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# Simple Bayesian and Ex-Post Equilibria in Combinatorial Auctions\*

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## Abstract

This paper derives Bayesian equilibria under general conditions for a broad family of sealed-bid combinatorial auction formats under incomplete information. Equilibria are constructed by linking an equilibrium of the combinatorial auction format to an equilibrium of its single-unit version, thereby providing a closed-form expression if the single-unit version has one. Auction formats that share the same single-unit version, like the Vickrey auction and all bidder-optimal core-selecting auctions, have common equilibria for any number of items and bidders. Furthermore, if an ex-post equilibrium exists in the single-unit version of an auction then the auction format has an ex-post equilibrium in the combinatorial auction. We characterize the ex-post equilibria for a subfamily of auctions.

*Key words:* Combinatorial auction, Bayesian equilibrium, equilibrium existence, *ex-post* equilibrium

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## 1. Introduction

Equilibria in combinatorial auctions with incomplete information are often hard to find.<sup>1</sup> This paper identifies equilibria under general conditions for a broad family of sealed-bid combinatorial auction formats. Equilibria are derived by linking an equilibrium in the corresponding single-unit auction to an equilibrium of the combinatorial auction. Thus, equilibria in combinatorial auctions exist for any number of bidders and items if they exist in the corresponding single-unit auction.

Our construction works as follows. Assume that a seller can restrict the set of bundles on which bids are allowed to any set that contains the bundle of all items. Take a combinatorial

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<sup>1</sup>There are exceptions. For example, in Vickrey auctions with private values, truthful bidding is a weakly dominant strategy, and ex-post equilibria in such auctions have been characterized by Holzman and Monderer (2004). In core-selecting auctions, Bayesian equilibria for small numbers of bidders and items under specific assumptions on valuations have been identified, e.g., by Sano (2011, 2012), Beck and Ott (2013), Goeree and Lien (2016), Bosshard et al. (2017), and Ausubel and Baranov (2018).

auction format that can be applied to any such restriction on bids. Take any assumption under which the auction has an equilibrium when the items are sold only as a bundle, i.e., when only bids on the bundle of all items are allowed. Such an auction is a single-unit auction. This paper shows how to construct an equilibrium for any restriction on bids from the equilibrium of this single-unit version of the auction. In such an equilibrium, bidders bid only on the bundle of all items, although they are allowed to bid on further bundles. The findings imply that combinatorial auctions with identical single-unit version share equilibria for any restriction on bids.

Furthermore, the ex-post equilibrium property transfers from the single-unit version of an auction to the combinatorial auction. That is, if an ex-post equilibrium exists in the single-unit auction, then an ex-post equilibrium exists also in the combinatorial auction for any restriction on bids. This permits identification of a simple equilibrium that is an ex-post equilibrium in a family of combinatorial auction formats that includes Vickrey and bidder-optimal core-selecting auctions.

Our equilibria run counter to one of the motivations for conducting a combinatorial auction with package bids. Bidders do not submit bids on subsets of items, thereby removing their opportunity for complementing one another. Such equilibria may appeal to bidders if evaluating bundles is costly, if spite motives make bidders refuse to support a potential opponent, or if bidders' payoffs are higher than in other equilibria. The efficiency of the equilibria that we construct is generically low because bidders whose valuations complement each other do not express this in their bids.

The results hold under mild assumptions. We require that a bidder values the full set of items no less than subsets thereof, i.e., we require a weak version of free disposal. In addition, our analysis focuses on auctions that request a bidder's bid on the bundle of all items to be no lower than his bids on other bundles, assign items by maximizing the sum of bids, and apply a payment rule with the following properties. It determines the same sum of payments (i.e., revenue) for all tied assignments and charges no payment if the bid associated with the assigned bundle is zero.

We exemplarily apply our results to the pay-as-bid auction and to a family of auctions that includes bidder-optimal core-selecting auctions and the Vickrey auction. For this family of auctions we derive a necessary and sufficient condition for our simple equilibrium to be an ex-post equilibrium. To do so, we apply new necessary or sufficient conditions for ex-post equilibria in this family of auctions and in all core-selecting auctions.

Reducing the gap of knowledge about combinatorial auctions and single-unit auctions is

important because combinatorial auctions are widely applied. Auctions that are analyzed in this paper have been used, for example, to procure bus or flight services (Cantillon and Pesendorfer, 2006; Letchford, 1996; Lunander and Lundberg, 2012), generation capacity (Moreno et al., 2010; Mastropietro et al., 2014), school meal provision (Epstein et al., 2002; Olivares et al., 2012), and transportation services (Caplice and Sheffi, 2003), as well as to assign telecommunications spectrum licenses (Ausubel and Baranov, 2017).

## 2. Model

We consider a setting with one seller of a set of items  $K = \{1, \dots, k\}$ ,  $k \geq 1$ , and a set of bidders  $N = \{1, \dots, n\}$ ,  $n \geq 2$ . The seller, 0, chooses an auction format and determines a set of bundles  $P \subseteq 2^K \setminus \emptyset$  on which bidders are allowed to bid.<sup>2</sup> We assume that bids on the bundle of all items,  $K$ , are always allowed:  $K \in P$ .

Before the auction, each bidder  $i$  receives a signal  $\theta_i \in \Theta_i \subseteq [0, 1]^{2^K}$ , where  $\Theta_i$  denotes  $i$ 's type space. Let  $\theta = (\theta_1, \theta_2, \dots, \theta_n)$ ,  $\theta \in \Theta = \times_{i \in N} \Theta_i$ .

Each bidder  $i$  has a valuation  $v_i(y, \theta)$  for each bundle  $y \in 2^K$ , with the normalization  $v_i(\emptyset, \theta) = 0$  for all  $\theta \in \Theta$ . Let  $v_i(\theta) = (v_i(y, \theta))_{y \in 2^K}$  and denote the matrix of these valuations by  $v(\theta) = (v_1(\theta), v_2(\theta), \dots, v_n(\theta))$ . We impose the assumptions that the empty bundle has the lowest value and the bundle of all items has the highest value,  $v_i(\emptyset, \theta) \leq v_i(y, \theta) \leq v_i(K, \theta)$  for all  $y \in 2^K$ ,  $\theta \in \Theta$ ,  $i \in N$ .<sup>3</sup> Bidder  $i$ 's utility is  $u_i(v_i(y, \theta) - p_i)$  from receiving bundle  $y \in 2^K$  at the price  $p_i$ , and  $u'_i > 0$ .

A bidder  $i$  places bids on the bundles in  $P$ . For ease of notation, bids on all other bundles are defined as being equal to zero:  $b_i(y) = 0$  for all bundles  $y \notin P$ ,  $i \in N$ .<sup>4</sup> Denote the vector of  $i$ 's bids by  $b_i = (b_i(y))_{y \in 2^K}$ . Let  $b = (b_1, b_2, \dots, b_n)$  denote the matrix of all bidders' bids and let  $b_S$  denote the matrix of bids of the subset of bidders  $S \subseteq N$ , with the simplified notation  $b_{-i} = b_{N \setminus \{i\}}$ . Let  $b(y) = (b_1(y), b_2(y), \dots, b_n(y))$  denote the vector of bids for the bundle  $y \in 2^K$ . Feasible bids on all bundles  $y \in P$  are required to satisfy the condition  $0 \leq b_i(y) \leq b_i(K)$ .<sup>5</sup> Let  $B$  denote the set of feasible bid matrices  $b$  if  $P = 2^K \setminus \emptyset$ . Let  $B(P) \subseteq B$  denote the set of feasible bids when a restriction  $P$  applies, where the only

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<sup>2</sup>Our results extend to the case that different bidders are allowed to bid on different bundles.

<sup>3</sup>This is a weak version of *free disposal*:  $v_i(y, \theta) \leq v_i(y', \theta)$  for all  $y \subset y'$ ,  $y, y' \in 2^K$ ,  $\theta \in \Theta$ ,  $i \in N$ .

<sup>4</sup>Payoffs and revenue in all our auctions will be the same whether we restrict assignments to bundles in  $P$  or we allow only zero bids on bundles not in  $P$ .

<sup>5</sup>A more restrictive and more common requirement on the bidding language is that bids for supersets of items are weakly increasing (e.g. Nisan, 2006).

difference between  $B$  and  $B(P)$  is that bids on bundles  $y \notin P$  are set to zero.

The assignment  $x = (x_0, x_1, x_2, \dots, x_n)$  assigns bundles to bidders and the seller. The set of feasible assignments in an auction with restriction  $P$  is written as

$$X(P) = \{x \mid x_i \in P \cup \emptyset \forall i \in N, x_i \cap x_j = \emptyset \forall i \neq j \in N \cup \{0\}, \cup_{i=0}^n x_i = K\}.$$

The set of optimal assignments  $\hat{X}$  given bids by a set of bidders  $S \in 2^N \setminus \emptyset$  is written as

$$\hat{X}(b_S) = \arg \max_{x \in X(P)} \sum_{i \in S} b_i(x_i).$$

A *sealed-bid combinatorial auction format*  $A$  is a mechanism that can be applied to any restriction on allowed bids  $P$ . For all  $P \subseteq 2^K \setminus \emptyset$ ,  $K \in P$ , the auction format  $A(P) = (x(\cdot), p_1(\cdot), p_2(\cdot), \dots, p_n(\cdot))$  collects bids  $b \in B(P)$  and consists of the assignment rule  $x(b) = (x_1(b), x_2(b), \dots, x_n(b))$  that chooses an assignment in  $X(P)$  and of a payment rule that determines the payment by bidder  $i$ ,  $p_i(b)$ .<sup>6</sup> Apart from potential randomization to break ties between assignments, auctions  $A$  are deterministic mechanisms. Thus, the payment  $p_i(b)$  is deterministic once the assignment is chosen. In slight abuse of notation, if necessary, we address the payment that would result if a tie was broken in favor of assignment  $x$  by  $p_i(b, x)$ .

We analyze the following family  $\mathcal{A}$  of sealed-bid combinatorial auction formats.

**Definition 1.** *The family of auction formats  $\mathcal{A}$  consists of all sealed-bid combinatorial auction formats  $A$  that, for every set of bundles  $P \subseteq 2^K \setminus \emptyset$  and every  $b \in B(P)$ , randomly select an assignment  $x(b) \in \hat{X}(b)$ , and have a payment rule with the following properties:*

(a) *Take any  $b \in B(P)$  and  $x, x' \in \hat{X}(b)$ . Then,  $\sum_{i \in N} p_i(b, x) = \sum_{i \in N} p_i(b, x')$ .*

(b) *If  $x$  is the chosen assignment and  $b_i(x_i) = 0$  then  $p_i(b, x) = 0$ .*

Auction formats in family  $\mathcal{A}$  determine successful bids by maximizing the sum of bids,  $x(b) \in \hat{X}(b)$ , and by breaking ties by randomizing over assignments in  $\hat{X}(b)$ .<sup>7</sup> Furthermore,

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<sup>6</sup>Note that the auction cannot use information on valuations  $v_i(y, \theta)$  to determine assignments or payments.

<sup>7</sup>Our results hold for any way of breaking ties.

(a) revenues are independent of how ties are resolved and (b) bidders that receive a bundle for which they bid zero, including losing bidders, pay nothing.

The family  $\mathcal{A}$  contains most sealed-bid combinatorial auction formats that have been studied in the literature. Define the function  $w$  as the maximum reported value that the coalition of the bidders  $S \subseteq N$  and the seller can generate by trading the items in  $K$ :

$$w(b_\emptyset) = 0 \text{ and } \forall S \in 2^N \setminus \emptyset : \quad w(b_S) = \max_{x \in X(P)} \sum_{i \in S} b_i(x_i).$$

The most famous auction formats in family  $\mathcal{A}$  are the *Vickrey auction*, for which

$$\forall i \in N : \quad p_i^V(b) := w(b_{-i}) - \sum_{j \neq i} b_j(x_j(b)),$$

and the *pay-as-bid* (or first-price) auction, for which

$$\forall i \in N : \quad p_i^{PAB}(b) := b_i(x_i(b)).$$

Increasingly popular are *core-selecting auctions*.<sup>8</sup> Core-selecting auction formats select outcomes such that reported quasi-linear payoffs  $b_i(x_i(b)) - p_i(b)$  (i.e., payoffs calculated assuming valuations equal bids) are in the reported core, which holds iff  $x(b) \in \hat{X}(b)$  and payments fulfill<sup>9</sup>

$$\forall S \in 2^N \setminus \emptyset : \quad \sum_{j \in S} b_j(x_j(b)) \geq \sum_{j \in S} p_j(b) \geq w(b_{N \setminus S}) - \sum_{i \in N \setminus S} b_i(x_i(b)). \quad (1)$$

An important subclass of core-selecting auctions are the *bidder-optimal core-selecting (BOCS) auctions*. A BOCS auction selects payments  $p^{BOCS}(b)$  such that there does not exist any  $p$  that fulfills conditions (1) with  $p_i \leq p_i^{BOCS}(b)$  for all  $i \in N$  and the inequality being strict for at least one  $i \in N$ .

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<sup>8</sup>For their motivation and for design variants, see, e.g., Parkes and Ungar (2000); Ausubel and Milgrom (2002); Day and Raghavan (2007); Day and Milgrom (2008); Erdil and Klemperer (2010); Day and Cramton (2012). Goeree and Lien (2016) call them core\*-selecting auctions to emphasize that they may not have the core-selecting property with respect to true values in equilibrium.

<sup>9</sup>For a more detailed derivation see, e.g., Beck and Ott (2018). Note that the pay-as-bid auction is a core-selecting auction and is the bidder-pessimal core-selecting auction because it selects payments at the bidder-pessimal frontier of the reported core. Also note that not all core-selecting auctions are members of family  $\mathcal{A}$  because some core-selecting payment rules violate condition (a) of Definition 1.

Let  $\beta_i(\theta_i) = (\beta_i(y, \theta_i))_{y \in 2^K}$  denote a strategy of bidder  $i$ , which provides a bid for each type  $\theta_i \in \Theta_i$  and each bundle  $y \in P$  and zero bids for the bundles  $y \notin P$ . Let  $\beta(\theta) = (\beta_1(\theta_1), \beta_2(\theta_2), \dots, \beta_n(\theta_n)) \in B(P)$  for all  $\theta \in \Theta$ . The equilibrium concepts used are the Bayesian(-Nash) equilibrium, which is a profile of bidding strategies  $\beta_i(\theta_i)$  that are mutual best responses, and the ex-post equilibrium, which is a profile of bidding strategies  $\beta_i(\theta_i)$  that are mutual best responses at the interim stage and even after bidders learn the others' bids (for formal definitions see, e.g., Krishna, 2009).

### 3. Bayesian and ex-post equilibria of combinatorial auctions

The first result provides an equilibrium of auctions in family  $\mathcal{A}$  for any set of allowed bids  $P$  under mild conditions. It allows for interdependent values and affiliated signals. Appendix A contains all proofs.

**Proposition 1.** *Take any auction format  $A \in \mathcal{A}$ . A profile  $\beta$  is a Bayesian equilibrium in auction format  $A$  for all  $P \subseteq 2^K \setminus \emptyset$ ,  $K \in P$ , if for  $P = \{K\}$   $\beta$  is a Bayesian equilibrium with  $|\hat{X}(\beta(\theta))| = 1$  for almost all  $\theta \in \Theta$ .*

Proposition 1 demonstrates how to derive an equilibrium for any restriction  $P$  on bids from knowledge about an equilibrium in the single-unit version,  $P = \{K\}$ , of an auction. These equilibria have a simple structure, with positive bids only for the bundle  $K$ . In what follows, denote by  $\tilde{\beta}$  strategy profiles of the form

$$\forall i \in N, \forall \theta_i \in \Theta_i : \quad \tilde{\beta}_i(y, \theta_i) = \begin{cases} 0 & \text{for } y \in 2^K \setminus K \\ \tilde{\beta}_i(K, \theta_i) & \text{for } y = K, \end{cases} \quad (2)$$

i.e., profiles with  $\tilde{\beta}_i(y, \theta_i) = 0$  for all  $y \in 2^K \setminus K$  and  $\theta \in \Theta$ .

The bidding strategies  $\tilde{\beta}$  essentially reduce the combinatorial auction to a single-unit auction of the bundle  $K$ , thereby solving the coordination problem of bidders in a combinatorial auction. If his opponents bid only on  $K$ , a bidder has no partner with whom to complement to be a stronger competitor for the remaining bidders. He can win a subset of  $K$  on his own only at the same price as  $K$ . As a consequence, a bid only on  $K$  is among his best responses to the others' bids. Strategies of the type  $\tilde{\beta}_i$  run counter to one of the motives for conducting combinatorial auctions, which is to allow bidders to join forces

to win items, but they make use of another motive, which is to allow bidders to express preferences for bundles of items.

Proposition 1 involves a no-tie condition,  $|\hat{X}(\beta(\theta))| = 1$  for almost all  $\theta \in \Theta$ . If a tie had positive probability in the equilibrium  $\beta$ , then a bidder  $i$  in an auction with  $P \neq \{K\}$  facing the opponents' strategies  $\tilde{\beta}_{-i}$  might want to bid to tie with his own bid, e.g., by bidding  $\tilde{\beta}_i(y, \theta_i) = \beta_i(K, \theta_i)$  for some bundle  $y \neq K$  with  $v_i(y, \theta) = v_i(K, \theta)$ , thereby increasing his chance to get one of his most valuable bundles. If  $v_i(y, \theta) < v_i(K, \theta)$  for all  $y \in 2^K \setminus K$  applies or if  $i$  does not know  $v_i(y, \theta)$  but assigns positive probability to  $v_i(y, \theta) < v_i(K, \theta)$ , then Proposition 1 holds without assuming no ties.

The decision in an equilibrium  $\tilde{\beta}$  to bid zero for all  $y \neq K$  is ex-post optimal. That is, if  $\tilde{\beta}_i$  is not an ex-post best response to  $\tilde{\beta}_{-i}$ , an optimal ex-post revision of  $\tilde{\beta}_i$  requires only a change of  $\tilde{\beta}_i(K, \theta)$ . The ex-post property of an equilibrium in a single-unit auction,  $P = \{K\}$ , therefore transfers to  $\tilde{\beta}$  in any associated combinatorial auction with  $P \subseteq 2^K \setminus \emptyset$ .

**Proposition 2.** *Take any auction format  $A \in \mathcal{A}$ . A profile  $\beta$  is an ex-post equilibrium in auction format  $A$  for all  $P \subseteq 2^K \setminus \emptyset$ ,  $K \in P$ , if for  $P = \{K\}$   $\beta$  is an ex-post equilibrium with  $|\hat{X}(\beta(\theta))| = 1$  for almost all  $\theta \in \Theta$ .*

Auction formats with ex-post equilibria of the form  $\tilde{\beta}$  include those in the following family, which contains the Vickrey auction and all BOCS auctions. Let  $\hat{p}_i^{BOCS}(b)$  be payments chosen by some BOCS auction. Denote by  $\mathcal{B}$  the family of auction formats in  $\mathcal{A}$  that choose payments  $p_i(b) \in [p_i^V(b), \hat{p}_i^{BOCS}(b)]$  for all  $b \in B(P)$ , all  $P$ , and all  $i \in N$ . All auction formats in family  $\mathcal{B}$  have the same single-unit version, the second-price sealed-bid auction. Assume the bidders' valuations for the bundle of all items depend only on the own signal,  $v_i(K, \theta) = v_i(K, \theta_i)$ . Then, in all auctions in  $\mathcal{B}$ , the following is a symmetric ex-post equilibrium:

$$\forall i \in N, \forall \theta_i \in \Theta_i : \quad \tilde{\beta}_i(y, \theta_i) = \begin{cases} 0 & \text{for } y \in 2^K \setminus K \\ v_i(K, \theta_i) & \text{for } y = K. \end{cases} \quad (3)$$

#### 4. Properties of the equilibrium $\tilde{\beta}$

This section gives examples of equilibria  $\tilde{\beta}$  of the form in (2) in pay-as-bid auctions and discusses properties and appeal of equilibria  $\tilde{\beta}$  in pay-as-bid, core-selecting, and Vickrey auctions.



*Applications to the pay-as-bid auction.* Assume each bidder  $i$ 's valuation for the bundle  $K$  depends only on a component  $\theta_{i,K}$  of  $i$ 's signal, i.e.,  $v_i(K, \theta) = v_i(K, \theta_{i,K})$  for all  $\theta \in \Theta$ , and that the bidders' signals  $\theta_{i,K}$  are independently distributed according to the same absolutely continuous distribution,  $F := F_i = F_j$ , with full support on  $[0, \bar{\theta}]$  for all  $i, j \in N$ . Proposition 1 and results on the single-unit first-price auction (e.g., by Riley and Samuelson, 1981) imply that when bidders are risk-neutral, an equilibrium  $\tilde{\beta}$  in a combinatorial pay-as-bid auction is

$$\forall i \in N, \forall \theta_i \in \Theta_i : \quad \tilde{\beta}_i(y, \theta_i) = \begin{cases} 0 & \text{for } y \in 2^K \setminus K \\ v_i(K, \theta_{i,K}) - \frac{\int_0^{v_i(K, \theta_{i,K})} F_{1,n-1}(z) dz}{F_{1,n-1}(v_i(K, \theta_{i,K}))} & \text{for } y = K, \end{cases}$$

where  $F_{1,n-1}$  denotes the highest order statistics of  $n - 1$  distributions  $F$ .<sup>10</sup>

More generally, Proposition 1 extends the existence conditions for monotone pure-strategy equilibria in the first-price auction with asymmetric bidders and independent private values or affiliated one-dimensional signals and interdependent values (e.g. Lebrun, 1996; Reny and Zamir, 2004) to the combinatorial pay-as-bid auction.

*Revenue equivalence.* Proposition 1 implies that two combinatorial auctions in  $\mathcal{A}$  have equilibria with the same expected revenue if their single-unit versions have revenue equivalent equilibria.

*Efficiency.* An undesirable property of equilibria  $\tilde{\beta}$  is their typically low efficiency. Call an assignment  $\hat{x} \in \arg \max_{x \in X(P)} \sum_{i \in N} v_i(x_i, \theta)$  *constrained ex-post efficient*. Such assignments are associated with efficient payoffs given the constraints imposed by  $P$ .

**Corollary 1.** *An equilibrium  $\tilde{\beta}$  of the form in (2) is not constrained ex-post efficient if*

$$\exists \theta \in \Theta : \quad \left\{ \hat{x} \mid \hat{x} \in \arg \max_{x \in X(P)} \sum_{i \in N} v_i(x_i, \theta) \right\} \cap \left\{ \hat{x} \mid \exists i : \hat{x}_i = K \right\} = \emptyset.$$

The probability of an efficient allocation in the equilibrium  $\tilde{\beta}$  may be zero. Consider an example with  $n \geq k$  bidders in which each bidder has a strictly positive value for one and only one of the  $k$  items and each item is desired by at least one bidder. Assigning all items to one bidder is not efficient, but only such assignments occur in the equilibrium  $\tilde{\beta}$ .

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<sup>10</sup>This equilibrium is symmetric even when bidders are asymmetric with respect to  $v_i(y, \theta_i)$  for  $y \neq K$ .

*Equilibrium selection and coordination on  $\tilde{\beta}$ .* Equilibrium strategies of the form  $\tilde{\beta}_i$  may be dominated or undominated for  $P \neq K$ , depending on the auction format and the valuation setting. In the Vickrey auction with private values, reporting  $\tilde{\beta}_i$  in (3) is weakly dominated by reporting the true values, whereas in the pay-as-bid auction, whether equilibrium strategies  $\tilde{\beta}_i$  are dominated depends on the valuation structure. If there exists a bundle  $y \in P$  with  $v_i(y, \theta_i) = v_i(K, \theta_i)$ , then an equilibrium strategy  $\tilde{\beta}_i$  in a pay-as-bid auction is weakly dominated by a strategy that replaces the bid  $\tilde{\beta}_i(y, \theta_i) = 0$  on  $y$  by a bid of size  $\tilde{\beta}_i(K, \theta_i)$ . If, however,  $v_i(K, \theta_i) - \tilde{\beta}_i(K, \theta_i) > v_i(y, \theta_i)$  for all  $y \neq K$ , then there is no strategy that differs only in the bids on bundles  $y \neq K$  that makes bidder  $i$  better off than  $\tilde{\beta}_i$ , and, because any dominating bid must be a best response to  $\tilde{\beta}_{-i}$ , a bid that involves  $\beta_i(K, \theta_i) \neq \tilde{\beta}_i(K, \theta_i)$  cannot dominate  $\tilde{\beta}_i(K, \theta_i)$  if any best response to  $\tilde{\beta}_{-i}$  involves  $\tilde{\beta}_i(K, \theta_i)$ . With interdependent values, bidders cannot report their (unknown) values in the Vickrey auction, they do not have a dominant strategy, and, if  $v_i(y, \theta) < v_i(K, \theta)$  for all  $y \in P$  and  $\theta$ , placing positive bids on bundles  $y$  may result in not winning  $K$  although it is the desired bundle given the opponents' strategies and although  $\tilde{\beta}_i$  would win  $K$ .

In the equilibria  $\tilde{\beta}$ , bidders believe that opponents bid positively only on  $K$ . Assume a bidder  $i$  cannot exclude that the opponents tremble, but he believes that they do not bid sufficiently much on bundles other than  $K$  to successfully complement with one of them. Then, choosing  $\tilde{\beta}_i$  is also a best response for bidder  $i$  if  $v_i(y, \theta) < v_i(K, \theta)$  for all  $y \in P$  and  $\theta$ , so  $\tilde{\beta}_i$  is robust to such trembles by the opponents.

In some settings, coordination on an equilibrium of the form  $\tilde{\beta}$  appears natural. If bidders are symmetric in their valuations for the bundle  $K$  but otherwise asymmetric, with diverse assignments possibly being efficient, the symmetry in  $K$  may act as a coordination device on a symmetric equilibrium  $\tilde{\beta}$ .

Coordination on  $\tilde{\beta}$  is more likely if the equilibrium appeals to bidders, for example because it is costly to evaluate bundles, bidders have spite motives and want to avoid supporting a potential opponent's success in the auction, or bidders have payoff advantages from playing  $\tilde{\beta}$ .<sup>11</sup>

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<sup>11</sup>For example, three bidders with values  $(v_i(\{A\}, \theta_i), v_i(\{B\}, \theta_i), v_i(\{A, B\}, \theta_i))_{i \in \{1, 2, 3\}} = ((1, 1, 1), (1, 1, 1), (1, 1, 2))$  pay a total of 2 in the equilibrium with truthful reporting in any auction  $A \in \mathcal{B}$  whereas the total payments are 1 and all bidders are weakly better off in the equilibrium  $\tilde{\beta}(\theta) = ((0, 0, 1), (0, 0, 1), (0, 0, 2))$ .

*Ex-post equilibria.* We will identify properties of strategy profiles that are necessary or sufficient for an ex-post equilibrium in auction formats in family  $\mathcal{A}$  and will use these properties to provide a necessary and sufficient condition such that strategies  $\tilde{\beta}$  form an ex-post equilibrium in auction formats in family  $\mathcal{B}$ .

The properties that we will use are the *bidders-are-substitutes condition (BAS)* by Bikhchandani and Ostroy (2002) on bids,

$$\forall S \subset N : \quad w(b) - w(b_{N \setminus S}) \geq \sum_{i \in S} [w(b) - w(b_{-i})], \quad (\text{BAS})$$

and the *constrained ex-post efficiency condition (CE)* on the assignment, given bids  $b \in B(P)$ :

$$\forall i \in N : \quad x(b) \in \arg \max_{x \in X(P)} \left( v_i(x_i, \theta) + \sum_{j \neq i} b_j(x_j, \theta) \right). \quad (\text{CE})$$

In an assignment that fulfills CE, each bidder contributes his maximum to the total value when the other bidders' contribution is measured by their reported values. Note, an assignment may fulfill CE although being inefficient. Efficiency is constrained both by being conditional on values only for bundles in  $P$  and by being conditional on opponents' bids rather than values. We say a strategy profile  $\beta$  satisfies BAS or CE if it satisfies the condition for all  $\theta \in \Theta$ .

**Lemma 1.** *Any strategy profile  $\tilde{\beta}$  of the form in (2) satisfies BAS.*

The following lemma clarifies for core-selecting auction formats and auction formats in family  $\mathcal{B}$  the relationship between the ex-post equilibrium property, BAS, and CE.

**Lemma 2.** *Take any auction format  $A \in \mathcal{A}$  with a restriction  $P$  and any feasible strategy profile  $\beta$  in this auction.*

- (i) *If  $A$  is a core-selecting auction and  $\beta$  an ex-post equilibrium then  $\beta$  satisfies BAS and CE.*
- (ii) *If  $A \in \mathcal{B}$  and  $\beta$  satisfies BAS and CE then  $\beta$  is an ex-post equilibrium. If  $\beta$  violates CE, then it is not an ex-post equilibrium.*
- (iii) *If  $A$  is a Vickrey auction, then  $\beta$  is an ex-post equilibrium iff it satisfies CE.*

A BOCS auction is a core-selecting auction and a member of  $\mathcal{B}$ , therefore,  $\beta$  is an ex-post equilibrium in a BOCS auction if and only if it satisfies BAS and CE. Furthermore, if  $\beta$  is

an ex-post equilibrium in a BOCS auction then it is an ex-post equilibrium in every BOCS auction.

For strategy profiles  $\tilde{\beta}$ , Lemmas 1 and 2(ii) imply the following.

**Proposition 3.** *Take any auction format  $A \in \mathcal{B}$  with a restriction  $P$ . Strategy profile  $\tilde{\beta}$  of the form in (2) is an ex-post equilibrium iff it satisfies CE.*

This finding implies that all equilibria of the single-unit Vickrey auction have an associated ex-post equilibrium  $\tilde{\beta}$  in all auctions in family  $\mathcal{B}$  for all  $P$ .<sup>12</sup>

The BAS is relevant in the auction formats of family  $\mathcal{B}$  because if and only if bids satisfy the BAS, then Vickrey payments fulfill the core constraints (1). If Vickrey payments fulfill (1), then they are the unique bidder-optimal payments in the reported core (Ausubel and Milgrom, 2002; Bikhchandani and Ostroy, 2002). This implies that any BOCS auction with a bidding space  $B(P)$  such that all bids  $b \in B(P)$  satisfy the BAS is equivalent to the Vickrey auction with the same bidding space  $B(P)$ . Therefore, with such a bidding space, truthful bidding – only on the bundles in  $P$  and only if feasible under  $B(P)$  – is another ex-post equilibrium (in this case, a dominant strategy equilibrium) in all auction formats in family  $\mathcal{B}$ .<sup>13</sup>

## 5. Conclusion

We conclude that findings on equilibrium existence in single-unit auctions transfer to combinatorial auctions under mild assumptions. Furthermore, a combinatorial auction has an ex-post equilibrium if the associated single-unit auction has one. To show this, we make use of equilibria with a specific structure. In these equilibria, all bidders – although some of them may be complements to each other by their valuations – claim through their bids that they are substitutes. These equilibria have generically a low efficiency but may have payoff advantages for the bidders. Rejecting these prevalent equilibria may require stronger criteria than dominance or ex-post best responses.

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<sup>12</sup>For private values, Blume and Heidhues (2004) characterize all equilibria of the single-unit Vickrey auction.

<sup>13</sup>Relatedly, Sano (2011) shows for a private valuations setting with single-minded bidders (i.e., each bidder desires one bundle, which can be the same or a different bundle than that desired by competitors) that if and only if a “my rival’s rival is my rival” condition is fulfilled for each bidder for all feasible bids, then all bidders have a weakly dominant strategy to bid their valuation in any BOCS auction.

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## A. Proofs

**Proof of Proposition 1:** Denote by  $\hat{\beta}$  a Bayesian equilibrium in auction format  $A \in \mathcal{A}$  with positive bids allowed only on the single unit,  $P = \{K\}$ , and with no tie for almost all  $\theta \in \Theta$ , which exists by assumption. We have to show that  $\hat{\beta}$  is a Bayesian equilibrium in auction format  $A$  with  $P \subseteq 2^K \setminus \emptyset$ ,  $K \in P$ . Note that  $\hat{\beta}$ , by being feasible when  $P = \{K\}$ , has the form

$$\forall i \in N, \forall \theta_i \in \Theta_i : \hat{\beta}_i(y, \theta_i) = \begin{cases} 0 & \text{for } y \in 2^K \setminus K \\ \hat{\beta}_i(K, \theta_i) & \text{for } y = K. \end{cases}$$

Suppose  $P \neq \{K\}$  and bidders choose  $\hat{\beta}$ . Because  $\hat{\beta}(\theta)$  has positive bids only for  $K$  and there is no tie for almost all  $\theta \in \Theta$ , the auction assigns for all  $\theta$  all items to one of the bidders and for almost all  $\theta$  there is no tie. Furthermore, bidder  $i$ 's expected utility from  $\hat{\beta}(\theta)$  is the same in auctions  $A(\hat{P})$  and  $A(P)$ . Because  $\hat{\beta}_i(\theta_i)$  is bidder  $i$ 's best response to  $\hat{\beta}_{-i}(\theta_{-i})$  in  $A(\hat{P})$ ,  $\hat{\beta}_i(\theta_i)$  must give bidder  $i$  in  $A(P)$  his highest expected utility among all strategies with  $\beta_i(y, \theta_i) = 0$  for all  $y \neq K$  and all other strategies with that he wins  $K$  or nothing. If bidder  $i$  in  $A(P)$  deviates unilaterally to a strategy  $\beta'_i(\theta_i) \neq \hat{\beta}_i(\theta_i)$  with  $\beta'_i(\theta_i) \in B(P)$  for all  $\theta_i$ , he may be assigned a bundle  $y \notin \{\emptyset, K\}$  only if  $\beta'_i(y, \theta_i) = \beta'_i(K, \theta_i)$  for some  $y \neq K$ . Then,  $\beta'_i$  is successful (i.e., bidder  $i$  receives a bundle) iff  $\hat{\beta}_i$  is successful, and if  $\beta'_i$  is successful then there is a tie between assignments in which  $i$  receives  $K$  or subsets thereof. His payment is the same for any bundle he might be assigned in such a tie because he is the only bidder that may have a strictly positive payment as  $\hat{\beta}_j(x_j(\hat{\beta}(\theta)), \theta_j) = 0$  implies  $p_j(\hat{\beta}(\theta), x(\hat{\beta}(\theta))) = 0$  for all  $j \neq i$  (by property (b) from Definition 1) and because total payments from tied assignments are the same (by property (a) from Definition 1). Given that his payments are the same, bidder  $i$  is not better off when receiving  $y$  than when receiving  $K$  because  $v_i(y, \theta) \leq v_i(K, \theta)$  for all  $\theta \in \Theta$ , and the unilateral deviation cannot be profitable. ■

**Proof of Proposition 2:** Suppose auction format  $A \in \mathcal{A}$  with bids only on the single unit  $P = \{K\}$  has an ex-post equilibrium  $\hat{\beta}$  in which ties have zero probability. By Proposition 1,  $\hat{\beta}$  is a Bayesian equilibrium of auction format  $A$  if  $P \neq \{K\}$ . Suppose  $P \neq \{K\}$  and

bidders choose  $\hat{\beta}$ . To prove that  $\hat{\beta}$  is an ex-post equilibrium, repeat the arguments from the proof of Proposition 1 replacing “best-response” by “ex-post best response” and “expected utility” by “ex-post utility”. ■

**Proof of Lemma 1:** Applied to a strategy profile  $\tilde{\beta}$  of the form in (2), BAS is

$$\begin{aligned} \forall \theta \in \Theta, S \subset N : \quad & w(\tilde{\beta}(\theta)) - w(\tilde{\beta}_{N \setminus S}(\theta)) \geq \sum_{i \in S} [w(\tilde{\beta}(\theta)) - w(\tilde{\beta}_{-i}(\theta))] \\ \iff \forall \theta \in \Theta, S \subset N : \quad & \max_{i \in N} \tilde{\beta}_i(\theta) - \max_{i \in N \setminus S} \tilde{\beta}_i(\theta) \geq \sum_{i \in S} [\max_{j \in N} \tilde{\beta}_j(\theta) - \max_{j \in N \setminus \{i\}} \tilde{\beta}_j(\theta)]. \quad (4) \end{aligned}$$

This holds because if  $\exists k : k \in S \wedge k \in \arg \max_{i \in N} \tilde{\beta}_i(\theta)$  then (4) is

$$\tilde{\beta}_k(\theta) - \max_{i \in N \setminus S} \tilde{\beta}_i(\theta) \geq \tilde{\beta}_k(\theta) - \max_{j \in N \setminus \{k\}} \tilde{\beta}_j(\theta) \iff \max_{i \in N \setminus S} \tilde{\beta}_i(\theta) \leq \max_{j \in N \setminus \{k\}} \tilde{\beta}_j(\theta)$$

and if  $\nexists k : k \in S \wedge k \in \arg \max_{i \in N} \tilde{\beta}_i(\theta)$  then (4) is

$$\max_{i \in N} \tilde{\beta}_i(\theta) - \max_{i \in N \setminus S} \tilde{\beta}_i(\theta) = 0 = \sum_{i \in S} [\max_{j \in N} \tilde{\beta}_j(\theta) - \max_{j \in N \setminus \{i\}} \tilde{\beta}_j(\theta)].$$

■

**Proof of Lemma 2:** A profile  $\beta$  is an ex-post equilibrium iff  $\beta(\theta)$  is a full-information Nash equilibrium for every  $\theta \in \Theta$ . Fix  $\theta = \hat{\theta}$ . In all auction formats in (i) to (iii), given  $\beta_{-i}(\hat{\theta}_{-i})$ , bidder  $i$ 's bid is a best response iff  $x(\beta(\hat{\theta})) \in \arg \max_{x \in X(P)} (v_i(x_i, \hat{\theta}) + \sum_{j \neq i} \beta_j(x_j, \hat{\theta}_j))$  and he pays  $p_i^V(\beta(\hat{\theta}))$ . We show this by generalizing Theorem 1(a) by Beck and Ott (2018) to any restriction  $P$ . Then, we connect these two conditions to CE and BAS for the auction formats in (i) to (iii).

Consider bidder  $i$  and opponents' bids  $\beta_{-i}(\hat{\theta}_{-i})$ . In all auction formats in (i) to (iii), bidder  $i$ 's payoff is bounded above by  $\max_{x \in X(P)} (v_i(x_i, \hat{\theta}) + \sum_{j \neq i} \beta_j(x_j, \hat{\theta}_j)) - w(\beta_{-i}(\hat{\theta}_{-i}))$ .



For every  $\beta_i(\hat{\theta}_i)$  with  $\beta(\hat{\theta}) \in B(P)$ ,

$$v_i(x_i(\beta(\hat{\theta})), \hat{\theta}) - p_i(\beta(\hat{\theta})) \leq v_i(x_i(\beta(\hat{\theta})), \hat{\theta}) - p_i^V(\beta(\hat{\theta})) \quad (5)$$

$$= v_i(x_i(\beta(\hat{\theta})), \hat{\theta}) + \sum_{j \neq i} \beta_j(x_j(\beta(\hat{\theta})), \hat{\theta}_j) - w(\beta_{-i}(\hat{\theta}_{-i})) \quad (6)$$

$$\leq \max_{(b_i, \beta_{-i}(\hat{\theta}_{-i})) \in B(P)} \left[ v_i(x_i(b_i, \beta_{-i}(\hat{\theta}_{-i})), \hat{\theta}) + \sum_{j \neq i} \beta_j(x_j(b_i, \beta_{-i}(\hat{\theta}_{-i})), \hat{\theta}_j) \right] - w(\beta_{-i}(\hat{\theta}_{-i})) \quad (7)$$

$$= \max_{x \in X(P)} \left[ v_i(x_i, \hat{\theta}) + \sum_{j \neq i} \beta_j(x_j, \hat{\theta}_j) \right] - w(\beta_{-i}(\hat{\theta}_{-i})) \quad (8)$$

Inequality (5) holds because, given a  $\beta(\hat{\theta}) \in B(P)$ , payments in all auction formats in (i) to (iii) weakly exceed Vickrey payments. Equality (6) holds by the definition of Vickrey payments. Inequality (7) is by maximization, and (8) holds because all auction formats in (i) to (iii) choose a bid-maximizing assignment among feasible assignments in  $X(P)$ .

The inequality in (5) holds with equality iff  $p_i(\beta(\hat{\theta})) = p_i^V(\beta(\hat{\theta}))$ . The inequality in (7) holds with equality iff  $x(\beta(\hat{\theta})) \in \arg \max_{x \in X(P)} (v_i(x_i, \hat{\theta}) + \sum_{j \neq i} \beta_j(x_j, \hat{\theta}_j))$ . Therefore, bidder  $i$ 's payoff equals the upper bound iff these two conditions hold.

We will show that a bidder can get arbitrarily close to the upper bound. Let  $\hat{v}_i(y, \hat{\theta}) = v_i(y, \hat{\theta}) \forall y \in P$  and  $\hat{v}_i(y, \hat{\theta}) = 0 \forall y \notin P$ . Note,  $x(\beta(\hat{\theta})) \in \hat{X}(\hat{v}_i(\hat{\theta}), \beta_{-i}(\hat{\theta}_{-i}))$  is equivalent to  $x(\beta(\hat{\theta})) \in \arg \max_{x \in X(P)} (v_i(x_i, \hat{\theta}) + \sum_{j \neq i} \beta_j(x_j, \hat{\theta}_j))$ . Consider  $\beta_i(\hat{\theta}_i)$  with  $\beta_i(K, \hat{\theta}_i) = w(\beta_{-i}(\hat{\theta}_{-i})) - \sum_{j \neq i} \beta_j(x_j(v_i(\hat{\theta}), \beta_{-i}(\hat{\theta}_{-i}))) + \varepsilon$  for  $\varepsilon \geq 0$ ,  $\beta_i(x_i(\hat{v}_i(\hat{\theta}), \beta_{-i}(\hat{\theta}_{-i})), \hat{\theta}_i) = \beta_i(K, \hat{\theta}_i)$ , and  $\beta_i(y, \hat{\theta}_i) = 0$  for all  $y \in P \setminus \{K, x(\hat{v}_i(\hat{\theta}), \beta_{-i}(\hat{\theta}_{-i}))\}$ . That is, bidder  $i$  bids zero for every bundle except the bundle  $K$  and the bundle he would win, given  $P$ , if he bid truthfully. With this bid,  $x(\hat{v}_i(\hat{\theta}), \beta_{-i}(\hat{\theta}_{-i}))$  is in  $\hat{X}(\beta(\hat{\theta}))$  if  $\varepsilon = 0$  and is the unique optimal assignment (i.e.,  $x(\hat{v}_i(\hat{\theta}), \beta_{-i}(\hat{\theta}_{-i})) = \hat{X}(\beta(\hat{\theta}))$ ) if  $\varepsilon > 0$ . When bidder  $i$  wins  $x_i(\hat{v}_i(\hat{\theta}), \beta_{-i}(\hat{\theta}_{-i}))$ , his payment must be within  $\varepsilon$  of his Vickrey payment because his bid for this bundle, which is the upper bound on his payment in any auction in (i) to (iii), equals his his Vickrey payment plus  $\varepsilon$ .

Since bidder  $i$  can get a payoff arbitrarily close to the upper bound (8) for all  $\beta_{-i}(\hat{\theta}_{-i})$  in all auctions in (i) to (iii),  $\beta_i(\hat{\theta}_i)$  is a best response iff the two conditions  $p_i(\beta(\hat{\theta})) = p_i^V(\beta(\hat{\theta}))$  and  $x(\beta(\hat{\theta})) \in \arg \max_{x \in X(P)} (v_i(x_i, \hat{\theta}) + \sum_{j \neq i} \beta_j(x_j, \hat{\theta}_j))$  are satisfied. Thus,  $\beta$  is an ex-post equilibrium iff the auction chooses such assignment and payments for all  $i \in N$  and

all  $\theta \in \Theta$ . The requirement on the assignment is exactly CE.

In a Vickrey auction, payments equal  $p_i^V(\beta(\theta))$  for all  $\beta$  and all  $\theta$  by the rules of the auction. This proves part (iii). It remains to show the connection between BAS and payments  $p_i^V(\beta(\theta))$  in the auctions in (i) and (ii). A core-selecting auction must satisfy all constraints in (1). Payments  $p_i^V(\beta(\theta))$  for all  $i \in N$  satisfy the constraints in (1) iff BAS holds:  $\forall \theta \in \Theta, S \subset N$

$$\begin{aligned} \sum_{i \in S} \left[ w(\beta_{-i}(\theta)) - \sum_{j \in N \setminus \{i\}} \beta_j(x_j(\beta(\theta)), \theta_j) \right] &\geq w(\beta_{N \setminus S}(\theta)) - \sum_{i \in N \setminus S} \beta_i(x_i(\beta(\theta)), \theta_i) \\ \iff w(\beta(\theta)) - w(\beta_{N \setminus S}(\theta)) &\geq \sum_{i \in S} [w(\beta(\theta)) - w(\beta_{-i}(\theta))]. \end{aligned}$$

This proves part (i). If payments  $p_i^V(\beta(\theta))$  for all  $i \in N$  satisfy the constraints in (1), they are the unique bidder-optimal payments in the reported core. (To see this, combine Ausubel and Milgrom (2002), Theorem 6, with the equivalence of reported payoffs in the core and constraints in (1) plus  $x(\beta(\theta)) \in \hat{X}(\beta(\theta))$ .) Thus, any BOCS auction chooses payments  $p_i^V(\beta(\theta))$  for all  $i \in N$  iff these payments satisfy all constraints in (1). Any auction in  $\mathcal{B}$  must then choose payments  $p_i^V(\beta(\theta))$  for all  $i \in N$  because  $p_i^V(\beta(\theta)) = p_i^{BOCS}(\beta(\theta))$  for all  $i \in N$  and all BOCS auctions. This proves part (ii). ■