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Income polarization and stagnation in a stochastic model of growth - when the demand side matters

Thomas Gries*

August 26, 2019

Abstract

This contribution is motivated by two stylized observations, the slow-down in growth, and a simultaneous income polarization in many advanced economies. While mainstream neoclassical and endogenous growth theory cannot sufficiently explain the nexus, we argue that the demand side plays a role. We suggest a hybrid model that is based on a standard product-variety model with intermediate goods, and which is frequently used in endogenous growth theory. Even if there are stochastic frictions, prices are generally flexible. As we want to look at polarization effects, we model two income groups, labor and financial wealth owners. Further, we obtain a sustainable steady-state equilibrium by suggesting an unconventional equilibrium concept in a stochastic environment. We define equilibrium as a stationary no-expectation-error equilibrium. This equilibrium concept relates to the Nash idea of individual stationary behavior as long as all expected values are the effectively realized values. Even if potential growth is generated only by the supply side, endogenous effective demand restricts level and growth rate of the income path. The growth rate is semi-endogenous and determined by entrepreneurs' success of market entry of new technologies. Our hybrid model bridges a gap between Keynesian and neoclassical (and endogenous) ideas of economic growth.

JEL classifications: D33, O40, E10

Keywords: product-variety model, demand-constrained growth, semi-endogenous growth

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Abstract

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1 Introduction

Even if this contribution intends to rather generally bridge the gap between mainstream supply-side driven growth theory and a thinking that suggests that the demand side matters (even in long-run growth), this contribution is motivated by two stylized observations. First, the slowdown in growth, and second the simultaneously observed income polarization in many advanced economies.¹ While both phenomena seem to coincide it is not clear if they are connected and what the mechanism might be. A potentially important channel linking both phenomena may go through the demand side. However, yet we do not know, and we should not rule out any channel in advance. Therefore, we cannot purely rely on standard supply side driven growth theory, and thus, we take this occasion to suggest a general hybrid model of growth in which the demand side plays an important role. Further, we apply this model to a discussion of the income polarization-growth nexus and obtain unconventional results.

In today's mainstream growth theory, there is no role for the demand side. Innovations and/or factors that can be accumulated determine potential production, and customers absorb whatever is potentially available in the economy. Economists often refer to this mechanism as Say's Law. As there is never a lack of demand, production capacity defines the limits of consumption, income, savings, and growth. With the automatic existence of sufficient demand for consumable goods, the choice to save transforms to investments, and investments are spent on the factors that generate capacity and production growth. In this approach, the greatest economic problem is the intertemporal decision of how much we should save of today's income to be invested in tomorrow's consumption. The answer to this question is given by the Cass-Koopmans-Ramsey² rule for the aggregate economy which closes the basic story of (constant return to scale) *neoclassical growth* mechanics. As long as technology growth is exogenous, savings restrict investments and hence the level of the growth path.

This story still holds in the debate around *endogenous growth theory*, which is relevant for this paper. Endogenous growth theory started with the models by Romer (1986, 1987, 1990) or Lucas (1988) and has since become the dominant growth approach. The most prominent attribute of endogenous growth theory is the ability to generate a sustainable constant growth rate via various mechanics (often scale economies) and to select this sustainable growth rate as an optimal intertemporal choice. As excellent reviews and descriptions of these mechanisms are given in Jones (1999), Aghion & Durlauf (2005), or Aghion & Howitt (1998, 2009), there is no need to repeat their comprehensive detailed discussion in this paper. A major implication of these models is that a society only needs to choose its desired rate of growth according to a Cass-Koopmans-

¹With "long-term slow down in growth" we mean the discussion related to Gordon (2012, 2015), Summers (2015), Plosser (2014), or Fritz et al. (2019). Examples for the identification of income polarization include contributions like Autor et al. (2003, 2006, 2008), Autor & Dorn (2013), Spitz-Oener (2006), Goos et al. (2014), or Van Reenen (2011).

²See Cass (1965) and Koopmans (1965).

Ramsey type of aggregate intertemporal choice. However, semi-endogenous approaches have appeared in the course of this debate, expressing concern about the specific mechanics needed to obtain scale economies in order to maintain a fully endogenous growth mechanism. This has produced a further strand of literature which attempts to modify the models and eliminate this rather specific outcome. These *semi-endogenous growth* models suggested a combination of exogenous and endogenous elements that determine the growth rate of an economy. Among others, Jones (1995), Kortum (1997), or Segerstrom (1998) develop models in which per-capita growth is directly connected to a given population growth. Research-promoting measures affect the level of the income growth path, but they cannot permanently and sustainably determine the long-term growth rate at a desired level. However, for the purpose of this paper we can state that in this strand of the literature, the major mechanisms are again driven by elements of the supply-side only. The demand side does not play a role.

How can we be so sure that only the supply side determines growth? Why is it difficult to think of another mechanism that allows the demand side to play a role? In the recent debate on the Great Recession and Secular Stagnation, we find some arguments that go beyond a discussion of the supply side. E.g., Summers (2014) has concerns about a systematic savings-investment mismatch that may be a reason for secular stagnation. Gordon (2012, 2015) emphasizes six headwinds caused by an aging population, education, increasing inequality, and an increasing ratio of federal government debt to GDP, all of which impede growth. While most of these headwinds clearly address the supply side, increasing inequality may also relate to effects on the demand side, as we will show in section 3. Further, Chamley (2013) describes an adjustment process in a general equilibrium model by which aggregate supply creates the income that generates corresponding demand, which is exactly what Say's Law suggests. However, this mechanism may not work in a general equilibrium with decentralized markets and savings in bonds or money, as convergence to that state is slow.

Despite this clear focus of mainstream growth theory on the supply side, there is indeed a strand of literature in the Keynesian tradition that focuses on the demand side. This literature started with the pioneering work of Harrod (1939) and Domar (1946) and continues up to the more recent work of, e.g., Dutt (2006) or Palley (1996, 1997, 2014). Unlike the neoclassical and endogenous growth approach and the large amount of variation therein, this literature is characterized by a number of rather heterogeneous models which do not have a kind of commonly accepted and consistent frame of modelling. This heterogeneity – together with the fact that there are almost no empirical studies focusing on elements of the demand side as growth determining factors – is likely to be the reason why the demand side is not part of the mainstream discussion.

Due to the appealing consistency of neoclassical and endogenous growth theory, and at the same time due to the general doubt that the demand side has no role to play in a long-term growth process, we suggest a hybrid model covering

both elements – a modified (semi-)endogenous growth model. In this model we depart from Romer’s (1987, 1990) product-variety model and modify it in six ways. (i) Firms are in an uncertain environment with imperfect information and perceived stochastic frictions in final goods markets. (ii) We introduce an alternative stationary equilibrium concept which we refer to as the no-expectation-error equilibrium (n-e-ee). With this concept we define equilibrium as rational stationary behavior (similar to Nash) which is possible at any level of income or production. (iii) Even if we allow for flexible prices and wages, an endogenous demand side may restrict GDP and GDP growth. (iv) We move away from a representative household and introduce two separate income groups, labor and owners of financial assets. The two groups have different intertemporal choices which allow for group-specific consumption and savings pattern. Further, market entry conditions, specifically the match between potential new technologies and market opportunities determine investments. Thus consumption and endogenous investments define an endogenous demand side. (v) Further, growth of new implemented technologies is driven by the market entry of these technologies. Market entry is determined not only by the given growth of ideas, but also by a match between endogenous market opportunities and new products offered, and thus, by demand-side elements. (vi) Stochastic market frictions perceived by firms can be countered by shifting resources to a more successful sales process. This extra effort can close the gap between expected (demand-side related) sales in the market and effective supply. With these modifications of an otherwise mainstream model, we obtain rather different economic mechanisms with very different policy implications for the resulting growth process.

Unlike the economic mechanism in mainstream growth approaches, the narrative of this approach starts on the demand side. The demand side exhibits limited expenditure of the two demand-side elements of consumption and investments. Consumption is restricted by consumer income which, broadly speaking, is labor income, while investment expenditure is restricted by investment opportunities and potential technology growth. Even if consumers would potentially like to consume more, their labor income limits consumption expenditure. In the standard world of the representative consumer, this is impossible. However, more realistically (in particular in a polarized economy) we assume different households with different consumption and savings behavior. Thus, if we introduce different Euler equations for each group of households, we can derive different intertemporal decisions and different consumption and savings behavior. With some simple assumptions and taking long-term limits, the consumption rate of labor households as well as the savings rate of wealth holders turn towards one; and consumption becomes restricted by labor income. Further, investment expenditure is driven by new firms and their intention to launch new technologies on the market. Thus, market opportunities and innovative potentials determine the match of such new products and the required investment expenditure. There are no automatic investments just because sufficient savings are available (Say’s law is not applied), neither in the short nor in the long run. As a result, total demand can become a constraining factor. Further, firms observe that the market process is not perfect. They have a perception

about market frictions related to their individual sales. Consequently, firms observe that effective sales may currently be less than their supply to the market. Unlike the standard model, they can use a fraction of their potential output to counter the effects of these market frictions and experience that this strategy improves their sales. As a result, the excess supply of each firm declines until expected sales equal effective supply. Thus, in this equilibrium firms expect what they observe, and no need for further adjustments is required. This market equilibrium is also a stationary equilibrium, even if it is established below potential GDP. Stationary equilibria can be established at all levels below potential output. Therefore, a demand restricted growth equilibrium refers to stationary equilibria in which an endogenous demand does not support a supply of final goods at the potential level of GDP. There is a gap between potential and effective GDP.

2 The model

As general frame for the model we suggest a product-variety approach and hence depart from the models introduced by Romer (1987, 1990). Except for the introduction of stochastic friction in equations (2) and (3), the largest part of the model described by (1) and (4) to (11) is close to the textbook types of product variety model as described e.g. by Aghion & Howitt (2009, ch. 3). Thus, we just give a brief sketch of this model part.

Final-goods-producing firms

Production of final goods: A representative firm $i \in \mathcal{F}$ produces with labor L_i and the number $N_i(t)$ of differentiated intermediate inputs $x_{ji}(t)$ offered by $N(t)$ small firms. Total potential production of firm i 's good $Q_i^p(t)$ is

$$Q_i^p(t) = L_i^{1-\alpha} \sum_{j=1}^{N_i} x_{ji}^\alpha(t) = N_i(t) L_i^{1-\alpha} x_i^\alpha(t). \quad (1)$$

with $x_i(t)$ being a representative variation of the intermediate goods $x_{ji}(t)$. Total potential production is also the potential supply of final goods to the final goods market.

Market frictions, countermeasures and expected sales: However, firms are in a stochastic environment with random shocks, imperfect information, and frictions. Not all production directly and instantaneously finds a customer. When they offer to the market, they observe that only the share $\Phi_i(t)$ of potential production is instantly sold. Thus, $\Phi_i(t)$ is firm i 's *effective sales ratio*. Firms' interpretation of $\Phi_i \leq 1$ is that these deviations are generated randomly by a stochastic market friction $\delta_i(t)$, and that they can invest a fraction of their potential output θ_i to improve the chance of placing the remaining goods in the market. In general, then, the share of potential production that

a firm i expects to sell in the market is a function of the expected stochastic friction $E[\delta_i]$ and the countermeasure θ_i ,

$$E[\Phi_i] = E[\Phi_i(E[\delta_i], \theta_i)].$$

Further, at this point and from the perspective of the single firm, it is sufficient to assume that the function $E[\Phi_i(\theta_i, E[\delta_i])]$ is monotonic increasing in θ_i and decreasing in $E[\delta_i]$.

$$\frac{\partial E[\Phi_i(t)]}{\partial E[\delta_i]} < 0, \quad \frac{\partial E[\Phi_i(t)]}{\partial \theta_i(t)} > 0. \quad (2)$$

Thus, effective production and supply $Q_i(t)$ of final goods in final goods markets is less than potential supply, and $1 - \theta_i(t)$ is the *effective supply ratio*

$$Q_i(t) = L_i^{1-\alpha} N_i(t) x_i^\alpha(t) (1 - \theta_i(t)). \quad (3)$$

Note, in section 3 we find an equilibrium value for $\theta_i(t)$ for each firm. Further, firms use this fraction of potential output for market placement, and to reduce sales problems which are caused by frictions according to their individual perceptions and expectations about these frictions. Thus, at the micro level, firms have opportunity costs θ_i due to exogenous subjectively expected frictions. In section 3 we show that opportunity costs (a loss of final good output) are the result of a reallocation of labor from production to sales promoting activities. In section 3 we also show how individual expectations and the effort devoted to the sales and placement process relates to the aggregate economy.

Demand for labor and wages: Real wages $w(t)$ are determined according to the marginal productivity of labor. Wages and intermediate goods prices are expressed in units of the final output good. With $p_x(t)$ denoting the prices of firm i 's representative intermediate inputs $x_i(t)$, the final goods-producing firm's profit is $\Pi_{Q_i} = N_i L_i^{1-\alpha} x_i^\alpha (1 - \theta_i) - N_i p_x x_i - w L_i$. From the first-order condition we can derive wages

$$w(t) = (1 - \alpha) \frac{Q_i(t)}{L_i}. \quad (4)$$

Demand for intermediate goods: Demand for intermediate goods is also determined by marginal productivity. Prices are set by innovative firms as they offer a specific monopoly-like unique good. Using the first-order conditions, we can derive the demand for each intermediate input,³ namely

$$x_i(t) = \left(\frac{\alpha(1 - \theta_i(t))}{p_x(t)} \right)^{\frac{1}{1-\alpha}} L_i. \quad (5)$$

³F.O.C is $\frac{d\Pi_i}{dx_i} = \alpha(1 - \theta_i) x_i^{\alpha-1} L_i^{1-\alpha} - p_x = 0$, thus $p_x = \alpha(1 - \theta_i) L_i^{1-\alpha} x_i^{-(1-\alpha)} \Leftrightarrow x_i^{1-\alpha} = \alpha \frac{(1-\theta_i)}{p_x} L_i^{1-\alpha}$.

Intermediate-goods-producing firms

Market entry of new monopolistic firms: The intermediate good firm is a monopoly because it sells a unique innovative product created by an innovation. c_x is the cost to produce one unit of x_i in units of the final output. Profits of an intermediate goods firm are $\pi_x = (p_x - c_x)x$. Being a monopoly, intermediate goods firms consider the demand function (5), and plugging in $p_x = \alpha(1 - \theta)L^{1-\alpha}x^{-(1-\alpha)}$ we arrive at

$$\pi_x(t) = \alpha(1 - \theta(t))L^{1-\alpha}x(t)^{-(1-\alpha)}x(t) - c_x x(t). \quad (6)$$

From the first-order conditions of the profit-maximizing intermediate goods firms, and using (5) we obtain the optimal price policy and determine⁴ the production of $x(t)$,

$$p_x = \frac{c_x}{\alpha}. \quad (7)$$

$$x(t) = (1 - \theta(t))^{\frac{1}{1-\alpha}} \left(\frac{\alpha^2}{c_x} \right)^{\frac{1}{1-\alpha}} L, \quad (8)$$

With (8) and (7), we eventually arrive at the maximum profit $\pi_x(t)$,

$$\pi_x(t) = \left(\frac{1}{\alpha} - 1 \right) (c_x)^{\frac{-\alpha}{1-\alpha}} \alpha^{\frac{2}{1-\alpha}} (1 - \theta(t))^{\frac{1}{1-\alpha}} L \quad (9)$$

As (7) gives us the flow of profits, and the intermediate good firm has an infinite lifetime, the present value of this profit flow - discounted with the interest rate $r(t)$ - is

$$V_x(t) = \frac{1}{r(t)} \pi_x(t) = \int_t^{\infty} \pi_x(t) e^{-r(v,t)(v-t)} dv. \quad (10)$$

Up to this point – except for the introduction of real output costs $(1 - \theta)$ related to stochastic frictions – this model is close to the standard textbook versions of variety models as presented, e.g., by Aghion & Howitt (2009, ch. 3). Now we turn to the second major economic difference in this model. While $\frac{1}{r}\pi_x$ is the present value of profits per innovation, $\frac{1}{r}\pi_x \dot{N}$ is the total profit of the startup when introducing $\dot{N}(t)$ new goods. Launching new goods on the market needs investment. Therefore, we assume that an investment ν is needed for the market entry of each new good. Thus, the total entry costs of the startup with innovation rate \dot{N} is $\dot{N}\nu$, and these entry costs determine investments $I_x = \dot{N}\nu$. Due to competition, the net rent of a new firm turns to zero and the net present value of a new firm just about covers the total startup costs

$$\frac{1}{r(t)} \pi_x(t) \dot{N}(t) - I_x(t) = 0.$$

⁴F.O.C. is $\frac{\partial \pi_x}{\partial x} = \alpha^2(1 - \theta_i)L^{1-\alpha}x^{\alpha-1} - c_x = 0$, thus $c_x = \alpha^2(1 - \theta_i)L^{1-\alpha}x^{\alpha-1} \Leftrightarrow x^{1-\alpha} = (c_x)^{-1}\alpha^2(1 - \theta_i)L^{1-\alpha}$.

Further, plugging in $\dot{N}\nu = I_x$ we arrive at the well-known result that the return on an investment with infinite lifetime is profit flow over investment costs⁵

$$r(t) = \frac{\pi_x(t)}{\nu}. \quad (11)$$

Market entry of new technologies and goods: In this model, we do not intend to endogenize original technical change. There is a long-standing debate about this issue, the latest broad empirical finding in this respect is Bloom et al. (2017). Instead, we want to focus on market entry of new technologies. $\dot{A}(t) = \frac{dA(t)}{dt}$ is the exogenous number of new technologies invented at t , which we refer to as innovative products. These innovative products are launched on the market by entrepreneurs and startups; however, they are not automatically successful in the market. According to the literature on startups and entrepreneurship, in general between 19 and 22 percent of firms exit the market in their first year of existence and thus fail to bring an idea to the market. In this approach, we model the process of market entry as an *aggregate matching* process.⁶ New, innovative goods $\dot{A}(t)$ are offered on the market and startup firms try to find buyers for their innovative products. The final goods sector provides opportunities for successful startups. Specifically, we assume that the number of new products successfully entering the market \dot{N} is a function of two elements, (i) the exogenous number of new, innovative products $\dot{A}(t)$ potentially ready for market entry, and (ii) the opportunities for market entry that entrepreneurs discover. These opportunities are determined by the capacity of a market to absorb new intermediate products. This absorption capacity for intermediate goods is indicated by the total effective demand of intermediate goods in the economy $X^{eD}(t) = N(t)x(t)$. A large and booming economy with large effective demand offers more opportunities to the sometimes referred to as "opportunity entrepreneurs" to launch new products and technologies.⁷ These two elements are combined in the aggregate matching function, $\dot{N} = f(\dot{A}, X^{eD})$. The number of surviving products are the new technologies that match the requirements of the market. They add to the implemented products or technologies of the economy. For simplicity, we assume a matching technology with constant economies to scale and obtain for the number of new products in

⁵Note: formally $\dot{N}\nu = I_x$ looks rather similar to the expression $\dot{N} = \frac{1}{\nu}I_x$ in the Romer-like reference model. However, the economic interpretation is reverse. Here, the number of new technologies in the economy is (semi-)exogenous and determined in a market matching process which we describe in the next section. Thus, causality runs from new goods entering the market, and this process leads to respective start-up investments ν . That is, when entering the market, the "market entry process leads to investments ($\dot{N}\nu \rightarrow I_x$)". In the Romer model, the interpretation of this relation is "investments generate innovation ($\dot{N} \leftarrow \frac{1}{\nu}I_x$)" and " $\frac{1}{\nu}$ " is the linear spending efficiency of the R&D sector. However, both interpretations lead to the same result for returns on investment of the innovative firm, as given in (11).

⁶For a micro-foundation of this process, see Gries & Naude (2011) and Gries et al. (2016).

⁷In a recent working paper, Fairlie & Fossen (2018) summarize these ideas and contrast the activities of "opportunity entrepreneurs" with those of "necessity entrepreneurs" who are pushed into the market during economic downturns as a means of survival.

the market

$$\dot{N}(t) = (X^{eD}(t))^\gamma (\dot{A}(t))^{1-\gamma}, \quad (12)$$

where γ describes the contribution of market opportunities to this process. Although this assumption of a macro matching process is rather basic and simplifying, it describes the main idea behind the mechanism.

Equation (12) makes clear that implementing technical change is a semi-endogenous process. The plain number of new technologies available A is exogenous given and not explained, but the number of these new technologies that are indeed implemented in the economy N is endogenous. Thus, we refer to this as a semi-endogenous process of technical progress implemented in the economy.

Aggregate production, income and budget constraints

Production and income constraints: Effective output of the representative firm and of the total economy is generated by the two factors of production, thus the budget constraint for total production is

$$Q(t) = N(t)p_x(t)x(t) + w(t)L. \quad (13)$$

However, effective output is not GDP or income. As x is produced by using c_x units of final goods, net final output, and thus *GDP* or *income* is

$$Y(t) = Q(t) - N(t)x(t)c_x. \quad (14)$$

Further, from (13) and (14) we obtain $Q - Nxc_x = Np_x x - Nxc_x + wL_i$. With the definition of profits in the intermediate goods sector (6), the income constraint turns into

$$Y(t) = N(t)\pi_x(t) + w(t)L. \quad (15)$$

This constraint is already very familiar as it states that total income consists of profits and labor income. Further, equation (11) states that the return on an investment with an infinite lifetime is profit over investment costs $r = \frac{\pi_x}{\nu}$, and thus, $Y = rN\nu + wL$. Further, $N\nu$ is the value of all debt ever issued as all new products are always financed by newly issued financial assets, $\dot{N}(t)\nu = \dot{F}(t)$. Thus, with $N(t)\nu = F(t)$ we see that all profits are channeled to financial investors. Financial investors have financed the process and now obtain a return on these investments

$$N(t)\pi_x(t) = r(t)F(t). \quad (16)$$

Using (16), we obtain the familiar income decomposition of the GDP

$$Y(t) = r(t)F(t) + w(t)L. \quad (17)$$

Income generated by innovative intermediate firms eventually generates a return for the financial asset owners. The growth process is implicitly a process of financial wealth accumulation through financing new products.

Income share of labor: As this approach explicitly refers to distributional aspects, we derive the income shares of the two groups, labor income and income of financial investors. In appendix A.1 we show that labor share of income can be derived by using (4) and (14) and is

$$\frac{w(t)L}{Y(t)} = \frac{1}{1+\alpha} < 1. \quad (18)$$

Income share of financial wealth: Profit share can also be explicitly determined using (14), (16) and (6). In the appendix A.1 we show how to arrive at

$$\frac{N_i(t)\pi_x(t)}{Y(t)} = \frac{\alpha}{1+\alpha} < 1. \quad (19)$$

Aggregate income and expenditure

The last aspect in which this model deviates from the standard endogenous growth approach is the assumption that we look at the consumption and savings behavior of two different groups of individuals. This is because we currently observe a polarization in income distribution in many countries,⁸ and we want to investigate the growth effects of this phenomenon. Therefore, a *representative* intertemporal choice with the help of a *representative* household's "Euler equation"⁹ is not an adequate procedure for this problem. It assumes away the effects of different intertemporal decisions of rich and poor households on aggregate consumption and savings. Thus the "*representative* household assumption" would not allow to properly study polarization effects on growth. Thus, we suggest another procedure. From income decomposition (17) we identify a group that earns wage income wL , and another group that earns income from financial assets rF . Each group exhibits its own expenditure pattern.

Consumption expenditure and labor income: According to (18) the share of labor income is $\frac{wL}{Y} = \frac{1}{1+\alpha}$. If we define group specific intertemporal choice models, and assume plausible group specific parameters for the choice problem we can suggest that total wage income is fully consumed, and that labor income is the only source of consumption expenditure in the economy. While this is a traditional assumption in Keynesian growth models (inter alia by Dutt, 1984, Kaldor & Mirrlees, 1962, and Kalecki, 1968), we show in appendix A.2 that once

⁸Examples include Autor et al. (2003, 2006, 2008) or Autor & Dorn (2013); for Germany, Spitz-Oener (2006), Dustmann et al. (2009); for the UK, Goos et al. (2014) and Van Reenen (2011) for various advanced countries.

⁹ $\frac{\dot{C}}{C} = \frac{rD-p}{\eta_U}$ with ρ denoting the representative agent's time preference rate and η_U the intertemporal elasticity of substitution. In appendix A.2 we give examples for specific intertemporal choices at individual or group level. Further, if group preferences are diverse, they may lead to diverse consumption and savings behavior which is consistent with the suggestions in our model. Thus, we do not intend to assume away the idea of rational intertemporal choices. However, what we do assume away is the idea of a simple aggregation rule like a *representative* household.

we move away from the representative household approach, it is not difficult to motivate this assumption by group-specific optimal intertemporal choices. The important assumption is that groups are different and have different expenditure behavior. Thus, with these assumption, effective consumption is

$$C^e(t) = w(t)L = cY^e(t), \quad \text{with } c = \frac{1}{1+\alpha} \quad (20)$$

as being the economy's marginal and average rate of consumption.

Investments and market entry of new goods: In this model, investments are determined by innovations and market entry of new technologies. Bringing a new technology to the market requires investments. We assume that these investments are identical for each innovation and thus total start-up investments $I^e(t)$ are

$$I^e(t) = \nu \dot{N}(t) + \varepsilon_I, \quad (21)$$

with ε_I being a random variable. Economically ε_I indicates that sometimes there may be more, sometimes less investments necessary for market entry; however, the expected value is $E[\varepsilon_I] = 0$. The idea that technological change relates to investments also links up with the Keynesian traditions and started with Kaldor (1957).

Effective demand and Keynesian income-expenditure equilibrium:

By the definition of national accounting, effective income Y^e can be used for effective consumption C^e and investment I^e . Thus, effective demand is defined by $Y^{eD} \equiv C^e + I^e$. While the consumption rate (determined by 20) is a constant fraction of total effective income, investments are driven only by market entry of new goods \dot{N} . With a given consumption rate the Keynesian income-expenditure mechanism can be applied to determine effective spending and thus effective total demand Y^{eD} . Therefore, in income-expenditure equilibrium, aggregate effective demand equals effective income

$$Y^e(t) \stackrel{!}{=} Y^{eD}(t) \equiv C^e(t) + I^e(t), \quad (22)$$

and we obtain the well-known Keynesian income-expenditure multiplier for the effective demand in aggregate goods market. Defining $\mu = \frac{\nu}{1-c}$ as the standard multiplier gives

$$Y^{eD}(t) = \frac{1}{1-c} I^e(t) = \mu \dot{N}(t) + \frac{\varepsilon_I}{1-c}, \quad \text{with } \mu = \frac{\nu}{1-c}. \quad (23)$$

3 Solving the model

In order to completely solve the model we have to determine the equilibrium values for the remaining six variables, namely the semi-endogenous growth rate g_N , the interest rate r , the (demand determined) *expected ratio of market absorption* $E[\lambda]$, the equilibrium *supply ratio* $1 - \theta_i$, firms' *expected effective sales*

ratio $E[\Phi(t)]$, and the expected size of the friction $E[\delta]$ individual firms observe. Thus, we need to derive six simultaneous equations that allow to solve for these variables.

Further, we have to introduce the concept of stationary equilibrium which we call "no-expectation-error equilibrium". This concept must be applied in order to argue that firms may remain stationary in their behavior at a level of supply of final goods which could be below the potential supply-level of final goods.

Defining stationary no-expectation-error equilibrium

Related to stochastic modeling we suggest a different concept for a stationary equilibrium. At the firm level, we have a stationary behavior if all expected values are the effectively realized values. Thus, this condition defines an equilibrium that implies no (need) for change in behavior. We will call this "no-expectation-error equilibrium" (n-e-ee). As a firm gets what the firm expects as result of optimal decisions, the firm has no reason for further adjustments. This concept of stationarity can be regarded as the result of an adjustment process in a stochastic environment, and it is not far away from the Nash idea of individual stationary equilibrium behavior. Thus in this model, we always start discussions by looking at the perception of each firm, and first define this stationary equilibrium from the perspective of each firm. Then we turn to the aggregate economy.

Definition 1

The expectation error $E[\Omega_i(t)]$ is the difference between effective supply and the expected effective sales

$$E[\Omega_i(t)] = Q_i(t) - E[Q_i(t)].$$

For each firm, "no-expectation-error equilibrium" (n-e-ee) is the state when expectation error $E[\Omega_i(t)]$ is zero. That is, (i) firm i expects that firm i 's effective supply in the market $Q_i(t)$ is indeed on average absorbed by the market $E[Q_i(t)]$, and (ii) this holds for all firms

$$\begin{aligned} (i) & : E[\Omega_i(t)] = Q_i(t) - E[Q_i(t)] = 0, \\ (ii) & : E[\Omega(t)] = Q(t) - E[Q(t)] = 0. \end{aligned} \tag{24}$$

Solving for the growth equation.

As the first step, we determine the endogenous growth rate of new products successfully entering and staying in the market $g_N = \dot{N}/N$. This growth rate is the result of market opportunities. Equation (12) describes the aggregate matching process for new technologies successfully entering the market. Thus, the growth rate of implemented technologies depends on effective demand for intermediate goods, and thus, the effective ratio of final goods production $1 - \theta$,

and is¹⁰

$$g_N = \frac{\dot{N}(t)}{N(t)} = \left((1 - \theta(t))^{\frac{1}{1-\alpha}} \left(\frac{\alpha^2}{c_x} \right)^{\frac{1}{1-\alpha}} L \right)^\gamma (g_A)^{1-\gamma}. \quad (25)$$

We call this process semi-endogenous, as the rate g_N is basically driven by the exogenous g_A . However, to what extent the exogenous innovative process g_A becomes usable and implemented in the economy is endogenous. As g_N is still depending on $\theta(t)$, equation (25) is the first of the six simultaneous equations we need.

Further, it is interesting to note that in this kind of economy the return on investment is equal to the growth rate. This implication can be easily derived. According to (21) investments are $I_x(t) = \dot{N}(t)\nu$ and from (11) we know $r(t) = \frac{\pi_x(t)}{\nu}$. With all profits being saved we obtain $\dot{N}(t)\nu = I_x(t) = S(t) = N(t)\pi_x(t)$ and plugging in (11) shows that in this kind of model returns on investments equalize to the growth rate of new products in the market¹¹

$$r = \frac{\pi_x(t)}{\nu} = \frac{\dot{N}(t)}{N(t)} = g_N. \quad (26)$$

While (26) is a result we also find that in other mainstream modeling, the interpretation of causality is different. In this model g_N is the driver for r . With more products entering the market, profits improve and return on investments increase.

Determining the aggregate demand restriction.

Further, to determine the ratio of effective demand λ possible under current demand conditions we need to add up total demand for Q . In this model total demand for final output goods is the demand for GDP, $Y^{eD}(t) \equiv C^e(t) + I^e(t)$, and the demand in terms of input goods taken from final goods sector $N(t)x(t)c_x$. From the Keynesian income-expenditure mechanism (23) we know that effective aggregate demand for GDP is $\frac{\nu\dot{N} + \varepsilon L}{1-c}$. As result, effective demand is an endogenous, but somehow autonomous fraction of potential output (see 27), $\lambda(t)$ describes the share of potential output that can be absorbed by effective demand. Thus, the effective demand ratio $\lambda(t)$ determines the (demand restricted) sales

¹⁰From (12) and (8) we obtain $X^{eD} = Nx = N\alpha^{\frac{2}{1-\alpha}}(1-\theta)^{\frac{1}{1-\alpha}}(c_x)^{-\frac{1}{1-\alpha}}L$ and $\dot{N} = \left(\alpha^{\frac{2}{1-\alpha}}(1-\theta)^{\frac{1}{1-\alpha}}(c_x)^{-\frac{1}{1-\alpha}}L \right)^\gamma N^\gamma (\dot{A})^{1-\gamma}$. Rearranging gives $\frac{\dot{N}(t)}{N(t)} = \left(\alpha^{\frac{2}{1-\alpha}}(1-\theta)^{\frac{1}{1-\alpha}}(c_x)^{-\frac{1}{1-\alpha}}L \right)^\gamma \left(\frac{\dot{A}(t)}{A(t)} \right)^{1-\gamma}$ for $N(t) = A(t)$.

¹¹For a check that this relation is fully consistent with all other results of the model, see appendix A.3.

possible. We obtain for the *expected ratio of market absorption*

$$\frac{\overbrace{\nu\dot{N}(t) + E[\varepsilon_I(t)]}^{\text{effective total demand for product } Q^D}}{1 - c} + N(t)x(t)c_x = E[\lambda(t)], \quad \text{with } E[\varepsilon_I(t)] = 0. \quad (27)$$

$$\underbrace{N(t)L^{1-\alpha}(x(t))^\alpha}_{\text{potential output of product } Q^P}$$

In other words, the demand side restricts the possible sales to fraction $\lambda(t)$ of potential output. Further, as $\varepsilon_I(t)$ is a random variable and $\lambda(t)$ is directly determined by $\varepsilon_I(t)$, $\lambda(t)$ is also a random variable.

Aggregate market and sales equilibrium.

Assuming identical firms, equilibrium in the final goods market would require that the demand determined ratio of market absorption $\lambda(t)$ is equal to the ratio of real effective supply $1 - \theta(t)$ for each firm and on aggregate. Therefore, the aggregate market equilibrium is reached in expected values if the expected effective demand ratio equals the effective supply ratio

$$E[\lambda(t)] = 1 - \theta(t). \quad (28)$$

This market equilibrium condition is actually not a new independent condition, it is implied by the already introduced no-expectation-error equilibrium condition (Definition 1). However, it is important to show that in this model the aggregate market condition is satisfied. We can easily show that this equilibrium condition translates in the well known total market equilibrium

$$\frac{\overbrace{\nu\dot{N}(t)}^{\text{effective demand } Y^{eD}}}{1 - c} = Y^e(t) = \overbrace{(1 - \theta(t)) N(t)L^{1-\alpha}(x(t))^\alpha - N(t)x(t)c_x}^{\text{effective supply } Y^{eS}}, \quad (29)$$

$$\text{with } x_j(t) = \alpha^{\frac{2}{1-\alpha}} (1 - \theta_i(t))^{\frac{1}{1-\alpha}} L(c_x)^{-\frac{1}{1-\alpha}} \rightarrow x_j \quad \text{and } g_N = \frac{\dot{N}(t)}{N(t)}.$$

However, two important questions remain: How can we reach such an equilibrium sales ratio? - And, how can an effective sales ratio below potential output represent a long-term stationary equilibrium ratio?

Firm's expectations and adjustment of supply.

From the discussion of equation (2), we know that whenever a firm observes market frictions which lead to a lower sales ratio [$\frac{\partial E[\Phi_i(t)]}{\partial E[\delta_i]} < 0$] it can potentially counter the effects of these frictions by using fraction $\theta_i(t)$ of potential final output and allocate it to sales promotion and market placement activities [$\frac{\partial E[\Phi_i(t)]}{\partial \theta_i(t)} > 0$]. While potential output is $Q_i^p(t)$, current effective production and supply is $Q_i(t) = (1 - \theta_i(t)) Q_i^p(t)$. Thus, there is a reallocation of potential output towards sales activities. Further, as we would like to explicitly

determine solutions we need to suggest a specification for the firm's effective sales ratio which shows the characteristics described in (2). Thus, the *expected effective sales ratio* is specified as

$$\begin{aligned}\Phi_i(t) &= 1 - \delta_i(t) + b\theta_i(t), \quad b > 0 \\ E[\Phi_i(t)] &= 1 - E[\delta_i(t)] + b\theta_i(t).\end{aligned}\tag{30}$$

Firm's effective sales are reduced by friction $\delta_i(t)$, but they can be countered by $\theta_i(t)$. Further, as long as the offered supply ratio of final goods $1 - \theta_i(t)$ is still larger than the *expected effective sales ratio* $E[\Phi_i(t)]$, the firm produces and supplies more than it expects to sell. Consequently, the firm expects a loss $E[\Lambda_i(t)] = [(1 - \theta_i(t)) - E[\Phi_i]] Q_i^p(t)$. Further, firm i minimize its expected loss $E[\Lambda_i]$ by reducing output and improving expected effective sales.¹² Using (30) we obtain

$$\min_{\theta_i(t)} : E[\Lambda_i(t)] = [(1 - \theta_i(t)) - E[\Phi_i]] Q_i^p(t) \quad \text{for} \quad [1 - \theta_i > E[\Phi_i]].$$

With increasing θ_i , firm i reduces effective supply and increase sales promotion. Using (30), we see that the firm reduces expected losses until the expected excess supply is fully eliminated¹³

$$\frac{dE[\Lambda_i(t)]}{d\theta_i(t)} = \begin{cases} -(1+b) Q_i^p(t) < 0 & \text{for } 1 - \theta_i > E[\Phi_i] \\ 0 & \text{for } 1 - \theta_i = E[\Phi_i] \end{cases}.$$

Further, while the discussion above indicates that $\theta_i(t)$ is an instrument to obtain maximum profits, the adjustment reaction for countermeasures $\theta_i(t)$ can be described by a dynamic mechanism that suggests an increase in placement effort ($\dot{\theta}_i(t) > 0$) as long as the offered supply ratio of final goods $1 - \theta_i(t)$ is still larger than the *expected effective sales ratio* $E[\Phi_i(t)]$ according to market observations and including market frictions

$$\dot{\theta}_i(t) = Q_i(t) - E[Q_i(t)] = [(1 - \theta_i(t)) - (1 - E[\delta_i(t)] + b\theta_i(t))] Q_i^p(t).\tag{31}$$

According to (31), $\dot{\theta}_i(t) > 0$ would reduce the gap between effective supply and *expected effective sales* in the final goods market. Definition 1 states that a firm arrives in no-expectation-error equilibrium when $E[Q_i(t)] = Q_i(t)$.

$$1 - \theta_i(t) = E[\Phi_i(\theta_i, \delta_i)] \quad \text{see (24 i),}$$

¹²In footnote 15 we show that this reallocation is done via labor reallocation from output production to sales promotion.

¹³In case of an excess demand, effective expected sales could be higher than the effective production and supply, such that the expected loss of profit in this case is $-E[\Lambda_i(t)] = [E[\Phi_i] - (1 - \theta_i(t))] Q_i^p(t)$. In this case firms would minimize their expected loss of profits, $\min_{\theta_i(t)} : -E[\Lambda_i(t)] = [E[\Phi_i] - (1 - \theta_i(t))] Q_i^p(t)$, for $1 - \theta_i < E[\Phi_i]$. With $\frac{dE[-\Lambda_i(t)]}{d\theta_i(t)} = \begin{cases} (1+b) Q_i^p(t) > 0 & \text{for } 1 - \theta_i < E[\Phi_i] \\ 0 & \text{for } 1 - \theta_i = E[\Phi_i] \end{cases}$ firms increase their profits by reducing their sales effort until equilibrium ($1 - \theta_i = E[\Phi_i]$) is reached.

respectively when the ratio of potential output a firm expects to sell $E[\Phi_i(t)]$ equals the ratio of effective supply $1 - \theta_i(t)$. Thus, the stationary θ_i directly relates to the expected friction¹⁴

$$\theta_i(t) = \frac{1}{1+b} E[\delta_i(t)]. \quad (32)$$

Firm's expectations and adjustment of expectation error.

If firm i 's supply ratio of final goods $1 - \theta_i$ is currently larger than the mean of effective sales $E[\Phi_i(\theta_i, \delta_i)]$, they have a mistake in their expectations and planning. The error in effective and expected values is

$$E[\Omega_i(t)] = Q_i(t) - E[Q_i(t)] = 1 - \theta_i(t) - 1 + E[\delta_i(t)] - b\theta_i(t). \quad (33)$$

Fraction θ_i is not sufficient, and as described by (31) an adjustment of θ_i corrects for this mistake. The firm would increase placements efforts $\dot{\theta}_i(t) > 0$, and the expectation error would decline

$$E[\dot{\Omega}_i(t)] = -(1+b)\dot{\theta}_i(t) < 0. \quad (34)$$

This adjustment of $\theta_i(t)$ and the change of the expectation error $E[\dot{\Omega}_i(t)]$ continues until $\dot{\theta}_i(t) = 0$, and n-e-ee has reached at individual firm level, $E[\dot{\Omega}_i(t)] = 0$.

Aggregate market and expected friction.

So far, we looked at activities from the perspective of single firms. Firms have an individual perception of market frictions δ_i which they relate to their individual market conditions. They use fraction θ_i of potential output on placement and reduce their individual sales problems accordingly.

In this paragraph, we suggest that the firm's sales problems are not only individual problems. They are in fact also a result of aggregate market conditions. Thus we need to aggregate. As $\theta_i(t)$ is spent by each firm and $E[\Phi_i(\theta_i, \delta_i)]$ are the expected effective sales that include individual frictions $E[\delta_i(t)]$, aggregation to a representative market variable is not easy. This holds even more, as we still do not know much about these frictions. What are the sources of these sales problems perceived as market frictions? In order to answer this question we need to discuss the random variable $\delta_i(t)$ in more detail. What is behind perceived market frictions $\delta_i(t)$? We assume that $\delta_i(t)$ is driven by two components, (i) aggregate market conditions, and (ii) an idiosyncratic component for each individual firm. The first source is the aggregate market, $1 - \lambda(t)$ describes the share of potential final output that currently cannot be absorbed by aggregate demand. Hence, $1 - \lambda(t)$ indicates an aggregate market tightness, so that we can suppose that this tightness affects the sales problem of all firms in

¹⁴ $1 - \theta_i(t) = 1 - E[\delta_i(t)] + b\theta_i(t) \iff 0 = -E[\delta_i(t)] + (b+1)\theta_i(t) \iff E[\delta_i(t)] = (1+b)\theta_i(t)$.

the same way. Second and in addition, if $\lambda(t)$ is the ratio of possible successful sales in the market, there are further sales problems. These sales problems are firm i specific or idiosyncratic obstacles in each firm's market and described by the random variable ε_{Fi} , with $1 > E[\varepsilon_{Fi}] > 0$. Therefore, possible sales $\lambda(t)$ are further reduced by this idiosyncratic source, $\lambda(t)(1 - \varepsilon_{Fi})$. Thus, total frictions perceived by each firm i can be described as

$$\delta_i(t) = 1 - \lambda(t)(1 - \varepsilon_{Fi}).$$

If we now assume that ε_{Fi} are i.i.d for $i \in \mathcal{I}$, we can aggregate ($\varepsilon_{Fi} = \varepsilon_F$) and obtain as general or representative perceived friction

$$\begin{aligned} \delta(t) &= 1 - \lambda(t)(1 - \varepsilon_F), \quad \text{and} \\ E[\delta(t)] &= 1 + E[\varepsilon_F]E[\lambda(t)] + Cov(\varepsilon_F, \lambda(t)) - E[\lambda(t)]. \end{aligned} \quad (35)$$

Further, de facto only the idiosyncratic component of the friction can be countered by individual firm's measures $\theta(t)$. As we assume perfect countermeasures by each firm, all idiosyncratic elements of the friction are compensated by these countermeasure, such that

$$\theta(t)b = E[\varepsilon_F]E[\lambda(t)] + Cov(\varepsilon_F, \lambda(t)). \quad (36)$$

Thus, after successfully eliminating individual obstacles, the expected sales ratio turns into

$$E[\Phi(t)] = E[\lambda(t)] - E[\varepsilon_F]E[\lambda(t)] - Cov(\varepsilon_F, \lambda(t)) + b\theta(t) = E[\lambda(t)]. \quad (37)$$

As result, expected sales are fully determined by the aggregate market. Finally, recalling the dynamic adjustment process for the aggregate economy

$$\dot{\theta}(t) = (1 - \theta(t)) - (1 - E[\delta(t)] + b\theta(t)),$$

and solving for stationary equilibrium ($\dot{\theta}(t) = 0$), we obtain

$$1 - \theta(t) = E[\Phi(\theta, \delta)] = E[\lambda(t)] \quad \text{see (28 and 24 ii)}.$$

This condition is identical to the aggregate version of what we define as no-expectation-error equilibrium for the total economy. Thus, in n-e-ee the *expected effective sales ratio* of firms is the *expected ratio of market absorption* of the economy in market equilibrium. All planing is on average consistent with effective real market conditions. If all firms obtain what they expect, there is no need for further action and the economy remains stationary in this position.

Stationary equilibrium solution.

To summarize, apart from stationarity condition (24) in n-e-ee we have six equations, (25) to (28), (30) and (35) to solve for the six variables, the *effective supply ratio* $1 - \theta_i$, the *expected ratio of market absorption* $E[\lambda]$, the *semi-endogenous growth rate* g_N , *expected effective sales ratio* $E[\Phi(\theta, \delta)]$, the *expected level of friction* $E[\delta]$, and the *interest rate* r .

Proposition 2 *No-expectation-error equilibrium (definition 24 equation 24) and equations (25), (27), and (28) determine*

(ii) *the equilibrium stationary effective supply ratio*¹⁵

$$1 - \tilde{\theta} = \left(\frac{\mu}{1 - \alpha^2} \right)^{\frac{1-\alpha}{1-\gamma}} \left(\frac{\alpha^2}{c_x} \right)^{\frac{\gamma-\alpha}{1-\gamma}} \left(\frac{g_A}{L} \right)^{(1-\alpha)}, \quad \frac{d\tilde{\theta}}{d\mu} < 0, \quad (38)$$

(i) *the equilibrium stationary expected ratio of market absorption*

$$E[\tilde{\lambda}] = 1 - \tilde{\theta}, \quad (39)$$

and (iii) with (35) firms' perceived friction in equilibrium is

$$E[\tilde{\delta}] = 1 - E[\tilde{\lambda}] (1 - E[\varepsilon_F]) + Cov(\varepsilon_F, \lambda), \quad (40)$$

(iv) *the growth rate of income (GDP) as being determined by $1 - \tilde{\theta}$ and the exogenous rate of technical progress g_A becomes*

$$\tilde{g}_Y = \frac{\dot{Y}^e(t)}{Y^e(t)} = g_N = \left(\left(1 - \tilde{\theta} \right) \frac{\alpha^2}{c_x} \right)^{\frac{\gamma}{1-\alpha}} L^\gamma (g_A)^{1-\gamma}, \quad (41)$$

(v) *the real rate of return on financial investment is*

$$\tilde{r} = \tilde{g}_Y, \quad (42)$$

and (vi) *the effective income at each point in time, and hence the level of the growth path is*

$$\tilde{Y}^e(t) = N(t) (1 - \alpha^2) \left(1 - \tilde{\theta} \right)^{\frac{1}{1-\alpha}} \left(\frac{\alpha^2}{c_x} \right)^{\frac{\alpha}{1-\alpha}} L. \quad (43)$$

For a proof, see appendix A.3.¹⁶

¹⁵While $1 - \tilde{\theta}$ and the other equilibrium results are sufficient to solve the model, we can complete the mechanism by explaining how potential final output is transformed and used to promote sales activities θ . The reduction of output to level $Q^P (1 - \tilde{\theta})$ is the result of a reallocation of labor. With total labor normalized to one $L = 1$, labor is either used in production with share L_Q or in sales promotion with share L_θ . Assuming for the production of a unit sales-promoting service $\theta = L_\theta$, and using the equilibrium level $L_\theta = \tilde{L}_\theta$, we can determine the labor share in production that leads to this equilibrium production: Using $x(t) = \left(\frac{\alpha^2}{c_x} \right)^{\frac{1}{1-\alpha}} L_Q$, physical production is $Q = N L_Q^{1-\alpha} x^\alpha = N \left(\frac{\alpha^2}{c_x} \right)^{\frac{\alpha}{1-\alpha}} L_Q$. This physical production takes place at the level $Q^P (1 - \tilde{\theta})$ of potential output $Q^P = N \left(\frac{\alpha^2}{c_x} \right)^{\frac{\alpha}{1-\alpha}} L$. Thus, $Q = Q^P (1 - \tilde{\theta})$. Plugging in and using the labor market constraint shows that this labor real-

location is a consistent mechanism in this model: $N \overbrace{\left(\frac{\alpha^2}{c_x} \right)^{\frac{\alpha}{1-\alpha}} L_Q}^Q = N \overbrace{\left(\frac{\alpha^2}{c_x} \right)^{\frac{\alpha}{1-\alpha}} (1 - \tilde{\theta})}^{Q^P} \Leftrightarrow \tilde{L}_Q = L_Q = 1 - \tilde{L}_\theta$.

¹⁶Using (14) and taking the time derivative in equilibrium, with $\lambda = \tilde{\lambda}$ and $x = \tilde{x}$ we obtain $\dot{Y}(t) = \dot{N}(t) (\lambda L^{1-\alpha} \tilde{x}^\alpha - \tilde{x} c_x)$ and thus $g_Y = \frac{\dot{Y}(t)}{Y(t)} = \frac{\dot{N}(t)}{N(t)}$.

Interpretation of individual firm's behavior: For each firm, perceived stochastic market frictions δ_i reduce the expected share of sales $E[\Phi_i(\delta_i, \theta_i)]$ in the market (see 2). Thus, each firm reallocates potential output towards sales activities (θ_i) to counter frictions and increase the expected share of sales $E[\Phi_i]$. This reallocation ($\theta_i > 0$) from final goods supply to reducing market frictions continues to improve the expected effective sales ratio as long as the firm still expects an oversupply of its final goods in the final goods market (31). At the same time – due to these activities – the expectation error declines (34), so that in no-expectation-error equilibrium firm i 's effective supply ratio of final goods equals the expected effective sales ratio $E[\Phi_i(\delta_i, \theta_i)] = 1 - \theta_i$ (Definition 1 i). Since δ_i are the firm's perceived frictions, fraction θ_i is the best individual response to this stochastic friction.

Interpretation of aggregate markets and coordinated market behavior: While – from the perspective of each individual firm i – frictions were regarded exogenous and identified by the random variable δ_i , equation (35) reveals that individually perceived frictions are driven by idiosyncratic and aggregate market sources. First, we assume symmetrical firms with the same idiosyncratic stochastic variations (ε_{Fi} are i.i.d., $E[\varepsilon_{Fi}] = E[\varepsilon_F]$). Further, all these firms face the same stochastic aggregate (demand-restricted) *expected ratio of market absorption* $E[\lambda]$, and together with the idiosyncratic component this leads to one *effective expected sales ratio* $E[\Phi]$. Further, we assume that firms are able to completely compensate their idiosyncratic component. The firms' expected sales are fully determined by the aggregate expected (demand-determined) ratio of market absorption $E[\lambda(t)]$ (36 and 37). What was perceived by each firm as individual friction is in fact generated by the aggregate demand conditions $E[\lambda(t)]$. Therefore, aggregate demand coordinates expectations of firms' perceived frictions, and determines the equilibrium $1 - \tilde{\theta} = E[\Phi] = E[\lambda(t)]$ (28 and 37). With equilibrium $\tilde{\theta}$, the *effective supply ratio* $1 - \theta$ and the *effective expected sales ratio* $E[\Phi]$ (37) have sufficiently adjusted such that there is no further action necessary. The economy is in stationary equilibrium and neither firms nor consumers or investors have incentives to leave this stationary equilibrium (see $\dot{\theta}_i(t) = 0$ in (31), and Definition 1).

Furthermore, from (38) we learn that the effective supply ratio $1 - \tilde{\theta}$ is driven by the Keynesian demand-income-expenditure multiplier μ . Thus, the equilibrium effective supply ratio depends on demand-side elements. Further, as the level of the growth path $Y^e(t)$ is also determined by effective demand, the demand side matters and restricts the process as long as effective demand is smaller than potential GDP: $Y^{eD}(t) < Q^p(t) - N(t)x(t)c_x$. The model is consistently closed.

Interpretation of growth mechanism: While this mechanism describes the equilibrium level, there is also an effect on the growth rate. According to (41), GDP growth g_Y is (semi-)endogenously determined. First, there are new

ideas or technologies. In this model the growth rate of these new ideas is exogenous. Second, the market entry of these ideas is endogenous. The market entry process is determined by an aggregate entrepreneurial market matching process. The better the market conditions, the easier it is for a new product to survive the first phase of market entry and the higher the rate of survival of new technologies. Not only the growth rate of inventions of potential technologies is crucial, but also market opportunities which are represented by aggregate demand. As a result, in this hybrid long-term growth approach, the demand side matters. Thus, the model does neither neglect the demand side, nor the supply side. Indeed, the supply side is the most important one and remains the growth-generating engine. However, even if growth is driven by the supply side and cannot be generated by the demand side, it can be sustainably restricted by the demand side. Further, it is also interesting that interest rates are determined by the market entry rate of new innovative products. Not the savings decision determines the investments and the rate of return, the rate of return is determined by innovations and investment opportunities. Savings will adjust. In this interpretation, a low innovation rate is responsible for the low interest rate and the low level of investments.

Comparative statics and implications of a technology change $d\alpha$:

For comparative static analysis, we consider a change in the elasticity of production α . It is well known that due to the properties of the production function, α affects income distribution and the labor share of income $\frac{d(wL/Y)}{d\alpha}$. In other words, income polarization can be considered as result of a shift in technology. What is new is that we look at both, supply and demand side channels. The change in income distribution affects the level of the effective income path $\frac{d\tilde{Y}^e}{d\alpha}$, and the semi-endogenous growth rate $\frac{d\tilde{q}_N}{d\alpha}$. Specifically, with two distinct income groups and distinct consumption behavior, we can analyze the role of income polarization on the demand side and the total growth process. With all labor income being consumed¹⁷ the aggregate consumption rate is $c = \frac{1}{1+\alpha}$. Thus, changes in technology that affect income distribution, affect the consumption rate and may affect growth. Further, the supply side is affected as there is a shift of each factor's contribution in production.

Why is this interesting? Turning to the discussion of "secular stagnation" as mentioned in the motivation of this paper and looking at Gordon's (2012) headwinds, we can easily analyze the effects of an increasing inequality that Gordon mentions. According to recent studies, e.g., Karabarbounis & Neiman (2014) or Lawrence (2015), there has been a decline in the share of labor income over recent decades. In proposition 3, we state the implications of this stylized fact on the growth path, as an increase in α leads to a reduction of the income share of labor.

Proposition 3 *A technology shift in favor of intermediate goods $d\alpha > 0$ leads to*

¹⁷See the discussion in section 2 and in appendix A.2.

(i) a decrease in the income share of labor and a decrease in the consumption rate

$$\frac{d(wL/Y)}{d\alpha} = \frac{dc}{d\alpha} = -\frac{1}{(1+\alpha)^2} < 0,$$

(ii) is generally ambiguous with respect to the effective supply ratio $(1 - \tilde{\theta})$.

However, for a large set of plausible conditions $(1 - \tilde{\theta})$ declines,

$$\frac{d(1 - \tilde{\theta})}{d\alpha} < 0. \quad (44)$$

While (iii) the effect on the level of the income path $Y^e(t)$ is in general ambiguous, we find a large set of plausible conditions for an upward shift of the path level

$$\frac{d\tilde{Y}^e}{d\alpha} > 0, \quad (45)$$

(iv) the semi-endogenous growth rate of the economy, and (v) the real rate of return declines if α is sufficiently large

$$\frac{d\tilde{g}_Y}{d\alpha} < 0, \quad \frac{d\tilde{r}}{d\alpha} < 0. \quad (46)$$

For a proof, see appendix A.5.

Figure 1 illustrates the effects graphically. In figure 1 an increase of α , and thus a lower income share of labor affects the dynamics of the growth path via the demand-side.

Due to the fact that all labor income is consumed, the consumption rate $c = \frac{1}{1+\alpha}$ is directly affected by the change of α . With $\tilde{Y}^e(0)$ the economy would start at the level $\ln \tilde{Y}^e(0)$ and grow at rate \tilde{g}_N . At $t = 1$ we consider a shock on α that affects the demand side by a decreasing share of labor income, leading to a decrease in consumption rate c . As result, the equilibrium expected ratio of market absorption declines (44). However, even if the effective ratio of market absorption declines, there is a second effect on potential output. Almost the same conditions that imply a reduction in the effective ratio of market absorption lead to an increase in potential supply (45). The net effect is positive, as the supply side effect dominates. This positive total effect shifts the income path upwards in figure 1.

Further, as result of a lower expected effective supply ratio the equilibrium semi-endogenous growth rate \tilde{g}_Y declines (46). With \tilde{g}'_Y the economy also has a lower speed of growth. The systematic decline of opportunities in the market for new ideas and technologies reduces the number of newly implemented technologies.

The combination of both effects gives an interesting interpretation, quite different to the standard narrative. The supply side effect generated by an

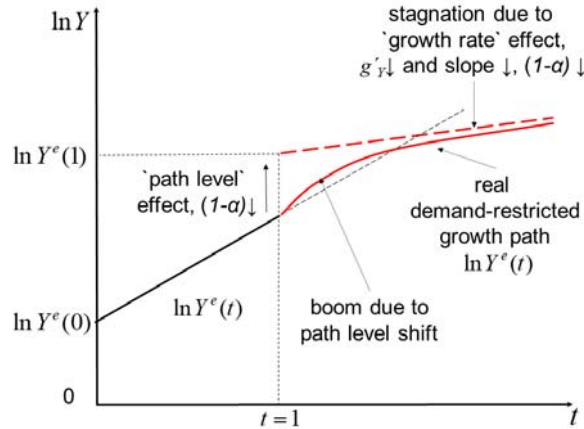


Figure 1: Path of an economy with a demand-restricted growth process, and a declining consumption rate c

increase in production elasticity of intermediate goods allows for a higher production. However, the negative demand side effect due to a decreasing share of consuming labor income affects the market entry of new goods negatively and thus, reduces the growth rate. Therefore, we obtain a supply side caused boom at the beginning of the process and a demand side driven depression of the growth rate which dominates the growth path in the long-term and reduces growth opportunities.

Further, this decline in investment opportunities also reduces the profits of innovative firms and leads to a decline in real returns on investments (46). Growth, profits, real returns and investments decline. These result are consistent with current observations and puzzles.

4 Summary and conclusions

Mainstream growth theory is dominated by variations of neoclassical and endogenous growth models. In these approaches, growth is explained fully by elements of the supply side. In light of a long-term slow down in growth – with the latest manifestation, the 2008 crisis and the Great Recession – and a simultaneous income polarization starting in the 90s, it is fair to ask why are we so sure that only the supply side determines growth. Instead of following a fully supply-side-driven neoclassical or endogenous growth approach, we suggest a hybrid approach that allows for growth restrictions induced by demand-side elements. We depart from an endogenous growth product-variety approach and modify the model. The four most important modifications are that (i) firms

face stochastic frictions in the market of their final goods, so that their effective sales ratio is below the potential. (ii) The second modification is the notion of "no-expectation-error equilibrium" as a concept for stationary equilibrium behavior. (iii) The third modification is an independent demand side that is not driven by inner solutions of representative intertemporal choices (Euler Equation) and Say's Law. Instead, the demand side is determined by investments in the market entry of new products and technologies. (iv) Finally, the growth rate is semi-endogenous, that is, technology growth is exogenous, but the implementation of resulting new products in the economy is determined by endogenous market opportunities.

As result of these variations changes in demand affect the effective sales ratio as well as profits and returns in the final and intermediate goods sector. The economy is not pinned down to a unique equilibrium path. When independent demand increases, excess demand in the markets for final products drives total production and final goods firms' input of intermediate goods. Further, intermediate goods profits increase and the return on financial investments in intermediate-good startups rises. The economy remains at this new higher level of activity as a new no-expectation-error equilibrium will be established. The new stationary equilibrium can be established, because firms and households have no reason for further adjustments. Their expectations, behavior and experiences are consistent. Furthermore, in this new equilibrium the growth rate increases because higher demand leads to more opportunities for market entry of new products, so the success rate in the market entry matching process improves. With more market entries the growth rate of implemented technologies increases. Potential growth is driven by technology growth and thus by the supply side; but the realized effective income and income growth is limited if demand is insufficient.

To address income polarization, we consider a technology shock account of the labor share of income and in favor of financial wealth holders. Even if the resulting effects are not completely unambiguous we may obtain an upward shift in the path level of income, and a reduction in the long-term growth rate. In other words, we obtain a short-term supply side driven boom and a long-term demand-side caused stagnation in growth.

The policy implications of such a growth process are clearly different than those of standard mainstream growth theory. According to standard theory, policies that encourage an increase in savings would make more resources available for growth (including R&D expenditure etc.). In our approach, the economy is bounded by investment opportunities. Policies that encourage higher demand instead of higher savings would improve these investment opportunities and thus lower the restrictions on the growth process. Thus, this hybrid model allows for both mechanisms and may bridge the gap between two hitherto distinct views: the view of mainstream growth focusing on the supply side, and the Keynesian demand-side-oriented view.

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A Appendix:

A.1 Determining the labor and profit share of income

Income Y (GDP) and total production Q . Before we determine the income shares, we determine the simple relation between income Y (GDP) and total production Q . According to (14)

$$Y(t) = Q(t) - N(t)x(t)c_x = \left(1 - \frac{Nxc_x}{Q}\right) Q(t).$$

Applying (8) gives

$$\frac{Nxc_x}{Q} = \frac{Nc_x}{NL^{1-\alpha}\alpha^{\frac{-(1-\alpha)^2}{1-\alpha}}(1-\theta_i)^{\frac{-(1-\alpha)}{1-\alpha}}(c_x)^{-\frac{-(1-\alpha)}{1-\alpha}}L^{-(1-\alpha)}(1-\theta)} = \frac{c_x}{\alpha^{-2}c_x} = \alpha^2,$$

and for $Y(t)$ we obtain

$$Y(t) = (1 - \alpha^2) Q(t) = (1 - \alpha^2) N(t)\alpha^{\frac{2\alpha}{1-\alpha}}(1 - \theta)^{\frac{1}{1-\alpha}}(c_x)^{-\frac{\alpha}{1-\alpha}}L. \quad (47)$$

Labor share of income: Labor share of income is defined as $\frac{wL}{Y} = \frac{(1-\alpha)}{(1-\alpha^2)}$. The fact that $(1 - \alpha^2) = (1 - \alpha)(1 + \alpha)$ and applying (8) gives $\frac{wL}{Y} = \frac{1}{1+\alpha}$.

Profit share of income: Profit share of income is defined as $\frac{N_i\pi_x}{Y} = \frac{N\alpha(1-\theta_i)L^{1-\alpha}x^\alpha - Nc_x x}{NL^{1-\alpha}x^\alpha(1-\theta_i) - Nc_x x} = \frac{\alpha - \frac{Nc_x x}{N(1-\theta_i)L^{1-\alpha}x^\alpha}}{1 - \frac{Nc_x x}{NL^{1-\alpha}x^\alpha(1-\theta_i)}} = \frac{\alpha - \alpha^2}{1 - \alpha^2} = \frac{\alpha}{1 + \alpha}$.

A.2 Intertemporal choices for labor and capital owners

In standard models, aggregate consumption expenditure and savings are determined by a representative household conducting an optimal intertemporal choice according to the Euler equation

$$\frac{\dot{C}}{C} = \frac{r - \rho}{\eta_U}.$$

However, this assumption of a *representative household* is *rather restrictive*. Moreover, for modeling a polarized economy it makes sense to substitute these assumptions by proposing two groups of households differing with respect to their consumption and savings behavior. We also argue that our assumptions are not in contrast to individual optimal intertemporal choices. In fact, in this appendix we would like to give a simple example that indicates that individual intertemporal choices can be integrated in this kind of modeling. However, since the main focus of the paper is to describe mechanisms of demand-restricted growth in a polarized economy, we reduce complexity by skipping over an explicit modeling of intertemporal choice mechanisms in the main text. The following example however indicates how our simplifying assumptions can be related to intertemporal choice modeling.

This is the example: (i) We assume that workers with wage income represent the "low per capita income" group. The second group, the owners of financial assets F , represent the "high per capita income" group. For these households returns r are the only source of income. (ii) Households in each group make their own intertemporal choices. We generally suggest that both, ρ and η_U vary across low and high income households. a) Low income, wage earning households: If we assume that the time preference rate of low income households is high, and if we do not allow for household debt, then wage earning households would

consume their full income. The result of individual intertemporal choice – even when applying the Euler equation $\frac{\dot{C}_L}{C_L} = \frac{r - \rho_L}{\eta_{U_L}}$ – is that these households do not intend to shift intertemporal consumption and simply consume what they earn from wage income. The consumption rate is $c_L = 1$. b) High income households: High income households obtain their total income from returns on financial assets $F(t)r$. Thus, the budget constraint of high income, financial asset owners is $S_F(t) \leq F(t)r - C_F(t)$. As savings are used to purchase newly issued financial assets $\dot{F}(t)$ and these assets finance investments (21), we obtain

$$\begin{aligned} \dot{N}(t)\nu &= I^e(t) = \dot{F}(t) = F(t)r - C_F(t) \\ \frac{\dot{N}(t)}{N(t)}\nu &= \frac{F(t)}{N(t)}r - \frac{C_F(t)}{N(t)}. \end{aligned} \quad (A1)$$

If the household takes the interest rate as given, we obtain from (26) that $r = g_N$ and hence, $g_N\nu = \frac{F(t)}{N(t)}g_N - \frac{C_F(t)}{N(t)}$. From the perspective of the individual investor the interest rate r is given such that for a stationary interest rate together with the Euler equation for financial investors would lead to a constant consumption growth for this group, $g_{C_F} = \frac{\dot{C}_F}{C_F} = \frac{r - \rho_F}{\eta_{U_F}}$. Thus, we can rewrite the budget constraint

$$g_N\nu = \frac{F(0)e^{g_F t}}{N(0)e^{g_N t}}g_N - \frac{C_F(0)e^{g_{C_F} t}}{N(0)e^{g_N t}},$$

rearranging shows that the path of financial income will finance the path of investments and the path of the household's consumption

$$\begin{aligned} g_N F(0)e^{g_F t} &= g_N N(0)e^{g_N t}\nu + C_F(0)e^{g_{C_F} t}, \text{ and} \\ F(0)e^{g_F t} &= \left[\nu + \frac{C_F(0)e^{g_{C_F} t}}{g_N N(0)e^{g_N t}} \right] N(0)e^{g_N t}. \end{aligned} \quad (A2)$$

Further, in our example we assume that financial investors' desired optimal consumption growth rate (Euler equation) is below the economy's growth rate $g_N > g_{C_F}$. With this assumption and the fact that the growth rate of the financial asset g_F is the result of the household's decisions and each period's budget constraint, it will adjust over time, $g_F = g_F(t)$. Thus, we first consider the initial condition,

$$\begin{aligned} \lim_{t \rightarrow 0} F(0)e^{g_F(t)t} &= \lim_{t \rightarrow 0} \left[\nu g_N + \frac{C_F(0)}{N(0)} e^{-(g_N - g_{C_F})t} \right] \frac{1}{g_N} N(0)e^{g_N t}, \\ \frac{F(0)}{N(0)} &= \left[\nu + \frac{C_F(0)}{g_N N(0)} \right], \quad \frac{N(0)}{F(0)} = \frac{1}{\left[\nu + \frac{C_F(0)}{g_N N(0)} \right]}. \end{aligned}$$

That is, we find initial conditions that show that financial investors start with positive consumption. Then they walk on their optimal consumption path according to $\frac{\dot{C}_F}{C_F} = \frac{r - \rho_F}{\eta_{U_F}}$. Further, how can we describe the long term behavior?

What is the long term consumption and savings rate? The assumption that $g_N > g_{C_F} = \frac{r - \rho_F}{\eta_{U_F}}$ implies that the investor's consumption rate converges to zero in the long term: $\lim_{t \rightarrow \infty} \frac{C_F(0)}{N(0)} e^{-(g_N - g_{C_F})t} = 0$, and for the consumption rate $\lim_{t \rightarrow \infty} c_F = \frac{C_F(t)}{Y(t)} = \frac{C_F(t)}{N(t)((1-\tilde{\theta})L^{1-\alpha}\tilde{x}^\alpha - \tilde{x}c_x)} = \frac{C_F(t)}{N(t)} \left((1-\tilde{\theta})L^{1-\alpha}\tilde{x}^\alpha - \tilde{x}c_x \right)^{-1} = 0$, for steady state values $\tilde{\theta}$ and \tilde{x} . Further, from (A2) we obtain the relation between the path of the financial asset $F(t)$ and innovations $N(t)$:

$$\begin{aligned} F(0) e^{g_F(t)t} &= \left[\nu + \frac{C_F(0) e^{g_{C_F}t}}{g_N N(0) e^{g_N t}} \right] N(0) e^{g_N t}, \\ \lim_{t \rightarrow \infty} \frac{F(0) e^{g_F(t)t}}{N(0) e^{g_N t}} &= \lim_{t \rightarrow \infty} \left[\nu + \frac{C_F(0)}{N(0)} e^{-(g_N - g_{C_F})t} \right]. \end{aligned}$$

Relation of the growth rates $g_F(t)$ and g_N :

$$\begin{aligned} e^{g_F(t)t} &= \left[\nu + \frac{C_F(0)}{N(0)} e^{-(g_N - g_{C_F})t} \right] \frac{N(0)}{F(0)} e^{g_N t} \\ g_F(t)t &= \ln \left[\nu + \frac{C_F(0)}{N(0)} e^{-(g_N - g_{C_F})t} \right] + \ln \frac{N(0)}{F(0)} + g_N t \\ g_F(t) &= \ln \left[\nu + \frac{C_F(0)}{N(0)} e^{-(g_N - g_{C_F})t} \right] \frac{1}{t} + \frac{1}{t} \ln \frac{N(0)}{F(0)} + g_N \\ \lim_{t \rightarrow \infty} g_F(t) &= \frac{1}{t} \ln [\nu] + \frac{1}{t} \ln \frac{N(0)}{F(0)} + g_N = g_N. \end{aligned}$$

Thus, we obtain for the ratio of the two paths

$$\begin{aligned} \lim_{t \rightarrow \infty} \frac{F(0) e^{g_N t}}{N(0) e^{g_N t}} &= \frac{F(0)}{N(0)}, \quad \lim_{t \rightarrow \infty} \left[\nu + \frac{C_F(0)}{N(0)} e^{-(g_N - g_{C_F})t} \right] = \nu \\ \lim_{t \rightarrow \infty} \frac{F(t)}{N(t)} &= v. \end{aligned}$$

At the limit, this intertemporal choice model would converge towards the simplifying assumptions of household behavior of the two different groups in our model. In other words, in the choice model financial investors would start with both, consumption and savings. The consumption rate would converge towards one for wage earning households $c_L = 1$; the savings rate would converge towards one for financial investors $c_F = 0$, and $\lim_{t \rightarrow \infty} \frac{F(t)}{N(t)} = v$. As $c_L = 1$ and $c_F = 0$ are the simplifying assumptions in our model, we find in appendix A.4 that $\frac{F(0)}{N(0)} = v$ is the start value, and that $g_F = g_N$ are the growth rates required for dynamic consistency.

A.3 Proof of proposition 2:

Equilibrium $(1 - \tilde{\theta})$: Equation (27) gives the expected effective demand as ratio to potential output

$$\frac{\overbrace{\frac{\nu \dot{N}(t) + E[\varepsilon_I(t)]}{1-c} + N(t)x(t)c_x}^{\text{effective total demand for product } Q^D}}{\underbrace{N(t)L^{1-\alpha}(x(t))^\alpha}_{\text{potential output product } Q^P}} = E[\lambda(t)], \quad \text{with } E[\varepsilon_I(t)] = 0$$

with equilibrium condition (28), we describe market equilibrium. Expected effective demand equals effective supply

$$\begin{aligned} \overbrace{\frac{\nu \dot{N}(t)}{1-c}}^{\text{effective demand } Y^{eD}} &= \overbrace{(1-\theta(t))N(t)L^{1-\alpha}(x(t))^\alpha - N(t)x(t)c_x}_{\text{effective supply } Y^{eS}}, \\ \frac{\nu}{1-c} \frac{\dot{N}(t)}{N(t)} &= (1-\theta(t))L^{1-\alpha}(x(t))^\alpha - x(t)c_x \\ \mu g_N &= ((1-\theta(t))L^{1-\alpha}x^\alpha - xc_x), \quad \text{with } x = \alpha^{\frac{2}{1-\alpha}}((1-\theta(t)))^{\frac{1}{1-\alpha}}L(c_x)^{-\frac{1}{1-\alpha}}, \end{aligned}$$

as described by (29). We can now determine the equilibrium values for $E[\lambda]$ and g_N . Using (25) and plugging in for x gives

$$\begin{aligned} g_N &= \left(\alpha^{\frac{2}{1-\alpha}}((1-\theta(t)))^{\frac{1}{1-\alpha}}(c_x)^{-\frac{1}{1-\alpha}}L \right)^\gamma (g_A)^{1-\gamma} \\ \frac{\nu}{1-c} \frac{\dot{N}(t)}{N(t)} &= (1-\theta(t))L^{1-\alpha}(x(t))^\alpha - x(t)c_x \\ \mu g_N &= (1-\theta(t))(1-\theta(t))^{\frac{\alpha}{1-\alpha}}L^{1-\alpha} \left(\alpha^{\frac{2}{1-\alpha}}L(c_x)^{-\frac{1}{1-\alpha}} \right)^\alpha \\ &\quad - (1-\theta(t))^{\frac{1}{1-\alpha}} \left(\alpha^{\frac{2}{1-\alpha}}L(c_x)^{-\frac{1}{1-\alpha}} \right) c_x \\ \mu \left(\alpha^{\frac{2}{1-\alpha}}(c_x)^{-\frac{1}{1-\alpha}}L \right)^\gamma (g_A)^{1-\gamma} &= \left[\begin{array}{l} L^{1-\alpha} \left(\alpha^{\frac{2}{1-\alpha}}L(c_x)^{-\frac{1}{1-\alpha}} \right)^\alpha \\ - \left(\alpha^{\frac{2}{1-\alpha}}L(c_x)^{-\frac{1}{1-\alpha}} \right) c_x \end{array} \right] (1-\theta(t))^{\frac{1}{1-\alpha} - \frac{\gamma}{1-\alpha}} \\ (1-\theta)^{\frac{1-\gamma}{1-\alpha}} &= \mu \frac{\left(\alpha^{\frac{2}{1-\alpha}}(c_x)^{-\frac{1}{1-\alpha}}L \right)^\gamma (g_A)^{1-\gamma}}{\left[L^{1-\alpha} \left(\alpha^{\frac{2}{1-\alpha}}L(c_x)^{-\frac{1}{1-\alpha}} \right)^\alpha - \left(\alpha^{\frac{2}{1-\alpha}}L(c_x)^{-\frac{1}{1-\alpha}} \right) c_x \right]} \\ &= \mu \frac{\left(\alpha^{\frac{2}{1-\alpha}}(c_x)^{-\frac{1}{1-\alpha}}L \right)^\gamma (g_A)^{1-\gamma}}{\left[L^{1-\alpha} - \left(\alpha^{\frac{2}{1-\alpha}}(c_x)^{-\frac{1}{1-\alpha}}L \right)^{1-\alpha} c_x \right] \left(\alpha^{\frac{2}{1-\alpha}}(c_x)^{-\frac{1}{1-\alpha}}L \right)^\alpha} \\ &= \mu \frac{\left(\alpha^{\frac{2}{1-\alpha}}(c_x)^{-\frac{1}{1-\alpha}} \right)^{\gamma-\alpha} (g_A)^{1-\gamma}}{[1-\alpha^2]} L^{\gamma-\alpha-1+\alpha} \end{aligned}$$

$$\begin{aligned}
(1 - \theta) &= \left(\frac{\mu}{1 - \alpha^2} \right)^{\frac{1-\alpha}{1-\gamma}} \left(\frac{\alpha^2}{c_x} \right)^{\frac{\gamma-\alpha}{1-\gamma}} \frac{(g_A)^{(1-\alpha)}}{L^{(1-\alpha)}}, \quad \text{and} \\
\frac{d(1 - \theta(t))}{dL} &= -(1 - \alpha) \left(\frac{\mu}{1 - \alpha^2} \right)^{\frac{1-\alpha}{1-\gamma}} \left(\frac{\alpha^2}{c_x} \right)^{\frac{\gamma-\alpha}{1-\gamma}} \frac{(g_A)^{(1-\alpha)}}{[1 - \alpha^2] L^{2-\alpha}}
\end{aligned}$$

Equilibrium mean of firms' perceived friction $E[\delta]$: As (35) describes the expected total perceived frictions, we obtain in equilibrium

$$E[\delta(t)] = 1 - E[\lambda(t)](1 - E[\varepsilon_F]) + Cov(\varepsilon_F, \lambda(t)).$$

This is also fully consistent with market equilibrium (28) and the stationary conditions of no-expectation-error (24). From (31) and the stationary condition $\dot{\theta}(t) = 0$, we know $1 - \theta(t) = 1 - E[\delta(t)] + b\theta(t)$. Rewriting leads to (32), $\theta(t) = \frac{1}{1+b}E[\delta(t)]$. If we plug in (35) and use (36), it leads to

$$\begin{aligned}
\theta_i(t) &= \frac{1}{1+b} [1 + E[\varepsilon_F] E[\lambda(t)] + Cov(\varepsilon_F, \lambda(t)) - E[\lambda(t)]] \\
(1+b)\theta_i(t) &= [1 + \theta(t)b - E[\lambda(t)]] \\
1 - \theta_i(t) &= E[\lambda(t)].
\end{aligned}$$

Equilibrium level of income path $\tilde{Y}^e(t)$: In appendix A.1, we have determined with equation (47) income or GDP level as

$$Y(t) = (1 - \alpha^2) N(t) \alpha^{\frac{2\alpha}{1-\alpha}} (1 - \theta)^{\frac{1}{1-\alpha}} (c_x)^{-\frac{\alpha}{1-\alpha}} L.$$

Plugging in the equilibrium values for $1 - \tilde{\theta}$, we obtain (43).

Equilibrium rate of return \tilde{r} : In this section we want to show that the interest rule $r = g_N$ is fully consistent with the other findings of the model. It shows the general consistency of all elements. We can plug in (11) and (9) [$\tilde{r} = \frac{\pi_x}{\nu} = \frac{1}{\nu} (\frac{1}{\alpha} - 1) (c_x)^{-\frac{\alpha}{1-\alpha}} \alpha^{\frac{2}{1-\alpha}} (1 - \tilde{\theta})^{\frac{1}{1-\alpha}} L$] and (25) [$g_N = \frac{\dot{N}(t)}{N(t)} = \left(\alpha^{\frac{2}{1-\alpha}} (1 - \tilde{\theta})^{\frac{1}{1-\alpha}} (c_x)^{-\frac{1}{1-\alpha}} L \right)^\gamma (g_A)^{1-\gamma}$] and obtain the solution for $E[\lambda]$ and $\tilde{\theta}$ as derived in (38).

A.4 Dynamic consistency:

- **Consistent motion of the demand and supply side in n-e-ee:**

From (27) and (28), we obtain

$$\frac{\overbrace{\nu \dot{N}(t) + E[\varepsilon_I(t)]}^{\text{effective total demand for product } Q^D}}{1-c} + N(t)x(t)c_x}{\underbrace{N(t)L^{1-\alpha}(x(t))^\alpha}_{\text{potential output product } Q^P}} = E[\tilde{\lambda}] = 1 - \theta, \quad \text{with } E[\varepsilon_I(t)] = 0,$$

$$\text{with } \tilde{x}_j = \alpha^{\frac{2}{1-\alpha}} \left((1 - \tilde{\theta}) \right)^{\frac{1}{1-\alpha}} L(c_x)^{-\frac{1}{1-\alpha}},$$

and in equilibrium

$$\frac{\nu}{1-c} \dot{N}(t) = N(t) \left(L^{1-\alpha} \tilde{x}^\alpha (1 - \tilde{\theta}) - \tilde{x}c_x \right).$$

A change over time is described by

$$\frac{1}{1-c} \nu \ddot{N}(t) = \dot{N}(t) \left(L^{1-\alpha} \tilde{x}^\alpha (1 - \tilde{\theta}) - \tilde{x}c_x \right)$$

$$\frac{1}{1-c} \nu \frac{\ddot{N}(t)}{\dot{N}(t)} = \left(L^{1-\alpha} \tilde{x}^\alpha (1 - \tilde{\theta}) - \tilde{x}c_x \right).$$

Second, the growth rate of innovation related investments is $\frac{\dot{N}}{N} = g_N$ for exponential growth ($N(t) = e^{g_N t}$) and

$$\frac{1}{1-c} \nu g_N = \left(L^{1-\alpha} \tilde{x}^\alpha (1 - \tilde{\theta}) - \tilde{x}c_x \right).$$

q.e.d.

- **Consistent start values of Financial and Technology stocks:**

Finally, we can show consistency by deriving the savings, and demonstrate that these savings indeed can finance the process from the start. Financial wealth income is $rF(t)$. According to our discussion in section 2 financial asset owners only save, and as these savings finance the investments for newly introduced goods, we obtain

$$rF(t) = S(t) = \dot{F}(t) = \dot{N}(t) \nu.$$

For this debt and technology growth mechanism, we need to show that savings in deposits and financing investments are consistent in their stock and flow

mechanism, and we can derive a relation for the start period $F(0)/N(0)$ that leads to this consistent growth process ¹⁸

$$\begin{aligned}
r \frac{F(t)}{N(t)} &= \nu \frac{\dot{N}(t)}{N(t)} \iff \\
rF(0) e^{(g_N)t} &= \nu N(0) e^{g_N t} g_N \iff \\
rF(0) &= \nu N(0) g_N \\
\frac{F(0)}{N(0)} &= \frac{\nu}{r} g_N = \nu.
\end{aligned}$$

q.e.d.

A.5 Proof of proposition 3:

Derivative of the effective supply ratio: $\frac{d(1-\tilde{\theta})}{d\alpha}$:

$$\begin{aligned}
(1-\tilde{\theta}) &= \left(\frac{\mu}{1-\alpha^2} \right)^{\frac{(1-\alpha)}{1-\gamma}} \left(\frac{\alpha^2}{c_x} \right)^{\frac{\gamma-\alpha}{(1-\gamma)}} \left(\frac{g_A}{L} \right)^{(1-\alpha)} \\
&= \left(\frac{\mu}{\alpha^{-2}-1} \right)^{\frac{(1-\alpha)}{1-\gamma}} \alpha^{-\frac{2(1-\alpha)}{1-\gamma}} \alpha^{2\frac{(\gamma-\alpha)}{(1-\gamma)}} (c_x)^{-\frac{\gamma-\alpha}{(1-\gamma)}} \left(\frac{g_A}{L} \right)^{(1-\alpha)} \\
&= \left(\frac{\nu}{\alpha^{-2}-1} \frac{(1+\alpha)}{\alpha} \right)^{\frac{(1-\alpha)}{1-\gamma}} \alpha^{-\frac{2(1-\alpha)}{(1-\gamma)}} \alpha^{2\frac{\gamma-\alpha}{(1-\gamma)}} (c_x)^{-\frac{\gamma-\alpha}{(1-\gamma)}} \left(\frac{g_A}{L} \right)^{(1-\alpha)} \\
&= \left(\frac{\nu}{\alpha^{-2}-1} \frac{(1+\alpha)}{\alpha} \right)^{\frac{(1-\alpha)}{1-\gamma}} \alpha^{-2} (c_x)^{-\frac{\gamma-\alpha}{(1-\gamma)}} \left(\frac{g_A}{L} \right)^{(1-\alpha)} \\
&= \left(\frac{\nu}{\alpha^{-2}-1} \frac{(\alpha^{-1}+1)\alpha}{\alpha} \right)^{\frac{(1-\alpha)}{1-\gamma}} \alpha^{-2} (c_x)^{-\frac{\gamma-\alpha}{(1-\gamma)}} \left(\frac{g_A}{L} \right)^{(1-\alpha)}
\end{aligned}$$

$$G = \left[\left(\nu \frac{(\alpha^{-1}+1)}{(\alpha^{-2}-1)} \right)^{(1-\alpha)} \alpha^{-2(1-\gamma)} (c_x)^{-(\gamma-\alpha)} \left(\frac{g_A}{L} \right)^{(1-\alpha)(1-\gamma)} \right]^{\frac{1}{(1-\gamma)}}$$

¹⁸ $r \frac{F(t)}{N(t)} = \nu \frac{\dot{N}(t)}{N(t)} \iff rF(0) e^{(g_N)t} = \nu N(0) e^{g_N t} g_N \iff rF(0) = \nu N(0) g_N$

Show that $\frac{d \ln G}{d \alpha} < 0$

$$\begin{aligned}
\ln G &= \frac{1}{(1-\gamma)} \ln \left[\left(\nu \frac{(\alpha^{-1}+1)}{(\alpha^{-2}-1)} \right)^{(1-\alpha)} \alpha^{-2(1-\gamma)} (c_x)^{-(\gamma-\alpha)} \left(\frac{g_A}{L} \right)^{(1-\alpha)(1-\gamma)} \right] \\
&= \frac{1}{(1-\gamma)} \left[(1-\alpha) \ln \left(\frac{\nu(\alpha^{-1}+1)}{(\alpha^{-2}-1)} \right) - 2(1-\gamma) \ln \alpha - (\gamma-\alpha) \ln c_x \right. \\
&\quad \left. + (1-\alpha)(1-\gamma) \ln \left(\frac{g_A}{L} \right) \right] \\
&= \frac{1}{(1-\gamma)} \left[(1-\alpha) (\ln \nu + \ln(\alpha^{-1}+1) - \ln(\alpha^{-2}-1)) \right. \\
&\quad \left. - 2(1-\gamma) \ln \alpha - (\gamma-\alpha) \ln c_x \right. \\
&\quad \left. + (1-\alpha)(1-\gamma) \ln \left(\frac{g_A}{L} \right) \right] \\
\frac{d \ln G}{d \alpha} &= \frac{1}{(1-\gamma)} \left[- \left(\ln \nu + \ln(\alpha^{-1}+1) - \ln(\alpha^{-2}-1) \right) + (1-\alpha) \left(\frac{-\alpha^{-2}}{(\alpha^{-1}+1)} - \frac{-2\alpha^{-3}}{(\alpha^{-2}-1)} \right) \right. \\
&\quad \left. - 2(1-\gamma) \frac{1}{\alpha} + \ln c_x - (1-\gamma) \ln g_A + \ln L - \gamma \ln L \right] \\
&= \frac{1}{(1-\gamma)} \left[- \left(\ln \nu - \ln L + \ln(\alpha^{-1}+1) - \ln(\alpha^{-2}-1) \right) \right. \\
&\quad \left. + (1-\alpha) \left(\frac{-\alpha^{-2}}{(\alpha^{-1}+1)} - \frac{-2\alpha^{-3}}{(\alpha^{-2}-1)} \right) - 2(1-\gamma) \frac{1}{\alpha} \right. \\
&\quad \left. + \left(\ln c_x - (1-\gamma) \ln g_A - \gamma \ln L \right) \right]
\end{aligned}$$

First element (1) > 0 : $(\ln \nu - \ln L)$ is the investment per capita and per growth rate of new products. This must be a large absolute number. Further,

$$\begin{aligned}
\ln(\alpha^{-1}+1) - \ln(\alpha^{-2}-1) &> 0 \\
\ln(\alpha^{-1}+1) &> \ln(\alpha^{-2}-1) \\
\text{for } \alpha &> 1/2
\end{aligned}$$

Second element and third element: (2) + (3) < 0

$$\begin{aligned}
(1-\alpha) \left(\frac{-\alpha^{-2}}{(\alpha^{-1}+1)} - \frac{-2\alpha^{-3}}{(\alpha^{-2}-1)} \right) - 2(1-\gamma) \frac{1}{\alpha} &< 0 \\
\frac{-\alpha^{-2}}{(\alpha^{-1}+1)} + \frac{2\alpha^{-3}}{(\alpha^{-2}-1)} &< 2 \frac{(1-\gamma)}{(1-\alpha)} \frac{1}{\alpha} \\
\frac{-\alpha^{-1}}{(\alpha^{-1}+1)} + \frac{2\alpha^{-2}}{(\alpha^{-2}-1)} &< 2 \frac{(1-\gamma)}{(1-\alpha)} \\
\frac{-1}{\alpha(\alpha^{-1}+1)} + \frac{2}{\alpha^2(\alpha^{-2}-1)} &< 2 \frac{(1-\gamma)}{(1-\alpha)} \\
\frac{-1}{(1+\alpha)} + \frac{2}{(1-\alpha^2)} &< 2 \frac{(1-\gamma)}{(1-\alpha)}
\end{aligned}$$

$$\begin{aligned}
\frac{-1(1-\alpha)}{(1+\alpha)} + \frac{2(1-\alpha)}{(1-\alpha^2)} &< 2(1-\gamma) \\
\frac{-1(1-\alpha)}{(1+\alpha)} + \frac{2(1-\alpha)}{(1-\alpha)(1+\alpha)} &< 2(1-\gamma) \\
\frac{-1(1-\alpha)}{(1+\alpha)} + \frac{2}{(1+\alpha)} &< 2(1-\gamma) \\
\frac{-2+\alpha}{(1+\alpha)} + \frac{2}{(1+\alpha)} &< 2(1-\gamma)
\end{aligned}$$

$$\begin{aligned}
\frac{\alpha}{(1+\alpha)} &< 2(1-\gamma) \\
\frac{1}{2(1-\gamma)} &< \frac{1+\alpha}{\alpha} \\
\frac{1-2(1-\gamma)}{2(1-\gamma)} &< \frac{1}{\alpha} \\
\alpha &< \frac{2(1-\gamma)}{1-2(1-\gamma)} \\
\alpha &< \frac{1-\gamma}{\gamma-\frac{1}{2}}
\end{aligned}$$

$\gamma > 1/2$ must be sufficiently low.

Fourth element, (4), $-(1-\gamma)\ln g_A + \ln c_x - \gamma\ln L$: With g_A roughly estimated between 0.05 and 0.1, and $\gamma > 1/2$ (see one line above) we obtain a value of less than 1.5 for $-(1-\gamma)\ln g_A$. This small value together with log of output costs of intermediate goods, $\ln c_x$, can be assumed to be smaller in absolute terms than the log of total labor force, $\gamma\ln L$. Thus, the conditions for a negative reaction $(1-\tilde{\theta})$ when α increases are not particularly restrictive.

Derivative of the growth rate: $\frac{dg_Y}{d\alpha}$

$$\begin{aligned}
g_Y &= \alpha^{\frac{2\gamma}{1-\alpha}} (1-\tilde{\theta})^{\frac{\gamma}{1-\alpha}} c_x^{-\frac{\gamma}{1-\alpha}} L^\gamma (g_A)^{1-\gamma} \\
\ln g_Y &= \frac{2\gamma}{1-\alpha} \ln \alpha + \frac{\gamma}{1-\alpha} \ln (1-\tilde{\theta}) - \frac{\gamma}{1-\alpha} \ln c_x + \gamma \ln L + (1-\gamma) \ln g_A \\
\frac{d \ln g_Y}{d\alpha} &= -\frac{\gamma}{(1-\alpha)^2} (-) \left[2 \ln \alpha + \ln (1-\tilde{\theta}) - \ln c_x \right] + \frac{2\gamma}{1-\alpha} \frac{1}{\alpha} + \frac{\gamma}{1-\alpha} \frac{d \ln (1-\tilde{\theta})}{d\alpha} \\
\text{with } \frac{\gamma}{1-\alpha} \frac{d \ln (1-\tilde{\theta})}{d\alpha} &< 0 \text{ and } -\ln c_x < 0 \text{ we need to check only for}
\end{aligned}$$

$$= \frac{\gamma}{(1-\alpha)^2} \left[2 \ln \alpha + \ln (1-\tilde{\theta}) \right] + \frac{2\gamma}{1-\alpha} \frac{1}{\alpha}$$

$$\begin{aligned}
&= \frac{\gamma}{(1-\alpha)^2} \ln(1-\tilde{\theta}) + \frac{\gamma}{(1-\alpha)^2} 2 \ln \alpha + \frac{2\gamma}{1-\alpha} \frac{1}{\alpha} < 0 \\
&= \frac{\gamma}{(1-\alpha)^2} \left[\ln(1-\tilde{\theta}) + 2 \left[\ln \alpha + \frac{(1-\alpha)}{\alpha} \right] \right] < 0
\end{aligned}$$

as we know that $\lim_{\alpha \rightarrow 1} (\ln \alpha + \frac{1-\alpha}{\alpha}) = 0$, and $\ln(1-\tilde{\theta}) < 0$ the $\frac{d \ln g_Y}{d\alpha}$ turns negative at a sufficiently large α .

Derivative of the GDP level, $\frac{dY(t)}{d\alpha}$:

$$Y(t) = (1-\alpha^2) N(t) \alpha^{\frac{2\alpha}{1-\alpha}} (1-\theta)^{\frac{1}{1-\alpha}} (c_x)^{-\frac{\alpha}{1-\alpha}} L$$

$$\begin{aligned}
\ln Y(t) &= \ln(1-\alpha^2) + \ln N(t) + \frac{2\alpha}{1-\alpha} \ln \alpha + \frac{1}{1-\alpha} \ln(1-\theta) - \frac{\alpha}{1-\alpha} \ln(c_x) + \ln L \\
&= \ln(1-\alpha^2) + \ln N(t) + \frac{1}{1-\alpha} [2\alpha \ln \alpha - \alpha \ln(c_x) + \ln(1-\theta)] + \ln L
\end{aligned}$$

$$\frac{d \ln Y(t)}{d\alpha} > 0 :$$

$$\begin{aligned}
\frac{d \ln Y(t)}{d\alpha} &= \frac{-2\alpha}{(1-\alpha^2)} + \frac{-1}{(1-\alpha)^2} [2\alpha \ln \alpha - \alpha \ln(c_x) + \ln(1-\theta)] \\
&\quad + \frac{1}{1-\alpha} \left[2 + 2 \ln \alpha + \frac{d \ln(1-\theta)}{d\alpha} - \ln(c_x) \right] \\
&= \frac{-2\alpha(1-\alpha)}{(1-\alpha^2)(1-\alpha)} + 2 \frac{(1-\alpha^2)}{(1-\alpha)(1-\alpha^2)} + \frac{1}{1-\alpha} \frac{d \ln(1-\theta)}{d\alpha} + \frac{2 \ln \alpha (1-\alpha^2)}{(1-\alpha)(1-\alpha^2)} - \frac{\ln c_x}{1-\alpha} \\
&\quad - \frac{1}{(1-\alpha)^2} [2\alpha \ln \alpha - \alpha \ln(c_x) + \ln(1-\theta)] \\
&= \frac{2}{(1-\alpha^2)} + \frac{2 \ln \alpha}{1-\alpha} + \frac{1}{1-\alpha} \frac{d \ln(1-\theta)}{d\alpha} - \frac{1}{(1-\alpha)^2} [2\alpha \ln \alpha - \alpha \ln(c_x) + \ln(1-\theta)] - \frac{\ln c_x}{1-\alpha}
\end{aligned}$$

$$\begin{aligned}
&= \frac{2}{(1-\alpha^2)} + \frac{(1-2\alpha)2\ln\alpha}{(1-\alpha)^2} + \frac{1}{1-\alpha} \frac{d\ln(1-\theta)}{d\alpha} - \frac{1}{(1-\alpha)^2} [-\alpha \ln(c_x) + \ln(1-\theta)] - \frac{\ln c_x}{1-\alpha} \\
&= \frac{2(1-\alpha)^2}{(1-\alpha^2)(1-\alpha)^2} + \frac{(1-\alpha^2)(1-2\alpha)2\ln\alpha}{(1-\alpha^2)(1-\alpha)^2} + \frac{1}{1-\alpha} \frac{d\ln(1-\theta)}{d\alpha} - \frac{1}{(1-\alpha)^2} [(1-2\alpha)\ln(c_x) + \ln(1-\theta)] \\
&= \frac{2-2\alpha-2\alpha+2\alpha\alpha}{(1-\alpha^2)(1-\alpha)^2} + \frac{(1-\alpha^2-2\alpha+2\alpha\alpha^2)2\ln\alpha}{(1-\alpha^2)(1-\alpha)^2} \\
&\quad + \frac{1}{1-\alpha} \frac{d\ln(1-\theta)}{d\alpha} - \frac{1}{(1-\alpha)^2} [(1-2\alpha)\ln(c_x) + \ln(1-\theta)] \\
&= \frac{2-4\alpha+2\alpha\alpha}{(1-\alpha^2)(1-\alpha)^2} + \frac{(1-\alpha^2-2\alpha+2\alpha\alpha^2)2\ln\alpha}{(1-\alpha^2)(1-\alpha)^2} \\
&\quad + \frac{1}{1-\alpha} \frac{d\ln(1-\theta)}{d\alpha} - \frac{1}{(1-\alpha)^2} [(1-2\alpha)\ln(c_x) + \ln(1-\theta)] \\
&= \frac{2(1+\ln\alpha)+2\alpha^2(1-2\ln\alpha)-2\alpha(2+2\ln\alpha)+2\alpha^3 2\ln\alpha}{(1-\alpha^2)(1-\alpha)^2} \\
&\quad + \frac{1}{1-\alpha} \frac{d\ln(1-\theta)}{d\alpha} - \frac{1}{(1-\alpha)^2} [(1-2\alpha)\ln(c_x) + \ln(1-\theta)] \\
&= \frac{2[(1-2\alpha)(1+\ln\alpha)+\alpha^2+2(-\alpha^2+\alpha^3)\ln\alpha]}{(1-\alpha^2)(1-\alpha)^2} \\
&\quad - \frac{1}{(1-\alpha)^2} [(1-2\alpha)\ln(c_x) + \ln(1-\theta)] + \frac{1}{1-\alpha} \frac{d\ln(1-\theta)}{d\alpha}
\end{aligned}$$

First element (1) $\frac{2[(1-2\alpha)(1+\ln\alpha)+\alpha^2+2(-\alpha^2+\alpha^3)\ln\alpha]}{(1-\alpha^2)(1-\alpha)^2} > 0$ for $\alpha > 1/2$

Second element (2), $-\frac{1}{(1-\alpha)^2} [(1-2\alpha)\ln(c_x) + \ln(1-\theta)] > 0$:

$$\begin{aligned}
(1-2\alpha)\ln(c_x) + \ln(1-\theta) &< 0 \\
\ln(1-\theta) &< -(1-2\alpha)\ln(c_x)
\end{aligned}$$

Third element (3) $+\frac{1}{1-\alpha} \frac{d\ln(1-\theta)}{d\alpha} < 0$: If conditions in the section above hold, this element is negative. The dominating elements cannot be derived unambiguously, and it is not easy to find reasons for one or the other effect to be the dominant one.