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Working Paper

The impact of long-range dependence in the capital stock on interest rate and wealth distribution

Frank Calisse*

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Abstract

Macroeconomic modeling in the context of a stochastic continuous-time environment has become more popular in recent years. Most of these models are based on stochastic differential equations to describe macroeconomic dynamics and stochastic uncertainty is mostly modeled by Brownian motions or Poisson processes. However, these assumptions neglect the statistical evidence of long-range dependence in macroeconomic time series such as inflation rates, GDP, unemployment rates and interest rates. Based on Brunnermeier and Sannikov's contribution to the Handbook of Macroeconomics 2016, we present a small and quite simple model where the uncertainty is modeled by an approximated Liouville fractional Brownian motion. With this approach we are able to consider the effects of correlated shocks as well as the impact of long-range dependence in capital stock on the rate of interest and the distribution of wealth.

JEL Classification Codes: C63, E44, G00, G11

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1 Introduction

The question, whether macroeconomic time series are persistent over time (in the sense that they show a long-range dependence or long memory behavior as a slowly decaying autocorrelation function), is discussed controversially in the literature. Early results from Diebold and Rudebusch [1] indicate that aggregate output is persistent over time. Based on an autoregressive fractionally integrated moving average (ARFIMA) process, they found statistical evidence for long-range dependence in U.S. data of aggregate output. Although the results of Diebold and Rudebusch were refuted by [2], the use of ARFIMA models is still common in the literature for modeling long memory properties. See [3] for general applications and an overview of long-range dependence in time series of US real GNP and US Consumer Price Index. See [4] for an investigation of long memory in US real output per capita and [5] for an investigation of the persistence of the monthly EURIBOR rate. Škare and Stjepanović found evidence for long-range dependence in Croatian aggregate output, see [6]. Caporale and Škare found evidence for long memory in UK real GDP in [7]. Gil-Alana et al. found evidence for persistence in Turkish unemployment rates in [8]. It, therefore, seems to be necessary to account for long-range dependence in macroeconomic models.

Tarasov and Tarasova point out in [9] that long-range dependence effects cannot be included in classical macroeconomic models which are based on differential equations. They argue that derivatives of a function describe the behavior of this function only in an infinitesimal neighborhood of specific points and thus they are unable to describe long-memory effects. In their meaning these models can only describe economic environments in which agents have “total amnesia”. Therefore, Tarasov and Tarasova consider a deterministic setting based on fractional derivatives to introduce memory effects in different macroeconomic models (see also their related works given in the references of [9]).

There have been arising new classes of macroeconomic models based on stochastic differential equations in the recent literature, see for instance [10] and the various works of Brunnermeier et al. [11–15]. These types of models combine two concepts. The first is the idea of stochastic modeling which is already realized and well-understood in discrete time settings via classical DSGE models to consider effects of random shocks to the economy. The second concept is to model in continuous-time frameworks in order to describe the whole dynamics of a variable instead of linear approximations around a steady state. This paradigm of continuous-time stochastic modeling is not quite new, since it is already used in classical finance models, see for instance [16]. So the idea behind these new macro models of Brunnermeier and others is to apply the well-developed continuous-time stochastic framework from financial mathematics and finance to macroeconomic problems. They investigate the effects of financial frictions on the wealth distribution within the economy and go thereby further than classical finance issues like option pricing.

To realize this, Brunnermeier and Sannikov describe the evolution of capital in the economy with the following stochastic differential equation

$$\frac{dk_t}{k_t} = (\Phi(\iota_t) - \delta) dt + \sigma dW_t \quad (1.1)$$

where capital grows with investment minus depreciation $(\Phi(\iota_t) - \delta)$, for details see Section 4. The term dW_t is interpreted as a macroeconomic shock modeled as stochastic disturbance generated by a Brownian motion $W = (W_t)_{t \geq 0}$. This stochastic differential equation is very

similar to the one of the Black-Scholes model that describes the evolution of stock prices. This can be seen if we set $\Phi(\iota_t) - \delta$ as the drift and σ as the volatility of the stock.

Most procedures are similar in this kind of models: After assuming some dynamics of endogenous variables and deriving equilibrium conditions, an equilibrium is defined where all endogenous variables are assumed to be functions of state variables. This leads to Markov-structures in an equilibrium which can be solved explicitly or numerically with stochastic dynamic programming, Hamilton-Jacobi-Bellman equations or Fokker-Planck-equations, see for instance Isohätälä et al. [17]. Although this approach leads to mathematically tractable models, there are some shortcomings due to the simplification of only accounting for Markov equilibria. Therefore, it might be more reasonable to assume that endogenous variables depend not only on current values of state variables, but also on the history of shocks to depict the statistical evidence of long-range dependence mentioned above. This cannot be captured by the structure of these models, since increments of a Brownian motion are stochastically independent and thus macroeconomic shocks in these models are uncorrelated. Hence, considering a specific point in time, we cannot gain further information if we additionally know the entire history of shocks that led to the present situation. Or, to put in a nutshell, the future evolution of a process is only determined by its present state and not by past states. This is a heuristic description of the so called Markov property of a stochastic process. For a rigorous mathematical definition, see [18, Section 17.1]. Therefore, these models do not account for long-range dependence.

So we consider a slightly different approach for modeling macroeconomic shocks to include long-range dependence effects. Based on Thao et al. [19], we consider an approximated Liouville fractional Brownian motion $Z^{H,\varepsilon} = (Z_t^{H,\varepsilon})_{t \geq 0}$ as the source of uncertainty in our model. This process can also be regarded as a continuous-time analogue of an ARFIMA process which we mentioned above. See Section 3 for the definition of $Z^{H,\varepsilon}$ and [3, Section 3.1] as well as [19] for the relation between an ARFIMA process and $Z^{H,\varepsilon}$. The parameter $H \in (0, 1)$ denotes the Hurst-index and controls for the correlation of shocks and the degree of long-range dependence. As benchmark model we use the simple real economy model of Brunnermeier and Sannikov [15, Section 2].

We introduce the long-range dependence effect through replacing the Brownian motion in (1.1) by the process $Z^{H,\varepsilon}$, i.e.

$$\frac{dk_t}{k_t} = (\Phi(\iota_t) - \delta) dt + \sigma dZ_t^{H,\varepsilon}.$$

Together with an AK-production technology this generates long-range dependence in the log differenced output series, which was mentioned in the literature review above.

Accounting for the dependence of macro shocks on the history leads to non-Markov structures in an equilibrium. Starting with the simple model of Brunnermeier and Sannikov, we provide proofs which we adopt from the classical finance theory based on martingale techniques. Interesting is that the original model is a special case of our extension, since it can be recovered if we set $H = 1/2$ in our context. The main question is whether there are any effects on output, prices, interest rates and wealth distribution when the shocks are correlated and thus macroeconomic shocks show a long-range dependence behavior over time. We obtain good comparability with the baseline model and can analyze those effects. As we will see, the interest rate is mainly influenced by the history of shocks and it depends on the whole path of the stochastic process. This shows that such models depend strongly on the stochastic

nature of shocks and stated relations do not hold any longer if long memory effect are taken into consideration. Furthermore, we provide a relation between long memory in the growth rates of output and long memory in interest rates.

The paper is organized as follows: In Section 2, we give a short overview of the concept of long-range dependence in the classical finance context. In Section 3, we define the shock generating process based on [19] which drives the dynamics of the capital in our model. In Section 4, we specify the model economy and agent's preferences. A definition and solution of an equilibrium is then given in Section 5. In Section 6, we state our results and compare them with the baseline model. Section 7 concludes.

2 Long-range dependence in the finance literature

In [15], Brunnermeier and Sannikov set up a macroeconomic model in which the evolution of capital follows a geometric Brownian motion. This approach is similar to that from Black and Scholes in [20] which model the price of a stock as a geometric Brownian motion. From an empirical point of view there is evidence that the model of Black and Scholes is unsuitable to model the dynamics of stock prices, since most stylized facts about stock prices cannot be captured by this model, see for instance [21]. To solve this discrepancy there was a detailed and controversy discussion in the finance literature in the 2000s whether or not it is useful to replace the Brownian motion in the Black-Scholes model by a fractional Brownian motion (hereafter: fBm), a stochastic process which was already introduced and suggested for financial modeling by Mandelbrot and van Ness in [22] to include long-range dependence effects. Regarding the empirical literature review in Section 1 it is questionable how far the geometric Brownian motion is a suitable approach for modeling capital in a macroeconomic context. Before we highlight the role of fBm in financial modeling, we give the formal definition of this process and summarize a few properties.

Definition 2.1

The continuous and real-valued Gaussian process $B^H = (B_t^H)_{t \geq 0}$ defined on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is called fractional Brownian motion with Hurst-index $H \in (0, 1)$ if

$$\mathbb{E}B_t^H = 0 \quad \text{for all } t \geq 0 \quad (2.1)$$

and

$$\mathbb{E}(B_s^H B_t^H) = \frac{1}{2} \left(s^{2H} + t^{2H} - |s - t|^{2H} \right) \quad \text{for all } s, t \geq 0. \quad (2.2)$$

×

In the case of $H = 1/2$ in Definition 2.1, the process $(B_t^{1/2})_{t \geq 0}$ is a Brownian motion (hereafter: Bm). The following properties of fBm are crucial for our considerations. An overview and formal proofs can be found in [23, Chapter 1]. If $H > 1/2$, the increments are positively correlated and exhibit long-range dependence. The reverse is the case if $H < 1/2$, i.e. the increments are negatively correlated and are short-range dependent. In the finance literature of the 2000s the idea arose to consider the fractional analogue of the classical Black-Scholes

market. That means Brownian motion in the dynamics of the stock price process is replaced by a fBm, i.e.

$$dS_t = \mu S_t dt + \sigma S_t dB_t^H \quad (2.3)$$

which is the shorthand notation for the integral equation

$$S_t = S_0 + \int_0^t \mu S_s ds + \int_0^t \sigma S_s dB_s^H \quad (2.4)$$

for $H \in (0, 1)$, $\mu \in \mathbb{R}$ and $\sigma > 0$. Since fBm is not a semimartingale for $H \neq 1/2$ (see for instance [23, Section 1.8]), the Itô integration calculus cannot be applied to the stochastic integral in (2.3). For details of the Itô calculus see for instance [24]. Therefore, other attempts have been made for defining a stochastic integral and a related stochastic differential equation. A model based on the Wick-Itô-Skorohod integration calculus is given by Biagini et al. in [23, Chapter 4 and 7]. It contains the case if $0 < H < 1/2$ as well and therefore it is an extension of the original model which was introduced by Hu et al. in [25]. Biagini et al. indicate in [23, Section 7.2] that the interpretation of S_t as the stock price at time t is difficult since the underlying Wick-Itô-Skorohod integration calculus cannot be interpreted well in an economic manner. However, they prove in [23, Theorem 7.2.6] that the corresponding fractional Black-Scholes market is free of arbitrage. Since the absence of arbitrage is related to the definition of the wealth process of self-financing portfolios, the No-Arbitrage condition depends mainly on the underlying integral calculus. Björk et al. criticize in [26] that the model introduced in [25] is indeed free of arbitrage in a mathematical manner, but the definition of self-financing portfolios in the model has no relation to economic reality. They construct a portfolio in the Wick-Itô-Skhorod based model of Hu et al. which has a negative value on a set with positive probability. Three years before Necula developed in [27] a formula for the price of a European call option in a Wick-Itô-Skhorod based Black-Scholes model. Meng et al. consider in [28] this formula and the original one in the context of foreign exchange rates. A critical review of the formula stated by Necula is given by Rostek et al. in [29]. Other approaches regard (2.3) in a pathwise sense and Salopek constructs in [30] an arbitrage opportunity in such a setting. For a more detailed discussion see [31]. The absence of arbitrage is an important requirement for calculating option prices and, as mentioned above, depends on the definition of the wealth process of self-financing portfolios. These problems can be solved if other assumptions on the model are proposed, e.g. if transaction costs are introduced. Guasoni provides a model with transaction costs in [32] which is free of arbitrage in a setting of pathwise defined integrals. Transaction costs are also considered by Wang et al. in [33]. Gu et al. derive in [34] a formula for a European call option where the stock price follows a time-changed geometric fBm. Xiao et al. consider the pricing of currency options under a fBm with jumps in [35].

This short literature review shows that introducing fBm as noise in a stochastic differential equation leads to difficulties related to its definition and then to difficulties in the interpretation and comparability of the results between the considered models. Therefore, we follow an idea proposed by Thao in [19] and similar by Dung in [36] which allows us to introduce correlated noise into a stochastic differential equation and defining stochastic differential equations in the classical Itô integration setting. We explain this approach in the next section.

3 Definition of the shock generating process

Starting with an approximation of an autoregressive fractionally integrated moving average (ARFIMA) process, Thao et al. propose in [19] the usage of the process $Z^H = (Z_t^H)_{t \geq 0}$ defined by

$$Z_t^H = \int_0^t (t-s)^{H-1/2} dW_s$$

with $1/2 < H < 1$ for modeling the stochastic noise in a financial market. The process $(W_t)_{t \geq 0}$ is a standard Brownian motion. This process is related to fBm by the following integral representation which was proposed by Mandelbrot et al. in [22]. They stated the following representation of a fBm $(B_t^H)_{t \geq 0}$:

$$B_t^H = \frac{1}{\Gamma(H+1/2)} \left(\int_{-\infty}^0 [(t-s)^{H-1/2} - (-s)^{H-1/2}] dW_s + Z_t^H \right), \quad (3.1)$$

where $(W_t)_{t \in \mathbb{R}}$ is a Brownian motion defined on the real line. Since the first part of (3.1) is a stochastic process with absolutely continuous paths, Thao et al. argue that the process Z^H is appropriate to model the noise in financial markets. Since Z^H is again not a semimartingale, they proposed an approximation of this process which is defined as follows.

Definition 3.1

For $H \in (0, 1)$ and $\varepsilon > 0$, we define the stochastic process $Z^{H,\varepsilon} = (Z_t^{H,\varepsilon})_{t \geq 0}$ on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ by

$$Z_t^{H,\varepsilon} = \int_0^t (t-s+\varepsilon)^{H-1/2} dW_s. \quad (3.2)$$

Thereby, $(W_t)_{t \geq 0}$ denotes a standard Brownian motion defined on $(\Omega, \mathcal{F}, \mathbb{P})$. Furthermore, we define the filtration $\mathbb{F} = (\mathcal{F}_t)_{t \geq 0}$ with $\mathcal{F}_t = \sigma(W_s : 0 \leq s \leq t)$, which is assumed to satisfy the usual conditions, and the corresponding filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$. \times

Remark 3.2

The definition of the process $Z^{H,\varepsilon}$ is more general than in Thao et al. [19], since they consider only the case of $H > 1/2$. The more general case of $H \in (0, 1)$ is considered in [37] and [36]. The process $Z^{H,\varepsilon}$ can be regarded as aggregation of Brownian shocks weighted by the integral kernel $K(t+\varepsilon, s)$ which is given by $K(t, s) = (t-s)^{H-1/2}$.¹ \times

In the following we consider several properties of the process $Z^{H,\varepsilon}$. Our aim is to define a stochastic differential equation respect to the process $Z^{H,\varepsilon}$. As we mentioned above, this leads to difficulties if the underlying stochastic process is not a semimartingale. Whereas the processes B^H and Z^H are no semimartingales, the process $Z^{H,\varepsilon}$ is indeed a semimartingale. This result, stated in the following lemma, was proved by Thao in [37, Lemma 2.1] and in a more general setting by Dung in [36, Proposition 2.1].²

¹Dung considers another weighting function in [36].

²A similar approach based on a slightly different integral representation of fBm is given by Rogers in [38]. He also imposes sufficient conditions on the integral kernel for which the corresponding process becomes a semimartingale.

Lemma 3.3

The process $Z^{H,\varepsilon}$ defined by (3.2) is a semimartingale and has the following decomposition

$$\begin{aligned} Z_t^{H,\varepsilon} &= \int_0^t \int_0^s \alpha(s-u+\varepsilon)^{\alpha-1} dW_u ds + \varepsilon^\alpha W_t \\ &= \int_0^t \varphi_s^{\alpha,\varepsilon} ds + \varepsilon^\alpha W_t \end{aligned}$$

with $\alpha = H - 1/2$ and $\varphi_s^{\alpha,\varepsilon} = \int_0^s \alpha(s-u+\varepsilon)^{\alpha-1} dW_u$. Thus we can write

$$dZ_t^{H,\varepsilon} = \varphi_t^{\alpha,\varepsilon} dt + \varepsilon^\alpha dW_t.$$

This lemma allows us to consider a stochastic differential equation respect to the process $Z^{H,\varepsilon}$ in the classical Itô sense. If we set $H = 1/2$ in the definition above, then the process $Z^{1/2,\varepsilon}$ is a Brownian motion for every $\varepsilon > 0$.³ The main reason for considering a model driven by the process $Z^{H,\varepsilon}$ is the property of long and short memory mentioned above. As the increments of a Brownian motion are uncorrelated, the increments of $Z^{H,\varepsilon}$ are indeed correlated. Long memory refers to the property that the autocorrelation function decays very slowly to zero, i.e. there is a positive correlation between two far distant increments of $Z^{H,\varepsilon}$ in our case. This property is often defined via the non-summability of the autocorrelation function.⁴ To formalize this, we are following Dung in [36] and define the auto-variance function

$$\rho^{H,\varepsilon}(n) := \mathbb{E} \left(Z_1^{H,\varepsilon} (Z_{n+1}^{H,\varepsilon} - Z_n^{H,\varepsilon}) \right) \text{ for } n \in \mathbb{N}.$$

He proved the following proposition, see [39, Proposition 4.1].

Proposition 3.4

The auto-variance function defined above satisfies

$$\begin{aligned} \sum_{n=1}^{\infty} \rho^{H,\varepsilon}(n) &= \infty, \text{ if } H > 1/2 \text{ and} \\ \sum_{n=1}^{\infty} \rho^{H,\varepsilon}(n) &< \infty, \text{ if } H < 1/2, \end{aligned}$$

i.e. the process $Z^{H,\varepsilon}$ exhibits the long-range dependence property if $H > 1/2$ and the short-range dependence property if $H < 1/2$.

We have motivated the process $Z^{H,\varepsilon}$ twice. First via its linkage between time discrete ARFIMA models, which seems to be reasonable under an empirical point of view as we mentioned in Section 1, and second via its linkage to fBm and its usage in classical finance contexts. For these reasons, we introduce long-range dependence and persistence effects into the evolution of capital via the process $Z^{H,\varepsilon}$ instead of fBm.

³In [36] Dung derives a pricing formula for a European call option in a corresponding Black-Scholes model in such a setting.

⁴There are different definitions of long-range dependence in the literature. An overview is given in [23, Section 1.4].

4 The model setup

Our model is mainly based on Section 2 of [15] which itself extends the model presented by Basak and Cuoco in [40]. In the latter model the authors suppose a finite time horizon and consider a financial market with a risky asset and a riskless bond in zero net supply. The stochastic setting of Basak and Cuoco is more general than in the model of [15] since they allow for general stochastic processes and not only for state dependent ones. In [15], Brunnermeier and Sannikov link the risky asset in the model of Basak and Cuoco to a production technology in the sense that the risky asset of Basak and Cuoco is explicitly employed with this technology and produces output. This output corresponds to the dividend of the stock in the setting of Basak and Cuoco. We generalize the model of Brunnermeier and Sannikov in [15] in the sense that we account for long- and short-range dependencies in the dynamics of the risky asset. As Brunnermeier and Sannikov we suppose an infinite time horizon. We adopt the notation of Brunnermeier and Sannikov and we follow mainly the steps they propose in [15].

There are two kinds of agents called experts and households in the economy. Experts are allowed to hold (risky) capital to produce output. Furthermore, they can invest or lend money at a risk-free interest rate r_t which is determined in equilibrium. The financial friction for experts comes into effect as they only have the possibility to lend money from households to finance their investments in capital, i.e. they have to issue a risk-free asset with interest rate r_t . They cannot issue equity. Households are constrained as they cannot invest into the risky asset, i.e. they only can hold the risk-free asset/bond which is in zero net supply in the economy. As in the original model, we assume that all agents are small and behave as price-takers. In the following, we characterize the economy.

4.1 The production technology

Output is produced at time $t \geq 0$ according to the production technology

$$y_t = ak_t$$

where $a > 0$ is a productivity parameter and k_t is the amount of capital at time t . Capital is understood as capital in efficiency units rather than physical capital, i.e. it measures the future production potential of the capital instead of physical capital, see [12, p. 385]. The price of output is normalized to one and is treated as the numeraire.

Capital is held by experts and we assume that it evolves according to the stochastic differential equation:

$$\frac{dk_t}{k_t} = (\Phi(\iota_t) - \delta) dt + \sigma dZ_t^{H,\varepsilon} \quad (4.1)$$

where $(\iota_t)_{t \geq 0}$ is interpreted as the re-investment rate process, i.e. ι_t describes the amount of output that is re-invested into the capital stock expressed as investment rate per unit of capital.⁵ This can be done since output and capital are proportional. After the re-investment

⁵Throughout we assume that all stated stochastic processes are restrictedly progressively measurable, cf. [16, Definition 1.7.1], to ensure that they are adapted to the Filtration generated by the Brownian motion.

into capital there are $(a - \iota_t)k_t$ units of output left for consumption. Recall Definition 3.1 for the definition of $Z^{H,\varepsilon}$. As in the original model the function $\Phi : [0, 1] \rightarrow [0, \infty)$, $x \mapsto \Phi(x)$ is assumed to be twice differentiable and strictly concave with $\Phi(0) = 0$ and $\Phi'(0) = 1$. It represents adjustment costs of transforming output into capital, see [12, p. 384]. The depreciation rate of capital is given by δ and is assumed to be non-negative. The volatility parameter σ is assumed to be positive.

By Lemma 3.3, we can rewrite (4.1) and obtain

$$\frac{dk_t}{k_t} = (\Phi(\iota_t) - \delta + \sigma\varphi_t^{\alpha,\varepsilon}) dt + \sigma\varepsilon^\alpha dW_t \quad (4.2)$$

with $\alpha = H - 1/2$ and $\varphi_t^{\alpha,\varepsilon} = \int_0^t \alpha(t - u + \varepsilon)^{\alpha-1} dW_u$ and $(H, \varepsilon) \in (0, 1) \times (0, \infty)$. To clarify the main differences between the approach of Brunnermeier and Sannikov, c.f. (1.1) and our approach stated in (4.1) and (4.2), it is useful to consider the solutions of these two stochastic differential equations. For the sake of simplicity, we assume a constant re-investment rate $\iota_t \equiv \iota$ at the moment.⁶ To distinguish the two capital processes, we denote with k_t^{BS} the one of Brunnermeier and Sannikov and with k_t the capital in our setting. An application of the Itô formula and Lemma 3.3 leads to the solutions

$$k_t^{\text{BS}} = k_0^{\text{BS}} \exp\left(\left(\Phi(\iota) - \delta - \frac{\sigma^2}{2}\right)t + \sigma W_t\right) \quad (4.3)$$

$$\begin{aligned} k_t &= k_0 \exp\left(\left(\Phi(\iota) - \delta - \frac{(\varepsilon^\alpha \sigma)^2}{2}\right)t + \sigma Z_t^{H,\varepsilon}\right). \\ &= k_0 \exp\left(\left(\Phi(\iota) - \delta - \frac{(\varepsilon^\alpha \sigma)^2}{2}\right)t + \sigma \int_0^t \varphi_s^{\alpha,\varepsilon} ds + \sigma\varepsilon^\alpha W_t\right). \end{aligned} \quad (4.4)$$

Assume that we can observe the amount of output y_t at $N + 1$ points in time, i.e. we observe y_0, \dots, y_N . For $n = 1, \dots, N$, we obtain the log differenced times series in the setting of Brunnermeier and Sannikov as

$$R_n^{\text{BS}} := \log\left(\frac{y_n^{\text{BS}}}{y_{n-1}^{\text{BS}}}\right) = \log\left(\frac{k_n^{\text{BS}}}{k_{n-1}^{\text{BS}}}\right) = \Phi(\iota) - \delta - \frac{\sigma^2}{2} + \sigma(W_n - W_{n-1}) \quad (4.5)$$

and

$$R_n := \log\left(\frac{y_n}{y_{n-1}}\right) = \log\left(\frac{k_n}{k_{n-1}}\right) = \Phi(\iota) - \delta - \frac{(\varepsilon^\alpha \sigma)^2}{2} + \sigma\left(Z_n^{H,\varepsilon} - Z_{n-1}^{H,\varepsilon}\right) \quad (4.6)$$

as the log differenced times series in our setting. Considering those equations under a time series viewpoint, we can argue that Brunnermeier and Sannikov model the log-output series as a unit root process, since its first difference behaves as white noise, whereas we impose that the log-differenced output series behaves like a fractional differenced white noise process of order $d = H - 1/2 = \alpha$. The implications of these different approaches can be seen in Figure 1. In panel a) there is a white noise behavior of the autocorrelation function of the log differenced output series whereas we can see a slowly decaying autocorrelation function in panel b) indicating a long-range dependence behavior.

⁶It will turn out in Section 5, that ι_t is indeed constant in the equilibrium.

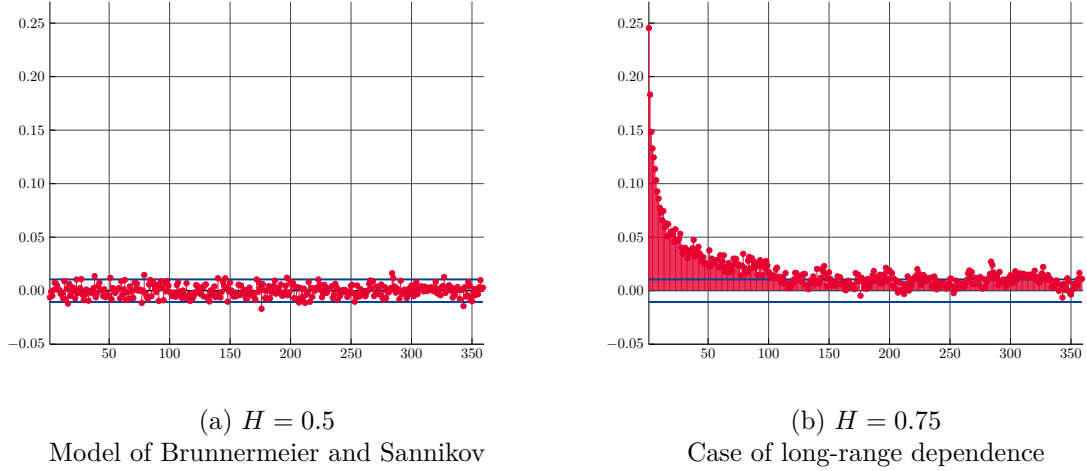


Figure 1: Sample autocorrelation function of the sequences $(R_n^{\text{BS}})_{n=1,\dots,360}$ and $(R_n)_{n=1,\dots,360}$ in panel a) and b), respectively. The parameters are $\iota = 0.04$, $\Phi(\iota) = \kappa^{-1} \log(\kappa\iota + 1)$ with $\kappa = 10$, $\sigma = 0.1$, $\delta = 0$ and $\varepsilon = 10^{-5}$. The Brownian shocks coincide in both cases.

We stress that the single random source in this economy is the Brownian motion $(W_t)_{t \geq 0}$ as in the original model. The autocorrelation enters into the model via a different aggregation approach of former Brownian shocks. To be more illustrative we consider the following identities

$$W_t = \int_0^t 1 dW_S \text{ and } Z_t^{H,\varepsilon} = \int_0^t (t-s+\varepsilon)^{H-1/2} dW_s,$$

c.f. Definition 3.1. Hence, Brunnermeier and Sannikov assume that all Brownian shocks enter into the development of the capital stock with a constant weight of 1 whereas we impose that the weight of the Brownian shock depends on the time when it happened. If we consider the impact of the Brownian shock that happened at time $s < t$ it is weighted with $(t-s+\varepsilon)^{H-1/2}$. Thus the impact of the past shock is mainly determined by the time difference $(t-s)$ between the times t and s . Figure 2 shows the weighting kernel $K(t,s) = (t-s+\varepsilon)^{H-1/2}$ as a function of the time difference $(t-s)$ for different parameter values of H . For $H > 1/2$, it can be seen that the weight of a shock increases with the time that has passed since the shock occurred. This highlights again the long memory effect. The higher the value of H , the higher is the weight on shocks that happened in the distant past and the higher is the degree of autocorrelation. As can be seen in Equations (4.2) and (4.4), the correlation of $Z^{H,\varepsilon}$ -shocks expressed in the parameters ε and α affects the dynamics of the capital twice: On the one hand we obtain the additional drift component $\sigma \varphi_t^{\alpha,\varepsilon}$, which can be interpreted as the weighted influence of past Brownian shocks, and on the other hand a risk component ε^α that occurs as multiplier in the volatility. Considering Equation (4.2), we can see that positive autocorrelations shift the stochastic uncertainty from the actual shock to the trend component of the capital stock: the impact of the actual shock dW_t decreases whereas the weight on the past shocks increases. This is intuitive since if shocks tend to be followed by shocks of the same sign (due to the positive correlation) the impact of the next shock will be reduced (volatility effect) since the former shocks have already determined the direction of the process (trend effect). This leads to more unpredictable long-run behavior. These

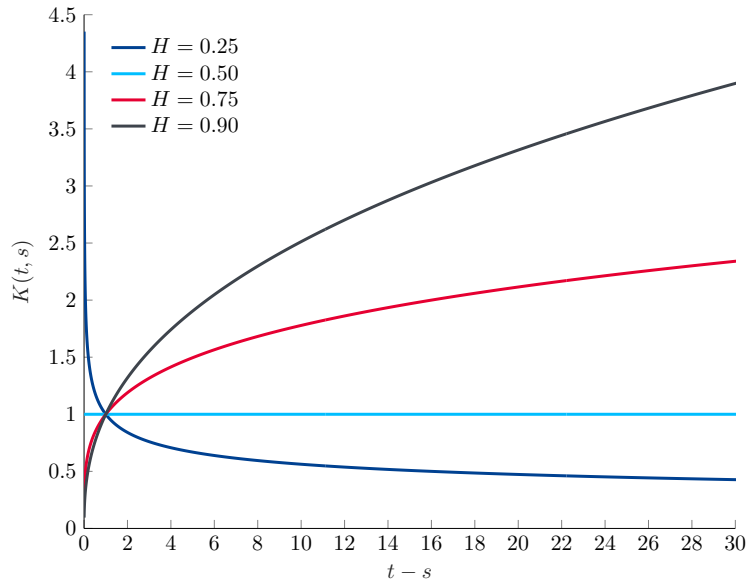


Figure 2: Weighting kernel $K(t, s) = (t - s + \varepsilon)^{H-1/2}$ as function of the time difference $(t - s)$ for different values of H with $\varepsilon = 10^{-5}$.

two effects can be observed directly in the capital stock: For $\varepsilon \leq 1$ it follows immediately from Equations (4.5) and (4.6) that the expected growth rate of output is higher in the case of long-range dependence $\alpha > 0$ than in the original model (volatility effect).⁷ However, it yields that $\text{Var}(\int_0^t \varphi_s^{\alpha, \varepsilon} ds) \neq 0$ indicating the trend uncertainty raising from past shocks (trend effect).

Admittedly, we consider these effects as exogenously given to imitate the empirical behavior in the growth rate of the output series but there are several approaches in the literature how long-memory can be explained endogenously. Following Henry and Zaffaroni [41], there are two main reasons that long-range dependence occurs in macroeconomic variables. The first reason is founded in the fact that economic variables are affected by geophysical processes like rain fall and river-flows which exhibit long-range dependencies (see the references in [41]). The major process then inherits these properties from the related processes. The second reason for long-range dependence in macroeconomic variables is aggregation. In [2], Haubrich and Lo consider a model in which long-range dependence is endogenously explained via aggregation over different sectors. In our case capital is measured in efficiency units so there could be an impact of the (not modeled) labor market on the dynamics of capital. In [8], Gil-Alana et al. provide empirical evidence for long-range dependence in Turkish unemployment rates. One could argue that the dynamics of the capital inherits this property from the labor market. Regarding to the AK-production technology, output then inherits the long-range dependence property from capital in our setting. Note that the dynamics of the capital proposed in the original model neglect those effects but we obtain their setting if we set $H = 1/2$ or $\alpha = 0$ in (4.2). Therefore, the dynamics given in (4.1) or (4.2) are more general than in the original

⁷This is due to the fact that $\mathbb{E}(W_t) = 0$ for all $t \geq 0$ and $(\varepsilon^\alpha \sigma)^2 < \sigma^2$.

model.

So the aim of this paper is rather to explain whether and how different variables are affected by these long-range dependence effects than to provide a full microfoundation. In the following, we introduce more details about the economy.

We denote the price process of the capital with $q = (q_t)_{t \geq 0}$, i.e. q_t is the price of one unit of capital at time t . We suppose that the price process is given as⁸

$$\frac{dq_t}{q_t} = \mu_t^q dt + \sigma_t^q dW_t. \quad (4.7)$$

4.2 Agents and Preferences

We assume that there is an infinite number of experts and households, each with total mass one. To be more precise, we denote with $\mathbb{I} = [0, 1]$ and $\mathbb{J} = (1, 2]$ the index set of experts and households, respectively.

We denote the consumption process of expert $i \in \mathbb{I}$ with $(c_t^i)_{t \geq 0}$ and the consumption process of household $j \in \mathbb{J}$ with $(\tilde{c}_t^j)_{t \geq 0}$.⁹ Both processes are assumed to be non-negative and restrictedly progressively measurable, cf. Footnote 5, satisfying $\int_0^T c_t^i dt < \infty$ for all $i \in \mathbb{I}$ and $\int_0^T \tilde{c}_t^j dt < \infty$ for all $j \in \mathbb{J}$ almost surely for every $T \in [0, \infty)$.

We assume that both types of agents have logarithmic utility and a constant discount rate of $\rho > 0$. Both types of agents are assumed to maximize the expected present value of total utility given by

$$\mathbb{E} \left[\int_0^\infty e^{-\rho t} \log(c_t^i) dt \right] \quad \text{and} \quad \mathbb{E} \left[\int_0^\infty e^{-\rho t} \log(\tilde{c}_t^j) dt \right]$$

for expert $i \in \mathbb{I}$ and household $j \in \mathbb{J}$, respectively. We assume that these expected values exist.¹⁰ In the following, we consider the maximization problems of experts and households more explicitly.

4.2.1 Experts

Applying Lemma 3.3, we can rewrite (4.1) to obtain the evolution of the capital holdings of an individual expert $i \in \mathbb{I}$ as

$$\frac{dk_t^i}{k_t^i} = \left(\Phi(\iota_t^i) - \delta \right) dt + \sigma dZ_t^{H,\varepsilon} \quad (4.8)$$

⁸On the first view this seems to be arbitrary because one would expect that the dynamics of q is affected by the same macro-shocks $dZ_t^{H,\varepsilon}$ as the capital. So if we assume that $dq_t/q_t = \hat{\mu}_t^q dt + \hat{\sigma}_t^q dZ_t^{H,\varepsilon}$ for some processes $(\hat{\mu}_t^q)_{t \geq 0}$ and $(\hat{\sigma}_t^q)_{t \geq 0}$, we obtain $dq_t/q_t = (\hat{\mu}_t^q + \hat{\sigma}_t^q \varphi_t^{\alpha,\varepsilon}) dt + \varepsilon^\alpha \hat{\sigma}_t^q dZ_t^{H,\varepsilon}$ by using Lemma 3.3. If we then define $\mu_t^q = (\hat{\mu}_t^q + \hat{\sigma}_t^q \varphi_t^{\alpha,\varepsilon})$ and $\sigma_t^q = \varepsilon^\alpha \hat{\sigma}_t^q$, we obtain the stated expression. Hence μ_t^q and σ_t^q can depend on H and ε . We will see that this issue doesn't matter in the equilibrium.

⁹In the following all individual specific processes for households are additionally marked with \sim to enclose them from experts' individual processes.

¹⁰Such an assumption is for example that the expected value of the negative part is finite, i.e. $\mathbb{E} \int_0^\infty \min\{0, e^{-\rho t} \log(c_t)\} dt < \infty$.

$$= \left(\Phi(\iota_t^i) - \delta + \sigma \varphi_t^{\alpha, \varepsilon} \right) dt + \sigma \varepsilon^\alpha dW_t \text{ with } k_0^i = \kappa_0^i, \quad (4.9)$$

where κ_0^i is the initial amount of capital held by expert i which is assumed to be given as a non-negative constant and $(\iota_t^i)_{t \geq 0}$ is the process of re-investment rates chosen by the expert $i \in \mathbb{I}$.¹¹

At time t , expert i has to decide how much of the generated output he should reinvest into the risky capital, i.e. he has to choose ι_t , and how much he wants to consume. Furthermore, he has to decide how much of his wealth is invested in capital or is held/lent at the interest rate r_t , i.e. he has to made the portfolio choice between capital and bonds. We denote the wealth process of expert i with $n^i = (n_t^i)_{t \geq 0}$ and the amount of bonds held by expert i at time t with π_t^i . Thus, the wealth of expert i is given by

$$n_t^i = q_t k_t^i + \pi_t^i B_t \quad (4.10)$$

where $dB_t = r_t B_t dt$ describes the dynamics of the value of the bond B_t with interest rate process $r = (r_t)_{t \geq 0}$. The initial wealth is given as $n_0^i = q_0 \kappa_0^i$. The evolution of wealth of expert i is affected by four sources:

After choosing ι_t^i the capital produces $(a - \iota_t^i)k_t^i$ units of output, cf. Section 4.1.

The value of the capital varies itself with an amount of $d(q_t k_t^i)$.

The expert earns (or has to pay) the interest rate on bonds, i.e. $\pi_t^i dB_t = \pi_t^i B_t r_t dt$.

The expert has consumption expenditures of c_t^i .

Thus we can express the dynamics of the wealth process n^i as:

$$dn_t^i = k_t^i(a - \iota_t^i) dt + d(q_t k_t^i) + \pi_t^i B_t r_t dt - c_t^i dt. \quad (4.11)$$

We assume that there are no other sources of income or expenditures and therefore we express capital and bond holdings as shares of wealth, i.e. we define the part of wealth of expert i that is invested in capital as

$$x_t^i = \frac{q_t k_t^i}{n_t^i}. \quad (4.12)$$

Combining (4.10) and (4.12), we can rewrite (4.11) and obtain

$$dn_t^i = x_t^i n_t^i \frac{(a - \iota_t^i)}{q_t} dt + d(q_t k_t^i) + (1 - x_t^i) n_t^i r_t dt - c_t^i dt. \quad (4.13)$$

To calculate the differential $d(q_t k_t^i)$ we use Itô-product formula and the dynamics of the price process q given in (4.7) and the dynamics of the capital holdings (4.9). We obtain

$$\frac{d(q_t k_t^i)}{q_t k_t^i} = \left(\Phi(\iota_t^i) - \delta + \sigma \varphi_t^{\alpha, \varepsilon} + \mu_t^q + \varepsilon^\alpha \sigma \sigma_t^q \right) dt + (\sigma \varepsilon^\alpha + \sigma_t^q) dW_t. \quad (4.14)$$

¹¹Note that the index i in (4.9) does not state that there are different assets for each expert in the economy. It states that the capital holdings of an individual expert are determined by the initial endowments κ_0^i and investment decisions $(\iota_t^i)_{t \geq 0}$.

Plugging this into (4.13), we obtain the dynamics of the wealth process

$$dn_t^i = x_t^i n_t^i \frac{(a - l_t^i)}{q_t} dt + \left[\left(\Phi(l_t^i) - \delta + \sigma \varphi_t^{\alpha, \varepsilon} + \mu_t^q + \varepsilon^\alpha \sigma \sigma_t^q \right) dt + (\sigma \varepsilon^\alpha + \sigma_t^q) dW_t \right] q_t k_t^i + (1 - x_t^i) n_t^i r_t dt - c_t^i dt.$$

After some rearrangements, we have

$$dn_t^i = \left[\frac{(a - l_t^i)}{q_t} + \left(\Phi(l_t^i) - \delta + \sigma \varphi_t^{\alpha, \varepsilon} + \mu_t^q + \varepsilon^\alpha \sigma \sigma_t^q \right) \right] x_t^i n_t^i dt + (1 - x_t^i) r_t n_t^i dt - c_t^i dt + (\sigma \varepsilon^\alpha + \sigma_t^q) x_t^i n_t^i dW_t. \quad (4.15)$$

Given the price process $(q_t)_{t \geq 0}$ and the risk-free rate $(r_t)_{t \geq 0}$, we are now able to state the utility maximization problem of the expert i :

$$\max_{x^i, c^i, l^i} \mathbb{E} \left[\int_0^\infty e^{-\rho t} \log(c_t^i) dt \right] \text{ s.t. } \begin{cases} x_t^i \geq 0 \text{ a.s. for all } t \geq 0. \\ c_t^i \geq 0 \text{ a.s. for all } t \geq 0 \\ n^i \text{ follows (4.15) with } n_0^i = q_0 \kappa_0^i. \\ n_t^i \geq 0 \text{ a.s. for all } t \geq 0 \end{cases}. \quad (4.16)$$

We solve this maximization problem in Section 5.

4.2.2 Households

We consider an individual household $j \in \mathbb{J}$. His wealth process is denoted by $\tilde{n}^j = (\tilde{n}_t^j)_{t \geq 0}$. Since households can hold the risk-free bond only, the wealth process is given as

$$\tilde{n}_t^j = \tilde{\pi}_t^j B_t \quad (4.17)$$

where $\tilde{\pi}_t^j$ denotes the amount of bonds held by household j at time t . The household earns the interest-rate from holding bonds and has consumption expenditures.¹² Therefore, the dynamics of wealth is given by

$$d\tilde{n}_t^j = \tilde{\pi}_t^j dB_t - \tilde{c}_t^j dt = \left(\tilde{\pi}_t^j B_t r_t - \tilde{c}_t^j \right) dt. \quad (4.18)$$

Inserting (4.17) leads to

$$d\tilde{n}_t^j = \left(\tilde{n}_t^j r_t - \tilde{c}_t^j \right) dt. \quad (4.19)$$

We assume that the initial wealth \tilde{n}_0^j is given as $\tilde{\nu}_0^j > 0$ for each household $j \in \mathbb{J}$, i.e. $\tilde{n}_0^j = \tilde{\nu}_0^j > 0$. We can now state the utility maximization problem of the household $j \in \mathbb{J}$ formally as

$$\max_{\tilde{c}^j} \mathbb{E} \left[\int_0^\infty e^{-\rho t} \log(\tilde{c}_t^j) dt \right] \text{ s.t. } \begin{cases} \tilde{c}_t^j \geq 0 \text{ a.s. for all } t \geq 0 \\ \tilde{n}^j \text{ follows (4.19) with } \tilde{n}_0^j = \tilde{\nu}_0^j. \\ \tilde{n}_t^j \geq 0 \text{ a.s. for all } t \geq 0 \end{cases}. \quad (4.20)$$

In the next section, we give the formal definition of the equilibrium in the economy.

¹²Like the experts, households are assumed to have non-negative wealth. So (4.17) implies that households hold a non-negative amount of bonds.

5 Definition and solution of the equilibrium

At first we give a formal definition of an equilibrium in the economy and state the equilibrium processes.

Definition 5.1

Let $(K_t)_{t \geq 0}$ denote the process of the aggregate capital stock in the economy, i.e. the total amount of capital in the economy at time $t \geq 0$ is given by

$$K_t = \int_{\mathbb{I}} \kappa_t^i di.$$

Given the initial capital endowments of experts $\{\kappa_0^i : i \in \mathbb{I}\}$ and initial wealth endowments of households $\{\tilde{v}_0^j : j \in \mathbb{J}\}$, such that

$$K_0 = \int_{\mathbb{I}} \kappa_0^i di.$$

An equilibrium is defined as the following families of restrictedly progressively measurable stochastic processes over $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$:

Price of capital and the risk-free rate $\{(q_t)_{t \geq 0}, (r_t)_{t \geq 0}\}$; expert's individual investment and consumption decisions, capital holdings and wealth processes

$$\left\{ (\iota_t^i)_{t \geq 0}, (c_t^i)_{t \geq 0}, (k_t^i)_{t \geq 0}, (n_t^i)_{t \geq 0} \right\}_{i \in \mathbb{I}};$$

household's individual consumption decisions and wealth processes

$$\left\{ (\tilde{c}_t^j)_{t \geq 0}, (\tilde{n}_t^j)_{t \geq 0} \right\}_{j \in \mathbb{J}};$$

such that ¹³

i) experts solve their maximization problem (4.16)

ii) households solve their maximization problem (4.20)

iii) the market of consumption goods clears, i.e.

$$\int_{\mathbb{I}} \dot{c}_t^i di + \int_{\mathbb{J}} \tilde{c}_t^j dj = \int_{\mathbb{I}} (a - \iota_t^i) k_t^i di \text{ for all } t \geq 0$$

iv) the risk-free bond is in zero net supply, i.e.

$$\int_{\mathbb{I}} \pi_t^i di + \int_{\mathbb{J}} \tilde{\pi}_t^j dj = 0 \text{ for all } t \geq 0.$$

×

¹³Regarding [16, Definition 1.7.2], we assume the following integrability conditions in order to solve the experts' and households' maximization problems: For every finite $T > 0$ it yields $\int_0^T |r_t| dt < \infty$, $\int_0^T |\mu_t^i| dt < \infty$, $\int_0^T |\iota_t^i| dt < \infty$ and $\int_0^T (\sigma_t^i)^2 dt < \infty$ almost surely.

We first solve the maximization problem of experts (4.16). To do so, we introduce the market price of risk process $(\vartheta_t^i)_{t \geq 0}$ for expert $i \in \mathbb{I}$ as:

$$\vartheta_t^i := \frac{\frac{a - l_t^i}{q_t} + \Phi(l_t^i) - \delta + \sigma \varphi_t^{\alpha, \varepsilon} + \mu_t^q + \varepsilon^\alpha \sigma \sigma_t^q - r_t}{\varepsilon^\alpha \sigma + \sigma_t^q}. \quad (5.1)$$

This corresponds to the Sharpe ratio in [15]. The numerator can be interpreted as the expected excess return of (nominal) capital $q_t k_t^i$ over the interest rate r_t , cf. (4.14), since $(a - l_t^i)/q_t$ is the output left after investment expressed in $q_t k_t^i$ units, cf. (4.15). Thus it can be interpreted as an additional dividend of the capital.

Proposition 5.2

Consider the problem stated in (4.16) and define the stochastic process $\xi^i = (\xi_t^i)_{t \geq 0}$ as

$$\xi_t^i = \exp \left(- \int_0^t r_s ds - \int_0^t \vartheta_s^i dW_s - \frac{1}{2} \int_0^t (\vartheta_s^i)^2 ds \right) \quad (5.2)$$

which follows the dynamics

$$d\xi_t^i = -r_t \xi_t^i dt - \vartheta_t^i \xi_t^i dW_t \text{ with } \xi_0^i = 1. \quad (5.3)$$

Then:

- i) The optimal re-investment rate process $(\tilde{l}_t^i)_{t \geq 0}$ satisfies $\Phi'(\tilde{l}_t^i) = \frac{1}{q_t}$ for all $t \geq 0$ as long as $\hat{\vartheta}_t^i > 0$ where $\hat{\vartheta}^i = (\hat{\vartheta}_t^i)_{t \geq 0}$ is the market price of risk process corresponding to $(\tilde{l}_t^i)_{t \geq 0}$.
- ii) The optimal consumption process is given by $\hat{c}_t^i = \rho e^{-\rho t} \frac{1}{\hat{\xi}_t^i}$ where $\hat{\xi}_t^i$ is given by (5.2) corresponding to $\hat{\vartheta}^i$.
- ii) The corresponding optimal wealth process is given by $\hat{n}_t^i = e^{-\rho t} \frac{1}{\hat{\xi}_t^i}$.
- iv) The optimal fraction of wealth to invest in the capital is given by $\hat{x}_t^i = \frac{\hat{\vartheta}_t^i}{\varepsilon^\alpha \sigma + \sigma_t^q}$.

PROOF

To solve the utility maximization problem of the expert i , we use the analogy of this problem to classical problems in finance and apply those finance results given by Karatzas and Shreve in [16, Section 1 and 3] to our context. Let initially $(\tilde{l}_t^i)_{t \geq 0}$ be a given process of re-investment rates. Then the utility maximization problem of expert i is the same as for an investor in a financial market with the following investment possibilities:

A risk-free bond S^0 , with price process given as

$$dS_t^0 = S_t^0 r_t dt$$

and a risky stock S^1 whose price process is given as

$$dS_t^1 = \mu_t^S S_t^1 dt + \sigma_t^S S_t^1 dW_t,$$

with

$$\mu_t^S = \left[\frac{(a - l_t^i)}{q_t} + \left(\Phi(l_t^i) - \delta + \sigma \varphi_t^{\alpha, \varepsilon} + \mu_t^q + \varepsilon^\alpha \sigma \sigma_t^q \right) \right] \text{ and } \sigma_t^S = (\sigma \varepsilon^\alpha + \sigma_t^q).$$

In the following, we consider a portfolio process $\pi_t = (\pi_t^0, \pi_t^1)$, where π_t^0 and π_t^1 denote the value of bonds and stocks, respectively held by the investor. Then following Karatzas and Shreve, the wealth process of the investor with consumption process $(c_t)_{t \geq 0}$ is given as

$$dn_t = n_t r_t dt + \pi_t^1 (\mu_t^S - r_t) dt + \pi_t^1 \sigma_t^S dW_t - c_t \text{ with } n_0 = q_0 \kappa_0^i.$$

Let $x_t = \pi_t^1 / n_t$ be the share of wealth invested into the stock. Then we have

$$dn_t = n_t r_t dt + (\mu_t^S - r_t) x_t n_t dt + \sigma_t^S x_t n_t dW_t - c_t$$

which corresponds exactly to (4.15). Before we are allowed to apply Theorem 3.9.11 in [16], we have to prove that the assumptions are fulfilled in our setting, i.e. the financial market has to be viable, standard and complete, cf. [16, Definition 1.7.3]. Therefore, we have to check that the process ϑ^i satisfies the Novikov-condition

$$\mathbb{E} \left[\exp \left(\frac{1}{2} \int_0^T \|\vartheta_t^i\|^2 dt \right) \right] < \infty \text{ for all } T \in [0, \infty).$$

This was proved by Thao in [37, Section 5]. Hence, we are able to apply Theorem 3.9.11 of [16] and obtain an optimal consumption process $(\hat{c}_t)_{t \geq 0}$ and an optimal wealth process $(\hat{n}_t)_{t \geq 0}$ as

$$\hat{c}_t = \rho e^{-\rho t} n_0 \frac{1}{\xi_t^i} \tag{5.4}$$

and

$$\hat{n}_t = \frac{1}{\xi_t^i} \mathbb{E} \left[\int_t^\infty \xi_u^i \hat{c}_u du \middle| \mathcal{F}_t \right]. \tag{5.5}$$

Plugging the expression of optimal consumption (5.4) into the expression of optimal wealth (5.5) we obtain:

$$\hat{n}_t = \frac{1}{\xi_t^i} \int_t^\infty n_0 \rho e^{-\rho u} du = e^{-\rho t} n_0 \frac{1}{\xi_t^i}. \tag{5.6}$$

Comparing (5.6) and (5.4), we see that it is optimal for the investor to consume the fraction ρ of his wealth, i.e.

$$\hat{c}_t = \rho \hat{n}_t. \tag{5.7}$$

Now we can apply Itô's Lemma to (5.6) and use (5.3) to obtain the dynamics of optimal wealth

$$d\hat{n}_t = -\rho \hat{n}_t dt - e^{-\rho t} n_0 \left(\frac{1}{\xi_t^i} \right)^2 d\xi_t^i + \frac{1}{2} e^{-\rho t} n_0 \frac{2}{(\xi_t^i)^3} d\langle \xi^i \rangle_t$$

$$\begin{aligned}
&= -\rho \hat{n}_t dt - e^{-\rho t} n_0 \left(\frac{1}{\xi_t^i} \right)^2 \left[-r_t \xi_t^i dt - \vartheta_t^i \xi_t^i dW_t \right] + e^{-\rho t} n_0 \frac{1}{(\xi_t^i)^3} (\vartheta_t^i)^2 (\xi_t^i)^2 dt \\
&= \left[(\vartheta_t^i)^2 + r_t - \rho \right] \hat{n}_t dt + \vartheta_t^i \hat{n}_t dW_t.
\end{aligned} \tag{5.8}$$

All expressions above yield for arbitrary re-investment rate processes $(\hat{c}_t^i)_{t \geq 0}$ that was assumed to be given for the investor in this financial market setting. It is then clear that these optimal consumption and wealth processes hold for expert i as well, since it is indeed the same market under a different point of view. Expert i chooses the optimal re-investment rate process $(\hat{c}_t^i)_{t \geq 0}$ that maximizes

$$\mathbb{E} \left[\int_0^\infty e^{-\rho t} \log(\hat{c}_t^i) dt \right]$$

under the conditions stated in (4.16). Since ρ does not depend on ι and consumption is proportional to wealth, cf. (5.7), this is equivalent to maximize

$$\mathbb{E} \left[\int_0^\infty e^{-\rho t} \log(\hat{n}_t^i) dt \right]. \tag{5.9}$$

Using (5.8) and applying Itô's Lemma to $\log(\hat{n}_t^i)$, we obtain

$$\log(\hat{n}_t^i) = \log(\hat{n}_0^i) + \int_0^t \left(\frac{1}{2} (\vartheta_s^i)^2 + r_s - \rho \right) ds + \int_0^t \vartheta_s^i dW_s.$$

Inserting this back in (5.9), neglecting some constants and applying Tonellis Theorem to switch expectation and integration, we obtain

$$\int_0^\infty e^{-\rho t} \int_0^t \mathbb{E} \left[\frac{1}{2} (\vartheta_s^i)^2 + r_s - \rho \right] ds dt.$$

Since all experts are assumed to be price-takers, the interest rate r_t is independent of \hat{c}_t^i . Therefore, we have to maximize

$$\mathbb{E} \left[\frac{1}{2} (\vartheta_t^i)^2 \right].$$

An ω -wise maximization leads to the first-order condition

$$\vartheta_t^i \left(\Phi'(\hat{c}_t^i) - \frac{1}{q_t} \right) = 0.$$

By the concavity of Φ it can be easily verified that the maximum is attained if

$$\Phi'(\hat{c}_t^i) = \frac{1}{q_t} \text{ or } \hat{c}_t^i = \Psi \left(\frac{1}{q_t} \right) \text{ as long as } \hat{c}_t^i > 0 \tag{5.10}$$

where Ψ is the inverse function of Φ' . Inserting the optimal re-investment rate process in (5.8) and (5.4), leads to the optimal wealth and consumption process of expert i . It remains to determine the optimal fraction of wealth that is invested into capital. On the one hand, we know that the dynamics of the optimal wealth process is given by (5.8) if we replace ϑ_t^i

by $\hat{\vartheta}_t^i$. On the other hand, inserting (5.7) and (5.10) into the dynamics of the wealth process given by (4.15), we obtain

$$dn_t^i = \left[\frac{(a - \hat{\imath}_t^i)}{q_t} + \left(\Phi(\hat{\imath}_t^i) - \delta + \sigma\varphi_t^{\alpha,\varepsilon} + \mu_t^q + \varepsilon^\alpha \sigma\sigma_t^q \right) \right] x_t^i n_t^i dt + (1 - x_t^i)r_t n_t^i dt - \rho n_t^i dt + (\sigma\varepsilon^\alpha + \sigma_t^q)x_t^i n_t^i dW_t. \quad (5.11)$$

A comparison of (5.8) with (5.11) leads to the equations

$$\left[(\hat{\vartheta}_t^i)^2 + r_t - \rho \right] n_t^i = \left[\frac{(a - \hat{\imath}_t^i)}{q_t} + \left(\Phi(\hat{\imath}_t^i) - \delta + \sigma\varphi_t^{\alpha,\varepsilon} + \mu_t^q + \varepsilon^\alpha \sigma\sigma_t^q \right) \right] x_t^i n_t^i + (1 - x_t^i)r_t n_t^i - \rho n_t^i$$

and

$$\hat{\vartheta}_t^i n_t^i = (\sigma\varepsilon^\alpha + \sigma_t^q)x_t^i n_t^i.$$

These lead to the optimal fraction of wealth, to invest in the capital:

$$x_t^i = \frac{\hat{\vartheta}_t^i}{(\sigma\varepsilon^\alpha + \sigma_t^q)} \quad \text{if } \hat{\vartheta}_t^i > 0 \text{ and } x_t^i = 0 \quad \text{otherwise.} \quad (5.12)$$

□

Remark 5.3

The choice of the optimal re-investment rate $\hat{\imath}_t^i$ which satisfies $\Phi'(\hat{\imath}_t^i) = 1/q_t$ if $\hat{\vartheta}_t^i > 0$ states that experts do not re-invest into capital if there is then a negative excess return of capital over the interest rate r_t . Equation (5.12) highlights that experts have to claim a positive excess return over the risk-free rate to invest at least a positive amount of wealth into capital.

Brunnermeier and Sannikov state in [15] that experts choose a re-investment rate that maximizes the expected return on capital. We have shown that this choice is indeed utility maximizing.

The proof of Proposition 5.2 sheds also some light on the connection between an equilibrium in our economy and the completeness of the treated financial market. The latter requires the existence of an equivalent probability measure under which the discounted price process of the stock becomes a martingale. If such a probability measure exists, Equation (5.2) describes the Radon–Nikodym derivative between these two probability measures which corresponds to the stochastic discount factor of Brunnermeier and Sannikov. To be more precise, we denote the stochastic discount factor of Brunnermeier and Sannikov with ξ_t^{BS} which is defined as

$$\xi_t^{\text{BS}} = e^{-\rho t} u'(n_t) \quad \text{with } u(x) = \log(x),$$

i.e. the stochastic discount factor is the marginal utility of wealth. Inserting (5.6) into this equation, we obtain:

$$\xi_t^{\text{BS}} = e^{-\rho t} u'(\hat{n}_t) = \frac{1}{n_0} e^{-\rho t} e^{\rho t} \xi_t^i = \frac{1}{n_0} \xi_t^i.$$

That means that the marginal utility of optimal wealth is proportional to the Radon–Nikodym derivative which is the transformation kernel between the real world probability measure and the risk neutral probability measure. ×

In the following Proposition we solve the maximization problem of households.

Proposition 5.4

Consider the maximization problem stated in (4.20). The optimal wealth process $(\hat{n}_t^j)_{t \geq 0}$ of household $j \in \mathbb{J}$ is given as

$$\hat{n}_t^j = \tilde{\nu}_0^j e^{-\rho t} \exp\left(\int_0^t r_u du\right)$$

and the corresponding consumption process $(\tilde{c}_t^j)_{t \geq 0}$ is given as

$$\hat{c}_t^j = \rho \hat{n}_t^j.$$

PROOF

Recall that the wealth process follows (4.19). Considering the discounted wealth process, defined by

$$\tilde{n}_t^j := \exp\left(-\int_0^t r_s ds\right) \hat{n}_t^j,$$

we obtain

$$d\tilde{n}_t^j = -r_t \tilde{n}_t^j dt + \exp\left(-\int_0^t r_s ds\right) (\tilde{n}_t^j r_t - \tilde{c}_t^j) dt = -\exp\left(-\int_0^t r_s ds\right) \tilde{c}_t^j dt. \quad (5.13)$$

This implies

$$\tilde{\nu}_0^j = \mathbb{E}\left[\tilde{n}_t^j + \int_0^t \exp\left(-\int_0^s r_u du\right) \tilde{c}_s^j ds\right] \geq \mathbb{E}\left[\int_0^t \exp\left(-\int_0^s r_u du\right) \tilde{c}_s^j ds\right],$$

since $\tilde{n}_0^j = \tilde{\nu}_0^j = \tilde{\nu}_0^j$ and $\tilde{n}_t^j \geq 0$. Applying the monotone convergence theorem, we obtain

$$\mathbb{E}\left[\int_0^\infty \exp\left(-\int_0^t r_s ds\right) \tilde{c}_s^j ds\right] \leq \tilde{\nu}_0^j.$$

This can be interpreted as a budget constraint in the sense that total expected discounted future consumption expenditures cannot be larger than the initial wealth. It is clear that for the optimal consumption process this constraint is binding. We can now set up a Lagrangian function with corresponding multiplier λ to solve the maximization problem:

$$\mathbb{E}\left[\int_0^\infty e^{-\rho t} \log(\tilde{c}_t^j) dt\right] - \lambda \left(\tilde{\nu}_0^j - \mathbb{E}\left[\int_0^\infty \exp\left(-\int_0^t r_s ds\right) \tilde{c}_s^j ds\right]\right).$$

Again an ω -wise maximization leads to the first-order condition

$$e^{-\rho t} \frac{1}{\tilde{c}_t^j} + \lambda \exp\left(-\int_0^t r_s ds\right) = 0$$

and thus

$$\hat{c}_t^j = -\frac{1}{\lambda} e^{-\rho t} \exp\left(\int_0^t r_s ds\right)$$

$$= \tilde{\nu}_0^j \rho e^{-\rho t} \exp\left(\int_0^t r_s ds\right)$$

where the last line uses the budget constraint to solve for $\lambda = -1/(\tilde{\nu}_0^j \rho)$. Inserting this back in (5.13), we obtain

$$\tilde{n}_t^j = \tilde{\nu}_0^j - \int_0^t n_0 \rho e^{-\rho s} ds = \tilde{\nu}_0^j e^{-\rho t}$$

and finally

$$\hat{n}_t^j = \tilde{\nu}_0^j e^{-\rho t} \exp\left(\int_0^t r_s ds\right),$$

which states that $\hat{c}_t^j = \rho \hat{n}_t^j$. □

Now we can solve the equilibrium and we can state the equilibrium risk-free rate r_t and the equilibrium price of capital q_t .

Optimal re-investment rate

From Proposition 5.2 we know that each expert $i \in \mathbb{I}$ chooses the re-investment rate process $(\hat{c}_t^i)_{t \geq 0}$ such that

$$\Phi'(\hat{c}_t^i) = \frac{1}{q_t} \text{ or } \hat{c}_t^i = \Psi\left(\frac{1}{q_t}\right) \text{ for all } t \geq 0$$

as long as $\hat{\vartheta}_t^i$ is positive. Recall that Ψ is the inverse function of Φ' . The market price of risk depends on the risk-free rate and we will see that in equilibrium $\hat{\vartheta}_t^i$ is indeed positive. The optimal re-investment rate does not depend on i , so we write in the following $\iota_t \equiv \hat{c}_t^i$. The same holds for the market price of risk, i.e. $\hat{\vartheta}_t \equiv \hat{\vartheta}_t^i$.

Expert's optimal wealth and consumption

Optimal wealth and consumption processes of experts are given by

$$dn_t^i = [(\hat{\vartheta}_t)^2 + r_t - \rho] n_t^i dt + \hat{\vartheta}_t n_t^i dW_t \text{ with } n_0^i = q_0 \kappa_0^i$$

and

$$c_t^i = \rho n_t^i$$

respectively. Inserting \hat{c}_t into (4.9) we obtain the equilibrium process of capital holdings $(k_t^i)_{t \geq 0}$ as

$$\frac{dk_t^i}{k_t^i} = (\Phi(\hat{c}_t) - \delta + \sigma \varphi_t^{\alpha, \varepsilon}) dt + \sigma \varepsilon^\alpha dW_t \text{ with } k_0^i = \kappa_0^i.$$

This implies that the aggregate capital stock K_t follows the dynamics

$$\frac{dK_t}{K_t} = (\Phi(\hat{c}_t) - \delta + \sigma \varphi_t^{\alpha, \varepsilon}) dt + \sigma \varepsilon^\alpha dW_t \text{ with } K_0 = \int_{\mathbb{I}} \kappa_0^i di.$$

Household's optimal wealth and consumption

For household $j \in \mathbb{J}$ the equilibrium wealth and consumption processes are given by

$$\tilde{n}_t^j = \tilde{\nu}_0^j e^{-\rho t} \exp\left(\int_0^t r_u du\right) \text{ with } \tilde{n}_0^j = \tilde{\nu}_0^j$$

and

$$\tilde{c}_t^j = \rho \tilde{n}_t^j,$$

respectively.

Goods market clearing

Since the risk-free bond is in zero net supply in the economy, the net worth of the economy is given by the total value of capital, i.e. $q_t K_t$. Plugging this into the goods market clearing condition, cf. iii) of Definition 5.1, we obtain

$$\int_{\mathbb{I}} \dot{c}_t^i di + \int_{\mathbb{J}} \tilde{c}_t^j dj = \int_{\mathbb{I}} (a - \hat{\iota}_t) k_t^i di \text{ and thus } \rho \left(\int_{\mathbb{I}} n_t^i di + \int_{\mathbb{J}} \tilde{n}_t^j dj \right) = (a - \hat{\iota}_t) K_t.$$

This implies

$$\rho q_t = (a - \hat{\iota}_t) \text{ or } \rho q_t = \left(a - \Psi\left(\frac{1}{q_t}\right) \right).$$

By the monotonicity of Ψ , recall that $\Phi'' < 0$, and since $a > 0$, the mean value theorem implies that the price $q_t \equiv q$ is uniquely determined by this equation. Furthermore, it is constant and thus $\mu_t^q = \sigma_t^q \equiv 0$. The consumption rate ρ can also be expressed as:

$$\rho = \frac{a - \hat{\iota}_t}{q} = \frac{a - \Psi(1/q)}{q}.$$

Inserting this into $\hat{\vartheta}_t$ leads to

$$\hat{\vartheta}_t = \frac{\rho + \Phi(\Psi(1/q)) - \delta + \sigma \varphi_t^{\alpha, \varepsilon} - r_t}{\varepsilon^{\alpha \sigma}}.$$

The optimal fraction of wealth that expert i invests into capital is given by (5.12) and does not depend on i in the equilibrium. We conclude, cf. (4.12), that

$$\frac{q K_t}{N_t} = \frac{\hat{\vartheta}_t}{\varepsilon^{\alpha \sigma}} \tag{5.14}$$

where $N_t = \int_{\mathbb{J}} n_t^j dj$ denotes the aggregate wealth of experts. Then

$$\eta_t := \frac{N_t}{q K_t} \in [0, 1]$$

denotes the wealth share of experts. Thus, η_t is a measure for the wealth distribution within the economy. The higher η_t , the higher is the wealth share of experts and the lower is the wealth share of households. Inserting this into (5.14), we obtain

$$\frac{1}{\eta_t} = \frac{\hat{\vartheta}_t}{\varepsilon^{\alpha \sigma}} = \frac{\rho + \Phi(\Psi(1/q)) - \delta + \sigma \varphi_t^{\alpha, \varepsilon} - r_t}{(\varepsilon^{\alpha \sigma})^2}. \tag{5.15}$$

This determines the equilibrium risk-free rate

$$r_t = \rho + \Phi(\Psi(1/q)) - \delta + \sigma\varphi_t^{\alpha,\varepsilon} - \frac{(\sigma\varepsilon^\alpha)^2}{\eta_t}. \quad (5.16)$$

Remark 5.5

Equation (5.14) clarifies that $\hat{\vartheta}_t$ is positive if and only if $\eta_t > 0$. The corresponding equation for expert $i \in \mathbb{I}$ is (5.12), i.e. an expert only invests into capital if he expects a positive excess return over the risk-free rate. If the higher risk of capital investment is not compensated with higher expected return, there will be no incentive for the experts to invest in capital. Therefore, the case of $\eta_t = 0$ is degenerative. \times

As in the original model, we can determine a law of motion of η_t given in the next lemma.

Lemma 5.6

The law of motion of η is given by

$$\frac{d\eta_t}{\eta_t} = \left(\frac{1 - \eta_t}{\eta_t} \right)^2 (\varepsilon^\alpha \sigma)^2 dt + \frac{(1 - \eta_t)}{\eta_t} \varepsilon^\alpha \sigma dW_t.$$

PROOF

Optimal aggregate wealth of experts follows the dynamics

$$\begin{aligned} dN_t &= \left[\hat{\vartheta}_t^2 + r_t - \rho \right] N_t dt + \hat{\vartheta}_t N_t dW_t \\ &= \left[\left(\frac{\varepsilon^\alpha \sigma}{\eta_t} \right)^2 + r_t - \rho \right] N_t dt + \frac{\varepsilon^\alpha \sigma}{\eta_t} N_t dW_t, \end{aligned}$$

cf. (5.8), and the aggregate value of capital qK_t follows

$$\begin{aligned} d(qK_t) &= (\Phi(\Psi(1/q)) - \delta + \sigma\varphi_t^{\alpha,\varepsilon}) qK_t dt + \sigma\varepsilon^\alpha qK_t dW_t \\ &= \left(r_t - \rho + \frac{(\varepsilon^\alpha \sigma)^2}{\eta_t} \right) qK_t dt + \sigma\varepsilon^\alpha qK_t dW_t. \end{aligned}$$

Applying Itô's Lemma leads to

$$\frac{d\eta_t}{\eta_t} = \left[\left(\frac{\varepsilon^\alpha \sigma}{\eta_t} \right)^2 + r_t - \rho - \left(r_t - \rho + \frac{(\varepsilon^\alpha \sigma)^2}{\eta_t} \right) + (\varepsilon^\alpha \sigma)^2 - \frac{(\varepsilon^\alpha \sigma)^2}{\eta_t} \right] dt + \left[\frac{\varepsilon^\alpha \sigma}{\eta_t} - \varepsilon^\alpha \sigma \right] dW_t.$$

Rearranging terms proves the statement. \square

6 Results and comparison with the benchmark model

In this section, we present the results of our considerations and highlight the differences between our approach and the one of the benchmark model. It turns out that different variables are affected by the long-range dependence in the growth rate of output in different

ways. Since our approach generalizes the original model, we focus on the effects of changes in the parameters H and ε . Recall that the original model corresponds to $H = 1/2$ and is independent of the value of ε . Values of $H < 1/2$ indicate negatively correlated shocks with short-range dependence whereas values of $H > 1/2$ indicate positively correlated shocks with long-range dependence.

We denote the drift and the volatility of η_t with $\mu_\eta^{H,\varepsilon}\eta_t$ and $\sigma_\eta^{H,\varepsilon}\eta_t$, respectively. Regarding Lemma 5.6 we write

$$\begin{aligned} d\eta_t &= \mu_\eta^{H,\varepsilon}\eta_t dt + \sigma_\eta^{H,\varepsilon}\eta_t dW_t \\ &= \frac{(1 - \eta_t)^2}{\eta_t}(\varepsilon^\alpha \sigma)^2 dt + (1 - \eta_t)\varepsilon^\alpha \sigma dW_t. \end{aligned}$$

Note that η_t does not depend on the history of shocks, i.e. η_t is independent from the trend effect, expressed by the term $\varphi_t^{\alpha,\varepsilon}$, mentioned in Section 4.1. This is because the net worth N_t as well as the value of the capital stock $q_t K_t$ depend both in the same way on the trend effect. This implies that the trend effect is canceled in the fraction of N_t and $q_t K_t$, i.e. η_t . However, η_t depends on the correlation of shocks expressed by the volatility effect ε^α , recall that $\alpha = H - 1/2$. The interest rate depends on both the volatility and the trend effect. Let η_t^{BS} and r_t^{BS} be the processes of experts' wealth share and interest rate in the model of Brunnermeier and Sannikov. They are given as

$$d\eta_t^{\text{BS}} = \frac{(1 - \eta_t^{\text{BS}})^2}{\eta_t^{\text{BS}}}\sigma^2 dt + (1 - \eta_t^{\text{BS}})\sigma dW_t$$

and

$$r_t^{\text{BS}} = \rho + \Phi(\Psi(1/q)) - \delta - \frac{\sigma^2}{\eta_t^{\text{BS}}}, \quad (6.1)$$

cf. (9) and (11) in [15]. This shows how correlations of shocks have impacts on the wealth share of experts. The dynamics of η and η^{BS} coincide for $H = 1/2$ or $\varepsilon = 1$. In the case of $H = 1/2$, both models are identical. In the case of $\varepsilon = 1$, the interest rate differs from the one in the original model since it depends on the trend effect $\varphi^{\alpha,\varepsilon}$. This underlines the fact that memory effects can influence different variables in different ways.

To illustrate the results and to obtain good comparability, we choose the same parameter values as in the original model. The results are presented in Figure 3 for $\varepsilon = 10^{-3}$. The effect of a variation of ε is shown in Figure 4. We see that the positive correlation of shocks implies a lower drift rate and a lower volatility of the wealth share of experts. This comes as experts are faced with a series of shocks with the same sign, i.e. a sequence of positive shocks or a sequence of negative shocks. This has a reducing effect on volatility as the behavior of shocks is more predictable (volatility effect, c.f. Section 4.1). Interesting is the behavior of the drift rate of η which is much lower in the case of $H > 1/2$ than in the original model. The reverse is the case if $H < 1/2$. Then the volatility is higher than in the original model which depicts the behavior of changing signs of shocks. In this case, the drift rate is higher. So on average the experts sector grows faster if the shocks on the economy are negatively correlated than if the shocks are positively correlated.

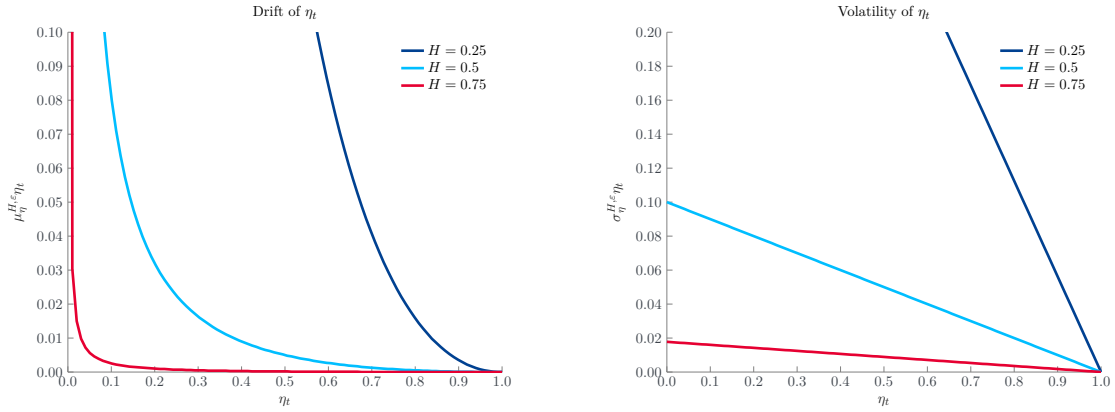


Figure 3: Drift and volatility of η as functions of η_t for different values of H with $\sigma = 0.1$ and $\varepsilon = 10^{-3}$.

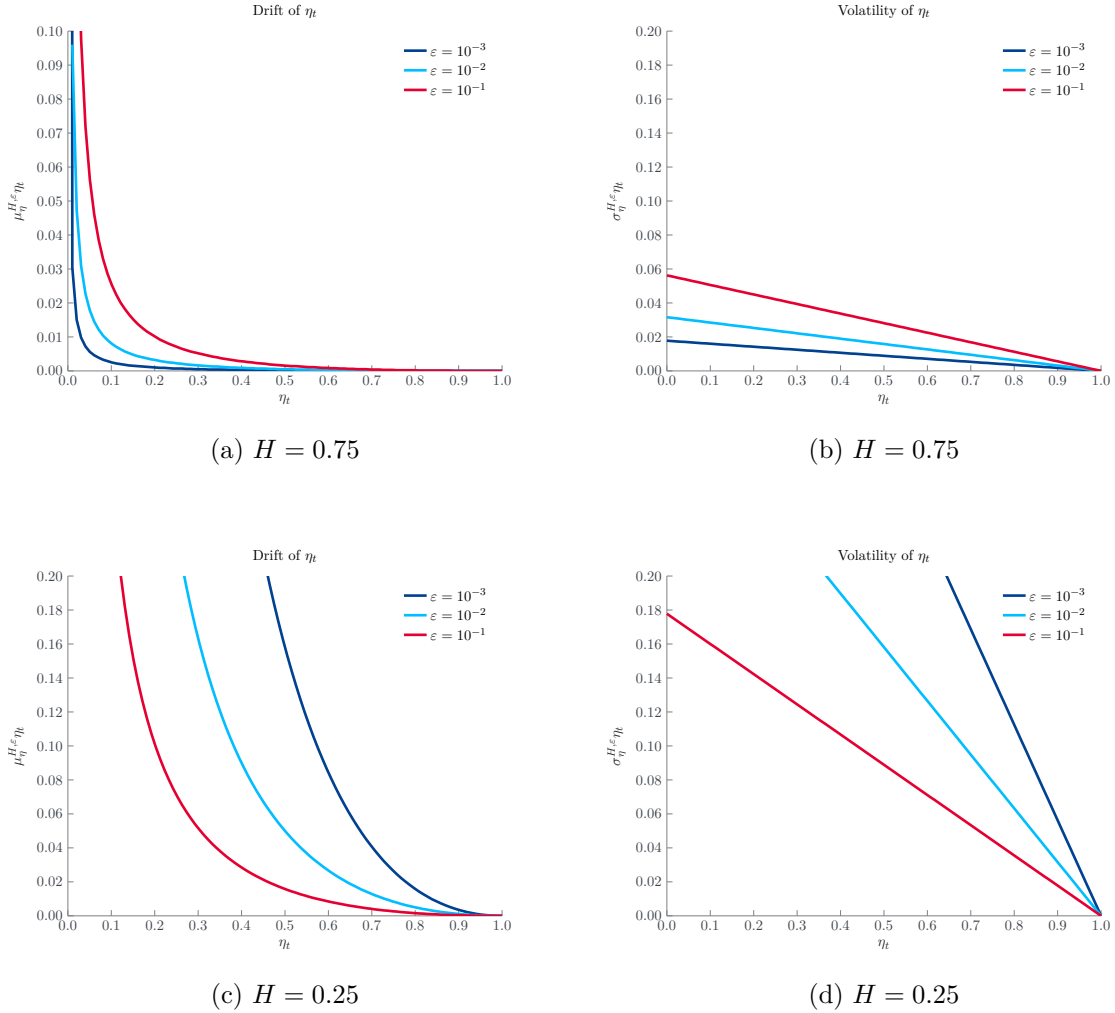


Figure 4: Drift and volatility of η as functions of η_t for different values of ε with $\sigma = 0.1$ and $H = 0.75$ in panels (a) and (b) and $H = 0.25$ in panels (c) and (d).

As in the original model, the experts' sector tends to overwhelm the households' sector since the drift rate is positive for all values of η_t , cf. Figure 3. Brunnermeier and Sannikov argue that this is due to the fact that experts have an advantage over households, see [15]. We can see from Lemma 5.6 that if the experts' wealth share reaches 1, it remains at this level. Hence, the state $\eta_t = 1$ can be regarded as a steady state value of η . It can be seen from Figure 5 how big the correlations effects are. There are different paths of η for different values of H and ε .¹⁴ The volatility effect and the reduced growth rate of η in the case of $H > 1/2$ can be seen clearly in panel (a) of Figure 5. In the case of $H = 0.25$, we see from panel (a) of Figure 5 that η reaches its steady state very quickly, so that all wealth of the economy is in the experts' sector. This is an artificial effect as we state in the following remark.

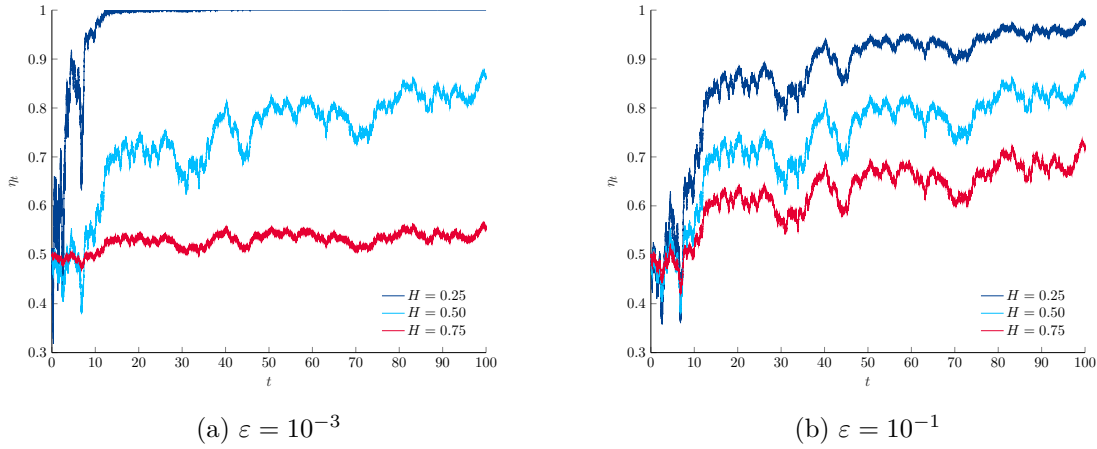


Figure 5: Sample paths of η over one hundred years under the assumption that there is one daily shock, i.e. $dt = 1/360$. The paths are plotted for different values of H with $\sigma = 0.1$, initial value $\eta_0 = 0.5$ and $\varepsilon = 10^{-3}$ in panel (a) and $\varepsilon = 10^{-1}$ in panel (b). To highlight the effects of parameter variation and therefore to exclude random effects, all six paths are generated by the same Brownian motion. The other parameters are $\iota = 0.04$, $\Phi(\iota) = \kappa^{-1} \log(\kappa\iota + 1)$ with $\kappa = 10$.

Remark 6.1

The task of visualizing an economic model is inherently connected with a proper choice of parameters. Regarding (4.6) we know that the expected value of the growth rate of the output is given as

$$\Phi(\iota) - \delta - \frac{(\varepsilon^\alpha \sigma)^2}{2}.$$

¹⁴We insinuate throughout that time is measured in years and that there is one Brownian shock each day, i.e. we choose $dt = 1/360$. With this approach we follow Di Tella [42, Appendix B] with the difference that in our setting the Brownian shocks dW_t are drawn from a centered normal distribution with variance dt instead of a binomial distribution. Measuring and interpreting time is more necessary in our context than in the original model since the weights in the definition of the process $Z^{H,\varepsilon}$ are determined by time differences and thus by the time between two consecutive shocks. The reduction to annual time measuring therefore is not necessary but simplifies the interpretation of the results.

We argue that there can occur shocks that erode the capital stock and create (temporarily) negative growth rates. However, the evolution of capital and production should not be constructed as having a general trend for negative growth rates. Therefore we should impose that the expected growth rate of output is at least non-negative. Assuming $0 < \varepsilon < 1$ leads to the condition

$$H \geq \frac{\log\left(\sigma^{-1}\sqrt{2(\Phi(\iota) - \delta)}\right)}{\log(\varepsilon)} + \frac{1}{2} \text{ or } \begin{cases} \varepsilon \leq \left(\frac{\sqrt{2(\Phi(\iota) - \delta)}}{\sigma}\right)^{1/(H-1/2)} & \text{if } H > 1/2 \\ \varepsilon \geq \left(\frac{\sqrt{2(\Phi(\iota) - \delta)}}{\sigma}\right)^{1/(H-1/2)} & \text{if } H < 1/2 \end{cases}. \quad (6.2)$$

Given the parameter values of Figure 5, we obtain that $H \geq 0.36$ in the case of $\varepsilon = 10^{-3}$ and $H \geq 0.09$ in the case of $\varepsilon = 10^{-1}$, i.e. condition (6.2) is not satisfied in panel (a) of Figure 5 for the cases of $H = 0.25$. This confirms the very high growth rate of η as shown in panel (c) of Figure 4 and in panel (a) of Figure 5: On the one hand experts earn higher risk premia if $H < 1/2$, i.e. their net worth increases. On the other hand the capital stock tends to erode through time since condition (6.2) above is not satisfied. Overall this leads to the fast increase of experts' wealth share. However, at the end of this process we have a situation where experts own everything but everything is nothing since the (expected) negative growth rate erodes the capital stock completely. In panel (b) of Figure 5, where all values of H fulfill condition (6.2), we can see a reduction in the growth rate in the case of $H = 0.25$ compared to the one of panel (a) of Figure 5 since in this case the capital stock grows in expectation. Given the parameter values of Figure 5, in the empirical more relevant case of $H > 1/2$ we obtain an upper bound for ε of approximately 6.73.¹⁵ In the case of $H > 1/2$, the choice of the parameter ε between 0 and 6.73 seems to be arbitrary. However, the shock generating process $Z^{H,\varepsilon}$ was chosen as an approximation of the process Z^H , cf. Section 3. From this point of view it is plausible to choose small values for ε , i.e. $\varepsilon \ll 1$. \times

To provide a better intuition for these effects it is helpful to consider the long-range dependence effects on the Sharpe ratio $\hat{\vartheta}_t$, i.e. the relation of risk premium that experts receive for holding capital and the risk that they have to take in order to hold capital. Equation (5.15) implies that

$$\hat{\vartheta}_t = \frac{\varepsilon^\alpha \sigma}{\eta_t} = \frac{\rho + \Phi(\Psi(1/q)) - \delta + \sigma \varphi_t^{\alpha,\varepsilon} - r_t}{(\varepsilon^\alpha \sigma)}.$$

Although the Sharpe ratio depends on the volatility and the trend effect, higher values of H reduce the Sharpe ratio for every wealth share of experts η_t , if we assume that $\varepsilon < 1$. So if the wealth share of experts is sufficiently high, there is only a small incentive for experts to invest in capital and Proposition 5.2 shows that experts indeed invest less into capital if $\hat{\vartheta}_t$ is low. This is due to the volatility effect: experts face a smaller risk of the actual shock and earn smaller risk premia than they would do in the original model. Thus, experts are paid for facing the risk of actual shocks and not for the (possibly negative) outcomes from previous shocks. As can be seen from the equation above or from (5.16), this can only happen if the

¹⁵This upper bound is calculated for $H = 1$ and holds for all values of $1/2 < H < 1$ by using monotonicity properties of (6.2) for the given standard parameters.

interest rate absorbs the trend effect expressed by $\varphi_t^{\alpha, \varepsilon}$. In our setting this implies that the interest rate depends strongly on the history of previous shocks. This is founded in the set up of this simple model: since experts have to hold all capital in the economy, the price of capital is constant (capital is not traded). So the only possible way for experts to react to long-range dependence effects is via their credit demand towards households which reflects the behavior of the interest rate r_t . This effect on the interest rate is the main difference between the two models. Comparing (5.16) and (6.1) we see that through the volatility effect the wealth distribution, expressed by η_t , has a smaller impact on the interest rate than in the original model. This can also be seen from panels (b) and (d) of Figure 6.

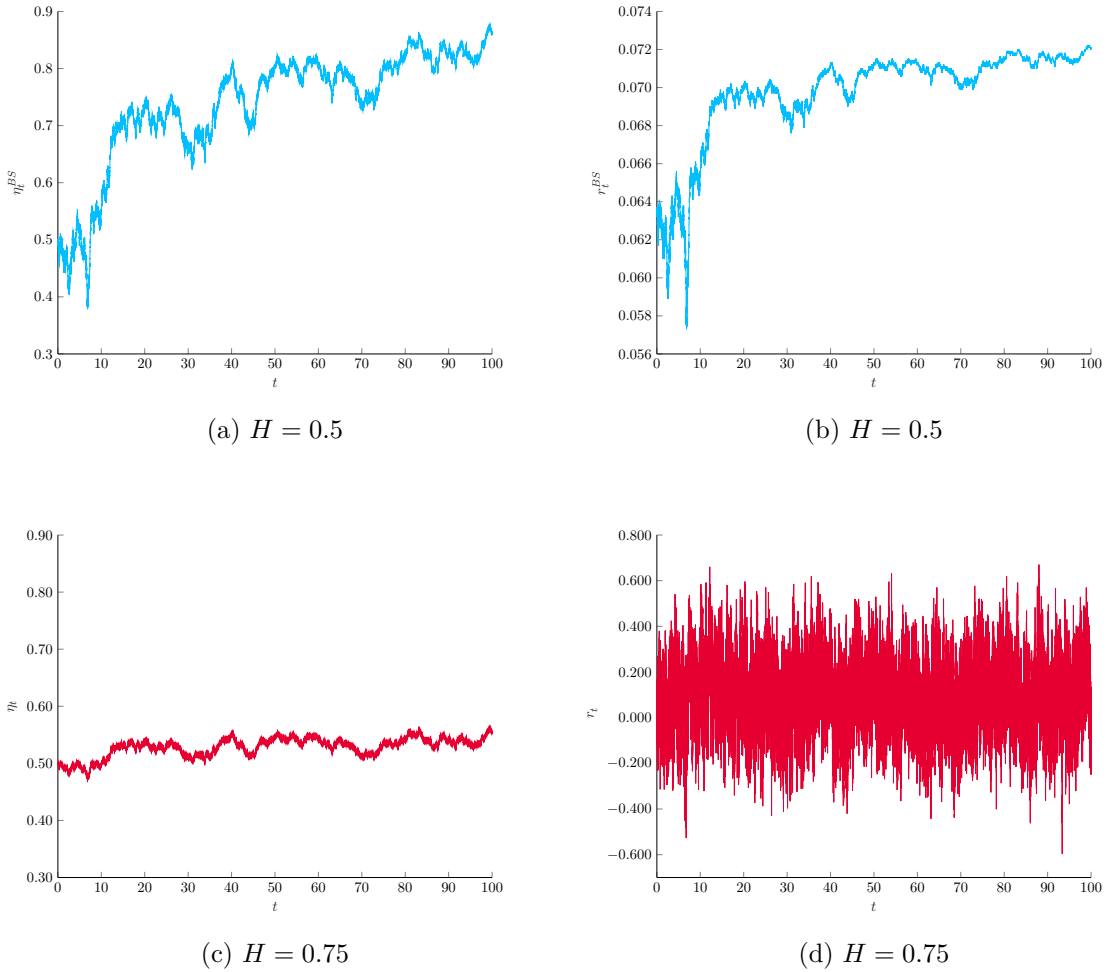


Figure 6: Sample paths of η and the interest rate over one hundred years under the assumption that there is one daily shock, i.e. $dt = 1/360$. The paths are plotted for $H = 0.5$ (panels (a) and (b)) and $H = 0.75$ (panels(c) and (d)) with $\varepsilon = 10^{-3}$ and initial value $\eta_0 = 0.5$. All four paths are generated by the same Brownian motion. The other parameters are $\sigma = 0.1$, $\iota = 0.04$, $\Phi(\iota) = \kappa^{-1} \log(\kappa \iota + 1)$ with $\kappa = 10$. Condition (6.2) is satisfied in all cases.

In the original model (panels (a) and (b) of Figure 6) the interest rate is mainly determined by the behavior of η . And as a consequence, if η reaches its steady state value $\eta_t = 1$, the interest

rate also reaches its steady state value $r_t^{\text{BS}} = \rho + \Phi(\Psi(1/q)) - \delta - \sigma^2$, cf. (6.1). In the case of long-range dependence (panels (c) and (d) of Figure 6) the interest rate fluctuates strongly around the path generated by the component $-(\sigma\varepsilon^\alpha)^2/\eta_t$. This is due to the additional term $\varphi^{\alpha,\varepsilon}$. As can be seen from panel (d) of Figure 6, this leads to an unrealistic strong oscillation of the interest rate. This is founded in the structure of $\varphi_t^{\alpha,\varepsilon}$:

$$\varphi_t^{\alpha,\varepsilon} = \int_0^t \alpha(t-s+\varepsilon)^{\alpha-1} dW_s \quad \text{with} \quad \alpha(t-s+\varepsilon)^{\alpha-1} = \left. \frac{dK(x)}{dx} \right|_{x=t-s}$$

where $K(x) = (x+\varepsilon)^\alpha = (x+\varepsilon)^{H-1/2}$ is the weighting kernel from Section 3 or Figure 2, respectively. Since

$$\lim_{\varepsilon \rightarrow 0} \left. \frac{dK(x)}{dx} \right|_{x=0} = \infty$$

for all $H > 1/2$, we can conclude that $\varphi^{\alpha,\varepsilon}$ is strongly affected by actual Brownian shocks which enters with a weight of $\alpha\varepsilon^{\alpha-1}$ in the equilibrium interest rate. This causes the high oscillation of the interest rate. So from this perspective it would be reasonable to choose higher values of ε in order to smooth the path of the interest rate. However, this contradicts our approximating argument from above which leads us to choose small values of ε . Anyway, it is still possible if we choose the values of ε such they are in the boundaries stated in Remark 6.1. Although, the parameter choice of ε is difficult, there is one special case that is worthy to note, namely the case of $\varepsilon = 1$, in which the volatility effect disappears and the dynamics of η^{BS} and η coincide. The corresponding paths of r^{BS} and r are shown in Figure 7. The path of η is the same as in panel (a) of Figure 6. The differences can be seen clearly: while there is a one to one relation between the interest rate and the wealth share of experts in the model of Brunnermeier and Sannikov, there is no such relation in our case. The trend effect is the main source which dominates the evolution of the interest rate.

Summarizing, we have shown that the evolution of various variables in this model is strongly dependent on the stochastic properties of shocks. Allowing for long-memory effects in the growth rates of output leads to an interest rate which itself depends on the history of shocks. Thus a main result of Brunnermeier and Sannikov, the one to one mapping between the wealth distribution and the interest rate, does not hold any longer. We have shown that in the case of $\varepsilon = 1$ the dynamics of the wealth distributions coincide in both models but the interest rates are completely different in their evolution in time. So this raises the question about the determinants of interest rates. Cajueiro and Tabak [43] found empirical evidence for long-range dependence in LIBOR interest rates for various maturities and currencies. Furthermore, Comte and Renault showed in [44, Section 2.3] that the process $\varphi^{\alpha,\varepsilon}$ indeed exhibits the long-range dependence property. Comte and Renault consider general continuous time fractionally integrated processes of order $d = \alpha$ and their long memory properties. It can easily be proven that the process $\varphi^{\alpha,\varepsilon}$ is such a process. So on the one hand we have shown that one possible source of long memory in the interest rate can be explained via long memory in the growth rates of output, in the sense that the interest rate inherits this property from capital or output. On the other hand, we have shown that this long memory component is the main determinant of the interest rate which seems reasonable under an empirical point of view. Furthermore for $\varepsilon \neq 1$, we have shown that including long- or short-range dependence effects into the simple real economy model of Brunnermeier and Sannikov does not change the steady

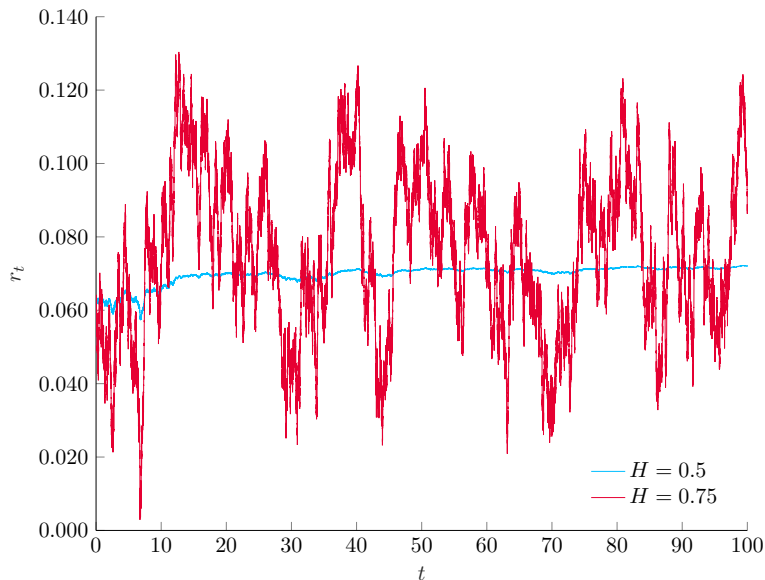


Figure 7: One sample path of r^{BS} (blue line) and one path of r with $H = 0.75$ and $\varepsilon = 1$ (red line) over one hundred years under the assumption that there is one daily shock, i.e. $dt = 1/360$. Both paths are generated by the same Brownian motion. The other parameters are $\sigma = 0.1$, $\iota = 0.04$, $\Phi(\iota) = \kappa^{-1} \log(\kappa\iota + 1)$ with $\kappa = 10$ and $\eta_0 = 0.5$. Condition (6.2) is satisfied in both cases.

state of the dynamics of η . However, there are several effects on the growth rate towards this steady state value, as we have outlined above. This coincides with the results of Tarasov and Tarasova in [9] as the growth rate of output is mainly affected by memory effects in their deterministic continuous time setting.

7 Conclusion

We have introduced long-range dependence and correlation effects via two additional parameters H and ε in the original model of Brunnermeier and Sannikov[15, Section 2] to capture empirical evidence of long-range dependence in the growth rates of output. In order to obtain those effects we have changed the shock generating process from a Brownian motion into an approximated Liouville fractional Brownian motion which is defined as a weighted integral over Brownian shocks. The fact, that shocks generated by this process can be decomposed in a trend effect $\varphi^{\alpha, \varepsilon}$ and a volatility effect ε^α , have allowed us to define stochastic differential equations respect to this process. We then have followed the steps proposed by Brunnermeier and Sannikov and have derived the dynamics of experts' wealth share with classical finance methods. The main insights of this model are that the wealth distribution between the two groups of agents is affected by long-range dependence effects although the evolution of the wealth distribution itself is independent from the history of shocks. Positively correlated shocks decrease the expected growth rate of the experts' sector and reduce the volatility due

to its effect on the risk premium. This results in a more stable economy since experts have weaker incentives to invest in capital. The results are exactly the opposite in the case of negatively correlated shocks.

The impacts on the interest rate are more complex. The equilibrium interest rate depends, due to the trend effect, on the whole history of macro shocks. This implies that the interest rate cannot be described only through the value of the state variable η . Or, to be more precise, this underlines that the one to one mapping between wealth distribution and interest rate stated by Brunnermeier and Sannikov is strongly dependent on the stochastic properties of the considered economic shocks. In our setting it is evident that the interest rate is mainly determined by the trend effect which exhibits the long-range dependence property, too. Thus, we have shown a relation between long-range dependence in growth rates of output and long-range dependence in interest rates. We have also seen that the long memory effect depends mainly on the choice of the weighting kernel. We have followed an approach which was discussed in the finance literature, however, we have seen that this leads to difficulties in the interpretation of the parameter ε . So it could be interesting to follow more general approaches like the one of Comte and Renault [44] which could provide a better intuition and estimation scheme due to their similarity to classical ARFIMA models. Of course, the considered model is not rich enough to explain these long-range dependence and memory effects in a wider economic context and they are assumed to be exogenously given in our setting. However, this exemplary consideration shows that it is important to account for such effects. With view of more complex models, in which capital is traded and thus additional price effects occur in the economy, it would be interesting to investigate the interaction of memory effects of different orders in a hopefully derivable equilibrium. This would open up the possibility to consider long-range dependence in inflation rates, which has already been empirically investigated, see e.g. [45], and to deduce their implications for other economic variables and processes. Nevertheless, we have shown that one task for future work is not only to account for stochastic shocks in an economy but also to find reliable ways for stochastic modeling in an economic context.

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