satisfies $\psi(e_t; \cdot) = \phi(e_t; \cdot)$, then $e_t^0 = 1$ and $\psi(1; \cdot) = \zeta_{t+1}^1 / \zeta_t^\alpha$ yields the value of ζ_{t+1} .

In Figure 2, we illustrate the determination of e_t^0 with the parameter values given above. We plot the difference between $\psi(e_t; \cdot)$ and $\phi(e_t; \cdot)$, which we call $G(e_t, \lambda_t)$, over the values of $e_t \in [0, 1]$ for the values of $\lambda_t \in \{1, 2, 3, \dots, 20\}$. An increase in λ_t shifts $G(e_t, \lambda_t)$ upwards, and so increases e_t^0 , the value of e_t satisfying $G = 0$. This explains how a poverty trap and a progressive state can coexist in our model. A state with a low level of human capital implies a low level of education and the opposite holds for a state with high values for λ_t . This means that once one generation is schooled sufficiently for human capital to increase, the next generation will receive even more schooling and the economy will converge to the progressive state.

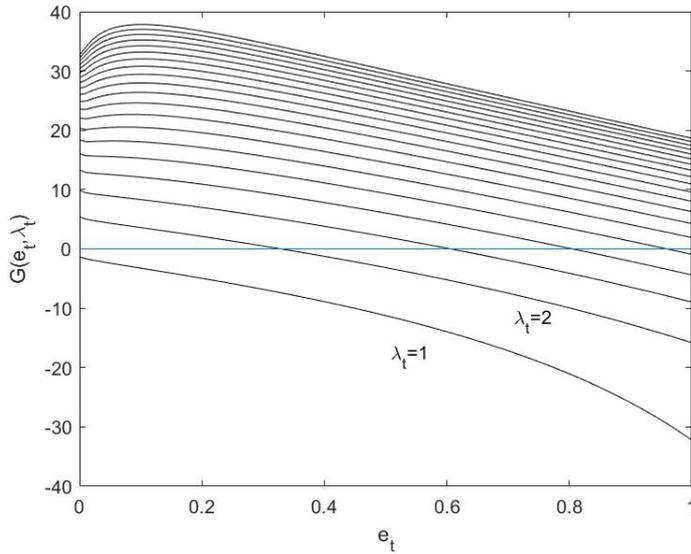


Figure 2: $G(e_t, \lambda_t)$ and the optimal values of $e_t(\lambda_t)$.

3.5 Stationary Paths With Incomplete Schooling

The possibility of paths with stationary levels of s_t and λ_t is of particular interest. If λ_t is stationary, $(1 - zh(e_t))\lambda_t = 1 \forall t$, where $zh(e_t) < 1$. We seek an $e^* \in (0, 1)$ such that $e_t = e^*$, $\lambda^* = 1/(1 - zh(e^*))$ and $s^* = s_t^0(\lambda^*)$. Substituting for λ^* in (22) and rearranging, we obtain

$$\frac{1}{s^*} = \frac{1+\alpha\delta(1-q^3)}{\alpha\delta(1-q^3)} \frac{n}{\sigma} \left(A \left(\frac{1-q^2-wne^*}{1-zh(e^*)} + n\gamma(1-e^*) \right)^{1-\alpha} - \rho A \left(\frac{1-q^2}{1-zh(e^*)+n\gamma} \right)^{1-\alpha} \right)^{-1}.$$

All pairs $(e^* \in (0, h^{-1}(1/z)), s^*)$ satisfying this equation are stationary configurations of the system. For any such pair to be an equilibrium, however, it must also satisfy (13). In the absence of altruism, the latter becomes

$$zh'(e_t) = \left(w + \frac{\gamma}{\lambda_t} \right) \frac{\sigma F_2[\bar{l}_{t+1}, \frac{\sigma s_t}{n}]}{F_1[\bar{l}_{t+1}, \frac{\sigma s_t}{n}]} F_1[l_t, \sigma s_{t-1}/n],$$

where $l_t = [(1 - q^2 - wne^*)/(1 - zh(e^*))] + n\gamma(1 - e^*)$, \bar{l}_t and $s_t = s^*$ are stationary. It is seen that if the derivative $h'(e^*)$ can be chosen independently of the level $h(e^*)$, there is considerable scope to satisfy both conditions. If, in contrast, h is linear or nearly so, then it is far from clear that there exists an equilibrium path with $e_t = e^* \in (0, 1)$. It appears, therefore, that some fairly strong restrictions must be imposed on h to ensure the existence of such stationary paths, with or without altruism.

4 A Generalisation: Isoelastic Functions

We now relax the assumptions on u and F regarding the existence of steady state paths with positive education. The state called progress is of central importance, so the social norm represented by ρ should not make it infeasible. A (weak) necessary condition for them to be compatible is $y_t(e = 1) > \rho \bar{y}_t$, or $F[(1 - q^2 - wn)\zeta, \frac{\sigma}{n}] > \rho F[(1 - q^2)\zeta, \frac{\sigma}{n}]$.

We now turn to (18). The numerator on the r.h.s. of the left weak inequality may be

written as a function of e and ζ . Where the latter is also constant along any such path:

$$D(e, \zeta) \equiv wF_1 \left[(1 - q^2 - wne)\zeta, \frac{\sigma}{n} \right] - \frac{(1 - q^2)zh'(e)}{\sigma} \cdot \frac{F_1 \left[(1 - q^2)\zeta, \frac{\sigma}{n} \right]}{F_2 \left[(1 - q^2)\zeta, \frac{\sigma}{n} \right]}, \quad (24)$$

and hence the ratio $v'(\lambda_{t+1})/u'(c_t^2)$ must be likewise. Condition (11) specialises to

$$\frac{u'(c_t^2)}{u'(c_{t+1}^3)} = \frac{\sigma\delta\rho[(1 - q^2) + \beta n]}{(1 - q^2)} \cdot F_2 \left[(1 - q^2)\zeta, \frac{\sigma}{n} \right], \quad (25)$$

so that $u'(c_{t+1}^3)/u'(c_t^2)$ must also be constant, a requirement that motivates the following restriction on preferences:

Assumption 2.

$$u(c_t) = c_t^{1-\xi}/(1 - \xi).$$

Hence, (25) may be written

$$F_2 \left[(1 - q^2)\zeta, \frac{\sigma}{n} \right] = \frac{(1 - q^2)[(1 + g(e))(c_t^3/c_t^2)]^\xi}{\delta\rho\sigma[(1 - q^2) + \beta n]}. \quad (26)$$

The following lemma enables comparisons to be made across steady-state growth paths.

Lemma 1.

Let e vary parametrically to yield steady-state growth paths. Then ζ is increasing in e for all F that are:

- (i) *Sufficiently close to Cobb-Douglas in form, provided $\xi \leq 1$; or*
- (ii) *Members of the CES family the absolute value of whose elasticity of substitution, $|(\epsilon - 1)^{-1}|$, is at most 1, provided $\xi + \epsilon \leq 1$.*

Proof. See Appendix 1.

Remark: The condition $\xi \leq 1$ in part (i) can be weakened to include values exceeding, but sufficiently close to, 1. Regarding part (ii), if, for example, $\epsilon = -1$, the elasticity of substitution is -0.5 , and the result holds for all $\xi \leq 2$.

Corollary 1.

If $h(e)$ is concave or sufficiently weakly convex, $D(e, \zeta(e))$ is increasing in e across paths.

Proof. See Appendix 1.

The whole yield of human capital in the next period, $zh(e)$, as well as the marginal yield $zh'(e)$ in optimisation, plays a central role. Let e^p denote the smallest value of e satisfying $zh(e) = 1$, where $e^p > 0$ in virtue of $h(0) = 0$. If $e^p \geq 1$, there will exist no steady-state growth path. Corollary 1 yields:

Corollary 2.

If h is strictly concave for all $e \in [e^p, 1]$ and $D(e^p, \zeta(e^p)) > 0$, the expression $D(e, \zeta(e))/[b[(1 - q^2) + \beta n]zh'(e)]$ on the r.h.s. of the left weak inequality in (18) will be continuous, positive and increasing in e for all $e \in (e^p, 1]$.

Under the hypothesis that growth is occurring at a steady rate, the said expression in (18) is a constant. If $e < 1$, the l.h.s. of the left weak inequality must be likewise. If, however, (18) holds as a strict inequality at $e = 1$, the behaviour of $v'(\lambda_{t+1})/u'(c_t^2)$ is not so restricted. That is to say, the requirement that c_t^2 and λ_t grow at the same rate imposes certain restrictions on both v and u .

Assumption 3.

The sub-utility function v is iso-elastic: $v(\lambda) = \lambda^{1-\eta}/(1 - \eta)$.

4.1 No Altruism

A special case of particular interest is the absence of altruism ($b = 0$), wherein v plays no role. Condition (17) then specialises to $0 \geq D(e)$, $e \leq 1$. In virtue of Corollary 1, this yields:

Proposition 4.

If F satisfies the conditions in Lemma 1 and $e^p < 1$, then in the absence of altruism, there are just three possibilities when e is parametric:

- (i) If $D(e^p) > 0$, there exists no steady-state growth path.*
- (ii) If $D(e^p) \leq 0 < D(1)$, there exists a unique, steady-state growth path such that $e \in (e^p, 1)$.*
- (iii) If $D(1) \leq 0$, the only such path is the progressive state.*

The direct costs of education, as represented by the parameter w , exert a strong influence on which of these holds. If w is sufficiently close to zero, it follows from (24) that $D < 0$, so that progress is the only possible outcome, a result which accords with intuition. In fact, the educational system is a fairly heavy user of its own output, so that the other outcomes are then distinctly possible.

The feasibility of growth at a steady rate $g(e)$ when z and h are such that $zh(e) > 1$ does not imply that families will choose to maintain the value of e everywhere along the path in question. Suppose the economy is on such a path. The pairwise marginal rates of transformation among c_t^2 , e_t and s_t are obtained from the budget constraint (8). For any value of $e_t \in [0, 1]$,

$$MRT_{ce} = -\frac{(1 - q^2) + \beta n}{n(w\lambda_t + \gamma)F_1\left[l_t, \frac{\sigma s_{t-1}}{n}\right]}, \quad (27)$$

where $F_1\left[l_t, \frac{\sigma s_{t-1}}{n}\right]$ is constant and terms involving γ can be neglected along the hypothesised path.

Total differentiation of (9) yields the corresponding marginal rate of substitution:

$$MRS_{ce} = -\frac{u'(c_t^2)}{n[\delta\rho F_1(\bar{l}_{t+1}, \frac{\sigma s_t}{n})u'(c_{t+1}^3) + bv'(\lambda_{t+1})]zh'(e_t)\lambda_t} \equiv -\frac{u'(c_t^2)}{Q_t zh'(e_t)\lambda_t} \equiv -R_t. \quad (28)$$

We now compare the levels of MRT_{ce} and MRS_{ce} along the path in question, noting that e_t and $F_1\left[\bar{l}_t, \frac{\sigma s_{t-1}}{n}\right]$ are constant. A continuous approximation yields:

$$ds_t/s_t = dk_t/k_t = d\lambda_t/\lambda_t = dc_t^2/c_t^2 = dc_t^3/c_t^3 = zh(e_t) - 1 = g(e_t). \quad (29)$$

In the absence of altruism ($b = 0$), it is seen from Assumption 2 that $\lambda_t R_t$ is constant, so that R_t is falling at the rate $g(e_t)$. From (27), the same holds for $|MRT_{ce}|$. Hence, if the optimality condition $|MRT_{ce}| \geq |MRS_{ce}|$, $e_t \leq 1$ is once established, it will hold thereafter. This yields the following result.

Proposition 5.

In the absence of altruism, the parametric growth paths defined by parts (ii) and (iii) of Proposition 4 will be sustained by families' optimal choices.

4.2 Altruism

Altruism introduces the additional term $bv'(\lambda_{t+1})$ into Q_t . Differentiating (28) totally and recalling (29) and Assumption 2, we obtain

$$\frac{dR_t}{R_t} = -\left(\xi + \frac{dQ_t}{d\lambda_t} \frac{\lambda_t}{Q_t} + 1\right) \frac{d\lambda_t}{\lambda_t}. \quad (30)$$

It is shown in Appendix 1 that the elasticity of Q_t w.r.t. λ_t can be expressed in the form

$$\frac{dQ_t}{d\lambda_t} \cdot \frac{\lambda_t}{Q_t} = -\frac{\xi A + \eta b \cdot B(1+g)^{-(\eta-\xi)}}{A + b \cdot B(1+g)^{-(\eta-\xi)}}, \quad (31)$$

where $A = \delta\rho F_1(\bar{l}_{t+1}, \frac{\sigma s_t}{n})$ is constant along such a path and B is also a positive constant.

The final step is to establish conditions under which the choice $e_t = e$, once attained, remains optimal as λ_t grows without bound at the rate $g(e)$. As noted above from (27), $|MRT_{ce}|$ falls at the rate $g(e)$. To maintain the optimality of $e_t = e$ when $e < 1$, however, the $|MRS_{ce}| (= R_t)$ must fall at exactly the same rate. If, in any period t , $|MRS_{ce}(e = 1)| < |MRT_{ce}(e = 1)|$, then $e_t^0 = 1$, which will remain optimal thereafter if $|MRS_{ce}(e = 1)|$ falls at least as fast as $|MRT_{ce}(e = 1)|$.

Rewriting (30) as

$$\frac{dR_t}{d\lambda_t} \frac{\lambda_t}{R_t} = -\xi + \frac{\xi A + \eta b \cdot B(1+g(e))^{-(\eta-\xi)}}{A + b \cdot B(1+g(e))^{-(\eta-\xi)}} - 1,$$

we obtain:

Lemma 2.

With altruism ($b > 0$), there exists a steady-state growth path with $e_t^0 = e$ if

$$1 \geq \frac{A + (\eta/\xi)b \cdot B(1+g(e))^{-(\eta-\xi)}}{A + b \cdot B(1+g(e))^{-(\eta-\xi)}}, \quad (32)$$

which must hold as an equality if $e < 1$.

If $e < 1$, (32) will hold iff $\xi = \eta$, whereas if $e = 1$, then $\eta \leq \xi$.

This leaves open what paths are possible if $\eta > \xi$. For if $|MRS_{ce}(e = 1)| < |MRT_{ce}(e = 1)|$ the former may still fall more slowly than the latter without necessarily violating the

condition itself. Suppose, therefore, that $e_t^0 = 1$, with growth proceeding at the steady rate $g(1) > 0$. For t sufficiently large, the child-labour parameter γ can be neglected, and $|MRS_{ce}(e = 1)| < |MRT_{ce}(e = 1)|$ if, and only if,

$$\frac{(1 - q^2) + \beta n}{wF_1[l_t, \frac{\sigma s_{t-1}}{n}]} > \frac{u'(c_t^2)}{[\delta \rho F_1(\bar{l}_{t+1}, \frac{\sigma s_t}{n})u'(c_{t+1}^3) + bv'(\lambda_{t+1})]zh'(1)}.$$

By hypothesis, c_t^2, c_t^3 and λ_t are all growing at the same steady rate, and the derivatives $F_1(\cdot)$ are constant. Thus the l.h.s. is constant. By assumptions 2 and 3, the r.h.s. may be written $A_1/[B_1 + b \cdot B_2(1 + g(1))^{(\xi - \eta)t}]zh'(1)$, where A_1, B_1 and B_2 are positive constants. If $\eta > \xi$, then in the limit as $t \rightarrow \infty$, the r.h.s. goes to $A_1/B_1zh'(1)$: the altruism term $b \cdot B(1 + g(1))^{(\xi - \eta)t}$ goes to zero. It then follows from Propositions 4 and 5 that the postulated path of progress can hold if, and only if, $D(1) \leq 0$. The same argument holds if the postulated steady-state growth path is such that $e_t < 1$, whereby $D(e^p) \leq 0 < D(1)$ must now hold.

The only remaining possible steady states are stationary ones, wherein $e_t^0 = e_{t-1}^0, \lambda_t = \lambda_{t-1}$ and $s_t^0 = s_{t-1}^0$ for all t . Denoting stationary values by a *, we have $\lambda^* = 1/[1 - zh(e^*)]$. In virtue of the assumption $zh(1) > 1$, there exists an $e^* \in (0, 1)$ satisfying the latter condition. If such an equilibrium exists, then $MRS_{ce}(e^*) = MRT_{ce}(e^*)$ and, from (13), $D(e^*) > 0$. The altruism term is now operative for all t , and $Q_t = n[\delta \rho F_1(\bar{l}^*, \frac{\sigma s^*}{n})u'(c^{3*}) + bv'(\lambda^*)]$ is constant. There may exist more than one such $e^* \in (0, 1)$.

These results are summarised as:

Proposition 6.

With iso-elastic preferences, the possible steady-state paths with positive education are as follows.

- (i) *If u and v differ, with $\eta < \xi$, progress ($e = 1$) is the sole steady-state path that can be supported by families' optimising decisions.*
- (ii) *If $\eta = \xi$, there may exist steady-state growth paths with a less than fully educated population ($e < 1$); but the state of progress is also possible as a limiting case.*
- (iii) *If $\eta > \xi$ and $D(1) \leq 0$, then progress is the sole steady-state path that can be supported by families' optimising decisions. If $D(1) > 0$, progress is ruled out. If $D(e^p) \leq 0 <$*

$D(1)$, then by part (ii) of Proposition 4, steady-state growth is possible with $e_t < 1$. If $D(e^p) > 0$, there is no steady-state growth path, but there may exist a stationary state wherein $e > 0$.

What is the intuition for these findings? If R_t is falling at the rate $g(e)$ and $e < 1$, both terms in Q_t must fall at the same rate to preserve $MRS_{ce} = MRT_{ce}$, which imposes $\xi = \eta$ – a very strong restriction on preferences. In relaxing it, consider the path $e = 1$, along which R_t may fall at a rate faster than $g(1)$ without violating the conditions for optimality. Investing in education provides both for old age and the children’s well-being in adulthood, as expressed by the two terms comprising Q_t . Since λ_{t+1} and c_{t+1}^3 are growing at the rate $g(1)$, it suffices that $v'(\lambda_{t+1})$ fall no faster than $u'(c_{t+1}^3)$. That is to say, if v is less strongly concave than u , then steady-state growth will be maintained. If parents are perfectly selfish, this consideration does not arise.

If, however, u is less strongly concave than v ($\eta > \xi$), potential difficulties arise; for along any postulated growth path, $v'(\lambda_{t+1})$ is falling faster than $u'(c_t^2)$ and $u'(c_{t+1}^3)$. With iso-elastic preferences, there is a common growth term in the numerator and denominator of MRS_{ce} , but with different exponents. If $\eta > \xi$, then upon division, the altruism term, which appears only in the denominator, goes asymptotically to zero as t becomes arbitrarily large, and the results of Section 4.1 apply.

We have already established that altruism can rule out backwardness, which is not surprising. It now emerges that it may also fail to yield steady-state growth when u and v are iso-elastic and v is more concave than u . For diminishing marginal returns then set in faster where the evaluation of the children’s human capital is concerned than that of own consumption, and growth renders altruism effectively inoperative over the long run. With steady growth ruled out when the direct costs of education are so high that $D(e^p) > 0$, the only remaining possible steady states are stationary ones.

5 War and Pestilence as Stochastic Events

In reality, mortality and destruction rates are, in some degree, stochastic; for the outbreak of war or a severe epidemic are events that cannot be forecast with certainty. This fact rules out steady-state growth, and if there is a poverty trap, such shocks may pitch a growing

economy into backwardness. In order to analyse this possibility, a preliminary step is needed, namely, to establish how such events influence consumption and investment. We formulate the shock as the actual outbreak of war, coupled with the (prior) probability of its occurrence. This prior is assumed to be sharp.⁵

Let $I_t \in \{0, 1\}$ denote the states of peace and war, respectively, in period t , and let $\pi_{t+1} = Pr(I_{t+1} = 0)$ denote the probability of peace in period $t + 1$. The survival rate of physical capital is $\sigma_t(I_t)$, where $\sigma_t(1) < \sigma_t(0) \leq 1$. Mortality rates \mathbf{q}_t are likewise dependent on I_t . It is almost surely the case that $q_t^a(1) > q_t^a(0)$ ($a = 2, 3$), and this much will be assumed. By assumption, I_t is known when decisions are taken in period t . Consumption in old age, denoted by $c_{t+1}^3(I_{t+1})$ for those who survive to enjoy it, now depends on the state ruling in period $t + 1$. The large extended family cannot provide insurance against this particular risk, which does not exist in a steady state.

The young adults' preferences now involve not only the compound lottery arising from the future state I_{t+1} , but also the current realisation of I_t if this affects q_t^2 .

$$V_t(I_t) = u(c_t^2) + \delta [\pi_{t+1}(1 - q_{t+1}^3(0))u(c_{t+1}^3(0)) + (1 - \pi_{t+1})(1 - q_{t+1}^3(1))u(c_{t+1}^3(1))] + \frac{b [\pi_{t+1}(1 - q_{t+1}^2(0)) + (1 - \pi_{t+1})(1 - q_{t+1}^2(1))]}{(1 - q_t^2(I_t))} \cdot nv(\lambda_{t+1}), \quad I_t = 0, 1. \quad (33)$$

Exploiting as before the assumption that F is homogeneous of degree one, we have

$$c_{t+1}^3(I_{t+1}; I_t) = \frac{\rho n \cdot F[(1 - q_{t+1}^2(I_{t+1}))\lambda_{t+1}(e_t) + n\gamma, \sigma_{t+1}(I_{t+1})s_t/n]}{(1 - q_t^2(I_t))(1 - q_{t+1}^3(I_{t+1}))}. \quad (34)$$

The budget constraint becomes

$$[(1 - q_t^2(I_t)) + \beta n]c_t^2 + s_t + \rho F \left[(1 - q_t^2(I_t))\lambda_t + n\gamma, \frac{\sigma_t(I_t)s_{t-1}}{n} \right] \leq F \left[(1 - q_t^2(I_t) - wne_t)\lambda_t + n\gamma(1 - e_t), \frac{\sigma_t(I_t)s_{t-1}}{n} \right], \quad I_t = 0, 1, \quad (35)$$

where the dependence of current decision variables on the current realised state can be (notationally) suppressed without ambiguity.

To analyze the economy's behaviour in the face of systemic shocks, we proceed essentially

⁵For a vigorous argument that rational actors must have sharp priors, see Elga (2010).

as before, noting that the choices of s_t and e_t determine the productive endowments in the next period and hence ζ_{t+1} . Let F be Cobb-Douglas. We provide the generalized versions of (22) and (23) in a stochastic setting:

$$\frac{\zeta_{t+1}}{\zeta_t^\alpha} = \left(\frac{zh(e_t) + \frac{1}{\lambda_t}}{\alpha\delta E_t[1 - q_{t+1}^3(I_{t+1})]} + 1 \right) \cdot \left(\frac{\sigma_t(I_t)}{n} \right)^{-\alpha}. \quad (36)$$

$$\left(A \left(1 - q_t^2(I_t) - wne_t + \frac{n\gamma(1 - e_t)}{\lambda_t} \right)^{1-\alpha} - \rho A \left(1 - q_t^2(I_t) + \frac{n\gamma}{\lambda_t} \right)^{1-\alpha} \right)^{-1} = \psi(e_t, I_t; \cdot)$$

and, if $e_t \in (0, 1)$,

$$\frac{\zeta_{t+1}}{\zeta_t^\alpha} = \left(\frac{\alpha\delta \left(w + \frac{\gamma}{\lambda_t} \right)}{zh'(e_t)} E_t[1 - q_{t+1}^3(I_{t+1})] A \left(1 - q_t^2(I_t) - wne_t + \frac{n\gamma(1 - e_t)}{\lambda_t} \right)^{-\alpha} \right. \\ \left. \left(\frac{\sigma_t(I_t)}{n} \right)^\alpha \right)^{-1} \cdot \left(E_t \left[\frac{(1 - q_{t+1}^3(I_{t+1}))(1 - q_{t+1}^2(I_{t+1}))}{1 - q_{t+1}^2(I_{t+1}) + n\gamma/\lambda_{t+1}} \right] \delta + \frac{\tilde{b}n}{1 - \alpha} \right) = \phi(e_t, I_t; \cdot), \quad (37)$$

where $\tilde{b} = b \cdot [\pi_{t+1}(1 - q_{t+1}^2(0)) + (1 - \pi_{t+1})(1 - q_{t+1}^2(1))]/(1 - q_t^2(I_t)) = b \cdot E_t[1 - q_{t+1}^2]/1 - q_t^2$. The forms of ψ and ϕ are highly complicated, even under the assumption that F is Cobb-Douglas, so it would be as well to untangle their elements, relying rather on intuition. We therefore discuss how the various factors in play influence the final outcome, but without the said restriction on F .

5.1 The Occurrence Of War

The first step is to examine how I_t and q_t^2 affect the set of current feasible choices, which is independent of π_{t+1} . This set is denoted by

$$S(I_t) = \{c_t^2, e_t, s_t : (35), c_t^2 \geq 0, e_t \in [0, 1], s_t \geq 0\}.$$

It is seen that if the ratio of survival rates, $\sigma_t(I_t)/(1 - q_t^2(I_t))$, is independent of the current state, then the outer frontier of the feasible set is affected only by the mortality rate $q_t^2(I_t)$, though the latter may certainly depend on the current state. If the said ratio is indeed independent of I_t , an increase in q_t^2 , whether associated with war or not, also makes c_t^2

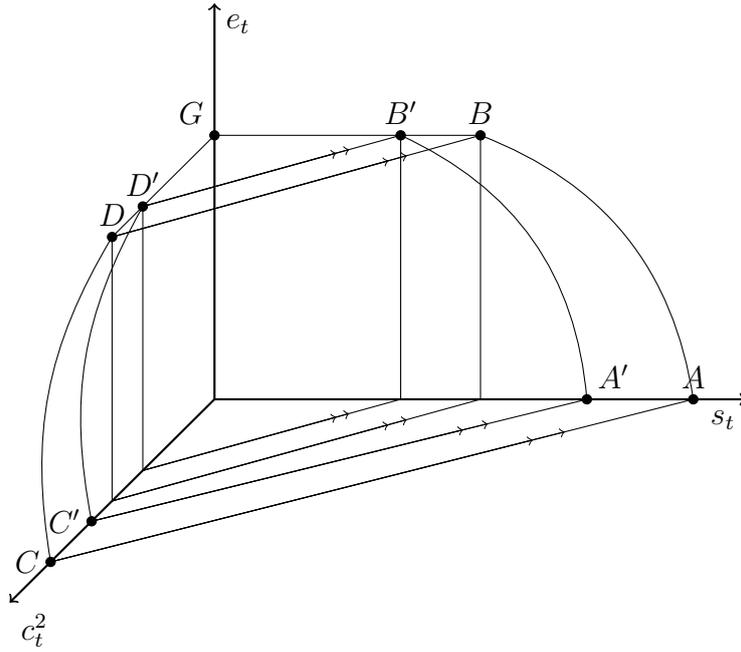


Figure 3: Feasible sets of consumption and investment

cheaper relative to s_t .

The extreme allocations of $S(I_t)$'s outer frontier, which are depicted as the points A, B, C and D, respectively, in Figure 3, are examined in Appendix 2. To summarise: A sufficient condition for $(dc_t^2/dq_t^2(I_t))_{e_t=s_t=0} < 0 \forall \lambda_t$ is $\beta > \gamma$, which is not a very strong requirement. If the ratio of survival rates, $\sigma_t(I_t)/(1 - q_t^2(I_t))$, is fixed for each current state I_t and $\beta > \gamma$, the outer frontier of the feasible set $S(I_t)$ will contract inwards everywhere as the mortality rate $q_t^2(I_t)$ rises. If the said ratio is the same for both states, the contraction from $S(0)$ to $S(1)$ represents the effects of an outbreak of war.

To complete the argument, consider the case where $e_t = 1$ is infeasible for sufficiently large values of $q_t^2(I_t)$. Suppose that when $q_t^2(I_t) = 0$, the maximal values of c_t^2 and s_t , respectively, are both positive, as depicted in the figure when ABCD corresponds to a zero level of such premature mortality. As $q_t^2(I_t)$ progressively increases, BD will shift towards G until the allocations B and D coincide at G ($e_t = 1, c_t^2 = s_t = 0$). Further increases in $q_t^2(I_t)$ will reduce the maximal feasible level of e_t below one, with the associated allocation moving downwards along the e_t -axis towards the origin O. Since AC also shifts progressively inwards towards O, the outer frontier of $S(I_t)$ contracts everywhere as $q_t^2(I_t)$ increases.

The contraction of the feasible set established above points to unambiguous income effects, c_t^2, e_t and s_t being all normal goods; but changes in survival rates also imply changes in marginal rates of transformation, which are now examined. As noted above, given the current state I_t , an increase in $q_t^2(I_t)$ makes c_t^2 cheaper relative to s_t , as does an outbreak of war if this event leaves the ratio of survival rates unchanged. Turning to the marginal rate of transformation between s_t and e_t , we have

$$MRT_{se}(I_t) = - \left([n(w\lambda_t + \gamma)]F_1 \left[(1 - q_t^2(I_t) - wne_t)\lambda_t + n\gamma(1 - e_t), \sigma_t(I_t) \cdot \frac{s_{t-1}}{n} \right] \right)^{-1}.$$

For any given I_t and e_t , an increase in $q_t^2(I_t)$ will increase F_1 and so reduce $|MRT_{se}(I_t)|$: s_t will become cheaper relative to e_t , as intuition would suggest.

An outbreak of war, however, has ambiguous effects on $MRT_{se}(I_t)$. Since F_1 is homogeneous of degree zero, $MRT_{se}(I_t)$ can be expressed in the form

$$MRT_{se}(I_t) = - \left([n(w\lambda_t + \gamma)]F_1 \left[\lambda_t + \frac{n[\gamma - (w\lambda_t + \gamma)e_t]}{1 - q_t^2(I_t)}, \frac{\sigma_t(I_t)}{1 - q_t^2(I_t)} \cdot \frac{s_{t-1}}{n} \right] \right)^{-1}.$$

Suppose, as before, that the ratio of survival rates is independent of I_t , but with $q_t^2(1) > q_t^2(0)$. It is seen that the associated increase in mortality reduces, or increases, the (normalised) input of human capital according as $e_t \gtrless \gamma/(\gamma + w\lambda_t)$. For sufficiently large values of λ_t , the latter ratio will be very small, so that war in the current period will make s_t cheaper relative to e_t for all values of e_t except those very close to zero. The converse holds when λ_t is close to one; for $\gamma/(\gamma + w\lambda_t)$ is then close to one, and the $n/(1 - q_t^2(I_t))$ children cared for by each surviving young adult constitute a potentially large pool of labour, relatively speaking. The outbreak of war reduces the opportunity cost of their labour and so makes investment in their education more attractive relative to investment in physical capital. We summarise our findings in Proposition 7:

Proposition 7.

The contraction of the feasible set caused by war in the current period reduces both current consumption and investment in both forms of capital. Consumption also becomes cheaper relative to investment in physical capital. Investment in education is likely to suffer especially when λ_t is large, but not when λ_t is small.

It seems rather unlikely that the associated changes in the marginal rates of transformation will offset the reduction in investment in general arising from the adverse income effect.

5.2 The Probability Of War

Intuition suggests that an increase in the prior probability of war in the future will depress investment in the present. It will now be demonstrated that this is indeed so in our framework provided an additional – and plausible – condition holds.

The feasible set in period t , as defined by (35), is independent of π_{t+1} , so that changes in the latter will affect decisions only through $V(I_t)$. Inspection of (33) reveals that the weight on the altruism term v is increasing in π_{t+1} , since $q_{t+1}^2(1) > q_{t+1}^2(0)$. Where the terms involving old age are concerned, the probability of surviving into full old age is increasing in the probability of peace, π_{t+1} . This does not, however, settle the matter; for the pay-off received by survivors depends on the number of claimants as well as the size of the common pot. From (34), it is seen that for any given (e_t, s_t, I_t) , $c_{t+1}^3(I_t; 0) \gtrless c_{t+1}^3(I_t; 1)$ according as

$$\frac{F \left[(1 - q_{t+1}^2(0))\lambda_{t+1}(e_t) + n\gamma, \frac{\sigma_{t+1}(0)s_t}{n} \right]}{F \left[(1 - q_{t+1}^2(1))\lambda_{t+1}(e_t) + n\gamma, \frac{\sigma_{t+1}(1)s_t}{n} \right]} > \frac{1 - q_{t+1}^3(0)}{1 - q_{t+1}^3(1)}.$$

The numerator on the l.h.s. is the level of full income in period $t+1$ when peace prevails, the denominator is the corresponding level when war does so. The r.h.s. is the corresponding ratio of survival rates into old age. Both ratios exceed 1, but it is very likely that the former ratio is the larger; for war is likely to take a proportionally heavier toll on young adults, and it will surely destroy some of the capital stock. It is highly plausible, therefore, that the condition

$$\frac{F \left[(1 - q_{t+1}^2(0))\lambda_{t+1}(e_t) + n\gamma, \frac{\sigma_{t+1}(0)s_t}{n} \right]}{F \left[(1 - q_{t+1}^2(1))\lambda_{t+1}(e_t) + n\gamma, \frac{\sigma_{t+1}(1)s_t}{n} \right]} \geq \frac{1 - q_{t+1}^3(0)}{1 - q_{t+1}^3(1)} \quad (38)$$

holds. This suffices to ensure that the second term on the r.h.s. of (33), which may be expressed as $E_{I_{t+1}} u[c_{t+1}^3(I_t; I_{t+1})]$, is increasing in π_{t+1} , $I_t = 0, 1$. Hence, we have:

Proposition 8.

If (38) holds, an increase in the (prior) probability that war will prevail in the next period will depress investment in favour of consumption in the current one.

Consumption in old age depends on, *inter alia*, the savings made when young that survive destruction at the start of old age. War and peace, respectively, affect the second arguments of F in the denominator and numerator on the l.h.s of condition (38). The prospect of heavy destruction of physical capital makes saving less attractive when war in the next period has rather small effects on the survival rate of those then entering old age, and hence on the number of such claimants. The argument in Section 5.1 indicates that how the balance between e_t and s_t will be affected thereby depends in a complicated way on the differences in survival rates between the two states. The weight on the altruism term $v(\lambda_{t+1}(e_t))$ in (33) is decreasing in the future mortality rate among young adults, and the stronger is altruism, as represented by b , the larger will be the absolute size of the reduction in the said weight. War would have to be extremely destructive of physical relative to human capital such that substitution between the two forms of investment could induce a net increase in e_t .

6 Shocks and Stability

The system's stability in the face of shocks will now be examined in some detail, drawing upon the above findings. The argument proceeds in a series of taxonomic steps. It rests on the claim that that the two extremes, i.e., backwardness with $e_t = 0$ and the progressive state with $e_t = 1$ are both locally stable steady states. Sufficient conditions for this claim to hold are given in the following proposition.

Proposition 9.

If $u = \ln c_t$ and F is Cobb-Douglas, both backwardness and the progressive state are locally stable steady states if:

$$(i) (1 - \alpha)\alpha\delta(1 - q^3) < (1 - q^2 + n\gamma)^{1-\alpha} - (1 - \alpha)n\gamma;$$

(ii) the production function for human capital $h(e_t)$ is concave or sufficiently weakly convex;

(iii) $1 - q^2 > \alpha wn$.

Remark. In the light of Section 4, the assumptions on u and F can be weakened. Condition (ii) has been discussed in Section 3.2. Condition (iii) is also easily fulfilled, as the term αwn is close to zero for plausible values of w , whereas q^2 is not close to 1. This leaves condition (i) as the only additional assumption we have to make to ensure convergence when shocks are not too large.

Suppose backwardness is a locally stable, long-run equilibrium, even when peace always reigns ($e_t^0 = 0, I_t = 0 \forall t$). Then once in backwardness, the economy is perpetually trapped in that state, be there war or peace thereafter.

Suppose there also exists, when peace always reigns, a set of stationary states with $e_t^0 \in (0, 1)$. Let $e^*(0)$ denote the smallest such value of e_t^0 , so that λ_t is stationary, at $\lambda^*(0)$, where $\lambda^*(0) = zh(e^*(0))\lambda^*(0) + 1$. Associated with $e^*(0)$ there is a stationary level of k_t , denoted by $k^*(0)$. Since the state of backwardness is locally stable, the equilibrium $(\lambda^*(0), k^*(0))$ is unstable. If, at time t , the state variables are such that $(\lambda_t, k_t) \ll (\lambda^*(0), k^*(0))$,⁶ a descent into permanent backwardness will certainly occur. This conclusion holds *a fortiori* if there is some chance of war. For given any $\pi < 1$, it is established in Section 5.1 that an outbreak of war in the current period will almost surely reduce current investment relative to the state of peace, and in Section 5.2 that an increase in the hazard rate $1 - \pi$ will do likewise, *cet. par.*

Taking a longer view where the economy's capacity to withstand shocks is concerned, a *robust* economy can be defined as one in which growth can occur even in a state of perpetual war: $e_t^0 = e^{**}(1), I_t = 1 \forall t$, where $\lambda_{t+1} = zh(e^{**}(1))\lambda_t + 1 > \lambda_t$. This requires, *inter alia*, that (32) hold at $\mathbf{q}_t = \mathbf{q}(1), \sigma_t = \sigma(1)$. If steady growth is possible in a state of perpetual war, growth will also be possible when peace sometimes rules, but it will not be steady.

For any growth path to be attained, the starting values of the state variables must be sufficiently favourable. The said values depend on the economy's particular history of war and peace. If, at time t' , the state variables (λ_t, k_t) are such that, should war become permanent, $e_t^0 \geq e^{**}(1) \forall t \geq t'$, then a sustained growth path will be attained for all π .

Analogously to $(e^*(0), k^*(0))$, suppose there is also a pair $(\lambda^*(1), k^*(1))$ in the state of

⁶The inequality $\mathbf{x} \ll \mathbf{y}$ indicates that each component of the vector \mathbf{y} exceeds its counterpart in \mathbf{x} .

perpetual war. Since survival rates are higher in peace, $(\lambda^*(0), k^*(0)) = (\lambda^*(1), k^*(1))$ cannot hold, and it is natural to conjecture that $(\lambda^*(0), k^*(0)) \ll (\lambda^*(1), k^*(1))$. If war and peace are both possible, i.e. $\pi \in (0, 1)$, this conjecture introduces chance into the final outcome in the long run if the initial conditions satisfy

$$(\lambda^*(0), k^*(0)) \ll (\lambda_0, k_0) \ll (\lambda^*(1), k^*(1)).$$

For suppose (λ_0, k_0) exceeds, but lies close to, $(\lambda^*(0), k^*(0))$. With some positive probability, the economy will enjoy an uninterrupted run of peace; and if long enough, this run could yield state variables exceeding $(\lambda^*(1), k^*(1))$, and hence ultimately, if the next stationary value of e_t^0 is such that $zh(e_t^0) > 1$, sustained growth. Then again there is the grim possibility that (λ_0, k_0) falls short of, but lies close to, $(\lambda^*(1), k^*(1))$, and that this initially tantalising prospect recedes ever farther away as the economy endures an unbroken run of wars, an event whose probability of occurrence is also strictly positive. If long enough, such a run could well yield state variables short of $(\lambda^*(0), k^*(0))$ and hence, ultimately, backwardness with certainty.

One consequence of the sharing rule for old-age provision, as specified by (7), in a stochastic environment is that if war is more destructive of property than life, then the aggregate payment to the old will fall disproportionately, though war will also exact an additional toll on the numbers of those making a claim on it. Consider, therefore, the variation in which all those who survive into old age are allocated, not a fixed share of total full income at that time, but each one of them a fixed proportion ρ' of the full income of each surviving young adult: $c_t^3 = \rho' \bar{Y}_t / (1 - q_t^2) N_t^2$. It is seen that if mortality rates are varying over time and sharply forecast, the difference in the respective denominators can yield different incentives to invest. Under the alternative rule, we have, instead of (7),

$$c_{t+1}^3 = \frac{\rho'}{(1 - q_{t+1}^2)} \cdot F [(1 - q_{t+1}^2) \lambda_{t+1}(e_t) + n\gamma, \sigma_{t+1} s_t / n].$$

The arguments of F are formally identical, embodying the decision (e_t, s_t) in the previous period. If the environment is stationary and non-stochastic, so that steady-state equilibria may exist, choose ρ' such that $\rho' = \rho n / (1 - q^3)$. The two rules then yield identical allocations.

An unexpected outbreak of war in period t after an uninterrupted state of peace will leave each of the surviving young adults less well equipped with physical capital than their elders had intended, thus producing an adverse income effect on both forms of investment in period t . The alternative sharing rule will relieve the current loss by reducing the payment to each of the old-age survivors. If, however, war takes a heavier toll on young adults than their elders, that rule will not necessarily make the economy more robust to such asymmetric shocks to life and property.

An unexpected outbreak of pestilence, such as the Black Death, is an asymmetric shock of another kind, carrying off much of the population, but leaving the capital stock untouched. This will be a windfall for the survivors, but it will avail them little if physical and human capital are poor substitutes in production – indeed, not at all if they are strict complements. If, in contrast, they are perfect substitutes, then the windfall will yield a correspondingly large income effect, which may be sufficiently strong to propel an economy out of backwardness onto a growth path, even with perpetual, but not unduly destructive warfare.⁷

7 Simulations

Establishing more precisely whether an economy will withstand a particular shock in the presence of a poverty trap involves some resort to simulations, whereby the initial conditions need to be specified in a tractable form. Suppose, therefore, that the economy has been proceeding along some steady-state growth path in the state of peace, when it is suddenly hit by war out of the blue. Agents then form some sharp prior $1 - \pi$ that war will also occur in the next period, and make their investment decisions accordingly.

If much time has elapsed along that path, both state variables will be very large indeed, so that extremely heavy loss rates will have to occur in order to reduce the normalised endowments to levels where even $e_t = 1$ will not be optimal, let alone a certain collapse into backwardness. Suppose, therefore, that both state variables are still relatively small. Given the resulting normalised endowments and the (sharp) prior π_{t+1} , the households

⁷For an analysis of this potentially liberating stroke, see Bell and Gersbach (2013), in which there is only human capital. The assumption that both inputs are necessary in production leaves the matter open.

will choose (e_t^0, s_t^0) . If the worst occurs again in period $t + 1$, the resulting normalised endowments will be, suppressing the time subscripts for n_t and q_t^2 ,

$$\bar{l}_{t+1} = (1 - q^2(1))\lambda_{t+1}(e_t^0) + n\gamma \text{ and } k_{t+1} = \sigma(1)s_t^0/[(1 - q^2(1))n].$$

If peace is confidently expected in period $t + 2$, (e_{t+1}^0, s_{t+1}^0) are chosen accordingly. Since peace also actually rules, the resulting normalised endowments in period $t + 2$ will be

$$\bar{l}_{t+2} = (1 - q^2(0))\lambda_{t+2}(e_{t+1}^0) + n\gamma \text{ and } k_{t+2} = \sigma(0)s_{t+1}^0/[(1 - q^2(0))n]. \quad (39)$$

Given the values of the state variables in period t , $(\lambda_t(e_{t-1}^0), s_{t-1}^0)$, where these conform to the ζ associated with the path under consideration, it can be checked numerically whether the economy will recover from what are, in effect, the new starting endowments under a regime of perpetual peace given by (39) or fail to do so.

There remains the alternative possibility that peace, not war, rules in period $t + 1$. In that event, the normalized endowments will be

$$\bar{l}_{t+1} = (1 - q^2(1))\lambda_{t+2}(e_t^0) + n\gamma \text{ and } k_{t+1} = \sigma(1)s_t^0/[(1 - q^2(1))n];$$

and the calculations for period $t + 2$ then proceed as before.

A particular limitation of the two-period phase t and $t + 1$ during which war can ever occur is that its influence on decisions *ex ante* is confined to period t . The only possible sequences are war-war and war-peace, each followed by permanent peace. Two consecutive adverse shocks are possible, but the certainty of peace from $t + 2$ onwards makes ultimate recovery more likely. It is desirable, therefore, to extend the said phase to three periods, thus yielding an *ex ante* influence in both periods t and $t + 1$. The possible sequences are

$$\{1, 1, 1\}, \{1, 1, 0\}, \{1, 0, 1\}, \{1, 0, 0\}, \{0, 1, 1\}, \{0, 1, 0\}, \{0, 0, 1\}, \{0, 0, 0\},$$

with all outcomes preceded by development in the environment $(\mathbf{q}(0), \sigma(0))$ up to $t = 0$. We concentrate on the grimmest outcome: three consecutive periods of war.

The constellation of parameter values in Table 1 must be extended to cover the states of war and peace. Let those values hold in the state of peace, so there is a poverty trap, even

with unbroken peace, $\{0,0,0\}$, as the actual outcome. The associated long-run value of ζ_t when ‘progress’ rules, $\zeta_t(0; e = 1)$, is 5.6. Let the prior probability of war in periods 1 and 2, $1 - \pi_{t+1}(t = 0, 1)$, be 0.5, and the mortality rates in that state be $q_t^1(1) = 0.25$, $q_t^3(1) = 0.35$, with $\sigma_t(1) = 0.4$. As noted above, the initial values of human and physical capital, λ_0 and k_0 , which are inherited from period $t = -1$, must be sufficiently small for a sequence of shocks even as heavy as $\{1,1,1\}$ to rule out any path to progress. Recalling Section 6, the stationary (critical) values of λ^* are now in play. Under perpetual peace, expected as well as realised, $\lambda^*(0) = 2.378$. With $\pi_{t+1} = 0.5$, the critical value of λ_0 when the realised sequence is indeed $\{0, 0, 0\}$ is 2.541, but 2.894 when the outcome is three periods of war, $\{1, 1, 1\}$. To complete the initial conditions, let $\zeta_0 = 2.8$; for physical capital plausibly forms the greater part of the whole endowment in the state of backwardness than that of progress.

The trajectories of λ_t and ζ_t for each of the values $\lambda_0 = 1, 2, 2.7, 3, 5$ are depicted in Figure 4. As intuition suggests, three periods of warfare generate an immediate and sharp upward spike in ζ_t , even when backwardness is the ultimate outcome. The two trajectories that closely bracket the critical value of $\lambda_0 = 2.894$, namely, those for $\lambda_0 = 2.7$ and 3, have more than one local extremum. The former follows the spike by first undershooting, and then converging from below to the value under backwardness; the latter goes on to attain a second local maximum, at $t = 12$, before converging from above to the value under progress. These oscillations indicate complex and long drawn-out transitional dynamics near critical values of the boundary conditions, though these dynamics are less apparent in the trajectories of λ_t . The latter for $\lambda_0 = 2.7$ recovers only slowly from the three consecutive episodes of war, whereas that for $\lambda_0 = 5$ is little affected.

8 Conclusions

It is not difficult to think of conditions that will keep a society in a state of backwardness. Unremitting warfare and endemic communicable diseases untrammelled by public health measures, together with the privation that accompanies warfare and disease, will almost surely suffice to bring about a Hobbesian existence, even when productive technologies are available. What we have established, however, is that there are constellations of unchanging

War (first 3 periods)

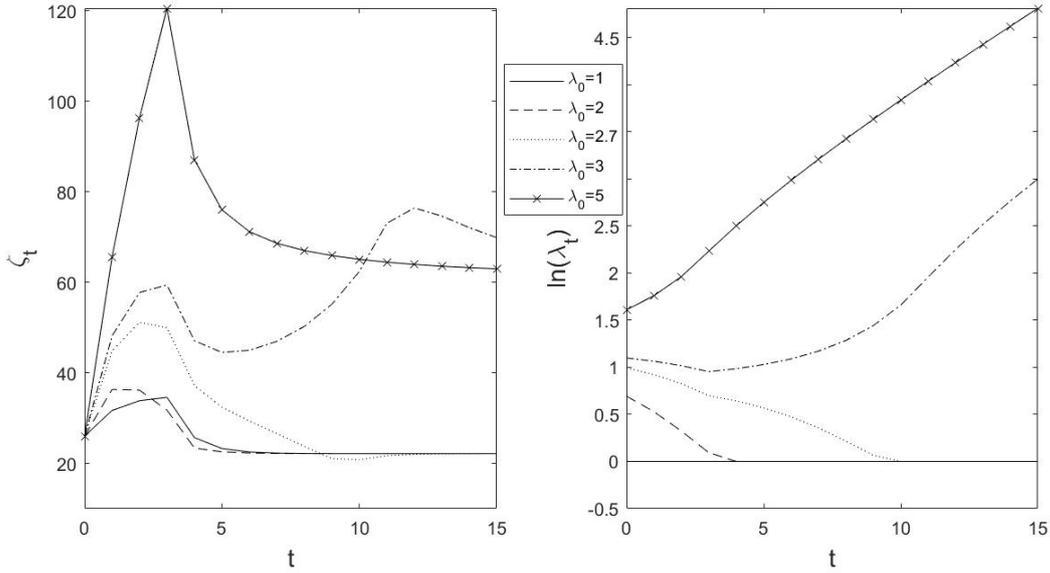


Figure 4: Three consecutive periods of war followed by peace

war losses and premature adult mortality such that backwardness, the state in which there is no investment in human capital through schooling, and steady growth with a fully educated population are both possible equilibria. The associated poverty trap is also precisely characterised.

Parents' altruism can exert a decisive influence on the outcome. If sufficiently strong, it can rule out backwardness in environments in which the hazards of destruction are such as to keep a selfish population in that condition for good. That is no great surprise. Where attaining – and maintaining – steady-state growth is concerned, however, altruism also comes into play in a different way. If parents' preferences are such that the sub-utility functions for their own consumption and their children's well-being in adulthood differ – which is highly likely – and the former is more concave than the latter, then the only steady-state path other than backwardness is progress: there is growth with a fully educated population. If, however, the sub-utility function for own consumption is less concave than that for the children's human capital, then steady-state growth paths with an incompletely educated population may exist, as may stationary paths. The same holds if parents are perfectly selfish, so that provision for old age is the sole motive for investment. Thus, not only does altruism tend to promote growth as an outcome, as expected, but it

may also lead to permanently faster growth.

The fact that outbreaks of war and pestilence are stochastic events introduces a central role for expectations. It also raises the question of whether a growing economy is sufficiently robust to withstand a series of adverse shocks. Mature economies that have experienced growth for long periods will have large per capita stocks of human and physical capital. They will be correspondingly robust, unless nuclear war destroys the environment itself. Economies at an earlier stage of development are more vulnerable. Numerical simulations in which the realised outcome is three consecutive periods of war followed by a confidently expected era of perpetual peace reveal how the boundary conditions at the start have a decisive influence on whether this series of shocks will pitch the economy into permanent backwardness.

Appendix 1. Proofs

The f.o.c. are, noting that $0 \leq e_t \leq 1$,

$$\frac{\partial \Phi_t}{\partial c_t^2} = u'(c_t^2) - \mu_t[(1 - q_t^2) + \beta n_t] = 0, \quad (40)$$

$$\frac{\partial \Phi_t}{\partial s_t} = \frac{\delta u'(c_{t+1}^3) \cdot n_t \rho}{(1 - q_t^2)} \cdot \frac{\partial \bar{y}_{t+1}}{\partial s_t} - \mu_t = 0, \quad (41)$$

$$\frac{\partial \Phi_t}{\partial e_t} = \frac{\delta u'(c_{t+1}^3) \cdot n_t \rho}{(1 - q_t^2)} \cdot \frac{\partial \bar{y}_{t+1}}{\partial \lambda_{t+1}} \frac{\partial \lambda_{t+1}}{\partial e_t} + \nu_t v'(\lambda_{t+1}) \frac{\partial \lambda_{t+1}}{\partial e_t} + \mu_t \frac{\partial y_t}{\partial e_t} \leq 0, \quad e_t \geq 0, \quad (42)$$

$$\frac{\partial \Phi_t}{\partial e_t} = \frac{\delta u'(c_{t+1}^3) \cdot n_t \rho}{(1 - q_t^2)} \cdot \frac{\partial \bar{y}_{t+1}}{\partial \lambda_{t+1}} \frac{\partial \lambda_{t+1}}{\partial e_t} + \nu_t v'(\lambda_{t+1}) \frac{\partial \lambda_{t+1}}{\partial e_t} + \mu_t \frac{\partial y_t}{\partial e_t} \geq 0, \quad e_t \leq 1, \quad (43)$$

where, recalling that F is homogeneous of degree 1 and $\zeta_t = \lambda_t/s_{t-1}$,

$$\begin{aligned} \frac{\partial \lambda_{t+1}}{\partial e_t} &= z_t h'(e_t) \lambda_t, \\ \frac{\partial \bar{y}_{t+1}}{\partial s_t} &= \frac{\sigma_{t+1}}{n_t} \cdot F_2 \left[(1 - q_{t+1}^2) \zeta_{t+1} + \frac{n_{t+1} \gamma}{s_t}, \frac{\sigma_{t+1}}{n_t} \right], \\ \frac{\partial y_t}{\partial e_t} &= -(\gamma + w \lambda_t) n_t \cdot F_1 \left[(1 - q_t^2 - w n_t e_t) \zeta_t + \frac{n_t \gamma (1 - e_t)}{s_{t-1}}, \frac{\sigma_t}{n_{t-1}} \right] \text{ and} \\ \frac{\partial \bar{y}_{t+1}}{\partial \lambda_{t+1}} &= (1 - q_{t+1}^2) F_1 \left[(1 - q_{t+1}^2) \zeta_{t+1} + \frac{n_{t+1} \gamma}{s_t}, \frac{\sigma_{t+1}}{n_t} \right]. \end{aligned}$$

Proof of Lemma 1

Since s_t and λ_t are growing at the steady rate $g(e) = zh(e) - 1 > 0$, we have

$$\frac{c_t^2}{c_{t+1}^3} = \frac{(1 - q^2)(1 - q^3)}{\rho n((1 - q^2) + \beta n)} \cdot \left(\frac{y_t - s_t}{\bar{y}_{t+1}} - \frac{\rho}{zh(e)} \right),$$

where

$$\frac{y_t - s_t}{\bar{y}_{t+1}} - \frac{\rho}{zh(e)} = \frac{1}{zh(e)} \left(\frac{F[(1 - q^2 - wne)\zeta, \sigma/n] - zh(e)}{F[(1 - q^2)\zeta, \sigma/n]} - \rho \right),$$

which is a constant for any given e . Substituting for c_t^2/c_{t+1}^3 from (26), we obtain

$$\begin{aligned} & \frac{\rho\delta\alpha(1-q^2+\beta n)F_2[(1-q^2)\zeta, \sigma/n]}{1-q^2} \\ &= \left(\frac{\rho n((1-q^2)+\beta n)zh(e)F[(1-q^2-wne)\zeta, \sigma/n]}{(1-q^2)(1-q^3)[F[(1-q^2-wne)\zeta, \sigma/n] - \rho F[(1-q^2)\zeta, \sigma/n] - zh(e)]} \right)^\xi, \end{aligned}$$

which may be rearranged as

$$\begin{aligned} & F[(1-q^2-wne)\zeta, \sigma/n] - \rho F[(1-q^2)\zeta, \sigma/n] \\ &= \left(1 + B' \frac{F[(1-q^2)\zeta, \sigma/n]}{(F_2[(1-q^2)\zeta, \sigma/n])^{1/\xi}} \right) zh(e), \end{aligned} \quad (44)$$

where

$$B' \equiv \frac{n}{(1-q^3)(\delta\alpha)^{1/\xi}} \left(\frac{\rho(1-q^2+\beta n)}{1-q^2} \right)^{1-1/\xi}$$

is a positive constant.

The assumption that F is homogeneous of degree 1, with both inputs are necessary in production, implies that ζ is differentiable in e when e is varied parametrically. For continuous changes in e produce continuous changes in the feasible set and the preference functional V_i ; and the isoquant map is smooth everywhere and strictly convex to the origin, and no isoquant intersects either axis.

Part (i)

By assumption, F is Cobb-Douglas: $y_t = A l_t^{1-\alpha} k_t^\alpha$. Substituting into (44) and collecting terms, we have

$$\begin{aligned} & A \left(\frac{\sigma}{n} \right)^\alpha [(1-q^2-wne)^{1-\alpha} - \rho(1-q^2)^{1-\alpha}] \zeta^{1-\alpha} \\ &= \left(1 + \left(\frac{n}{\alpha\sigma} \right)^{1/\xi} B' \cdot (F[(1-q^2)\zeta, \sigma/n])^{1-1/\xi} \right) zh(e). \end{aligned} \quad (45)$$

Differentiating (45) totally, noting that $\partial F/\partial \zeta = (1 - \alpha)F/\zeta$, and collecting terms, we obtain

$$\begin{aligned} & \left[A \left(\frac{\sigma}{n} \right)^\alpha [(1 - q^2 - wne)^{1-\alpha} - \rho(1 - q^2)^{1-\alpha}](1 - \alpha)\zeta^{-\alpha} \right. \\ & \left. - \left(1 - \frac{1}{\xi} \right) \left(\frac{n}{\alpha\sigma} \right)^{1/\xi} \frac{(1 - \alpha)B'}{\zeta} \cdot (F[(1 - q^2)\zeta, \sigma/n])^{1-1/\xi} \right] \cdot d\zeta \\ & = \left[\left(1 + \left(\frac{n}{\alpha\sigma} \right)^{1/\xi} B' \cdot (F[(1 - q^2)\zeta, \sigma/n])^{1-1/\xi} \right) zh'(e) \right. \\ & \left. + A \left(\frac{\sigma}{n} \right)^\alpha [(1 - q^2 - wne)^{-\alpha}]wn(1 - \alpha)\zeta^{1-\alpha} \right] \cdot de. \end{aligned}$$

Now, the condition $F[(1 - q^2 - wn)\zeta, \sigma/n] > \rho F[(1 - q^2)\zeta, \sigma/n]$ implies that $(1 - q^2 - wn)^{1-\alpha} > \rho(1 - q^2)^{1-\alpha}$, so that ζ is increasing in e if $\xi \leq 1$. By continuity, this result also holds for all F sufficiently close to Cobb-Douglas in form and for all ξ exceeding, but sufficiently close to, 1.

Part (ii)

By assumption, $y_t = A[b_1 l_t^\epsilon + b_2 k_t^\epsilon]^{1/\epsilon}$, $\epsilon \leq 1$: the elasticity of substitution is $(\epsilon - 1)^{-1}$, where $\epsilon = 0$ is Cobb-Douglas. Proceeding as before,

$$\begin{aligned} \frac{F[(1 - q^2)\zeta, \sigma/n]}{(F_2[(1 - q^2)\zeta, \sigma/n])^{1/\xi}} &= \frac{A[b_1((1 - q^2)\zeta)^\epsilon + b_2(\sigma/n)^\epsilon]^{1/\epsilon}}{[b_2(\sigma/n)^{\epsilon-1} A[b_1((1 - q^2)\zeta)^\epsilon + b_2(\sigma/n)^\epsilon]^{1/\epsilon-1}]^{1/\xi}} \\ &= B_1 [b_1((1 - q^2)\zeta)^\epsilon + b_2(\sigma/n)^\epsilon]^\psi, \end{aligned}$$

where $\psi = (1/\xi) + (1 - 1/\xi)/\epsilon$ and B_1 is a positive constant. Substituting into (44), noting the derivative of $[b_1((1 - q^2)\zeta)^\epsilon + b_2(\sigma/n)^\epsilon]^{1/\psi}$ w.r.t. ζ and rearranging as before in part (i), there are two terms on the l.h.s. The first is the partial derivative of $\{F[(1 - q^2 - wne)\zeta, \sigma/n] - \rho F[(1 - q^2)\zeta, \sigma/n]\}$ w.r.t. ζ , which is positive if $\epsilon \leq 0$ and $F[(1 - q^2 - wn)\zeta, \sigma/n] > \rho F[(1 - q^2)\zeta, \sigma/n]$. The second term has the sign of $\psi \cdot \epsilon$. Now, $\psi \cdot \epsilon \leq 0$ iff $\epsilon + \xi \leq 1$. Since both inputs are assumed to be necessary in production, $\epsilon \leq 0$, which yields the required result. Q.E.D.

Proof Of Corollary 1

Recalling that both inputs are necessary in production, it follows that for all such F , $F_1 [(1 - q^2)\zeta, \frac{\sigma}{n}]$ and $F_2 [(1 - q^2)\zeta, \frac{\sigma}{n}]$ are, respectively, decreasing and increasing in e . Now, $F_1 [(1 - q^2 - wne)\zeta, \frac{\sigma}{n}]$ decreases more slowly than $F_1 [(1 - q^2)\zeta, \frac{\sigma}{n}]$ as e increases. Hence, if $h(e)$ is concave or sufficiently weakly convex, it is seen that D is increasing in e across paths. Q.E.D.

Derivation Of Condition (20)

We simplify the denominator of (18) using

$$\begin{aligned} F_1[l_t, \frac{\sigma s_{t-1}}{n}] &= \frac{1 - \alpha}{(1 - q^2 - wn)\zeta} F[(1 - q^2 - wn)\zeta, \sigma/n], \\ \frac{F_1[\bar{l}_{t+1}, \frac{\sigma s_t}{n}]}{F_2[\bar{l}_{t+1}, \frac{\sigma s_t}{n}]} &= \frac{1 - \alpha}{\alpha} \cdot \frac{\sigma}{n} \cdot \frac{1}{(1 - q^2)\zeta}. \end{aligned}$$

Noting (19) and that (44) specialises to

$$F[(1 - q^2 - wn)\zeta, \frac{\sigma}{n}] - \rho F[(1 - q^2)\zeta, \frac{\sigma}{n}] = \left(1 + \frac{\sigma}{(1 - q^3)\delta}\right) zh(1),$$

we obtain

$$\begin{aligned} c_t^2 &= s_{t-1} \left(\left(1 + \frac{\sigma}{(1 - q^3)\delta}\right) zh(1) - \frac{s_t}{s_{t-1}} \right) \frac{1}{1 - q^2 + \beta n} \quad \text{and} \\ c_t^2 &= s_{t-1} \frac{\sigma}{(1 - q^3)\delta} \cdot \frac{zh(1)}{1 - q^2 + \beta n}. \end{aligned}$$

Substituting into (18) and rearranging terms, we have

$$\begin{aligned} \lambda_{t+1}^{-1} s_{t-1} \frac{\sigma zh(1)}{(1 - q^3)\delta} &\geq \\ \left(\frac{w(1 - \alpha)}{(1 - q^2 - wn)\zeta(1)} F^{[.p]} - \frac{zh'(1)}{\sigma} \left(\frac{(1 - \alpha)\sigma}{(1 - q^2)\zeta(1)n\alpha} \right) \right) &\cdot \frac{1}{b\sigma zh'(1)}, \end{aligned}$$

Since $\zeta_t = \lambda_t/s_{t-1}$ and $\lambda_{t+1}/\lambda_t \rightarrow zh(1)$ in the state of progress, a further rearrangement yields

$$zh'(1) \geq \left(\frac{w(1-\alpha)}{(1-q^2-wn)} F^{[.p]} - \frac{zh'(1)(1-\alpha)}{(1-q^2)n\alpha} \right) \cdot \frac{(1-q^3)\delta}{b\sigma zh(1)},$$

so that

$$zh'(1) \geq \frac{w(1-\alpha)}{(1-q^2-wn)} F^{[.p]} \cdot \left(\frac{b\sigma zh(1)}{(1-q^3)\delta} + \frac{1-\alpha}{(1-q^2)n\alpha} \right)^{-1}.$$

Derivation Of Equation (31)

Total differentiation of

$$Q_t = n(\delta\rho F_1(\bar{l}_{t+1}, \frac{\sigma s_t}{n})u'(c_{t+1}^3) + bv'(\lambda_{t+1}))$$

yields, noting (29) once more,

$$\begin{aligned} dQ_t &= n \left[\delta\rho \left((1-q^2)F_{11}(\bar{l}_{t+1}, \frac{\sigma s_t}{n})u'(c_{t+1}^3) + \frac{u''(c_{t+1}^3)c_{t+1}^3}{zh(e)\lambda_t} F_1(\bar{l}_{t+1}, \frac{\sigma s_t}{n}) \right) + bv''(\lambda_{t+1}) \right] \\ &\quad \cdot zh(e) d\lambda_t + \delta\rho\sigma F_{12}(\bar{l}_{t+1}, \frac{\sigma s_t}{n})u'(c_{t+1}^3) ds_t \\ &\equiv A' \cdot zh(e) \cdot d\lambda_t + \delta\rho\sigma F_{12}(\bar{l}_{t+1}, \frac{\sigma s_t}{n})u'(c_{t+1}^3) ds_t. \end{aligned} \quad (46)$$

We examine next the expression $\frac{dQ_t}{d\lambda_t} \cdot \frac{\lambda_t}{Q_t}$ on the r.h.s. of (30). From (46), we have

$$\frac{dQ_t}{d\lambda_t} \cdot \frac{\lambda_t}{Q_t} = \frac{A'zh(e) \cdot \lambda_t + \delta\rho\sigma F_{12}(\bar{l}_{t+1}, \frac{\sigma s_t}{n})u'(c_{t+1}^3)s_t}{n(\delta\rho F_1(\bar{l}_{t+1}, \frac{\sigma s_t}{n})u'(c_{t+1}^3) + bv'(\lambda_{t+1}))}.$$

Collecting terms in the numerator involving $u'(c_{t+1}^3)$, the multiplicand is

$$J \equiv n\delta\rho \left[(1-q^2)\lambda_{t+1}F_{11}(\bar{l}_{t+1}, \frac{\sigma s_t}{n}) - \xi F_1(\bar{l}_{t+1}, \frac{\sigma s_t}{n}) + \frac{\sigma s_t}{n} F_{12}(\bar{l}_{t+1}, \frac{\sigma s_t}{n}) \right].$$

Since $F_1(\bar{l}_{t+1}, \frac{\sigma s_t}{n})$ and $F_2(\bar{l}_{t+1}, \frac{\sigma s_t}{n})$ are homogeneous of degree zero, it follows from Euler's

Theorem that

$$[(1 - q^2)\lambda_{t+1} + n\gamma]F_{11}(\bar{l}_{t+1}, \frac{\sigma s_t}{n}) + \frac{\sigma s_t}{n}F_{12}(\bar{l}_{t+1}, \frac{\sigma s_t}{n}) = 0,$$

so that for sufficiently large λ_t , J reduces to $-n\delta\rho\xi F_1(\bar{l}_{t+1}, \frac{\sigma s_t}{n})$. Hence, recalling that v is iso-elastic, we obtain the elasticity of Q_t w.r.t. λ_t :

$$\frac{dQ_t}{d\lambda_t} \frac{\lambda_t}{Q_t} = \frac{-\xi\delta\rho F_1(\bar{l}_{t+1}, \frac{\sigma s_t}{n})u'(c_{t+1}^3) - \eta bv'(\lambda_{t+1})}{\delta\rho F_1(\bar{l}_{t+1}, \frac{\sigma s_t}{n})u'(c_{t+1}^3) + bv'(\lambda_{t+1})}.$$

By hypothesis, (λ_t, s_t) are growing at the rate $g = zh(e) - 1$. Hence, this elasticity can be expressed in the form

$$\frac{dQ_t}{d\lambda_t} \cdot \frac{\lambda_t}{Q_t} = -\frac{\xi A + \eta b \cdot B(1+g)^{-(\eta-\xi)}}{A + b \cdot B(1+g)^{-(\eta-\xi)}},$$

where $A = \delta\rho F_1(\bar{l}_{t+1}, \frac{\sigma s_t}{n})$ and $v'(\lambda_{t+1})/u'(c_{t+1}^3)$ are positive constants along the path in question.

Stability Analysis

Equations (22) and (23) provide our starting point. We take their total derivatives, given that the right-hand sides depend on λ_t and e_t . We then equate the left-hand sides and find the sign of $de_t/d\lambda_t$, that is, of the derivative $\partial e_t^0/\partial \lambda_t$: it turns out to be positive. Hence, it follows from (4) that a higher value of λ_t also increases λ_{t+1} through its indirect effect on the choice e_t^0 . Two further statements can be made. First, if an economy has reached the progressive state and is then hit by a sufficiently small shock, e_t^0 will fall, if at all, not much below 1, and the economy will remain in, or return to, that state. Second, if λ_t is close to some stationary level associated with $e_t^0 < 1$ and the economy is hit by a sufficiently adverse shock, there will be a descent into backwardness.

In the following, we take the derivatives of the aforementioned equations with respect to e_t and λ_t . We then analyze their signs when $(e_t, \lambda_t) = (0, 1)$ and $e_t = 1$ with λ_t large.

The derivative of the right hand side of (22) with respect to e_t is

$$\begin{aligned} & \left(A \left(\frac{\sigma}{n} \right) \right)^{-1} \left[\left((1 - q^2 - wne_t) + \frac{n\gamma(1 - e_t)}{\lambda_t} \right)^{1-\alpha} - \rho \left(1 - q^2 + \frac{n\gamma}{\lambda_t} \right)^{1-\alpha} \right]^{-1} \cdot \\ & \left\{ \frac{zh'(e_t)}{\alpha\delta(1 - q^3)} + \left(\frac{zh(e_t) + \frac{1}{\lambda_t}}{\alpha\delta(1 - q^3)} + 1 \right) \right\} \cdot \\ & \left. \frac{\left[(1 - q^2 - wne_t + \frac{n\gamma(1 - e_t)}{\lambda_t})^{1-\alpha} - \rho \left(1 - q^2 + \frac{n\gamma}{\lambda_t} \right)^{1-\alpha} \right]^{-1}}{\left(1 - q^2 - wne_t + \frac{n\gamma(1 - e_t)}{\lambda_t} \right)^\alpha} \cdot \frac{(1 - \alpha)(w + \gamma)n}{\lambda_t}} \right\}. \end{aligned}$$

When $e_t = 0$ and $\lambda_t = 1$, this yields

$$\begin{aligned} & \left[A \left(\frac{\sigma}{n} \right) (1 - q^2 + n\gamma)^{1-\alpha} (1 - \rho) \right]^{-1} \cdot \\ & \left\{ \frac{zh'(0)}{\alpha\delta(1 - q^3)} + \left(\frac{1}{\alpha\delta(1 - q^3)} + 1 \right) \frac{(1 - \alpha)(w + \gamma)n}{(1 - \rho)(1 - q^2 + n\gamma)} \right\}, \end{aligned}$$

which is positive.

When $e_t = 1$ and λ_t is large, we have

$$\begin{aligned} & \left[A \left(\frac{\sigma}{n} \right) (1 - q^2 + n\gamma)^{1-\alpha} (1 - \rho) \right]^{-1} \cdot \left\{ \frac{zh'(1)}{\alpha\delta(1 - q^3)} + \left(\frac{zh(1) + \frac{1}{\lambda_t}}{\alpha\delta(1 - q^3)} + 1 \right) \right\} \cdot \\ & \left. \frac{\left[(1 - q^2 - wn)^{1-\alpha} - \rho \left(1 - q^2 + \frac{n\gamma}{\lambda_t} \right)^{1-\alpha} \right]^{-1}}{(1 - q^2 - wn)^{1-\alpha}} (1 - \alpha) \left(w + \frac{\gamma}{\lambda_t} \right) n \right\}, \end{aligned}$$

which is also positive.

The derivative of (22) with respect to λ_t is

$$\begin{aligned} & \left(A \left(\frac{\sigma}{n} \right) \right)^{-1} \frac{1}{\lambda_t^2} \left[\left((1 - q^2 - wne_t) + \frac{n\gamma(1 - e_t)}{\lambda_t} \right)^{1-\alpha} - \rho \left(1 - q^2 + \frac{n\gamma}{\lambda_t} \right)^{1-\alpha} \right]^{-1} \cdot \\ & \left\{ - \frac{1}{\alpha\delta(1 - q^3)} + \frac{(1 - q^2 - wne_t + \frac{n\gamma(1 - e_t)}{\lambda_t})^{-\alpha} (1 - e_t) - \rho \left(1 - q^2 + \frac{n\gamma}{\lambda_t} \right)^{-\alpha}}{\left(1 - q^2 - wne_t + \frac{n\gamma(1 - e_t)}{\lambda_t} \right)^{1-\alpha} - \rho \left(1 - q^2 + \frac{n\gamma}{\lambda_t} \right)^{1-\alpha}} \right. \\ & \left. (1 - \alpha)n\gamma \left(\frac{zh(e_t) + \frac{1}{\lambda_t}}{\alpha\delta(1 - q^3)} + 1 \right) \right\}. \end{aligned}$$

Evaluating this expression at $e_t = 0$ and $\lambda_t = 1$ gives

$$\left(A \left(\frac{\sigma}{n} \right)^\alpha \left[(1 - q^2 + n\gamma)^{1-\alpha} (1 - \rho) \right] \right)^{-1} (w + \gamma)^{-1} \cdot \left\{ -\frac{1}{\alpha\delta(1 - q^3)} + \left(\frac{1}{\alpha\delta(1 - q^3)} + 1 \right) \frac{(1 - \alpha)n\gamma}{(1 - q^2 + n\gamma)^{1-\alpha}} \right\},$$

which can be negative if $(1 - \alpha)\alpha\delta(1 - q^3) < (1 - q^2 + n\gamma)^{1-\alpha} - (1 - \alpha)n\gamma$.

When $e_t = 1$ and λ_t is large, we have

$$- \left(A \left(\frac{\sigma}{n} \right) \right)^{-1} \frac{1}{\lambda_t^2} \left[\left((1 - q^2 - wn) + \frac{n\gamma}{\lambda_t} \right)^{1-\alpha} - \rho \left(1 - q^2 + \frac{n\gamma}{\lambda_t} \right)^{1-\alpha} \right]^{-1} \left(w + \frac{\gamma}{\lambda_t} \right)^{-1} \cdot \left\{ \frac{1}{\alpha\delta(1 - q^3)} + \left(\frac{zh(1) + \frac{1}{\lambda_t}}{\alpha\delta(1 - q^3)} + 1 \right) \frac{\rho \left(1 - q^2 + \frac{n\gamma}{\lambda_t} \right)^{-\alpha} (1 - \alpha)n\gamma}{(1 - q^2 - wn)^{1-\alpha} - \rho \left(1 - q^2 + \frac{n\gamma}{\lambda_t} \right)^{1-\alpha}} \right\},$$

which is clearly negative.

We repeat these steps in Equation (23). The derivative with respect to e_t is

$$\left(\alpha\delta(1 - q^3) A \left(\frac{\sigma}{n} \right)^\alpha \right)^{-1} zh'(e_t) \left(1 - q^2 - wne_t + \frac{n\gamma(1 - e_t)}{\lambda_t} \right)^\alpha \left(w + \frac{\gamma}{\lambda_t} \right)^{-1} \frac{zh'(e_t)}{\lambda_t^2} \cdot \left\{ \frac{\delta(1 - q^3)(1 - q^2)}{(1 - q^2 + \frac{n\gamma}{\lambda_{t+1}})^2} \frac{n\gamma\lambda_t}{\lambda_{t+1}} zh'(e_t) + \left(\frac{\delta(1 - q^3)(1 - q^2)}{1 - q^2 + \frac{n\gamma}{\lambda_{t+1}}} + \frac{bn}{1 - \alpha} \right) \cdot \left[\frac{h''(e_t)}{h'(e_t)} - \frac{\alpha n \left(w + \frac{\gamma}{\lambda_t} \right)}{1 - q^2 - wne_t + \frac{n\gamma(1 - e_t)}{\lambda_t}} \right] \right\}.$$

When $e_t = 0$ and $\lambda_t = 1$, we have

$$\left(\alpha\delta(1 - q^3) A \left(\frac{\sigma}{n} \right)^\alpha \right)^{-1} zh'(0) (1 - q^2 - wn)^\alpha \cdot \left\{ \frac{\delta(1 - q^3)(1 - q^2)}{(1 - q^2 + n\gamma)^2} n\gamma zh'(0) + \left(\frac{\delta(1 - q^3)(1 - q^2)}{1 - q^2 + n\gamma} + \frac{bn}{1 - \alpha} \right) \left[\frac{h''(0)}{h'(0)} - \frac{\alpha n(w + \gamma)}{1 - q^2 + n\gamma} \right] \right\}.$$

When $e_t = 1$ and λ_t is large, we have

$$\left(\alpha \delta (1 - q^3) A \left(\frac{\sigma}{n} \right)^\alpha \right)^{-1} z h'(1) (1 - q^2 - wn)^\alpha \cdot \left\{ \frac{\delta (1 - q^3) (1 - q^2)}{(1 - q^2 + \frac{n\gamma}{\lambda_{t+1}})^2} \frac{n\gamma}{\lambda_{t+1}^2} z h'(1) + \left(\frac{\delta (1 - q^3) (1 - q^2)}{1 - q^2 + \frac{n\gamma}{\lambda_{t+1}}} + \frac{bn}{1 - \alpha} \right) \left[\frac{h''(1)}{h'(1)} - \frac{\alpha n (w + \gamma)}{1 - q^2 - wn} \right] \right\}.$$

Both can be negative if h is concave or sufficiently weakly convex.

Finally, we examine the derivative of the right hand side of (23) with respect to λ_t :

$$\left(\alpha \delta (1 - q^3) A \left(\frac{\sigma}{n} \right)^\alpha \right)^{-1} (1 - q^2 - wne_t + \frac{n\gamma(1 - e_t)}{\lambda_t})^\alpha \left(w + \frac{\gamma}{\lambda_t} \right)^{-1} \frac{z h'(e_t)}{\lambda_t^2} \cdot \left\{ \frac{\delta (1 - q^3) (1 - q^2)}{(1 - q^2 + \frac{n\gamma}{\lambda_{t+1}})^2} \frac{n\gamma \lambda_t^2}{\lambda_{t+1}^2} z h'(e_t) + \left(\frac{\delta (1 - q^3) (1 - q^2)}{1 - q^2 + \frac{n\gamma}{\lambda_{t+1}}} + \frac{bn}{1 - \alpha} \right) \cdot \left[\left(w + \frac{\gamma}{\lambda_t} \right) \gamma - \alpha \left(1 - q^2 - wne_t + \frac{n\gamma(1 - e_t)}{\lambda_t} \right)^{-1} n\gamma(1 - e_t) \right] \right\}.$$

Plugging in $e_t = 0$ and $\lambda_t = 1$ yields

$$\left(\alpha \delta (1 - q^3) A \left(\frac{\sigma}{n} \right)^\alpha \right)^{-1} (1 - q^2 + n\gamma)^\alpha (w + \gamma)^{-1} z h'(0) \cdot \left\{ \frac{\delta (1 - q^3) (1 - q^2)}{(1 - q^2 + n\gamma)^2} n\gamma z h'(0) + \left(\frac{\delta (1 - q^3) (1 - q^2)}{1 - q^2 + n\gamma} + \frac{bn}{1 - \alpha} \right) \left[\frac{\gamma}{w + \gamma} - \frac{\alpha n\gamma}{1 - q^2 + n\gamma} \right] \right\},$$

where $1 - q^2 > \alpha wn$ is a sufficient condition for the derivative to be positive.

When $e_t = 1$ and λ_t is large, the derivative is

$$\left(\alpha \delta (1 - q^3) A \left(\frac{\sigma}{n} \right)^\alpha \right)^{-1} (1 - q^2 - wn)^\alpha \left(w + \frac{\gamma}{\lambda_t} \right)^{-1} \frac{z h'(1)}{\lambda_t^2} \cdot \left\{ \frac{\delta (1 - q^3) (1 - q^2)}{(1 - q^2 + \frac{n\gamma}{\lambda_{t+1}})^2} \frac{n\gamma \lambda_t^2}{\lambda_{t+1}^2} z h'(1) + \left(\frac{\delta (1 - q^3) (1 - q^2)}{1 - q^2 + \frac{n\gamma}{\lambda_{t+1}}} + \frac{bn}{1 - \alpha} \right) \left[\left(w + \frac{\gamma}{\lambda_t} \right)^{-1} \gamma \right] \right\},$$

which is positive.

In summary, we find that the derivative of (22) w.r.t. e_t is positive and that w.r.t. λ_t is negative. The opposite holds for (23), given the previously stated conditions. Equating

the total derivatives yields

$$[+] \cdot de_t + [-] \cdot d\lambda_t = [-] \cdot de_t + [+] \cdot d\lambda_t,$$

where $[+]$ and $[-]$ stand for the respective derivatives and their signs. It follows that $de_t/d\lambda_t > 0$.

In the cases where (23) holds as a strict inequality, we can make the following statements. If the right hand-side of (23) is larger than the left-hand side, then $de_t/d\lambda_t$ is larger than some positive value. In the opposite case, $de_t/d\lambda_t$ is smaller than this positive value, theoretically allowing for a negative relationship. Yet, if (23) holds almost as an equality, we can conclude from continuity considerations that the derivative will be positive.

Appendix 2. The Extreme Allocations of $S_t(I_t)$

In what follows, it will be useful to rewrite (35) in the form

$$\begin{aligned} & \left[1 + \frac{\beta n}{1 - q_t^2(I_t)} \right] c_t^2 + \frac{s_t}{1 - q_t^2(I_t)} + \rho F \left[\lambda_t + \frac{\gamma n}{1 - q_t^2(I_t)}, \frac{\sigma_t(I_t)}{1 - q_t^2(I_t)} \cdot \frac{s_{t-1}}{n} \right] \\ & \leq F \left[\left(1 - \frac{wn e_t}{1 - q_t^2(I_t)} \right) \lambda_t + \frac{\gamma n(1 - e_t)}{1 - q_t^2(I_t)}, \frac{\sigma_t(I_t)}{1 - q_t^2(I_t)} \cdot \frac{s_{t-1}}{n} \right], \quad I_t = 0, 1. \end{aligned} \quad (47)$$

Allocation A: $c_t^2 = e_t = 0$. Given I_t , s_t is maximal. From (35), we have $s_t = (1 - \rho)F \left[(1 - q_t^2(I_t))\lambda_t + n\gamma, \frac{\sigma_t(I_t)s_{t-1}}{n} \right]$, $I_t = 0, 1$. Given I_t , an increase in $q_t^2(I_t)$ will induce A to shift towards the origin O, as depicted by the point A'. Given that $q_t^2(1) > q_t^2(0)$, the allocations A and A' also represent those ruling under peace and war, respectively, in period t .

Allocation B: $c_t^2 = 0, e_t = 1$. Given I_t and maximum (full-time) investment in education, s_t is maximal. We have, for $I_t = 0, 1$,

$$s_t = F \left[(1 - q_t^2(I_t) - wn)\lambda_t, \frac{\sigma_t(I_t)s_{t-1}}{n} \right] - \rho F \left[(1 - q_t^2(I_t))\lambda_t + n\gamma, \frac{\sigma_t(I_t)s_{t-1}}{n} \right].$$

We begin by noting that the outer boundary of $S(I_t)$ in the plane defined by $c_t^2 = 0$, AB, is strictly concave in virtue of the strict concavity of F in each argument.

We next establish conditions under which the said value of s_t is positive, i.e., B lies to the right of G on the plane defined by $e_t = 1$. From (47), we have, for $I_t = 0, 1$,

$$(1 - q_t^2(I_t) - wn)^{-1} s_t = F \left[\lambda_t, \frac{\sigma_t(I_t)}{1 - q_t^2(I_t) - wn} \cdot \frac{s_{t-1}}{n} \right] \\ - \rho F \left[\lambda_t + \frac{n(w\lambda_t + \gamma)}{1 - q_t^2(I_t) - wn}, \frac{\sigma_t(I_t)}{1 - q_t^2(I_t) - wn} \cdot \frac{s_{t-1}}{n} \right].$$

The input of human capital in the second term on the r.h.s. is larger in the proportion $n(w + \gamma/\lambda_t)/(1 - q_t^2(I_t) - wn)$. This proportion is maximal when $\lambda_t = 1$. Since F is homogeneous of degree one, $\lambda_t = 1$ will therefore yield the best chance that $s_t < 0$, as intuition would suggest. Now, w is fairly small, say about $1/20$, and γ would be about 0.6 . In such a state of economic backwardness, $n = 3/2$ and $q_t^2 = 0.2$ are broadly plausible, so that the said proportion of inputs of human capital would be about $4/3$. Hence,

$$\lambda_t + \frac{n(w\lambda_t + \gamma)}{1 - q_t^2(I_t) - wn} \leq 7\lambda_t/3, \forall \lambda_t.$$

Observe, however, that F is strictly concave in each argument alone and ρ is unlikely to exceed $1/3$. Comparing the two terms on the r.h.s., inputs of human capital in the second are slightly more than double those in the first, but the share in the resulting output is at most one-third. It follows that, for plausible values of parameters and demographic variables, $s_t(c_t^2 = 0, e_t = 1) > 0$ for all values of λ_t , and points B and B' are correspondingly depicted in the diagram.

An increase in $q_t^2(I_t)$ induces a larger movement in B than in A. For the said difference in s_t is

$$F \left[(1 - q_t^2(I_t))\lambda_t + n\gamma, \frac{\sigma_t(I_t)s_{t-1}}{n} \right] - F \left[(1 - q_t^2(I_t) - wn)\lambda_t + n\gamma, \frac{\sigma_t(I_t)s_{t-1}}{n} \right], \quad I_t = 0, 1,$$

which is increasing in $q_t^2(I_t)$ in virtue of the strict concavity of F in each argument.

If the cross-derivative F_{12} is sufficiently small, it is seen that the same claim will hold concerning a comparison of peace and war, respectively.

Allocation C: $e_t = s_t = 0$. Given I_t , c_t^2 is maximal. From (35), we have

$$c_t^2 = \frac{1 - q_t^2(I_t)}{1 - q_t^2(I_t) + \beta n} \cdot (1 - \rho)F \left[\lambda_t + \frac{n\gamma}{1 - q_t^2(I_t)}, \frac{\sigma_t(I_t)}{1 - q_t^2(I_t)} \cdot \frac{s_{t-1}}{n} \right], \quad I_t = 0, 1.$$

Suppose the ratio of survival rates is fixed for each I_t . Then

$$\frac{dc_t^2}{dq_t^2(I_t)} = \frac{(1 - \rho)n}{(1 - q_t^2(I_t) + \beta n)^2} \cdot \left[-\beta F + \frac{\gamma(1 - q_t^2(I_t) + \beta n)F_1}{1 - q_t^2(I_t)} \right], \quad I_t = 0, 1,$$

and $(dc_t^2/dq_t^2(I_t))_{e_t=s_t=0} < 0$ iff

$$\beta > \frac{\gamma(1 - q_t^2(I_t) + \beta n)F_1}{(1 - q_t^2(I_t))F}.$$

The input of human capital is $\lambda_t + n\gamma/(1 - q_t^2(I_t))$. Its imputed share in output is $1 - \alpha \equiv (\lambda_t + n\gamma(1 - q_t^2(I_t))^{-1})F_1/F$, so the foregoing inequality can be written

$$\beta > \frac{(1 - \alpha)\gamma(1 - q_t^2(I_t) + \beta n)}{(1 - q_t^2(I_t))\lambda_t + \gamma n},$$

which certainly holds for all sufficiently large λ_t . The denominator takes a minimum under backwardness ($\lambda_t = 1$), when the inequality becomes

$$(\beta - (1 - \alpha)\gamma)(1 - q_t^2(I_t)) + \alpha\beta\gamma n > 0.$$

Since both inputs are necessary in production, F is strictly concave in both arguments and $\alpha \in (0, 1)$. It is plausible that $\alpha < 0.5$, but $n \geq 1$, so that the inequality may hold even if $\beta < \gamma$, as in Table 1, for which constellation the inequality holds.

Under the assumption that the ratio of survival rates is fixed for each I_t , we have established that the points C and C' relate to each other as depicted in the figure, which reveals that there is damage even under a mild mortality shock, given I_t . If the ratio of survival rates is the same in both states, the points C and C' also represent the respective allocations in peace and war.

Allocation D: $e_t = 1, s_t = 0$. Given I_t and maximum investment in (full-time) education, c_t^2 is maximal. Analogously to AB, the outer boundary of S in the plane defined by $s_t^2 = 0$, CD, is strictly concave in virtue of the strict concavity of F in each argument.

Given e_t and I_t , all pairs (c_t^2, s_t) on the outer frontier of S are linearly related and independent of e_t : $ds_t = -(1 - q_t^2(I_t) + \beta n) dc_t^2$. Hence, AC is parallel to BD, and A'C' to B'D'. An increase in $q_t^2(I_t)$ makes c_t^2 cheaper relative to s_t ; but since C' lies closer to O than does C, it follows that D' lies closer to G than does D. The same holds when the ratio of survival rates is the same in both states.

Appendix 3. Analysis for the Simulations

The optimization problem under uncertainty is specified by (33)-(35). The term $u(c_t^2)$ in the objective function is unchanged, but its derivatives with respect to s_t and e_t require close attention. We have

$$\begin{aligned} \frac{\partial u(c_{t+1}^3(I_{t+1}))}{\partial s_t} &= u'(c_{t+1}^3(I_{t+1})) \cdot \frac{\rho n}{(1 - q_t^2(I_t))(1 - q_{t+1}^3(I_{t+1}))} \frac{\sigma_{t+1}(I_{t+1})}{n} \\ &\quad F_2[\bar{l}_{t+1}, \sigma_{t+1}(I_{t+1})s_t/n] \\ \frac{\partial u(c_{t+1}^3(I_{t+1}))}{\partial e_t} &= u'(c_{t+1}^3(I_{t+1})) \cdot \frac{\rho n}{(1 - q_t^2(I_t))(1 - q_{t+1}^3(I_{t+1}))} (1 - q_{t+1}^2(I_{t+1})) \\ &\quad zh'(e_t)\lambda_t F_1[\bar{l}_{t+1}, \sigma_{t+1}(I_{t+1})s_t/n]. \end{aligned}$$

After defining $E_t[x_{t+1}] = \pi_{t+1}x_{t+1}(0) + (1 - \pi_{t+1})x_{t+1}(1)$ for some variable x and substituting for the above derivatives, we obtain the following two equations:

$$\begin{aligned} \frac{\delta \rho}{1 - q_t^2(I_t)} E_t [u'(c_{t+1}^3)\sigma_{t+1}(I_{t+1})F_2[\bar{l}_{t+1}, \sigma_{t+1}(I_{t+1})s_t/n]] &= \frac{u'(c_t^2)}{1 - q_t^2(I_t) + \beta n} \\ \delta E_t \left[\frac{u(c_{t+1}^3)\rho n(1 - q_{t+1}^2(I_{t+1}))}{1 - q_t^2(I_t)} F_1[\bar{l}_{t+1}, \sigma_{t+1}(I_{t+1})s_t/n] \right] zh'(e_t)\lambda_t \\ + \tilde{b}nv'(\lambda_{t+1})zh'(e_t)\lambda_t &= \frac{u'(c_t^2)(wn\lambda_t + n\gamma)}{(1 - q_t^2(I_t) + \beta n)} F_1[l_t, \sigma_t(I_t)s_{t-1}/n], \end{aligned}$$

where $\tilde{b} = b \frac{\pi_{t+1}(1 - q_{t+1}^2(0)) + (1 - \pi_{t+1})(1 - q_{t+1}^2(1))}{(1 - q_t^2(I_t))} = b \cdot E_t[1 - q_{t+1}^2]/(1 - q_t^2)$.

In the next step, we substitute the following expressions into the two equations:

$$u'(c_t^2(I_t)) = \left(\frac{A(l_t^{1-\alpha} - \rho \bar{l}_t^{1-\alpha}) \left(\frac{\sigma_t(I_t) s_{t-1}}{n} \right) - s_t}{1 - q_t^2(I_t) + \beta n} \right)^{-1},$$

$$u'(c_{t+1}^3(I_{t+1})) = \left(\frac{\rho n}{(1 - q_t^2(I_t))(1 - q_{t+1}^3(I_{t+1}))} F[\bar{l}_{t+1}, \sigma_{t+1}(I_{t+1}) s_t/n] \right)^{-1}.$$

Hence, the first equation can be written as

$$A(l_t^{1-\alpha} - \rho \bar{l}_t^{1-\alpha}) (\sigma_t(I_t) s_{t-1}/n)^\alpha - s_t = \frac{s_t}{\alpha \delta E_t[1 - q_{t+1}^3]},$$

which is equivalent to

$$\begin{aligned} & A[(1 - q_t^2(I_t) - wne_t)^{1-\alpha} - \rho(1 - q_t^2(I_t))^{1-\alpha}] \left(\frac{\sigma_t}{n} \right)^\alpha \zeta_t^{-\alpha} \\ &= \zeta_{t+1}^{-1} \frac{zh(e_t)}{\alpha \delta E_t[1 - q_{t+1}^3(I_{t+1})]} + \zeta_{t+1}^{-1}, \end{aligned}$$

where $\zeta_t = \lambda_t/s_{t-1} \forall t$. The second equation can be written as

$$\begin{aligned} & \delta(1 - \alpha) zh'(e_t) E_t[1 - q_{t+1}^3(I_{t+1})] \frac{1}{\lambda_{t+1}} + \tilde{b} n v'(\lambda_{t+1}) zh'(e_t) \\ &= \\ & \frac{\alpha \delta}{s_t} E_t[1 - q_{t+1}^3(I_{t+1})] w n F_1[l_t, \sigma_t(I_t) s_{t-1}/n], \end{aligned}$$

or equivalently as

$$\begin{aligned} & \left(E_t[1 - q_{t+1}^3(I_{t+1})] \delta + \frac{\tilde{b} n}{1 - \alpha} \right) \zeta_{t+1}^{-1} = \\ & \frac{\alpha \delta w n}{zh'(e_t)} E_t[1 - q_{t+1}^3(I_{t+1})] A^{-1} (1 - q_t^2(I_t) - wne_t)^{-\alpha} (\sigma_t(I_t)/n)^\alpha \zeta_t^{-\alpha}. \end{aligned}$$

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