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DEPARTMENT OF ECONOMICS

Working Paper

The long run dynamics of economic growth with environmental catastrophe

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UNIVERSITY OF MASSACHUSETTS AMHERST

The long run dynamics of economic growth with environmental catastrophe

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Abstract

The purpose of this paper is to consider the dynamics of growth in a two state variable and two control variable model where the environment is taken as a constraint. This captures some elements of environmental problems not covered in the cost approach. It also captures the idea that the environment may be an absolute barrier or have a catastrophe boundary. It show that, even though the environment is not a cost, it may be optimal to cut growth before the barrier is reached. It also shows that the technology of production has a strong non-linear affect on maximum attainable output.

JEL classifications: O4, Q54, Q58

1 Introduction.

Societies produce material goods by combining already produced goods with energy and other resources from the environment and it is often claimed that the existing stock of environment resources is being reduced below a level which is sustainable. At one level it seems to make sense the the environment is being run down. At another it is more difficult to see how this might be understood in terms of the dynamics of growth.

The environmental literature takes the environment in a general sense of the atmosphere, oceans, land, forests, marine life and so on and emphasizes linkages between say agricultural production and destruction of marine environments, or between fossil fuels and climate change. In this literature there is clearly a strong distinction between the natural environment and the production process. This is perhaps not so clear in some of the economic literature where natural resources are taken as an input into production sometimes treated as substitutable for other forms of capital as in chemical fertilizer for more land or plastic for wood. For our purposes the important distinction is between produced goods in the usual sense and environmental goods. This does not mean that environmental goods cannot be partially or indirectly produced. For example, clean air can be produced by scrubbing outputs from power generators, cars and the like or even the atmosphere.

Land fertility can be regenerated, marine environments can be rebuilt by cleaning the oceans or restocking. Even the planet can, maybe, be geo-engineered. Nonetheless, this process is not the same as the typical process for manufacturing capital and other goods.

Rather than be too specific it is assumed there is a meaningful distinction between manufactured capital and environmental goods. The paper then develops a model to study the characteristics of economic growth when environmental limits are taken into account? In order to make the question interesting it is assumed that the environment only exists as a constraint. It has no intrinsic value.

Treating the environment as a constraint captures a position between those who wish to attribute both a constraint and a value to the environment and those who value production without limit. It raises a refinement to the previous question. If the environment only exists as a constraint, would it be optimal to spend resources preserving it at any greater level than the minimum required?

Another consequence of this treatment is that the limits to the environment are hard. They are exogenous to the process of determining the optimal growth path and not determined by attempting to choose the amount of damage that optimizes a payoff function. This approach is common in economics and has some serious problems. Not the least of these is that it imposes an economic skin on a problem in physical science and needs to assign values to uncertain risks with possible catastrophic outcomes. It would seem at least as desirable to consider the best way of progressing within given limits. See, for example, Ackerman for a discussion [1].

No attempt will be made to include a process whereby the environment regenerates. Some regeneration is implicitly included in the lower limit to the stock of environmental good . The process of interest is where regeneration is facilitated by resource inputs such as rehabilitating land, scrubbing emissions, restoring waterways, geo-engineering and the like. In an indirect way a decision not to extract resources or to create reserves, for example, can also be thought of as providing an environmental input.

Technology is important in the rate of environmental degradation. It has been included in the analysis as a choice parameter and fixed for the life of the programme. This is not ideal, but it seems a better way of capturing the lumpiness of technology than treating it as either a continuous variable or as a discontinuous spike.

A complete review of the literature is beyond the scope of this paper. It takes a different approach from the cost benefit that has dominated much of the discussion by assuming the constraints are given and asking different questions. Perhaps the most well known work in this area is that of Nordhaus and Stern [**?**] [4]. It differs from the study of optimal growth with environmental inputs by Beltratti, Chichilnisky and Heal in that their paper the environment is an input into production and also regenerates [2]. In addition their only choice variable is the level of consumption which determines the rate of growth of capital. In this paper the level of investment and the level of resources used to regenerate the environment are both control variables.

Among the main results are that the planner either switches off expenditure on economic growth before the

constraint is reached and slows the rate at which the environment degrades or, in some circumstances, stops growth in production to rebuild the environment. In addition technology has a strong non-linear affect on the attainable long run level of production and consumption.

In the next section I develop the model. It is analyzed in section three.

2 The model

2.1. Dynamics of growth and the environment

Suppose we have some capital stock that produces a single all purpose good that can be used for producing consumer goods or more capital stock or to offset the degradation of environmental resources by the production process. The rate at which this degradation takes place depends on the technology. The planner wants to choose a technology and a level of investment in growth and the environment that maximizes a welfare function over an indefinite time horizon. The environment has no direct value. There is, however, a constraint created by the fact that the environmental resource needs to be able to provide the required inputs into production and consumption.

The stock of capital at time t is $k(t)$ and the production function is written $\varphi f(k(t)$ where the parameter φ is a function of the environmental damage for each unit of production and depends on the choice of technology given by ϖ. As ϖ decreases the environmental damage caused by a technology decreases. This means a smaller ϖ indicates an improved technology. On the other hand a smaller ϖ is more expensive. Reducing the first ten percent of damage is more expensive than reducing the last ten percent. It is assumed that ϖ is optimally chosen and remains constant throughout the programme. If it is assumed that a less damaging technology reduces the output for each unit of input in an increasing manner $\varphi = \varphi(\varpi)$ with $\varphi_{\varpi} > 0$ and $\frac{\partial \varphi_{\varpi}}{\varpi} < 0$. It also makes sense to put some lower bound on ϖ through the cost function. The idea is that no matter how good the technology there will be some damage to the environment regardless of how much is spent on the technology. To do this let $\varphi_{\varpi} > 0$ be arbitrarily large as $\varpi \to \epsilon$ for some $\epsilon > 0$. I will return to the question of how ϖ is chosen below.

The amount consumed at time t is $C(t)$ and the amount invested in the environmental good is $\beta(t)$. This gives the rate of change of the capital stock as $\dot{k} = \varphi f(k) - C - \beta$. For this problem the interesting controls are levels of investment rather than the level of consumption. Let $C = \varphi f(k)$ minus total investment and write investment in capital stock as $\alpha(t)$. This means that growth equation an be written

$$\dot{k} = \alpha \tag{1}$$

with $k(0) = k_0$ given. In what follows t is omitted to simplify the notation unless necessary.

The stock of the environmental good is s and this is expressed in the same units as the stock of the all purpose good. This is done by constructing a measure such that one unit of s is the same as one unit of k. I don't give any details except to note that underlying arguments that too much environment is being used or

that there is plenty left is an implicit measure. For example, consider the case where some amount of timber and fish and fuels are being extracted and various waste products are being dumped. In principle these have an economic value and similarly the stock of these things or the natural machinery that produces them can be imputed a value.

Changes in the stock of environmental good will depend on the level of production and the amount of the all purpose good used to reduce the rate of degradation. For a specific level of capital stock there will be a constant drain on environmental resources. It is assumed that the environment can be maintained in the sense of protecting it from further deterioration and that it can be rebuilt. It seems reasonable to guess that the cost of rebuilding the environment increases in a non linear manner. That is at early stages of deterioration programmes such as tree planting or protection marine environments may be relatively cheap. Beyond this they become more expensive. This means the dynamics for the environmental stock can be written

$$\dot{s} = h(\beta) - \varpi k \tag{2}$$

where h is is a continuously differentiable one to one function. Let $h_\beta > 0$ and $\frac{\partial h_\beta}{\partial \beta} < 0$. Let $s(0) = s_0$.

It is assumed that some minimum stock of the environmental good is required in order to keep the system in equilibrium, for example a minimum mass of fish to ensure reproduction or of forest to provide for habitats and soil enrichment and climate. This minimum depends on the total amount of production. It is given by

$$s \geq \sigma k \tag{3}$$

for some $\sigma > 0$ and $\sigma \geq \varpi$. It is assumed that $k_0 < \frac{s_0}{\sigma}$.

2.2. Planner's problem

The planner wants to choose the level of investment in capital growth and the environmental good that will maximize consumption over time given the dynamics of the system and the given constraints. The performance index is

$$J = \int_0^\infty e^{-\delta t} u(C) dt \tag{4}$$

$\delta > 0$ is the discount factor.

To simplify the analysis let $u = C$. Set $\varphi = 1$ for the time being without loss of generality.

To allow for consumption it is assumed that there is some upper limit to the amount of resource that can be used for investment in capital formation and in the environment at any time. This is $f(k) - \alpha - \beta \geq j$ for $j > 0$ some constant.

It is assumed that a solution exists. It is also assumed as a conjecture that there is a stationary state with $\dot{p}, \dot{q}, \dot{k}, \dot{s} = 0$ and hence $\alpha = 0$ and $\beta = \varpi k$ for all $t \geq c$. This conjecture is proven below.

3 Solution

3.1 Summary and method

The planner's programme only operates one control at any time before the environmental constraint is reached. There are two types of trajectories. The first trajectory starts from an initial stock of the environmental good sufficiently high and expenditure on capital growth is positive until some point $t = b$ where, in general, the constraint has not been reached. At this point expenditure on capital goods goes to zero and expenditure on the environmental good is greater than zero. The environment is allowed to degrade until the stationary state is reached. This is somewhat striking. It tells us that on the optimum trajectory economic growth should be shut down before the environmental stock is driven to a minimum. In the second investment in capital stock is positive until the constraint is reached and then the trajectory moves up this constraint until the stationary state is reached. This will typically occur when the initial level of the environmental stock is low. Roughly it is a poverty path. It is driven by a greater need to increase material production from low levels. These paths are set out in fig. 1.

To justify these assertions the necessary conditions will be derived using the Pontryagin maximum principle. Under the assumption a solution exists a path which satisfies these conditions must be optimal.

The Hamiltonian and Lagrangian functions are

$$H = f(k) - \alpha - \beta + p\alpha + q(h(\beta) - \varpi k) \tag{5}$$

with

$$L = H + \psi(f(k) - \alpha - \beta - j) + \theta(s - \sigma k)$$

where $p(t)$ and $q(t)$ are costate variables and $\psi \geq$ and $\theta \geq 0$ are multipliers associated with $(f(k) - \alpha - \beta - j)$ and $\theta(s - \sigma k) = 0$. Since it is obvious leave out the ψ multiplier.

The necessary conditions are that α and β maximize H subject to equation 3 and equations 1 and 2 together with

$$\dot{p} = \delta p - f_k + \varpi q + \theta\sigma \tag{6}$$

$$\dot{q} = \delta q - \theta \tag{7}$$

and this gives

$$\ddot{p} = \delta - \frac{f_k}{\partial k}\dot{k} + \varpi\delta q \tag{8}$$

The derivatives on the Hamiltonian are

$$H_\alpha = -1 + p \tag{9}$$

$$H_\beta = \frac{-1}{h_\beta} + q \tag{10}$$

A trajectory that satisfies the necessary conditions is optimal because $f - \alpha - \beta$ is concave or quasi-concave in (k, α, β), $p\alpha$ and $q(ln\beta - \varpi k)$ and $\sigma(s - \theta k)$ are concave or quasi-concave in (k, s, α, β).

It can now be shown that the maximum value of k has an upper bound. From the properties of h there is an inverse function $g : \beta = g(\varpi k)$. Even if the stationary point is not reached in finite time and $c \to \infty$ the system spends an arbitrarily large amount of time arbitrarily close to a stationary point the Hamiltonian is $f - g$. It follows that

$$k\ max = K \leq k : f_k = \varpi g_{\varpi k} \tag{11}$$

and since f_k is decreasing and $g_{\varpi k}$ increasing this has a unique solution from the intermediate value theorem.

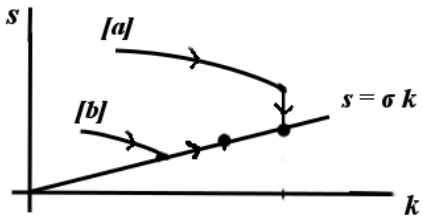

Figure 1. Examples of optimal trajectories

3.2 Trajectories

The first proposition shows that either the amount spent on economic growth or the amount spent on the environment must be zero in all measurable intervals when the environmental constraint is not active. I then prove the assertions.

Proposition 1: The optimal trajectory does not permit $\alpha > 0$ and $\beta > 0$ in any measurable interval in $(0, c)$ with $s < \sigma k$ for $t < c$.

Proof: Suppose there is an interior solution with $\alpha > 0$ and $\beta > 0$ in some measurable interval (a, b). Then from equations (9) and (10) we need $p = 1$ with $\dot{p} = 0$. From equation (6)

$$\dot{p} = \delta - f_k + wAe^{\delta t} = 0$$

and since either $f_k = 0$ or $f_k < 0$ the inequality cannot be maintained in any measurable interval. Contradiction.

□

An immediate consequence of this is that the conjecture about the existence of a stationary state is now proven. This is because $\dot{k} > 0$ and $\dot{s} \geq 0$ requires $\alpha > 0$ and $\beta > 0$. It is possible that the stationary state is not reached in finite time and $c \to \infty$. This cannot happen because $\beta = z(q)$ and from the properties of the function h we have $z_q > 0$ and $\frac{\partial z_q}{q} > 0$. This means $\beta = z(Ae^{\delta t})$ and hence $\dot{\beta} > 0$ and $\ddot{\beta} > 0$. It follows that it is not possible to hold β arbitrarily close to a fixed σk for an arbitrarily large period of time.

Proposition 2: The optimal trajectories are almost always of two types. [**a**] For s_0 sufficiently large and f_k evaluated at $k > k_0$ bounded $\alpha = \alpha\ max$ until $t = b$ where $s(b) > \sigma k(b)$. For $t > b$ we have $\alpha = 0$ and $\beta > 0$ until $t = c$ and $s(c) = \sigma k(c)$ at a stationary state. [**b**]. For f_k sufficiently large $\alpha = \alpha\ max$ until $t = b$ where $s(b) = \sigma k(b)$. In the interval (b, c) for $c \geq b$ we have $\alpha > 0$ and $\beta > 0$ with $s = \sigma k$ until $t = c$ and $s(c) = \sigma k(c)$.

Proof: The first thing is to show that $\alpha = \alpha\ max$ in the continuous interval $(0, b)$ and $\alpha = 0$ otherwise when $s > \sigma k$. Suppose not. This requires $\dot{p} > 0$. It is only possible for the multiplier θ to set $\dot{p} = 0$ when $s(c) = \sigma k(c)$ if $\dot{p} < 0$ at $t = c$. For this to be possible $\ddot{p} < 0$ for $\dot{p} = 0$. From equation (8) this requires

$$-\frac{df_k}{dk}\dot{k} + \varpi\delta q < 0$$

and the contradiction from the fact that $\frac{df_k}{dk} < 0$.
The second thing to show is that the stationary state cannot, in general, be reached at $t = b$ on a continuous trajectory without switching. Assume it can. This is only possible with $\alpha > 0$ at $t = b$ and hence $p \geq 1$. If $p > 1$ then $\beta = 0$ so $t = b$ cannot be a stationary state. This means $p = 1$. At $t = b$ equation (7) requires $\theta = \delta q$. From equation (10) we can write $q = y(\varpi k)$ at $t = c$ since h_β is a function of β alone and hence $\dot{p} = \delta - f_k + y(\varpi k)(\varpi + \delta\sigma)$. It follows from the fact that $\alpha = max\ \alpha$ that k is determined by $\int_0^c max\ \alpha + k_0$ and c is determined by $s(c) = \sigma k(c)$. This means that $\dot{p}$ is determined at $t = c$ by s_0 and k_0. This contradicts the requirement that $\dot{p} = 0$.
This leaves two trajectories.
Trajectory [**a**]. The first possibility is that $\alpha > 0$ until $t = b$ and $p < 1$ for $t > b$. This gives $\beta > 0$ and $\dot{\beta} > 0$ for $t \in (b, c)$ and the stationary state is reached at $t = c$. In this case p_0 is chosen such that $p = 1$ at $k(b) : f_k$ has the value at $t = c$ required for $\dot{p} = 0$ with $\theta : \dot{q} = 0$.
It might also be possible that there is a θ such that p jumps up to $p = 1$ at $t = c$ and $\dot{p} = 0$ in some interval $c, m)$. Since $\dot{p}$ is determined by the p_0 required for $p(b) = 1$ it would be necessary for $\theta(b)$ to have the value at $t = c$ for p to jump and for $\dot{p} = 0$ at $t = c$. In general this will not be possible.
[**b**]. The second possibility is that the constraint is reached at $t = b$ and the controls are $\alpha > 0$ and $\beta > 0$ with $s = \sigma k$ in the interval (b, c). This trajectory differs from the case where there is a stationary state at $t = b$ because θ is not determined. A stationary state is reached at $t = c$ where k and c depend on the value of α in the interval (b, c). For $\dot{p} = 0$ at $t = c$ it is necessary that $K : y(\varpi k)(\varpi + \delta\sigma) = f_k - \delta$. This is feasible if f_k is sufficiently large at $t = b$.
□

Proposition 2 says that for some given f with $\frac{\partial f_k}{\partial k} < 0$ and k_0 path [**a**] will occur for s_0 is large. This follows

from the fact that $k(c) = \int_0^c \alpha max$ and f_k will be too small at $t = c$ when $s = \sigma k$ for the adjustment in the value of k needed to set $\dot{p} = 0$ obtained by following path [**b**]. Conversely if s_0 is sufficiently low the time to reach $s = \sigma k$ is less and f_k can be large. If it were possible that a path would hit $k\ max$ before the switch it would be necessary to introduce the condition that $k < k\ max$. This would still be a path of type [**a**].

In different terms proposition 2 says that the programme anticipates the constraint in [**a**] and stops investment in economic growth before this constraint is met. Why do this? An after the fact intuition is that as k increases the cost of maintaining the stationary state out to the infinite time horizon will increases and it is better to switch rather than wait. This must be the case because there is an upper bound on the maximum stock of the capital good. This also gives something of a bridge between the position of the environmentalists and the advocates of growth. Even if the environment is treated as a free good the constraint forces early preservation measures.

It is difficult to see how this translates in any detail without information on the initial conditions and a way of constructing a metric. One possibility is that the system has already gone beyond the switching point at $t = b$ or beyond $s = \sigma k$. In these cases a variation to the model would be required to analyze the optimum trajectories.

3.3. Choice of technology and maximum value

The technology of production will affect the maximum value for k in equation (11). Taking the derivative of this equation with respect to ϖ gives

$$\frac{\partial f_k(max\ k)}{\partial \varpi} = k g_{\varpi}$$

and since $\frac{\partial g_{\varpi}}{\varpi} > 0$ ϖ has a non-linear effect on the the maximum attainable k.

It has been assumed that ϖ is optimally chosen. If it were cost free it would be set at zero to make $k\ max \to \infty$. This doesn't make much sense and it is now necessary to put the cost $\varphi(\varpi)$ into the performance index. To get the optimum ϖ we need to maximize the performance index in equation (4). This will be the same as maximising $\int (H - p\dot{k} - q\dot{s})$. Put this into Lagrangian form. Partial integration gives

$$v = \int_0^{\infty} (L + \dot{p}k + \dot{q}s)dt - (pk + qs)|_0^{\infty}$$

To get the total differential with respect to ϖ use the fact that along the optimal trajectory the partials of the Lagrangian with respect to the controls and the states will become zero. This leaves $\frac{\partial L}{\partial \varpi}$ inside the integral. It is also necessary to take onto account the discount rate when comparing the values of the integral terms across different intervals. Return to the present value formulation and express the present value co-states as $\bar{p}$ and $\bar{k}$. Using Leibnitz's rule and the fact that $(k+s)|_c^{\infty} = 0$ the optimum in the current value programme is

$$v_{\varpi} = \int_0^c (e^{-\delta t}\varphi_{\varpi} f(k) - \bar{q}k)dt - \bar{p}k(c)_{\varpi} - \bar{q}s(c)_{\varpi} + \int_c^{\infty} (e^{-\delta t}\varphi_{\varpi} f - kg(\varpi k)_{\varpi})dt = 0$$

where the $L(c)c_\varpi$ cancels since it is evaluated as a plus and a minus at the junction point $t = c$.

Using $\bar{p} = e^{-\delta t}p$ with a similar expression for q

$$v_\varpi = \int_0^c e^{-\delta t}(\varphi_\varpi f - qk)dt - e^{-\delta c}pk(c)_\varpi - e^{-\delta c}ps(c)_\varpi + \int_c^\infty e^{-\delta t}(\varphi_\varpi f - kg_\varpi)dt$$

Using the mean value theorem for integrals this can be rewritten

$$(1 - e^{-\delta c})c(\varphi_\varpi f(k(b)) - q(b)k(b)) + e^{-\delta c}(B - c)((\varphi_\varpi f(k(c)) - k(c)g_\varpi)) + \vartheta_\varpi$$

for $b < c$ and $\vartheta_\varpi = -e^{-\delta c}(pk(c)_\varpi + qs(c)_\varpi)$ and $B \to \infty$.

We cannot solve this without specifying the functions. To get an estimate note that for a fixed c the term involving B dominates the expression for v_ϖ as B becomes arbitrarily large. Let $k(c) = \bar{k}$. It follows that

$$\varphi_\varpi \bar{f} \approx \bar{k} g_\varpi \tag{12}$$

3.4 Examples

Example 1. The first example takes the case where $f(k) = k$ and $h(\beta) = 2\sqrt{\beta}$ to give $k\ max = \frac{2}{\varpi^2}$. Taking the second derivative $\frac{\partial k\ max}{\partial \varpi} = -\frac{4}{\varpi^3}$. This means that an improvement in the technology accelerates the maximum attainable k at a rate given by the inverse of ϖ^3.

Example 2. Now set $f(k) \neq k$. Then $k\ max$ is $k : f_k = \frac{\varpi^2 k}{2}$ and q at the stationary point is $\frac{\varpi k}{2}$. It is immediate from equation (6) evaluated at $\dot{p} = 0$ that $\frac{\partial f}{\partial k} > \frac{\varpi^2 k}{2}$. It follows that the optimum vale of k is less than $k\ max$.

Example 3. Continue with $\beta = \frac{(\varpi k)^2}{4}$ for all $t > c$ and let $\varphi = 2\sqrt{\varpi}$ with $f = k$. Using the approximate values this gives

$$\varphi_\varpi \bar{f} \approx \frac{\varpi \bar{k}^2}{2}$$

It follows that equation (12) becomes

$$\varpi = \bar{k}^{-\frac{2}{3}} \quad \text{and} \quad \varphi = \bar{k}^{-\frac{1}{3}}$$

and this will have a solution for all $\varphi : \frac{2\varphi_\varpi}{\varpi} > \frac{\bar{k}^2}{\bar{f}}$ as $\varpi \to 0$ and $\frac{2\varphi_\varpi}{\varpi} < \frac{\bar{k}^2}{\bar{f}}$as $\varpi \to B$.

4 Final remarks

The aim of the paper has been to get some understanding of the trajectories of capital growth and of environmental change when the environment acts as a constraint on production and consumption. Here is a summary.

The most striking feature of the model is that the optimal programme terminates capital growth short of the environmental constraint when initial stocks of the environmental good are sufficiently high. It then spends

resources slowing the rate of environmental degradation. In some ways this parallels the environmentalist argument. When initial stocks are low and the rate of return on investment in capital goods is high it may be best to rebuild the environment along the constraint. If a little speculation is allowed, it would appear that the market does not provide a mechanism for tracking the best trajectory. It may not even have the signals necessary to terminate on the constraint.

It was also shown that as the technology of production has an important role in determining the stable level of production and consumption in the long run. It increases the maximum attainable level of capital stock in a non-linear manner.

It is not clear where the system is in diagram 1. If it is to the left of the switching point in path [**a**] a different model is required to determine the path to go.

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