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**Working Paper**

## Equilibrium Directed Search with Multiple Applications

IZA Discussion Papers, No. 719

**Provided in Cooperation with:**

IZA – Institute of Labor Economics

*Suggested Citation:* Albrecht, James; Gautier, Pieter A.; Vroman, Susan (2003) : Equilibrium Directed Search with Multiple Applications, IZA Discussion Papers, No. 719, Institute for the Study of Labor (IZA), Bonn

This Version is available at:

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February 2003

# Equilibrium Directed Search with Multiple Applications

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Discussion Paper No. 719  
February 2003

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## **ABSTRACT**

### **Equilibrium Directed Search with Multiple Applications**

We analyze a model of directed search in which unemployed job seekers observe all posted wages. We allow for the possibility of multiple applications by workers and ex post competition among vacancies. For any number of applications, there is a unique symmetric equilibrium in which vacancies post a common wage. When workers apply to only one vacancy, a single wage is paid and the resulting equilibrium is efficient. When workers make multiple applications, there is dispersion in wages paid, and equilibrium may be inefficient. We show that our results also hold in a steady-state version of the model.

JEL Classification: J64, D83, J41

Keywords: job search, directed search, matching

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# 1 Introduction

In this paper, we construct an equilibrium model of directed search. Unemployed workers, observing the wages posted at all vacancies, direct their applications towards the vacancies they find most attractive. At the same time, (the owners of) vacancies post their wages taking into account that their posted wages influence the number of applicants they attract. In our model, each unemployed worker makes a fixed number of applications,  $a$ , where  $a \in \{1, 2, \dots, v\}$ . When workers apply for two or more jobs at the same time, i.e.,  $a \in \{2, \dots, v\}$ , there is a possibility that more than one vacancy will want to hire the same worker. In this case, we assume that the vacancies in question can compete for this worker's services.

When  $a = 1$ , our model is the same as that of Burdett, Shi, and Wright (2001) (hereafter BSW), albeit translated to a labor market setting. BSW derive a unique symmetric equilibrium in which all vacancies post a wage between zero (the monopsony wage) and one (the competitive wage). The value of this common posted wage depends on the number of unemployed,  $u$ , and the number of vacancies,  $v$ , in the market. When  $a \in \{2, \dots, v\}$ , our results are radically different from those of BSW. We show that all vacancies post the monopsony wage in the unique symmetric equilibrium. This leads to equilibrium wage dispersion. Some workers (those who receive exactly one offer) are employed at the monopsony wage, and some workers (those who receive multiple offers) have their wages bid up to the competitive level. When  $a = v$ , i.e., when each unemployed worker applies to every vacancy, our model is related to – but not the same as – that of Julien, Kennes, and King (2000) (hereafter JKK). JKK assume that each unemployed worker posts a minimum wage at which he or she is willing to work, i.e., a “reserve wage,” and that each vacancy then makes an offer to one worker. If more than one vacancy wants to hire the same worker, then, as in our model, there is *ex post* competition for that worker's services. JKK show that the unique, symmetric equilibrium reserve wage lies between the monopsony and competitive levels. There is thus also equilibrium wage dispersion in their model. Those workers who receive only one offer are employed at the reserve wage, while those who receive multiple offers are employed at the competitive wage.

In addition to subsuming BSW and (a model close to) JKK as special cases, we make several contributions by considering the general case of  $a \in \{1, 2, \dots, v\}$ . First, we contribute to the literature on the microfoun-

dations of the matching function,<sup>1</sup> an essential ingredient in much of the search literature (Pissarides 2000). The urn-ball model is a standard microfoundation for the matching function. In that model, each worker makes a single application, and there is a coordination problem among applicants because some vacancies can receive applications from more than one worker, while others receive none. With multiple applications, there is a second coordination problem, this time among vacancies. When workers apply for more than one job at a time, some workers can receive offers from more than one vacancy, while others receive none. Ultimately, a worker can only take one job, and the vacancies that “lose the race” for a worker will have wasted time and effort while considering his or her application. The matching function derived in BSW captures only the urn-ball friction, while the one derived in JKK captures only the multiple application friction. Our matching function incorporates both the urn-ball and the multiple application coordination frictions, and the interaction between these two frictions provides new insights.

Second, when  $a \in \{2, \dots, v\}$ , our model generates equilibrium wage dispersion with directed search, even though workers and vacancies are homogeneous.<sup>2</sup> Relative to the JKK result for the case of  $a = v$ , we (i) find wage dispersion so long as each worker makes at least two applications, (ii) derive our result under the (in our view more realistic) assumption that vacancies post wages, as opposed to workers posting reserve wages, and (iii) derive a two-point distribution in which some workers are paid the monopsony wage, while others are paid the competitive wage, as opposed to the reserve wage and the competitive wage in JKK. This last distinction is important because it makes our model quite tractable, as we will show below.

Our third contribution is to examine the normative question of whether vacancy creation in a labor market with wage posting and directed search is constrained efficient. That is, is the equilibrium, free-entry level of labor market tightness the same as the level that a social planner would choose? As  $u$  and  $v$  become arbitrarily large, the results of Moen (1997) on the

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<sup>1</sup>Our derivation of the matching function is taken from Albrecht, Gautier and Vroman (2003). Relative to that paper, our contribution here is to derive our matching function in an equilibrium setting.

<sup>2</sup>Postel-Vinay and Robin (2000) have a related result in an undirected, random search framework. In their model, as in Burdett and Mortensen (1998), wage offers arrive at Poisson rates to both the unemployed and the employed. If a worker who is already employed receives another offer, then that worker’s current employer and prospective new employer engage in Bertrand competition for his or her services. In the homogeneous worker/homogeneous firm version of their model, this leads to a 2-point equilibrium distribution of wages paid.

efficiency of competitive search equilibrium suggest an affirmative answer to this question. To look at this issue, we investigate a limiting version of our model. We first let  $u, v \rightarrow \infty$  with  $v/u = \theta$  and  $a$  fixed. We verify that the standard efficiency result holds when  $a = 1$ , but for each fixed  $a > 1$ , we show that the limiting equilibrium is inefficient; specifically, there is excess vacancy creation. We then let  $u, v \rightarrow \infty$  with  $v/u = \theta$  and  $a = \phi v$  with  $0 < \phi \leq 1$ ; that is, we let the number of applications grow at a rate proportional to the number of vacancies as  $v \rightarrow \infty$ . In this case, equilibrium is once again, i.e., as in the case of  $a = 1$ , constrained efficient.<sup>3</sup> In short, we show that the limiting equilibrium is constrained efficient in the extreme cases ( $a = 1$  and  $a$  proportional to  $v$ ) but constrained inefficient when workers make any fixed number of multiple applications.

In the next section we derive our basic positive results in a single-period framework, and in Section 3, we give our results on constrained efficiency. In Section 4, we present a steady-state version of our model for the case of  $a \in \{2, \dots, v\}$ . The key to the steady-state analysis is that a worker who receives only one offer in the current period has the option to reject that offer in favor of waiting for a future period in which more than one vacancy bids for his or her services. Allowing for free entry of vacancies, this leads to a tractable model in which labor market tightness and the equilibrium wage distribution are simultaneously determined. The normative results that we derived in the single-period model continue to hold in the steady-state setting. Finally, in Section 5, we conclude.

## 2 The Basic Model

We consider a game played by  $u$  homogeneous unemployed workers and (the owners of)  $v$  homogeneous vacancies, where  $u$  and  $v$  are given. This game has several stages:

1. Each vacancy posts a wage.
2. Each unemployed worker observes all posted wages and then submits  $a$  applications with no more than one application going to any one vacancy.
3. Each vacancy that receives at least 1 application randomly selects one to process. Any excess applications are returned as rejections.

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<sup>3</sup>When  $\phi = 1$ , our limiting equilibrium is the same as the one in JKK since in their model the reserve wage converges to the monopsony wage.

4. A vacancy with a processed application offers the applicant the posted wage. If more than one vacancy makes an offer to a particular worker, then those vacancies can bid against one another for that worker's services.
5. A worker with one offer can accept or reject that offer. A worker with more than one offer can accept one of the offers or reject all of them.

Workers who fail to match with a vacancy and vacancies that fail to match with a worker receive payoffs of zero. The payoff for a worker who matches with a vacancy is  $w$ , where  $w$  is the wage that he or she is paid. A vacancy that hires a worker at a wage of  $w$  receives a payoff of  $1 - w$ .

Before we analyze this game, some comments on the underlying assumptions are in order. First, this is a model of directed search in the sense that workers observe all wage postings and send their applications to vacancies with attractive wages and/or where relatively little competition is expected. We assume that vacancies cannot pay less than their posted wages. If they could, directed search would not make sense. Second, we are treating  $a$  as a parameter of the search technology; that is, the number of applications is taken as given. In general,  $a \in \{1, 2, \dots, v\}$ . Third, we assume that it takes a period for a vacancy to process an application. This is why vacancies return excess applications as rejections. This processing time assumption is important for our results. It captures the idea that when workers apply for several jobs at the same time, firms can waste time and effort pursuing applicants who ultimately go elsewhere. Finally, we assume that 2 or more vacancies that are competing for the same worker can engage in *ex post* Bertrand competition for that worker. This means that workers who receive more than one offer will have their wages bid up to  $w = 1$ , the competitive wage. There are, of course, other possible "tie-breaking" assumptions. For example, one might assume that vacancies hold to their posted wages, that is, refuse to engage in *ex post* bidding. This, however, would not be in the individual interest of vacancies.

We consider symmetric equilibria in which all vacancies post the same wage and all workers use the same strategy to direct their applications. We will show that for each  $(u, v, a)$  combination there is a unique symmetric equilibrium, and we will derive the corresponding equilibrium matching function and posted wage. Assuming (for the moment) the existence of a symmetric equilibrium, we begin with the matching function. The following result is from Albrecht et. al. (2003).



**Proposition 1** *The expected number of matches in symmetric equilibrium is*

$$M(u, v; a) = u(1 - (1 - \frac{v}{au}(1 - (1 - \frac{a}{v})^u))^a). \quad (1)$$

**Proof.** Let  $q$  be the probability that any one application leads to a job offer. This equals the number of vacancies with applications divided by the total number of applications, that is,  $q = pv/au$ , where  $p$  is the probability that a particular vacancy will receive at least one application. If all vacancies post the same wage, then the optimal mixed strategy for each unemployed (given that all other unemployed follow the same strategy) is to send applications to randomly selected vacancies. The number of applications received by any one vacancy is then a binomial random variable with parameters  $u$  and  $a/v$ ,

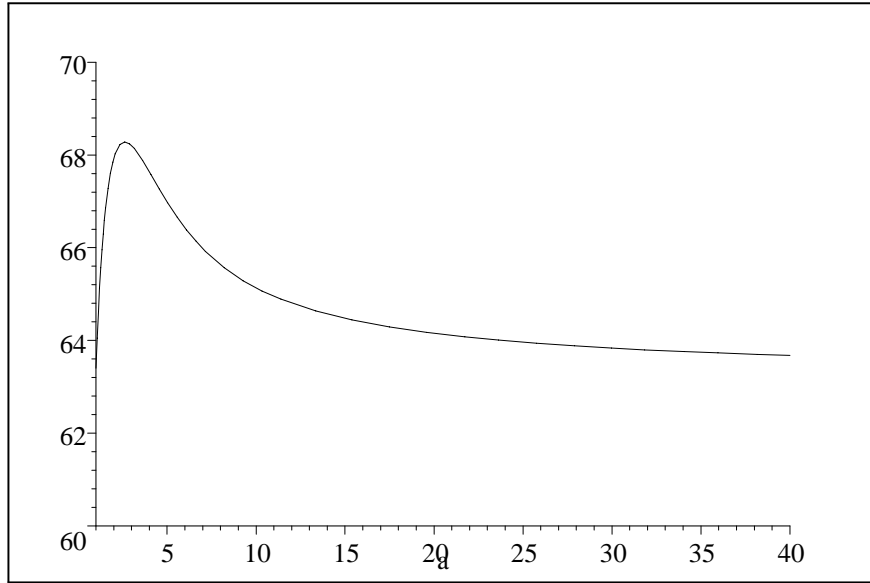
so  $p = 1 - (1 - \frac{a}{v})^u$  and  $q = \frac{1 - (1 - \frac{a}{v})^u}{au/v}$ . The probability that at least one of a worker's applications leads to a job offer is  $1 - (1 - q)^a$ ; so, the expected number of matches is  $u[1 - (1 - q)^a]$ . Substitution gives equation (1). ■

For  $a = 1$ , this result is analogous to the one given in Proposition 2 of BSW. That is, with  $a = 1$  (and with the notational change of  $m = v$  and  $n = u$ ) our results exactly match those of BSW. For  $a \in \{1, 2, \dots, v\}$ ,  $M(u, v; a)$  is increasing at a decreasing rate in both  $u$  and  $v$ . In addition,  $M(u, v; a)$  exhibits decreasing returns to scale in  $(u, v)$  for each fixed  $a$ . The basic point of these results is that the qualitative properties of the matching process in BSW hold for general values of  $a$ , i.e., not just for the special case of  $a = 1$ .

The properties of  $M(u, v; a)$  as a function of  $a$  are of more interest. With  $a = 1$ , the familiar urn-ball friction operates in the labor market. Some vacancies can receive more than one application, while others receive none, so the expected number of matches is less than the minimum of  $u$  and  $v$ . When workers submit more than one application, the urn-ball friction is reduced in the sense that the probability that any particular vacancy receives no applications decreases, but with  $a > 1$ , a new friction is introduced by the multiple applications. A worker who gets multiple offers can only accept one job. A vacancy that has processed a particular application may find at the end of the period that the worker whose application it processed takes a job elsewhere. The urn-ball friction results from a lack of coordination among job seekers; the multiple-application friction is due to a lack of coordination among vacancies. The urn-ball friction decreases with  $a$ ; the multiple-application friction increases with  $a$ . Figure 1 gives an example of the relationship between the number of matches and the number of appli-

cations treated as a continuous variable. When  $u = v = 100$ , the expected number of matches first increases with the number of applications and then decreases.

Figure 1  
 $M(100, 100, a)$



The special case of  $a = v$  is of particular interest. A matching process in which workers apply to all vacancies and each vacancy then randomly selects one applicant is an urn-ball process with the role of urn played by workers and that of ball played by vacancies. That is, the case of  $a = v$  is essentially the same as that of  $a = 1$ , except that the roles of workers and vacancies are reversed. The symmetry between the cases of  $a = 1$  and  $a = v$  can be seen in  $M(u, v; 1) = v(1 - (1 - \frac{1}{v})^u)$  and  $M(u, v; v) = u(1 - (1 - \frac{1}{u})^v)$ . The case of  $a = v$  is the one considered in JKK.

Proposition 1 and its implications are only interesting if a symmetric equilibrium exists. We now turn to the existence question.

**Proposition 2** *There is a unique symmetric equilibrium. When  $a = 1$ , all vacancies post a wage of*

$$w(u, v; 1) = \frac{u(v-1) \left(\frac{v-1}{v}\right)^u}{v(v-1) - \left(\frac{v-1}{v}\right)^u (u + v(v-1))}. \quad (2)$$

When  $a \in \{2, \dots, v\}$ , all firms post the monopsony wage of  $w = 0$ . As in

Proposition 1, let  $q = \frac{1 - (1 - \frac{a}{v})^u}{au/v}$ . Then a job seeker receives exactly one offer with probability  $aq(1 - q)^{a-1}$ . Such a worker is paid  $w = 0$ . The probability that a job seeker receives two or more offer is  $1 - (1 - q)^a - aq(1 - q)^{a-1}$ . In this case, the worker is paid  $w = 1$ .

**Proof.** Given in the Appendix. ■

To prove the existence of a symmetric equilibrium, we show that when  $a = 1$ , the wage given in equation (2) has the property that if all vacancies, with the possible exception of a “potential deviant,” post that wage, then it is also in the interest of the deviant to post that wage. When  $a \in \{2, \dots, v\}$ , then no matter what the common wage posted by other vacancies, it is always in the interest of the deviant to undercut that common wage. This forces the wage down to the monopsony level, which in our single-period model is  $w = 0$ .

The equilibrium wage for the case of  $a = 1$  is equal to one minus the price given in Proposition 2 in BSW – again with the appropriate notational change. The tradeoff that leads to a well-behaved equilibrium wage,  $w \in (0, 1)$ , when  $a = 1$  is the standard one in equilibrium search theory. As any particular vacancy increases its posted wage, holding the wages posted by other vacancies constant, the probability that it will attract at least one applicant also increases. At the same time, however, the profit that this vacancy generates conditional on attracting an applicant decreases. This tradeoff varies smoothly with  $u$  and  $v$ ; so the equilibrium wage varies smoothly between zero and one as  $v$  increases and/or  $u$  decreases. Thus, as emphasized in BSW (p. 1069), there is a sense in which frictions “smooth” the operation of the labor market.

When  $a \in \{2, \dots, v\}$ , matters are radically different. No matter what the values of  $u$  and  $v$ , so long as workers make more than one application, the posted wage collapses to the Diamond (1971) monopsony level. The intuition for this result is based on the change in the tradeoff underlying equilibrium wage determination. It is still the case that as any particular vacancy increases its posted wage, holding all other posted wages constant, the probability that at least one applicant will be attracted increases.<sup>4</sup> However, the profit that a vacancy generates conditional on attracting an applicant now

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<sup>4</sup>As  $a$  increases, the rate of increase in this probability decreases. In the extreme case of  $a = v$ , an increase in the posted wage cannot increase the probability of attracting an applicant since that probability is necessarily already one.

decreases for *two* reasons when the posted wage is increased. First, if the vacancy manages to employ the worker at the posted wage, then an increase in that wage obviously decreases profit. This is the same factor that limits increases in the posted wage when  $a = 1$ . Second, and this is the new factor, the probability that the applicant will have other offers increases. The reason is that in symmetric equilibrium, all workers respond to an increase in the wage posted by one vacancy by increasing the probability of applying for that job and decreasing the probability of applying to other jobs. That is, the probability that other workers will apply to the same vacancies as the other vacancies applied to by the chosen applicant decreases. The probability that the selected applicant will get multiple offers and so generate zero profit thus increases. This added factor is what drives the posted wage to the Diamond monopsony level.

Despite the fact that the posted equilibrium wage is zero when  $a \in \{2, \dots, v\}$ , there is still a sense in which “the wage” varies smoothly with  $u$  and  $v$ . The expected fraction of wages paid equal to one,

$$\gamma = \frac{1 - (1 - q)^a - aq(1 - q)^{a-1}}{1 - (1 - q)^a}, \quad (3)$$

increases with  $v$  and decreases with  $u$ , and in the limit, as  $v \rightarrow \infty$  holding  $u$  fixed (as  $u \rightarrow \infty$  holding  $v$  fixed),  $\gamma \rightarrow 1$  ( $\gamma \rightarrow 0$ ). Note that since the wage is either 0 (the posted wage) or 1 (the Bertrand wage),  $\gamma$  is the expected wage (paid, as opposed to posted).

Next, we look at the limiting properties of the labor market described above. We let the labor market get large in the standard way, namely, we let the number of unemployed and vacancies increase without limit in such a way that the ratio of vacancies to unemployed, i.e., labor market tightness, is held fixed. First, we carry out this limiting exercise holding the number of applications per worker fixed. To make this clear, we use the notation  $a \in \{1, \dots, A\}$ , where  $A$  is an integer greater than 1, which does not change as  $u$  and  $v$  go to infinity. Then we repeat the exercise, also allowing the number of applications to go to infinity.

**Proposition 3** *Let  $u, v \rightarrow \infty$  with  $v/u = \theta$  and  $a \in \{1, \dots, A\}$  fixed. The number of matches increases without limit, but the probability that any one worker finds a job converges to*

$$m(\theta; a) = 1 - \left(1 - \frac{\theta}{a}(1 - e^{-a/\theta})\right)^a. \quad (4)$$

In the case of  $a = 1$ , the wage converges to

$$w(\theta; 1) = \frac{(1/\theta) \exp(-1/\theta)}{1 - \exp(-1/\theta)}. \quad (5)$$

In the case of  $a \in \{2, \dots, A\}$ , the fraction of wages paid that equals one converges to

$$\gamma(\theta; a) = \frac{1 - (1 - \frac{\theta}{a}(1 - e^{-a/\theta}))^a - \theta(1 - e^{-a/\theta})(1 - \frac{\theta}{a}(1 - e^{-a/\theta}))^{a-1}}{1 - (1 - \frac{\theta}{a}(1 - e^{-a/\theta}))^a} \quad (6)$$

**Proof.** Given in the Appendix. ■

With  $a = 1$ , it is easy to verify that our limiting matching function and the limiting wage for the case of  $a = 1$  match the corresponding entities in a labor-market version of BSW. In general, i.e., for  $a \in \{1, 2, \dots, A\}$ , the limiting matching probability has the following properties:

- (i) For large values of  $u$  and  $v$ , the matching function exhibits approximate constant returns to scale for each fixed  $a$  in the sense that in the limit, the matching probability depends only on the ratio of  $v$  to  $u$ ;
- (ii)  $m(\theta; a)$  is increasing and concave in  $\theta$ ,  $\lim_{\theta \rightarrow 0} m(\theta; a) = 0$ , and

$$\lim_{\theta \rightarrow \infty} m(\theta; a) = 1;$$

- (iii)  $\frac{m(\theta; a)}{\theta}$  is decreasing in  $\theta$ ,  $\lim_{\theta \rightarrow 0} \frac{m(\theta; a)}{\theta} = 1$ , and  $\lim_{\theta \rightarrow \infty} \frac{m(\theta; a)}{\theta} = 0$ ;<sup>5</sup>

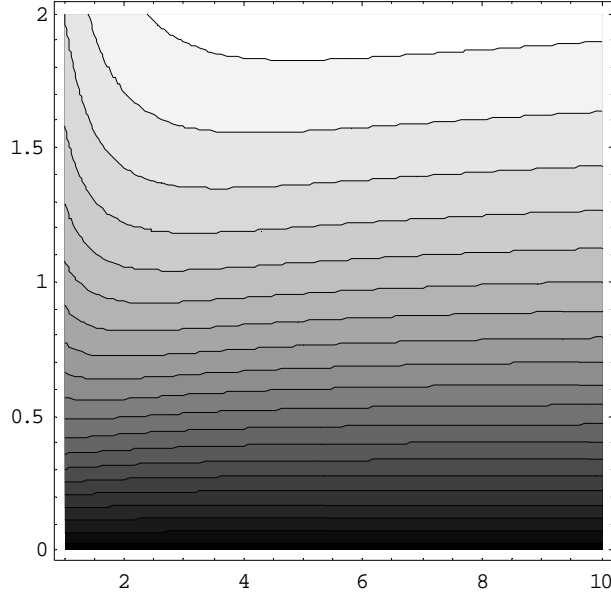
The proofs of these properties are straightforward and are available on request.

The effect of  $a$  on  $m(\theta; a)$  is less clearcut. We find that  $m_a(\theta; a) \geq 0$  as  $\frac{a}{1-q} \frac{\partial q}{\partial a} - \ln(1-q) \geq 0$  where  $q = \frac{\theta}{a}(1 - e^{-\frac{a}{\theta}})$ . This is illustrated in Figure 2, a contour map which shows how  $m(\theta; a)$  varies jointly with  $a$  and  $\theta$ . The lighter shaded areas denote larger values of  $m(\theta; a)$ . The figure suggests that for values of  $\theta$  below .5,  $m(\theta; a)$  decreases with  $a$ , but for values of  $\theta$  above .5,  $m(\theta; a)$  first increases and then decreases with  $a$ .

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<sup>5</sup>Interestingly,  $\frac{m(\theta; a)}{\theta}$  is not convex in  $\theta$ , as can be seen immediately by considering the case of  $a = 1$ . The properties of  $m(\theta; a)$  and  $\frac{m(\theta; a)}{\theta}$  given in (ii) and (iii) are the minimal ones required for our normative results in Sections 3 and 4 below.

Figure 2



For the case of  $a \in \{2, \dots, A\}$ , we also need to investigate the properties of the expected wage,  $\gamma(\theta; a)$ . We can show that  $\gamma(\theta; a)$  is increasing in  $\theta$  and in  $a$  and that  $\lim_{\theta \rightarrow 0} \gamma(\theta; a) = 0$  and  $\lim_{\theta \rightarrow \infty} \gamma(\theta; a) = 1$ . The result that  $\gamma$  is increasing in  $\theta$  is exactly as one would expect – as the labor market gets tighter, the chance that an individual worker gets multiple offers increases. To understand why  $\gamma$  is also increasing in  $a$ , it is important to remember that  $\gamma(\theta; a)$  is the expected wage for those workers who match with a vacancy; in particular, those workers who fail to match are not treated as receiving a wage of zero.

Proposition 3 describes the limiting properties of the labor market taking  $a$  as a constant. Alternatively, we could let the number of applications per worker become arbitrarily large as well. To do this while keeping  $a \leq v$ , we let  $a$  be a fixed fraction of  $v$  and then let  $u$  and  $v$  go to infinity in the standard way.

**Proposition 4** *Let  $a, u, v \rightarrow \infty$  with  $v/u = \theta$  and  $a/v = \phi$ , where  $0 < \phi \leq 1$ . Then the probability that any one worker finds a job converges to*

$$m(\theta) = 1 - e^{-\theta} \quad (7)$$

and the fraction of wages paid equal to one converges to

$$\gamma(\theta) = \frac{1 - e^{-\theta} - \theta e^{-\theta}}{1 - e^{-\theta}}. \quad (8)$$

**Proof.** Given in the Appendix. ■

These results are also derived in JKK for the case of  $\phi = 1$ . There are three points worth noting about Proposition 4. First, we can derive equations (7) and (8) if we start from Proposition 3, i.e., the situation in which  $u, v \rightarrow \infty$  with  $v/u = \theta$ , and take the limits of  $m(\theta; a)$  and  $\gamma(\theta; a)$  as  $a$  goes to infinity. Second,  $m(\theta)$  and  $\gamma(\theta)$  are both increasing and concave in  $\theta$ . Finally, neither  $m(\theta)$  nor  $\gamma(\theta)$  depend on  $\phi$ ; that is, the limiting matching probability and expected wage are the same in the limit whether each worker applies to every vacancy or whether each worker applies to, say, one vacancy in every thousand. The reason is that even if  $\phi$  is very small, eventually the total number of applications ( $au = \frac{\phi}{\theta}v^2$ ) swamps the total number of vacancies in the market. That is, in the limit, every vacancy receives at least one application with probability one.

### 3 Efficiency

We now turn to the question of constrained efficiency. As is usual, we examine this issue in a large labor market, i.e., one in which  $u, v \rightarrow \infty$  with  $v/u = \theta$ . The result suggested by the efficiency of competitive search equilibrium holds in our setting when  $a = 1$ ; however, when workers make a fixed number of multiple applications, this result breaks down. When we let  $a \rightarrow \infty$ , the equilibrium is again constrained efficient.

We start with  $a$  fixed and finite. We suppose that vacancies are set up at the beginning of the period and that each vacancy is created at cost  $c$ . The efficient level of labor market tightness<sup>6</sup> is determined as the solution to

$$\max_{\theta \geq 0} \{-c\theta + m(\theta; a)\}.$$

The first-order condition for an interior solution to this problem is

$$c = m_{\theta}(\theta^*; a). \quad (9)$$

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<sup>6</sup>In a finite labor market with  $u$  given, the social planner chooses  $v$  to maximize  $-cv + M(u, v; a)$ ; i.e., expected output (equal to the expected number of matches since each match produces an output of 1) minus the vacancy creation costs. Dividing the maximand by  $u$  and letting  $u, v \rightarrow \infty$  gives the maximand in the text.

The equilibrium level of labor market tightness is determined by free entry. When  $a = 1$ , this means

$$c = \frac{m(\theta^{**}; 1)}{\theta^{**}}(1 - w(\theta^{**}; 1)), \quad (10)$$

whereas for  $a \in \{2, \dots, A\}$ , the condition is

$$c = \frac{m(\theta^{**}; a)}{\theta^{**}}(1 - \gamma(\theta^{**}; a)). \quad (11)$$

Equations (10) and (11) reflect the condition that entry (vacancy creation) occurs up to the point that the cost of vacancy creation is just offset by the value of owning a vacancy. This value equals the probability of hiring a worker times the expected surplus generated by a hire – equal to 1 minus the posted wage when  $a = 1$  and to 1 minus the expected wage when  $a \in \{2, \dots, A\}$ .

Note that  $\theta^*$  denotes the constrained Pareto efficient level of labor market tightness and  $\theta^{**}$  denotes the equilibrium level of labor market tightness. At issue is the relationship between  $\theta^*$  and  $\theta^{**}$ .

**Proposition 5** *Let  $u, v \rightarrow \infty$  with  $v/u = \theta$  and  $a \in \{1, \dots, A\}$  fixed. For  $a = 1$ ,  $\theta^* = \theta^{**}$ . For  $a \in \{2, \dots, A\}$ ,  $\theta^{**} > \theta^*$ .*

**Proof.** Differentiating equation (4) with respect to  $\theta$  gives

$$m_\theta(\theta; a) = \left(1 - \frac{\theta}{a}(1 - e^{-a/\theta})\right)^{a-1} \left(1 - e^{-a/\theta} - \frac{a}{\theta}e^{-a/\theta}\right). \quad (12)$$

For the case of  $a = 1$ , equation (9) becomes

$$c = 1 - e^{-1/\theta} - \frac{1}{\theta}e^{-1/\theta}.$$

From equations (4) and (5), equation (10) is

$$\frac{m(\theta; 1)}{\theta}(1 - w(\theta; 1)) = 1 - e^{-1/\theta} - \frac{1}{\theta}e^{-1/\theta}.$$

Thus, equations (9) and (10) imply  $\theta^* = \theta^{**}$ .

When  $a \in \{2, \dots, A\}$ ,  $\theta^*$  solves

$$c = \left(1 - \frac{\theta}{a}(1 - e^{-a/\theta})\right)^{a-1} \left(1 - e^{-a/\theta} - \frac{a}{\theta}e^{-a/\theta}\right), \quad (13)$$



whereas, using equations (4) and (6),  $\theta^{**}$  solves

$$c = \left(1 - \frac{\theta}{a}(1 - e^{-a/\theta})\right)^{a-1}(1 - e^{-a/\theta}). \quad (14)$$

The right-hand sides of both (13) and (14) are decreasing in  $\theta$ . Since the right-hand side of (14) is greater than that of (13) for all  $\theta > 0$ , it follows that  $\theta^{**} > \theta^*$ . ■

Posting a vacancy has the standard congestion and thick-market effects in our model – adding one more vacancy makes it more difficult for the incumbent vacancies to find workers but makes it easier for the unemployed to generate offers. A striking result of the competitive search equilibrium literature is that adding one more vacancy causes the wage to adjust in such a way as to balance these external effects correctly. One way to interpret this is to say that competition leads to a wage equal to the one that would be dictated by the Hosios (1990) condition in a Nash bargaining model. Equivalently, one can say (Moen, 1997, p. 387) that the competitive search equilibrium wage has the property that the marginal rate of substitution between labor market tightness and the wage is the same for vacancies as for workers. The first part of Proposition 5 shows that this result holds when one uses an explicit urn-ball ( $a = 1$ ) microfoundation for the matching function. However, when workers make multiple applications, the result that  $\theta^{**} > \theta^*$  indicates that the equilibrium level of vacancy creation is too high. Equivalently, the equilibrium expected wage is below the level that would be indicated by the Hosios condition. The effects of the marginal vacancy are more complicated with multiple applications than in the urn-ball model. Adding one more vacancy makes it less likely that each incumbent vacancy will attract any applicants but, conditional on attracting an applicant, makes it more likely that the incumbent vacancy “wins the race” for that applicant. Adding another vacancy to the market puts upward pressure on the (expected) wage but not to the extent required to achieve the efficient level of entry.

Proposition 5 lets the labor market get large holding  $a$  fixed. In Proposition 6, we consider the question of efficiency letting  $a$  get arbitrarily large along with  $u$  and  $v$ . Of course, it is unreasonable to assume that each worker can apply to an infinite number of vacancies. We present Proposition 6 for the sake of completeness and to allow us to relate our efficiency results to those of JKK.

**Proposition 6** *Let  $a, u, v \rightarrow \infty$  with  $v/u = \theta$  and  $a/v = \phi$ , where  $0 < \phi \leq 1$ . Then  $\theta^{**} = \theta^*$ ; i.e., the equilibrium level of labor market tightness is constrained efficient.*

**Proof.** The result is shown by mimicking the proof of Proposition 5, that is, by comparing  $m_\theta(\theta)$  with  $\frac{m(\theta)}{\theta}(1 - \gamma(\theta))$ . Using equations (7) and (8),  $m_\theta(\theta) = e^{-\theta}$  as does  $\frac{m(\theta)}{\theta}(1 - \gamma(\theta))$ . ■

Proposition 6 is also shown in JKK (in their Proposition 2.5) for the case of  $\phi = 1$ . In a companion paper, Julien, Kennes, and King (2002) show that equilibrium in a finite labor market with  $a = v$  is also constrained efficient if one *assumes* a particular wage determination mechanism; namely, vacancies offering jobs to workers who have no other offers receive all of the surplus ( $w = 0$ ) but vacancies offering jobs to workers who do have other offers receive none of the surplus ( $w = 1$ ). Julien, Kennes, and King (2002) interpret this result in terms of what they call the Mortensen rule (Mortensen 1982) – that efficiency in matching is attained if the “initiator” of the match gets the total surplus.<sup>7</sup> Relative to their result, our contribution is to show that this assumed wage determination mechanism is in fact the symmetric equilibrium outcome in a directed search model with wage posting when  $a = v$ .

Returning to the limiting case, an intuition for why we find constrained efficiency with  $a = 1$  and as  $a \rightarrow \infty$  but not with a fixed, finite number of multiple applications is that with  $a = 1$  and as  $a \rightarrow \infty$ , only one coordination problem affects the operation of the labor market, whereas with a fixed  $a \in \{2, \dots, A\}$ , the urn-ball and the multiple applications coordination problems operate simultaneously. Adjusting the wage can only solve one coordination problem at a time.

## 4 Steady State

We now turn to steady-state analysis for a labor market with directed search and multiple applications. We work with the limiting case in which  $u, v \rightarrow \infty$

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<sup>7</sup>The intuitions for constrained efficiency (i) in a large labor market when  $a = 1$  and (ii) when  $a = v$  are thus quite different. When  $a = 1$ , constrained efficiency is a result of competition, and competition requires a labor market sufficiently large that individual vacancies have negligible market power. When  $a = v$ , constrained efficiency is a result of perfect monopoly power – the entire surplus goes to the vacancy if there is no competition for the applicant it selects and to the worker if he or she winds up having the monopoly power. The monopoly intuition does not require that the labor market be large.

with  $v/u = \theta$  and  $a \in \{2, \dots, A\}$  fixed. Since only the ratio of  $v$  to  $u$  matters in the limiting case, we normalize the labor force to 1; thus,  $u$  is interpreted as the unemployment rate.

In steady-state, workers flow into employment with probability  $m(\theta; a)$  per period. We assume that matches break up exogenously with probability  $\delta$ , giving the countervailing flow back into unemployment. Similarly, jobs move from vacant to filled with probability  $\frac{m(\theta; a)}{\theta}$  and back again with probability  $\delta$ . Steady-state analysis thus allows us to endogenize vacancies and unemployment. More importantly, moving to the steady state means that those unemployed who fail to find an acceptable job in the current period can wait and apply again in the future. In the case of  $a = 1$ , this isn't particularly interesting since, in equilibrium, there is no gain to waiting. However, with multiple applications, the ability of the unemployed to hold out for a situation in which vacancies engage in Bertrand competition for their services, albeit at the cost of delay, implies a positive reservation wage. This leads to a simple and appealing model in which labor market tightness and the reservation wage are simultaneously determined. On the one hand, the lower is the reservation wage of the unemployed, the more vacancies firms want to create. On the other, as the labor market becomes tighter, i.e., as  $\theta$  increases, the unemployed respond by increasing their reservation wage.

The analysis proceeds as follows. Suppose the unemployed set a reservation wage  $R$ . With multiple applications, the wage-posting problem for a vacancy is qualitatively the same as in the one-period game. Whatever common wage might be posted at other vacancies, each individual vacancy has the incentive to undercut. In the one-period game, this implies a monopsony wage posting of  $w = 0$ ; in the steady state, this same mechanism implies a dynamic monopsony wage posting of  $w = R$ . In addition, the probability that an unemployed worker finds a job in any period and the probability that he or she is hired at the competitive wage, conditional on finding a job, are the same as in the single-period model; i.e., equations (4) and (6) for  $m(\theta; a)$  and  $\gamma(\theta; a)$  continue to apply.

We begin by examining the value functions for jobs and for workers. A job can be in one of three states – vacant, filled paying the competitive wage, and filled paying  $R$ . Let  $V$ ,  $J(1)$ , and  $J(R)$  be the corresponding values. The value of a vacancy is

$$V = -c + \frac{1}{1+r} \left\{ \frac{m(\theta; a)}{\theta} [\gamma(\theta; a)J(1) + (1 - \gamma(\theta; a))J(R)] + \left(1 - \frac{m(\theta; a)}{\theta}\right)V \right\}.$$

Maintaining a vacancy entails a cost  $c$ , which is incurred at the start of

each period. Moving to the end of the period, and thus discounting at rate  $r$ , the vacancy has hired a worker with probability  $\frac{m(\theta; a)}{\theta}$ . With probability  $\gamma(\theta; a)$ , the worker who was hired had his or her wage bid up to the competitive level, thus implying a value of  $J(1)$ . With probability  $1 - \gamma(\theta; a)$  the worker was hired at  $w = R$ , thus implying a value of  $J(R)$ . Finally, with probability  $1 - \frac{m(\theta; a)}{\theta}$ , the vacancy failed to hire, in which case the value  $V$  is retained.

Free entry implies  $V = 0$ . Given  $V = 0$ , there is no incentive for vacancies competing for a worker to drop out of the Bertrand competition before the wage is bid up to  $w = 1$  (thus justifying the notation  $J(1)$ ). This in turn implies that we also have  $J(1) = 0$ . Inserting these equilibrium conditions into the expression for  $V$  gives

$$\frac{m(\theta; a)}{\theta}(1 - \gamma(\theta; a))J(R) = c(1 + r).$$

At the same time, the value of employing a worker at  $w = R$  is

$$J(R) = (1 - R) + \frac{1}{1 + r}[(1 - \delta)J(R) + \delta V].$$

Again using  $V = 0$ , we have

$$J(R) = \frac{1 + r}{r + \delta}(1 - R).$$

Combining these equations gives the first steady-state equilibrium condition,

$$c = \frac{m(\theta; a)}{\theta}(1 - \gamma(\theta; a))\frac{1 - R}{r + \delta}. \quad (15)$$

A worker also passes through three states – unemployed, employed at the competitive wage, and employed at  $R$ . The value of unemployment is defined by

$$U = \frac{1}{1 + r}\{m(\theta; a)[\gamma(\theta; a)N(1) + (1 - \gamma(\theta; a))N(R)] + (1 - m(\theta; a))U\},$$

where  $N(1)$  and  $N(R)$  are the values of employment at  $w = 1$  and  $w = R$ , respectively. These latter two values are in turn defined by

$$\begin{aligned} N(1) &= 1 + \frac{1}{1 + r}\{(1 - \delta)N(1) + \delta U\} \\ N(R) &= R + \frac{1}{1 + r}\{(1 - \delta)N(R) + \delta U\}. \end{aligned}$$

The reservation wage property, i.e.,  $N(R) = U$ , then implies

$$\begin{aligned} U &= \frac{1+r}{r}R \\ N(1) &= \frac{(1+r)}{r(r+\delta)}(r+\delta R). \end{aligned}$$

Inserting these expressions into the expression for  $U$  and rearranging gives the second steady-state equilibrium condition,

$$R = \frac{m(\theta; a)\gamma(\theta; a)}{r + \delta + m(\theta; a)\gamma(\theta; a)}. \quad (16)$$

The final equation for the steady-state equilibrium is the standard flow (Beveridge curve) condition for unemployment. Since the labor force is normalized to 1, this is

$$u = \frac{\delta}{\delta + m(\theta; a)}. \quad (17)$$

Equations (16) and (17) show that, as is common in this class of models, once labor market tightness ( $\theta$ ) is determined, the other endogenous variables – in this case,  $R$  and  $u$  – are easily determined. Using equation (16) to eliminate  $R$  from equation (15) gives the equation that determines the steady-state equilibrium value of  $\theta$ , namely,

$$c = \frac{m(\theta; a)}{\theta} \frac{1 - \gamma(\theta; a)}{r + \delta + m(\theta; a)\gamma(\theta; a)}. \quad (18)$$

Using our results on the properties of  $m(\theta; a)$  and  $\gamma(\theta; a)$ , we can show that the right-hand side of equation (18) equals  $\frac{1}{r+\delta}$  as  $\theta \rightarrow 0$ , that it goes to zero as  $\theta \rightarrow \infty$ , and that its derivative with respect to  $\theta$  is negative for all  $\theta > 0$ . Equation (18) thus has a unique solution for each  $c \in (0, \frac{1}{r+\delta}]$ .

The natural next step is to compare equilibrium steady-state labor market tightness with the constrained efficient value of  $\theta$ . The planner's problem is to choose the level of labor market tightness that maximizes the discounted value of output net of vacancy costs for an infinitely lived economy. That is, the planner's problem is to maximize

$$\sum_{t=0}^{\infty} \left( \frac{1}{1+r} \right)^t (1 - u_t - c\theta_t u_t)$$

subject to

$$u_{t+1} - u_t = m(\theta_t; a)u_t - \delta(1 - u_t)$$

with  $u_0$  given.

The current-value Hamiltonian for this problem is

$$H(\theta, u) = 1 - u - c\theta u + \lambda[m(\theta; a)u - \delta(1 - u)]$$

with necessary conditions

$$\begin{aligned} \frac{\partial H}{\partial \theta} &= -cu + \lambda m_\theta(\theta; a)u = 0 \\ r\dot{\lambda} &= -\frac{\partial H}{\partial u} = 1 + c\theta - \lambda[m(\theta; a) + \delta]. \end{aligned}$$

Evaluating at the steady-state, and eliminating  $\lambda$ , gives

$$c = \frac{(1 + c\theta)m_\theta(\theta; a)}{r + \delta + m(\theta; a)}. \quad (19)$$

Now we can compare the levels of labor market tightness implied by equations (18) and (19). Using equation (6), equation (18) can be rewritten as

$$c(r + \delta + m(\theta; a)) = (1 + c\theta)\left(1 - \frac{\theta}{a}(1 - e^{-a/\theta})\right)^{a-1}(1 - e^{-a/\theta}). \quad (20)$$

Using equation (12), equation (19) can be rewritten as

$$c(r + \delta + m(\theta; a)) = (1 + c\theta)\left(1 - \frac{\theta}{a}(1 - e^{-a/\theta})\right)^{a-1}\left(1 - e^{-a/\theta} - \frac{a}{\theta}e^{-a/\theta}\right). \quad (21)$$

As in the single period analysis, let  $\theta^*$  be the constrained efficient level of labor market tightness, i.e., the value of  $\theta$  that solves equation (21), and let  $\theta^{**}$  be the equilibrium level of labor market tightness, i.e., the value of  $\theta$  that solves equation (20). Comparing equations (20) and (21) yields the following:

**Proposition 7** *Let  $u, v \rightarrow \infty$  with  $v/u = \theta$  and  $a \in \{2, \dots, A\}$  fixed. Then in steady state,  $\theta^{**} > \theta^*$ .*

Proposition 7 indicates that, as in the single-period analysis, when the unemployed make a fixed number of multiple applications per period ( $a \in \{2, \dots, A\}$ ), equilibrium is constrained inefficient. Specifically, there is too much vacancy creation. This result holds even though the ability of the unemployed to reject offers in favor of waiting for a more favorable outcome in some future period implies a dynamic monopsony wage above the single-period monopsony wage of zero.

Finally, for the sake of completeness, we note that the steady-state analogue of Proposition 6 holds.

**Proposition 8** *Let  $a, u, v \rightarrow \infty$  with  $v/u = \theta$  and  $a/v = \phi$ , where  $0 < \phi \leq 1$ . Then in steady state, the equilibrium level of labor market tightness is constrained efficient, i.e.,  $\theta^* = \theta^{**}$ .*

Proposition 8 is proven by reworking the steady-state analysis using  $m(\theta)$  and  $\gamma(\theta)$  instead of  $m(\theta; a)$  and  $\gamma(\theta; a)$ , i.e., by using equations (7) and (8) in place of equations (4) and (6).

## 5 Concluding Remarks

In this paper, we construct an equilibrium search model in which workers, after observing all posted wages, submit a fixed number of applications,  $a \in \{1, \dots, v\}$ , to the vacancies that they find most attractive. We derive the symmetric equilibrium matching function and the common posted wage. When  $a = 1$ , our analysis is a labor market version of BSW. However, when  $a \in \{2, \dots, v\}$ , i.e., when workers make multiple applications, the symmetric equilibrium of our model is radically different. With multiple applications, the expected number of matches in our model reflects the interplay of two coordination failures – an urn-ball failure among workers and a multiple-application failure among vacancies – and our model thus offers new insights into the microfoundations of the matching function. In addition, when workers make more than one application, all vacancies post the monopsony wage, but there is dispersion in wages paid. Workers who receive only one job offer are paid the monopsony wage, but those who receive multiple offers get the competitive wage. The limiting equilibrium when workers make a single application is constrained efficient, but when workers make a finite number of multiple applications, too many vacancies are posted. These results, both positive and normative, carry over from the single-period model to a steady-state framework.

Directed search is an appealing way to model equilibrium unemployment and wage dispersion. In reality, workers do direct their applications to attractive vacancies, but unemployment nonetheless persists as a result of coordination failures on both sides of the labor market. In addition, those workers who are lucky enough to generate competition for their services do in fact have their wages bid up. The contribution of this paper is to show that these realistic features can be captured in a tractable equilibrium model.

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## 6 Appendix

### Proof of Proposition 2:

Let  $D$  denote the potential deviant, posting a wage of  $w^D$ , and let  $N$  denote the nondeviant vacancies, posting the common, putative equilibrium wage,  $w^N$ . Let  $k$  be the probability that any individual applies to  $D$ . In symmetric equilibrium,  $k$  must be the same for all workers. Let  $q^D$  be the probability that a worker is offered the  $D$  job, conditional on applying for that job, and let  $q^N$  be the probability that a worker is offered any particular  $N$  job, conditional on applying for that job.

The expected profit of  $D$  as a function of  $w^D$ , taking  $w^N$  as given, is

$$\pi(w^D; w^N) = (1 - w^D)(1 - (1 - k)^u)(1 - q^N)^{a-1}. \quad (\text{A1})$$

To understand (A1) note that when  $D$  posts a wage of  $w^D$ , there are 3 possible outcomes.

(1) No one applies to this vacancy. This occurs with probability  $(1 - k)^u$ . In this case,  $D$ 's profit is zero.

(2)  $D$  receives at least one application and the applicant to whom  $D$  offers its job has at least one other offer. This occurs with probability  $[1 - (1 - k)^u][1 - (1 - q^N)^{a-1}]$ . In this case, Bertrand competition bids the wage up to  $w = 1$ , and  $D$ 's profit is again zero.

(3)  $D$  receives at least one application and the applicant has no other offers. This occurs with probability  $[1 - (1 - k)^u](1 - q^N)^{a-1}$ . In this case, the applicant accepts  $D$ 's offer of  $w^D$ , leading to a profit of  $1 - w^D$ .

Using this notation, a symmetric equilibrium wage is a  $w$  such that

$$w \text{ solves } \max_{w^D \geq 0} \pi(w^D; w).$$

To proceed, we need explicit expressions for  $q^D$  and  $q^N$  and an implicit expression for  $k$ . The derivation of  $q^D$  is as follows. The probability that a particular worker is offered the  $D$  job is  $kq^D$ . At the same time, given that all workers choose the same value of  $k$ , each worker has an equal chance of being offered the  $D$  job, so the probability that the worker is offered this job equals the probability that this vacancy has at least one applicant divided by  $u$ . That is,  $kq^D = \frac{1 - (1 - k)^u}{u}$  or

$$q^D = \frac{1 - (1 - k)^u}{ku}. \quad (\text{A2})$$

$q^N$  is derived in a similar fashion. There are  $v - 1$   $N$  vacancies. Each worker sends  $a - 1$  applications to the  $N$  vacancies and sends his or her  $a^{\text{th}}$

application to an  $N$  vacancy with probability  $1 - k$ . The probability that a worker applies to any particular  $N$  vacancy is thus  $\frac{(a-1) + (1-k)}{v-1} = \frac{a-k}{v-1}$ , so the probability that an  $N$  vacancy has at least one applicant is  $1 - (1 - \frac{a-k}{v-1})^u$ . The probability that a worker gets a particular  $N$  job is  $\left(\frac{a-k}{v-1}\right) q^N = \frac{1 - (1 - \frac{a-k}{v-1})^u}{u}$ ; thus,

$$q^N = \frac{1 - (1 - \frac{a-k}{v-1})^u}{\left(\frac{a-k}{v-1}\right) u}. \quad (\text{A3})$$

For future reference, we note that

$$\frac{\partial q^D}{\partial k} = \frac{ku(1-k)^{u-1} - (1 - (1-k)^u)}{k^2 u} \quad (\text{A4})$$

and

$$\frac{\partial q^N}{\partial k} = \left(\frac{v-1}{u}\right) \frac{-u \left(1 - \frac{a-k}{v-1}\right)^{u-1} \left(\frac{a-k}{v-1}\right) + 1 - \left(1 - \frac{a-k}{v-1}\right)^u}{(a-k)^2}. \quad (\text{A5})$$

To derive an implicit expression for  $k$ , note that each worker has two possible application strategies:

1. Send  $a - 1$  applications to randomly selected  $N$  vacancies and also apply to  $D$ ;
2. Send all  $a$  applications to randomly selected  $N$  vacancies.

Note that if  $a = v$ , only the first strategy is possible and  $k = 1$ .

Given  $w^D$  and  $w^N$ , the expected payoff to the first strategy is

$$\begin{aligned} & q^D(1 - q^N)^{a-1}w^D + q^D(1 - (1 - q^N)^{a-1}) \\ & + (1 - q^D)(a - 1)q^N(1 - q^N)^{a-2}w^N \\ & + (1 - q^D)(1 - (1 - q^N)^{a-1} - (a - 1)q^N(1 - q^N)^{a-2}). \end{aligned}$$

The first term in this expression reflects the fact that a worker who follows the first strategy is offered only the  $D$  job with probability  $q^D(1 - q^N)^{a-1}$ ; in

this case, a payoff of  $w^D$  is realized. With probability  $q^D(1 - (1 - q^N)^{a-1})$ , the worker's application to  $D$  is accepted along with at least one of his or her applications to the  $N$  vacancies; in this case the worker's payoff is 1. With probability  $(1 - q^D)(a - 1)q^N(1 - q^N)^{a-2}$ , the worker is rejected at  $D$  and accepted at exactly one of the  $N$  vacancies; the resulting payoff is  $w^N$ . With probability  $(1 - q^D)(1 - (1 - q^N)^{a-1} - (a - 1)q^N(1 - q^N)^{a-2})$ , the worker is rejected at  $D$  but gets 2 or more offers at  $w^N$ ; in this case, a payoff of 1 is realized. The only other possibility is that all of the worker's applications are rejected, implying a payoff of zero.

The expected payoff to the second strategy is

$$aq^N(1 - q^N)^{a-1}w^N + [1 - (1 - q^N)^a - aq^N(1 - q^N)^{a-1}].$$

The first term reflects the fact that the probability of being offered only one  $N$  job is  $aq^N(1 - q^N)^{a-1}$ . In this case, the worker receives  $w^N$ . The second term gives the probability that the worker is offered more than one job in which case the worker receives 1. For  $k \in (0, 1)$ , the expected payoffs from the two strategies must be equal giving the indifference condition,

$$(1 - q^N)(q^D w^D - q^N + aq^N(1 - w^N)) - (1 - q^D)(a - 1)q^N(1 - w^N) = 0. \quad (\text{A6})$$

Holding  $w^N$  fixed, this can be differentiated with respect to  $w^D$ . Solving for  $\frac{\partial k}{\partial w^D}$  gives

$$\frac{\partial k}{\partial w^D} = \frac{-q^D(1 - q^N)}{M}, \quad (\text{A7})$$

where

$$M = \frac{\partial q^D}{\partial k}[w^D(1 - q^N) + (a - 1)q^N(1 - w^N)] + \frac{\partial q^N}{\partial k}[-(1 - 2q^N)(1 - a(1 - w^N)) - q^D w^D - (1 - q^D)(a - 1)(1 - w^N)]$$

$D$ 's choice of  $w^D$  can now be determined. The derivative of  $D$ 's expected profit (equation (A1)) with respect to  $w^D$  is

$$\frac{\partial \pi(w^D; w^N)}{\partial w^D} = -(1 - (1 - k)^u)(1 - q^N)^{a-1} + (1 - w^D)\frac{\partial k}{\partial w^D} \left( \begin{array}{l} u(1 - k)^{u-1}(1 - q^N)^{a-1} - \\ (a - 1)(1 - q^N)^{a-2}\frac{\partial q^N}{\partial k}(1 - (1 - k)^u) \end{array} \right) \quad (\text{A8})$$

Claim 1: For  $a = 1$ , there is a unique  $w \in (0, 1)$ , namely, the wage given by equation (2), such that  $\frac{\partial \pi(w; w)}{\partial w^D} = 0$ .

Claim 2: For  $a \in \{2, \dots, v\}$ ,  $\frac{\partial \pi(w; w)}{\partial w^D} < 0$  for all  $w \in [0, 1]$ .

To establish these claims,  $\frac{\partial \pi(w^D; w^N)}{\partial w^D}$  must be evaluated at  $w^D = w^N = w$ . When  $D$  posts the same wage as the other vacancies, we have

$$\begin{aligned} k &= \frac{a - k}{v - 1} = a/v \\ q^D &= q^N = \frac{1 - (1 - k)^u}{ku} \equiv q \\ \frac{\partial q^N}{\partial k} &= -\frac{v}{a(v - 1)}((1 - k)^{u-1} - q) \\ \frac{\partial q^D}{\partial k} &= -(v - 1)\frac{\partial q^N}{\partial k} \\ \frac{\partial k}{\partial w^D} &= \frac{-a(v - 1)q(1 - q)}{v^2((1 - k)^{u-1} - q)(w(1 - aq) + (a - 1)q)}. \end{aligned}$$

Note that (i)  $(1 - k)^{u-1} - q < 0$ <sup>8</sup> and (ii)  $w(1 - aq) + (a - 1)q > 0 \forall w \in [0, 1]$ <sup>9</sup>.

These inequalities, which are used below, imply  $\frac{\partial q^N}{\partial k} > 0$ ,  $\frac{\partial q^D}{\partial k} < 0$ , and  $\frac{\partial k}{\partial w^D} > 0$ , as expected.

When  $a = 1$ ,

$$\frac{\partial \pi(w; w)}{\partial w^D} = -(1 - (1 - k)^u) - \frac{1 - w}{w} \left( u(1 - k)^{u-1} \frac{(v - 1)q}{v^2((1 - k)^{u-1} - q)} \right). \quad (\text{A9})$$

Setting this expression equal to zero and substituting for  $k$  and  $q$  gives the wage in text equation (2).

The situation when  $a \in \{2, \dots, v\}$  is more complicated. In this case,

$$\begin{aligned} \frac{\partial \pi(w; w)}{\partial w^D} &= -(1 - (1 - k)^u)(1 - q)^{a-1} \\ &\quad + (1 - w) \frac{\partial k}{\partial w^D} \left( \begin{array}{c} u(1 - k)^{u-1}(1 - q)^{a-1} - \\ (a - 1)(1 - q)^{a-2} \frac{\partial q^N}{\partial k} (1 - (1 - k)^u) \end{array} \right) \end{aligned}$$

<sup>8</sup>**Proof:** Let  $X$  be a binomial random variable with parameters  $u > 1$  and  $k \in (0, 1)$ . Then  $(1 - k)^{u-1} - q = \frac{ku(1 - k)^{u-1} - (1 - (1 - k)^u)}{ku} = \frac{P[X = 0] + P[X = 1] - 1}{ku} < 0$ .

<sup>9</sup>**Proof:** The inequality holds at  $w = 0$  and at  $w = 1$ , and the expression is linear in  $w$ .

Using  $1 - (1 - k)^u = kuq$  and substituting for  $\frac{\partial k}{\partial w^D}$  and  $\frac{\partial q^N}{\partial k}$  gives

$$\frac{\partial \pi(w; w)}{\partial w^D} = kuq \left( -1 + \frac{1 - w}{v(w(1 - aq) + (a - 1)q)} \left( \frac{(v - 1)(1 - q)(1 - k)^{u-1}}{q - (1 - k)^{u-1}} - (a - 1)q \right) \right) \quad (\text{A10})$$

Since  $w(1 - aq) + (a - 1)q > 0 \forall w \in [0, 1]$ , the inequality we want to establish reduces to

$$v(w(1 - aq) + (a - 1)q) > (1 - w) \left( \frac{(v - 1)(1 - q)(1 - k)^{u-1}}{q - (1 - k)^{u-1}} - (a - 1)q \right) \quad (\text{A11})$$

Note that this inequality is (i) true for  $w = 1$  and (ii) linear in  $w$ . Thus, if this inequality holds at  $w = 0$ , then it is true  $\forall w \in [0, 1]$ . That is, we need to show

$$v(a - 1)q > \frac{(v - 1)(1 - q)(1 - k)^{u-1}}{q - (1 - k)^{u-1}} - (a - 1)q. \quad (\text{A12})$$

Since  $q - (1 - k)^{u-1} > 0$ , this can be rewritten as

$$(v + 1)(a - 1)q(q - (1 - k)^{u-1}) - (v - 1)(1 - q)(1 - k)^{u-1} > 0.$$

If this inequality holds for  $a = 2$ , then it holds for  $a \in \{3, \dots, v\}$ , so let  $a = 2$ . The inequality is then

$$(v + 1)q(q - (1 - k)^{u-1}) - (v - 1)(1 - q)(1 - k)^{u-1} > 0$$

or

$$v(q^2 - (1 - k)^{u-1}) + (1 - q)(1 - k)^{u-1} + q(q - (1 - k)^{u-1}) > 0.$$

Since the second and third terms on the left-hand side of this inequality are positive, it suffices to show  $q^2 - (1 - k)^{u-1} \geq 0$ , i.e.,

$$(1 - (1 - k)^u)^2 - k^2 u^2 (1 - k)^{u-1} \geq 0. \quad (\text{A13})$$

The proof of this final inequality is as follows.<sup>10</sup> First,

$$1 - (1 - k)^u = k(1 + (1 - k) + \dots + (1 - k)^{u-1}),$$

so the RHS of (A13) can be expressed as

$$k^2 \left( (1 + (1 - k) + \dots + (1 - k)^{u-1})^2 - u^2 (1 - k)^{u-1} \right).$$

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<sup>10</sup>We are extremely grateful to Harald Lang for this argument.

Next, since

$$\frac{1}{u}(1 + (1 - k) + \dots + (1 - k)^{u-1}) \geq (1 \times (1 - k) \times \dots \times (1 - k)^{u-1})^{\frac{1}{u}},$$

i.e., the arithmetic mean is at least as large as the geometric mean, and

$$1 \times (1 - k) \times \dots \times (1 - k)^{u-1} = (1 - k)^{\frac{u(u-1)}{2}}$$

we have

$$(1 + (1 - k) + \dots + (1 - k)^{u-1})^2 - u^2(1 - k)^{u-1} \geq 0. \quad \blacksquare$$

**Proof of Proposition 3:**

The probability that an unemployed worker finds a job is

$$\frac{M(u, v; a)}{u} = 1 - \left(1 - \frac{v}{au} \left(1 - \left(1 - \frac{a}{v}\right)^u\right)\right)^a.$$

Taking the limit as  $u, v \rightarrow \infty$  with  $v/u = \theta$  and  $a$  fixed gives

$$m(\theta; a) = 1 - \left(1 - \frac{\theta}{a} \left(1 - \lim_{u \rightarrow \infty} \left(1 - \frac{a}{\theta u}\right)^u\right)\right)^a = 1 - \left(1 - \frac{\theta}{a} \left(1 - \exp\left(-\frac{a}{\theta}\right)\right)\right)^a.$$

Similarly,

$$\begin{aligned} w(\theta; 1) &= \lim_{u, v \rightarrow \infty} \frac{\frac{1}{\theta} \frac{v-1}{v} \left(\frac{v-1}{v}\right)^u}{\frac{v-1}{v} - \left(\frac{v-1}{v}\right)^u \left(\frac{1}{v\theta} + \frac{v-1}{v}\right)} \\ &= \frac{(1/\theta) \exp(-1/\theta)}{1 - \exp(-1/\theta)}. \end{aligned}$$

The expression for  $\gamma(\theta; a)$  is derived using  $\gamma = \frac{1 - (1 - q)^a - aq(1 - q)^{a-1}}{1 - (1 - q)^a}$

and  $\lim_{u, v \rightarrow \infty} q = \frac{\theta}{a}(1 - e^{-a/\theta})$ , giving

$$\gamma(\theta; a) = \frac{1 - \left(1 - \frac{\theta}{a}(1 - e^{-a/\theta})\right)^a - \theta(1 - e^{-a/\theta})\left(1 - \frac{\theta}{a}(1 - e^{-a/\theta})\right)^{a-1}}{1 - \left(1 - \frac{\theta}{a}(1 - e^{-a/\theta})\right)^a}. \quad \blacksquare$$

**Proof of Proposition 4:**

The probability that an unemployed worker finds a job is

$$\lim_{a,u,v \rightarrow \infty} \frac{M(u,v;a)}{u} = \lim_{a,u,v \rightarrow \infty} [1 - (1 - \frac{\theta}{\phi v} (1 - (1 - \phi)^u))^{\phi v}] = 1 - e^{-\theta}.$$

Recall that  $\gamma = \frac{1 - (1 - q)^a - aq(1 - q)^{a-1}}{1 - (1 - q)^a}$  and  $q = \frac{1 - (1 - \frac{a}{v})^u}{au/v}$  and  $1 - q = 1 - \frac{1 - (1 - \phi)^u}{\phi u}$ . Using  $\lim_{a,u,v \rightarrow \infty} (1 - q)^a = e^{-\theta}$  and  $\lim_{a,u,v \rightarrow \infty} aq(1 - q)^{a-1} = \theta e^{-\theta}$ , we have

$$\gamma(\theta) = \frac{1 - e^{-\theta} - \theta e^{-\theta}}{1 - e^{-\theta}}. \quad \blacksquare$$

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