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A Model of Housing and Credit Cycles with Imperfect Market Knowledge

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Abstract

The paper presents a model of housing and credit cycles featuring distorted beliefs and comovement and mutual reinforcement between house price expectations and price developments via credit expansion/contraction. Positive (negative) development in house prices fuels optimism (pessimism) and credit expansion (contraction), which in turn boost (dampen) housing demand and house prices and reinforce agents' optimism (pessimism). Bayesian learning about house prices can endogenously generate self-reinforcing booms and busts in house prices and significantly strengthen the role of collateral constraints in aggregate fluctuations. The model can quantitatively account for the 2001–2008 U.S. boom-bust cycle in house prices and associated household debt and consumption dynamics. It also demonstrates that allowing for imperfect knowledge of agents, a higher leveraged economy is more prone to self-reinforcing fluctuations.

Keywords: Boom-Bust, Collateral Constraints, Learning, Leverage, Housing
JEL classifications: D83, D84, E32, E44

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"At some point, both lenders and borrowers became convinced that house prices would only go up. Borrowers chose, and were extended, mortgages that they could not be expected to service in the longer term. They were provided these loans on the expectation that accumulating home equity would soon allow refinancing into more sustainable mortgages. For a time, rising house prices became a self-fulfilling prophecy, but ultimately, further appreciation could not be sustained and house prices collapsed." (Bernanke (2010))

1 Introduction

The recent decade has witnessed a massive run-up and subsequent collapse of house prices, as well as the remarkable role of the interaction of housing markets and credit markets in aggregate fluctuations in the U.S. economy. Real house prices increased considerably in the decade before the recent financial crisis, as seen in the upper panel of figure 1.[1] They displayed relatively smaller variability before the year 2000 and increased by 35.9% from 2001 to 2006 in which house prices peaked. Associated with the price boom was a sharp increase in the household credit market debt/GDP ratio[2] and a consumption boom. As can be seen from the lower panel of figure 1, the household credit market debt/GDP ratio changed moderately before the year 2000 but increased from 45% in 2001 to 70% in 2006. Aggregate consumption[3] grew at about 3% per annum between 2001 and 2006, while its growth dropped sharply after house prices started to revert, as shown in figure 2.

A number of recent research document over-optimistic expectations about the future path of house prices and the comovement between the expectations and house price developments during the housing boom preceded the financial crisis. Cheng, Raina and Xiong (2014) study personal home transaction data of the mid-level Wall Street managers in securitized finance both on the buy and sell side, which is supposed to reveal their beliefs about the path of house prices. They document that the securitization agents held over-optimistic beliefs about future house prices and call for serious considerations of the role of beliefs in the financial crisis and the macroeconomic implications of their belief dynamics. Case, Shiller and Thompson (2012) document over-optimistic expectations of home buyers using the Case-Shiller home buyer survey implemented at four metropolitan areas of the U.S.. Based on the data from Michigan Survey of Consumers, Piazzesi and Schneider (2009) find that the "optimism" in the housing markets, i.e., the share of agents believing prices to increase further, co-moved positively with the house price level and peaked exactly when house prices reached its peak.

Deriving house price forecasts from the future markets for the Case-Shiller house price index (where only the data from 2006 onwards are available), Gelain, Lansing, and Mendicino (2013) find the comovement and mutual reinforcement between agents' pessimistic price

[1] The data is taken from the OECD. Its definition is the "national wide single family house price index". The real house price index is the nominal house price index deflated by CPI price index. It is normalized to a value of 100 in 2000. The price-to-rent ratio and price-to-income ratio display a similar pattern.

[2] The household credit market debt/GDP ratio is measured by the absolute value of the ratio of net credit market assets of US household and non-profit organizations to GDP. The data is from the Flow of Funds Accounts of the U.S. provided by the Board of Governors of the Federal Reserve System.

[3] The data is from Federal Reserve System. It is the Real Personal Consumption Expenditures (series ID: PCECC96).

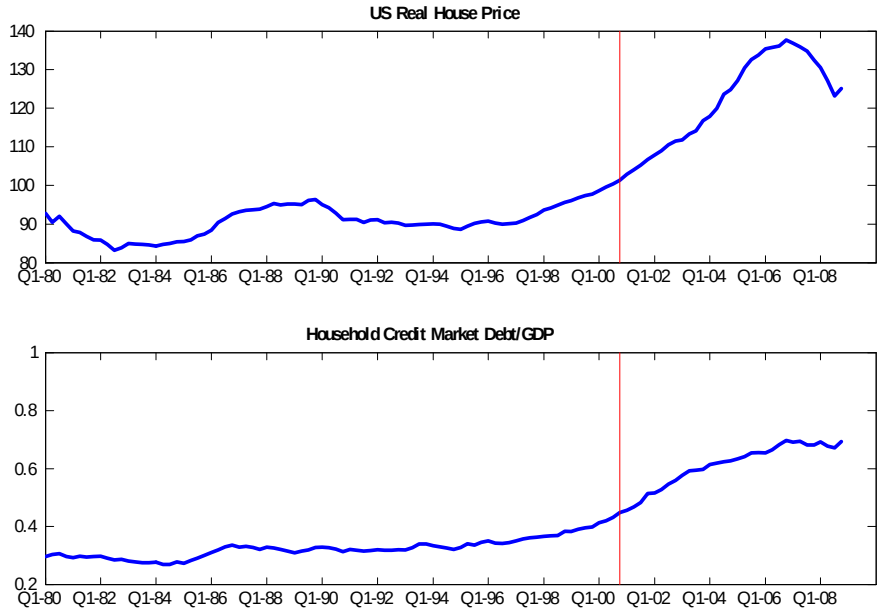

Figure 1: US Real House Prices and Household Credit Market Debt/GDP

beliefs and price realizations during the house price reversal, in particular, "the future market tends to overpredict future house prices when prices are falling" and persistent one-sided forecast errors.

The paper develops a model of housing and credit cycles with a housing collateral constraint à la Kiyotaki and Moore (1997, henceforth KM) but incorporating an explicit role for subjective beliefs consistent with the evidence in the work mentioned earlier. The model can quantitatively account for the 2001–2008 U.S. boom-bust cycle in house prices and associated debt and consumption dynamics following the strong fall in real interest rates after the year 2000.

In the model agents know their own objective, constraints and beliefs but have imperfect knowledge about the macroeconomy, such as other agents' preferences and prices beliefs unlike in the standard Rational Expectations (RE) modeling. Relaxing such informational assumption leads to agents' uncertainty about the equilibrium mapping between fundamentals (e.g. preference shocks, house holdings) of the economy and house prices, which is similar to that economists appear to be uncertain about the right model governing house prices. Following Adam and Marcet (2011), agents are assumed to be "Internally Rational," i.e., making optimal decisions under a completely specified and dynamically consistent subjective belief system about all payoff-relevant variables, including house prices. Internally rational agents do not understand how market prices are formed, so their subjective price beliefs need not be exactly the same as the objective price density as under RE. Yet their subjective beliefs are near-rational or close to the RE equilibrium beliefs. Optimal decisions imply that agents apply Bayes' law to equilibrium outcomes.

The Bernanke quotation before the Introduction can be viewed as a rough statement of the mechanism of the model and more details are as follows. The trigger of the price boom in the model is the persistently low interest rates after the year 2000. Responses of prices

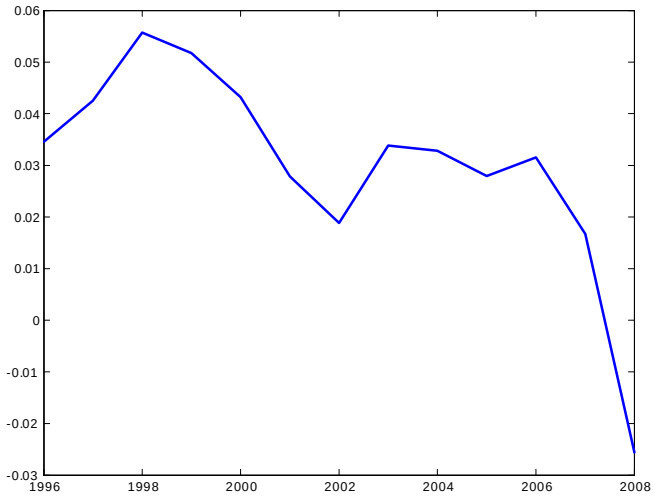

Figure 2: US Real Consumption Growth

and quantities in the model are drastically amplified due to the comovement and mutual reinforcement between agents' expectations and house price developments via credit expansion/contraction. Positive development or surprise in house prices brings about agents' belief revision and optimistic expectations. Optimism about future house prices induces credit expansion, which in turn boosts housing demand and house prices. The realized house prices partially validate the optimism, which leads to further optimism and persistent increases in house prices. Associated with the price boom is a widening household credit market debt/GDP ratio due to rising collateral values. Production and consumption amplification arise from shifts of collateral to more productive borrowers.

At some point, house prices will fall short of agents' expectations due to the combination of rising interest rates and endogenous dynamics, i.e., the dominance of the negative effect of excessive debt repayments. [4]This sets a self-reinforcing decline in motion. Agents revise their belief downward and become pessimistic about future house prices and collateral values, which contracts lending, housing demand and house prices. The realized prices reinforce agents' pessimism and leads to further pessimism, inducing periods of persistent and downward adjustments of beliefs and actual prices.

Consistent with Adam, Marcet and Nicolini (2011),[5] the paper emphasizes the role of the persistently low interest rates in the house price boom. Another policy implication of the paper is that a higher leveraged economy is more prone to self-reinforcing fluctuations allowing for imperfect knowledge about asset prices. Increasingly optimistic price beliefs endogenously enhance borrowing capacity, so that the drag of debt repayments based on less optimistic past beliefs on house prices becomes relatively smaller. The positive effect of the relative reduction of the debt burden on house prices will be sufficiently large when the leverage ratio is sufficiently large, so that realized house prices will be sufficiently large and can reinforce agents' optimism. Similarly for increasingly pessimistic beliefs. Simulations

[4]More generally than this particular episode, house price reversal in the learning model can be due to policy changes, or exogenous changes in fundamentals, or endogenous dynamics, or a mix of them.

[5]Section 4.4.1 provides a detailed discussion of the relation of the paper to Adam, Kuang and Marcet (2011).

show that the amplification of house prices and quantities responding to the interest rates changes in the learning model relative to RE is non-linear and increasing in the leverage ratio of the economy.

The rest of the paper is structured as follows. Section 1.1 reviews related literature. Section 2 presents the model and agents' optimality conditions. Section 3 discusses the equilibrium with imperfect knowledge, belief specification and optimal learning behavior of agents. The mechanism of the learning model is inspected in section 4. Quantitative results and further intuition of the learning model are presented in section 5. Section 6 concludes.

1.1 Related Literature

A strand of literature has studied the role of collateral constraints as an amplification mechanism transforming relative small shocks to the economy into large output fluctuations, such as KM, Kocherlakota (2000), Cordoba and Ripoll (2004), Iacoviello (2005) among others. More recently, a number of papers along this line attempt to understand the recent house price dynamics and its macroeconomic implications, such as Ferrero (2011), Justiniano, Primiceri, and Tambalotti (2014), and Hoffmann, Krause and Laubach (2012). The model can generate additional non-fundamental price fluctuations and strengthen the role of collateral constraints by allowing for agents' uncertainty about the equilibrium mapping.

Liu, Wang, and Zha (2013) find housing demand shocks in their model with land collateral constraints as the major driving force of land price fluctuations. Learning may be viewed as a structural interpretation of such shocks. Boz and Mendoza (2013) study the role of learning about the riskiness of a new financial environment. Burnside, Eichenbaum and Rebelo (2011) present a model in which a temporary house price boom emerges from infectious optimism that eventually dissipates once investors become increasingly certain about fundamentals.

Iacoviello and Neri (2010) estimate a DSGE model with a housing collateral constraint via Bayesian methods using data from 1965 to 2006. They find an important role of monetary factors in housing cycles over the whole sample and an increasing role during the recent housing cycle. In addition, they find a nonnegligible spillover effect from housing markets to consumption over the whole sample and increasing importance of the effect in the recent housing cycle. The transmission mechanism of the model is consistent with their findings.

The paper relates to the literature which explores the role of self-referential learning in business cycle fluctuations but differs by incorporating a specific form of financial frictions. For example, Eusepi and Preston (2011) present a business cycle model with learning which improves the internal propagation of business cycle shocks and is consistent with many features of observed survey expectations data. Huang, Liu and Zha (2009) study implications of adaptive expectations in a standard growth model and find them promising in generating plausible labor market dynamics. Milani (2011) estimates a New Keynesian Model with adaptive learning incorporating survey data on expectations and finds a crucial role of expectational shocks as a major driving force of the U.S. business cycle.

The paper is also related to papers which study the role of self-referential learning in asset pricing or asset price booms and busts. For example, Timmermann (1996) and Carceles-Poveda and Giannitsarou (2008) study asset pricing with learning in an endowment economy and a production economy, respectively. While they find a limited role of adaptive learning in asset pricing, section 4.4.2 discusses why the role of learning is strengthened in a credit-

constrained economy. Adam, Marcet, and Nicolini (2012) and Adam, Beutel, and Marcet (2014) develop learning models which can quantitatively replicate several major stock pricing facts, stock markets booms and busts and survey return expectations in the U.S.. Lansing (2010) examines a near-rational solution to Lucas-type asset pricing models and learning to generate intermittent stock bubbles and to match many quantitative features observed in the long-run U.S. stock market data. Branch and Evans (2011) examine stock market booms and crashes in an asset pricing model with learning about the conditional variance of a stock's return. Gelain, Lansing and Mendicino (2013) study a DSGE model with a housing collateral constraint and a subset of agents using moving-average rules to forecast future house prices. They find that this "significantly amplifies the volatility and persistence of house prices and household debt" and examines various policy options to dampen the excess volatilities.

2 The Model

The model builds on the basic version of the KM model with the major difference of expectation formation.

2.1 The Model Setup

There are two types of goods in the economy, durable assets, i.e., houses, and nondurable consumption goods, which are produced using houses but cannot be stored. Following KM and Iacoviello (2005), houses play a dual role: they are not only factors of production but also serve as collateral for getting loans. Houses are modeled as a producing factor because production activities usually need space. There are two types of infinitely lived risk-neutral agents, households and financial intermediaries, each of which has unit mass. Both produce and consume the nondurable goods. At each date t, there are two markets. One is a competitive spot market in which houses are exchanged for consumption at a price of q_t, while the other is a one-period credit market in which one unit of consumption at date t is exchanged for a claim to R_t units of consumption at date $t+1$.

The expected utility of a household i is

$$E_0^{\mathcal{P}^i} \sum_{t=0}^{\infty} (\beta^B(i))^t c_t^B(i) \tag{1}$$

where $\beta^B(i)$ is his subjective discount factor and $c_t^B(i)$ his consumption in period t. The operator $E_0^{\mathcal{P}^i}$ denotes household $i's$ expectation in some probability space $(\Omega, S, \mathcal{P}^i)$, where Ω is the space of payoff relevant outcomes that the household takes as given in his optimization problem. The probability measure $\mathcal{P}^i$ assigns probabilities to all Borel subsets S of Ω. It may or may not coincide with objective probabilities emerged in the equilibrium. Further details about Ω and $\mathcal{P}^i$ will be provided in the next section.

The household i produces with a constant return to scale technology. His production function is

$$y_{t+1}^B(i) = (a + \overline{e}) H_t^B(i) \tag{2}$$

where $H_t^B(i)$ is the amount of used houses. Only the $aH_t^B(i)$ component of the output is tradable in the market, while $\overline{e}H_t^B(i)$ is perishable and non-tradable. The introduction of non-tradable output is to avoid continual postponement of consumption by households.

The household's production technology is assumed to be idiosyncratic in the sense that it requires his specific labor input. He always has the freedom to withdraw his labor, or in the language of Hart and Moore (1994), the household's human capital is inalienable. The households are potentially credit-constrained. The financial intermediaries protect themselves against risks of default by collateralizing the households' houses. The household i can at most pledge collateral $(1-\tau)\, E_t^{\mathcal{P}^j} q_{t+1} H_t^B(i)$.[6] Thus his borrowing constraint is

$$b_t^B(i) \leq (1-\tau)\, \frac{E_t^{\mathcal{P}^j} q_{t+1}}{R_t} H_t^B(i) \tag{3}$$

where $b_t^B(i)$ is the amount of loans, $E_t^{\mathcal{P}^j} q_{t+1}$ the financial intermediary j's expectation about the collateral price in period $t+1$, and R_t gross interest rate between t and $t+1$. The borrowing constraint says that a household can get a maximum loan which is equal to a fraction of the discounted and expected liquidation value of his house holdings at $t+1$.

The household faces a flow-of-fund constraint

$$q_t(H_t^B(i) - H_{t-1}^B(i)) + R_{t-1} b_{t-1}^B(i) + c_t^B(i) \leq y_t^B(i) + b_t^B(i) \tag{4}$$

He produces consumption goods using houses and borrows from the credit market. He spends on consuming, repaying the debt, and investing in houses.

A financial intermediary j's preferences are specified by a linear utility function. She maximizes the following expected utility

$$E_0^{\mathcal{P}^j} \sum_{t=0}^{\infty} (\beta^L(j))^t A_t c_t^L(j) \tag{5}$$

where $\mathcal{P}^j$ is her subjective probability measure and $\beta^L(j)$ is her subjective discount factor. A_t is an i.i.d innovation to the financial intermediary's patience factor following a truncated normal distribution with a bounded support $[\underline{A}, \overline{A}]$ and $E[A_t] = 1$. She faces the following budget constraint:

$$q_t(H_t^L(j) - H_{t-1}^L(j)) + b_t^L(j) + c_t^L(j) \leq y_t^L(j) + R_{t-1} b_{t-1}^L(j) \tag{6}$$

where $H_t^L(j) - H_{t-1}^L(j)$ is her investment in collateral holdings. She uses a decreasing return to scale technology to produce, i.e., $y_{t+1}^L(j) = G^j(H_t^L(j))$, where $G^{j'} > 0$, $G^{j''} < 0$.

The aggregate supply of the collateral is assumed to be fixed at $\overline{H}$. Later I will assume that all households (financial intermediaries) have the same subjective discount factor $\beta^B = \beta^B(i)$ for $\forall i$ ($\beta^L = \beta^L(j)$ for $\forall j$) and households are less patient than financial intermediaries, i.e., $\beta^B < \frac{\beta^L}{\overline{A}}$.

[6] If borrowers repudiate their debt obligations, lenders can repossess borrowers' collateral by paying a transaction cost proportional to the expected liquidation value of the collateral $\tau E_t^{\mathcal{P}^j} q_{t+1} H_t^B(i)$. One explanation is that debt enforcement procedures in real world are significantly inefficient and some value is lost during such procedures, as documented by Djankov, Hart, Mcliesh and Shleifer (2008).

2.2 Optimality and Market Clearing Conditions

Individual household i's optimal decisions with respect to consumption, borrowing and collateral demand are similar to those in the original KM paper. Since the return to investment in collateral holding is sufficiently high,[7] he prefers to borrow up to the maximum, consume only the non-tradable part of his output and invest the rest in collateral holdings. His optimal consumption is

$$c_t^B(i) = \overline{e} H_{t-1}^B(i) \tag{7}$$

and optimal borrowing

$$b_t^B(i) = (1-\tau)\frac{E_t^{\mathcal{P}^j} q_{t+1}}{R_t} H_t^B(i) \tag{8}$$

The household uses both his own resources and external borrowing to finance collateral holdings. Given that the household consumes only the non-tradable output, his net worth at the beginning of date t contains the value of his tradable output $aH_{t-1}^B(i)$, and the current market value of the collateral held from the previous period $q_t H_{t-1}^B(i)$, net of the debt payment, $R_{t-1}b_{t-1}^B(i)$. The household i's demand on collateral can be derived from (2), (4), (7),and (8)

$$H_t^B(i) = \frac{1}{q_t - \frac{1}{R_t}(1-\tau)E_t^{\mathcal{P}^j} q_{t+1}}[(a+q_t)H_{t-1}^B(i) - R_{t-1}b_{t-1}^B(i)] \tag{9}$$

where $q_t - \frac{1}{R_t}E_t^{\mathcal{P}^j} q_{t+1}$ is the down-payment required to buy a unit of house.

Lagging equation (8) for one period yields the debt repayment at period t, $R_{t-1}b_{t-1}^B(i) = (1-\tau)\, E_{t-1}^{\mathcal{P}^j} q_t H_{t-1}^B(i)$, which is influenced by the price expectation formed at period $t-1$, i.e., $E_{t-1}^{\mathcal{P}^j} q_t$.[8] After plugging the debt repayment into (9), the collateral demand of the household i is derived as follows

$$H_t^B(i) = \frac{1}{q_t - \frac{1}{R_t}(1-\tau)E_t^{\mathcal{P}^j} q_{t+1}}(a + q_t - (1-\tau)\, E_{t-1}^{\mathcal{P}^j} q_t) H_{t-1}^B(i) \tag{10}$$

Note borrowers' collateral demand depends on expectations at two successive periods, $E_{t-1}^{\mathcal{P}^j} q_t$ and $E_t^{\mathcal{P}^j} q_{t+1}$: the inherited debt repayment and the downpayment. The dependence gives rise to interesting dynamics under learning, as analyzed later.

[7] Consider a marginal unit of tradable consumption at date t. The borrower could consume it and get utility 1. Alternatively he could invest it in collateral holding and produce consumption goods. In the next period, he will consume the nontradable part of production and invest further the tradable part, and so forth. Similar to KM, an assumption, i.e., $\frac{\overline{e}}{1-\frac{1}{R}(1-\tau)}\frac{(R-1)(1-\tau)}{aR} > \frac{1}{\beta^B} - 1$, is made to ensure that the discounted sum of utility of investing it at date t will exceed the utility of immediately consuming it (see Online Appendix G), which is 1. Assumption $\beta^B < \frac{\beta^L}{\overline{A}}$ ensures that the return to investment will also be larger than the alternative choice, saving it for one period and then investing. Hence the collateral constraint will always be binding.

The above argument is valid when the economy is in a neighborhood of the steady state under RE. Online Appendix G shows that the collateral constraint is binding in the quantitative analysis of the learning model when agents' expectations are formed based on subjective beliefs.

[8] For the initial period $R_0 b_0^B(i) = (1-\tau)\, E_0^{\mathcal{P}^j} q_1 H_0^B(i)$ is assumed to hold.

A financial intermediary j is not credit constrained and her demand for collateral is determined by the following optimality condition

$$\frac{1}{R_t}G^{j'}\left(H_t^L(j)\right) = q_t - \frac{1}{R_t}E_t^{\mathcal{P}^j}q_{t+1} \tag{11}$$

For a marginal unit of collateral, the financial intermediary could get $\frac{1}{R_t}G^{j'}\left(H_t^L(j)\right)$ by producing by herself. Alternatively, she can sell it, lend the proceeds to a household at rate R_t, and buy it back in period $t+1$ at the expected price $E_t^{\mathcal{P}^j}q_{t+1}$, which gives her $q_t - \frac{1}{R_t}E_t^{\mathcal{P}^j}q_{t+1}$. At the optimum, the two strategies give identical payoff at period t.

Note households are less patient than financial intermediaries because $\beta^B < \frac{\beta^L}{\overline{A}}$. In equilibrium the former will borrow from the latter and the interest rate will always be equal to the financial intermediaries' rate of time preference; that is

$$R_t = \frac{A_t}{\beta^L} \tag{12}$$

Assuming homogeneity among all borrowers and all lenders, symmetric equilibrium requires $H_t^B = H_t^B(i)$, $H_t^L = H_t^L(j)$, $b_t^B = b_t^B(i)$, and $b_t^L = b_t^L(j)$. Aggregation yields $H_t^B = \int_0^1 H_t^B(i)$, $H_t^L = \int_0^1 H_t^L(j)$, $b_t^B = \int_0^1 b_t^B(i)$, and $b_t^L = \int_0^1 b_t^L(j)$. Denote by y_t the aggregate output in period t, which is the sum of the production by borrowers and lenders

$$\begin{aligned} y_t &= \int_0^1 y_t^B(i) + \int_0^1 y_t^L(j) && (13)\\ &= (a+\overline{e})H_{t-1}^B + G(H_{t-1}^L) && (14)\end{aligned}$$

Market clearing implies $H_t^B + H_t^L = \overline{H}$ and $b_t^B = b_t^L$. In equilibrium, user cost of collateral, i.e., the opportunity cost of holding collateral for one more period, is

$$u_t^e = q_t - \frac{1}{R_t}E_t^{\mathcal{P}^j}q_{t+1}$$

and equals to the present value of the marginal product of collateral.

Due to zero net supply of loans and collateral assets, aggregate consumption c_t will equal to aggregate output y_t. Since aggregate investment is automatically zero in the model, I introduce a fixed, exogenous amount of autonomous investment following Boz and Mendoza (2013). This captures the investment and government absorption in the data. So the GDP in the model is the sum of aggregate consumption and investment

$$GDP_t = c_t + I \tag{15}$$

Denote $(Debt/GDP)_t$ the household credit market debt/GDP ratio, which is calculated by

$$(Debt/GDP)_t = b_t^B/GDP_t \tag{16}$$

Online Appendix A provides details on the steady state and the log-linearization of the model.

3 Equilibrium with Imperfect Knowledge

In the rational expectations equilibrium, agents are endowed with knowledge about the equilibrium mapping from the history of collateral holdings and lenders' preference shocks to collateral prices. Below I assume homogeneous expectations among all agents but relax the assumption that the homogeneity of agents is common knowledge, in particular, agents do not know other agents' discount factors and beliefs about future collateral prices. Relaxation of the informational assumption leads to agents in the model being uncertain about the equilibrium mapping between collateral prices and fundamentals.

3.1 The Underlying Probability Space and the Internally Rational Expectation Equilibrium

I now describe the probability space $(\Omega, S, \mathcal{P})$. Following Adam and Marcet (2011), I extend the state space of outcomes to contain not only the sequence of fundamentals, i.e., borrowers' collateral holdings and the shock to lenders' patience factor, but also other pay-off relevant variables: house prices. Both borrowers and lenders view the process for q_t, A_t and H_t^B as external to their decision problem and the probability space over which they condition their choices is given by $\Omega = \Omega_q \times \Omega_A \times \Omega_{H^B}$ where $\Omega_X = \Pi_{t=0}^{\infty} R_+$ and $X \in \{q, A, H^B\}$. The probability space contains all possible sequences of prices, lenders' preference shocks and borrowers' collateral holdings. Denote the set of all possible histories up to period t by $\Omega^t = \Omega_q^t \times \Omega_A^t \times \Omega_{H^B}^t$ and its typical element by $\omega^t \in \Omega^t$. The RE belief is nested as a special case in which the probability measure $\mathcal{P}$ features a singularity in the joint density of prices and fundamentals. Since equilibrium pricing functions are assumed to be known to agents under RE, conditioning their choices on the collateral price process is redundant.

The agents are assumed to be "Internally Rational"[9] as defined below, i.e., maximizing their expected utility under uncertainty, taking into account their constraints, and conditioning their choice variables over the history of all external variables. Their expectations about future external variables are evaluated based on their consistent set of subjective beliefs specified in the subsequent subsection, which is endowed to them at the outset.

Definition 1 *Internal Rationality*

a) A household i is "Internally Rational" if he chooses $(b_t^B(i), H_t^B(i), c_t^B(i)) : \Omega^t \to R^3$ to maximize the expected utility (1) subject to the flow-of-fund constraint (4), the collateral constraint (3) and his production function, taking as given the probability measure $\mathcal{P}^i$.

b) A financial intermediary j is "Internally Rational" if she chooses $(b_t^L(j), H_t^L(j), c_t^L(j)) : \Omega^t \to R^3$ to maximize the expected utility (5) subject to the flow-of-fund constraint (6) and her production function, taking as given the probability measure $\mathcal{P}^j$.

Note the internal rationality of agents is tied neither to any specific belief system nor to the learning behavior of agents. However, the belief system is usually specified with some near-rationality concept and it is natural to introduce learning behavior of agents.

Below I specify the equilibrium of the economy. Let (Ω_A, P_A) be a probability space over the space of histories of preference shocks Ω_A. Denote P_A the 'objective' probability measure

[9] This follows Adam and Marcet (2011).

for lenders' preference shocks. Let $\omega_A \in \Omega_A$ denote a typical infinite history of lenders' preference shocks.

Definition 2 *Internally Rational Expectations Equilibrium*

The Internally Rational Expectation Equilibrium (IREE) consists of a sequence of equilibrium price functions $\{q_t\}_{t=0}^{\infty}$ where $q_t : \Omega_A^t \to R_+$ for each t, contingent choices $(c_t^B(i), c_t^L(j), b_t^B(i), b_t^L(j), H_t^B(i), H_t^L(j)) : \Omega^t \to R^6$ and probability beliefs $\mathcal{P}^i$ for each household i and $\mathcal{P}^j$ for each financial intermediary j, such that
(1) all agents are internally rational, and
(2) when agents evaluate $(c_t^B(i), c_t^L(j), b_t^B(i), b_t^L(j), H_t^B(i), H_t^L(j))$ at equilibrium prices, markets clear for all t and all $\omega_A \in \Omega_A$ almost surely in P_A.

In the IREE, expectations about collateral prices are formed based on agents' subjective belief system, which are not necessarily equal to the 'objective' density. Collateral prices and borrowers' collateral holdings are determined by equations (10), (11) and market clearing conditions after agents' probability measures $\mathcal{P}$ are specified.

3.2 Agents' Belief System and Optimal Learning Behavior

This section describes agents' probability measure $\mathcal{P}$ and derives their optimal learning algorithm. Agents' belief system is assumed to have the same functional form as the RE solution. They believe collateral prices and borrowers' collateral holdings depend on past aggregate borrowers' collateral holdings.[10] It can be represented as following:[11]

$$\widehat{q}_t = \zeta^m + \zeta^p \widehat{H}_{t-1}^B + \epsilon_t \tag{17}$$

$$\widehat{H}_t^B = \varkappa^m + \varkappa^p \widehat{H}_{t-1}^B + \varrho_t \tag{18}$$

given $\widehat{H}_0^B$, where

$$\begin{pmatrix} \epsilon_t \\ \varrho_t \end{pmatrix} \sim iiN\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} \sigma_\epsilon^2 & 0 \\ 0 & \sigma_\varrho^2 \end{pmatrix}\right) \tag{19}$$

Unlike under rational expectations, they are assumed to be uncertain about the parameters and the shock precisions $(\zeta^m, \zeta^p, \frac{1}{\sigma_\epsilon^2}, \varkappa^m, \varkappa^p, \frac{1}{\sigma_\varrho^2})$, which is a natural assumption given that internally rational agents cannot derive the equilibrium distribution of collateral prices. Note agents' beliefs about $(\varkappa^m, \varkappa^p, \frac{1}{\sigma_\varrho^2})$ do not matter for equilibrium outcomes because only one-step ahead expectations about collateral prices enter the equilibrium under internal rationality in the model. So I omit belief updating equations for $(\varkappa^m, \varkappa^p, \frac{1}{\sigma_\varrho^2})$ for the rest of the paper.

[10] The shock to lenders' preferences is observable but not included in agents' regression. Including it will generate a singularity in the regression if initial beliefs coincide with the rational expectations equilibrium given it is the only shock in the model.

[11] This is analogous to learning the parameter linking prices and dividend in stock pricing models. Note the dividend here is the marginal product of lenders and a function of borrowers' collateral holding. After log-linearization, the (percentage deviation of) dividend is just a constant multiple of the (percentage deviation of) the borrowers' collateral holding.

Denote K the precision of the innovation ϵ_t, i.e., $K \equiv \frac{1}{\sigma_\epsilon^2}$. Agents' uncertainty at time zero is summarized by a distribution

$$(\zeta^m, \zeta^p, K) \sim f$$

The prior distribution of unknown parameters is assumed to be a Normal-Gamma distribution as follows

$$\begin{aligned} K &\sim G(\gamma_0, d_0^{-2}) && (20) \\ (\zeta^m, \zeta^p)' \mid K = k &\sim N((\theta_0^m, \theta_0^p)', (\nu_0 k)^{-1}) && (21) \end{aligned}$$

The residual precision K is distributed as a Gamma distribution, and conditional on the residual precision K unknown parameters (ζ^m, ζ^p) are jointly normally distributed. The deviation of this prior from the REE prior will vanish assuming agents' initial beliefs are at the RE value $\theta = \bar{\theta} = (\bar{\theta}^m, \bar{\theta}^p)'$, and they have infinite confidence in their beliefs about the parameters, i.e., $\gamma_0 \to \infty$, and $\nu_0 \to \infty$.

For the sake of notational compactness, I denote y_t and x_t the collateral price $\widehat{q}_t$ and $(1, \widehat{H}_{t-1}^B)$ in the rest of this section, respectively. $\theta_t \equiv (\theta_t^m, \theta_t^p)$ stands for the posterior mean of (ζ^m, ζ^p).

Given agents' prior beliefs (20) and (21), optimal behavior implies that agents' beliefs are updated by applying Bayes' law to market outcomes. Online Appendix B shows that the posterior distribution of unknown parameters is given by

$$\begin{aligned} K|\omega^t &\sim G(\gamma_t, d_t^{-2}) && (22) \\ (\zeta^m, \zeta^p)'|K = k, \omega^t &\sim N((\theta_t^m, \theta_t^p)', (\nu_t k)^{-1}) && (23) \end{aligned}$$

where the parameters $(\theta_t^m, \theta_t^p, \nu_t, \gamma_t, d_t^{-2})$ evolve recursively as following

$$\begin{aligned} \theta_t &= \theta_{t-1} + (x_t x_t' + \nu_{t-1})^{-1} x_t (y_t - x_t' \theta_{t-1}) && (24) \\ \nu_t &= \nu_{t-1} + x_t x_t' && (25) \\ \gamma_t &= \gamma_{t-1} + \frac{1}{2} && (26) \\ d_t^{-2} &= d_{t-1}^{-2} + \frac{1}{2}(y_t - x_t' \theta_{t-1})'(x_t x_t' + \nu_{t-1})^{-1} \nu_{t-1} (y_t - x_t' \theta_{t-1}) && (27) \end{aligned}$$

To avoid simultaneity between agents' beliefs and actual outcomes, I assume information on the data, i.e., prices and collateral holdings, is introduced with a delay in θ_t.[12] The following learning rule using lagged data is used[13]

$$\begin{aligned} \theta_t &= \theta_{t-1} + (x_{t-1} x_{t-1}' + \nu_{t-1})^{-1} x_{t-1} (y_{t-1} - x_{t-1}' \theta_{t-1}) && (28) \\ \nu_t &= \nu_{t-1} + x_{t-1} x_{t-1}' && (29) \end{aligned}$$

[12] Using current instead of lagged data in belief updating may give rise to multiple equilibria, i.e., high (low) price realizations associated with optimistic (pessimistic) beliefs. This is a potentially interesting avenue to explore asset prices boom and bust but is not pursued in the paper.

[13] A micro-founded belief system justifying the delay of information on the data can be provided following Adam, Beutel and Marcet (2014).

This timing convention is standard in the adaptive learning literature. Note while the forecast functions are predetermined, because beliefs are updated using lagged data, agents' expectations are not predetermined as they depend on period t information.

Equations (28) and (29) are equivalent to the following Recursive Least Square (RLS) learning algorithm

$$\theta_t = \theta_{t-1} + g_t S_t^{-1} x_{t-1}(y_{t-1} - x'_{t-1}\theta_{t-1}) \quad (30)$$
$$S_t = S_{t-1} + g_t(x_{t-1}x'_{t-1} - S_{t-1}) \quad (31)$$

when the initial parameter is set to $\nu_0 = NS_0$ and where $g_t = \frac{1}{t+N}$. Then it can be shown that for subsequent periods we have $\nu_t = (t+N)S_t$, for $\forall t \geq 1$. Therefore, N in the above equations measures the precision of initial beliefs.

The term $y_{t-1} - x'_{t-1}\theta_{t-1}$ in equation (30) is agents' price forecast error at period t. According to (30) and (31), a surprise in agents' price expectation will induce a revision of their beliefs or the parameters linking prices and fundamentals.

As standard in the literature, the learning rule with a small and constant gain sequence $g_t = g > 0$ is used in the quantitative exercise in section 5

$$\theta_t = \theta_{t-1} + gS_t^{-1} x_{t-1}(y_{t-1} - x'_{t-1}\theta_{t-1}) \quad (32)$$
$$S_t = S_{t-1} + g(x_{t-1}x'_{t-1} - S_{t-1}) \quad (33)$$

A Bayesian micro-foundation for this learning algorithm is provided in Online Appendix C. The learning rule (32)-(33) implies that agents discount past observations and give relatively more weight to new data when they are alert to possible structural changes in the economy.

I firstly studies the condition under which the learning process under the decreasing gain learning algorithm (30)-(31) converges towards the RE equilibrium or equivalently the Expectational Stability (E-stability) condition. This is a precursor of meaningful dynamics under the constant gain learning rule because the learning process under the constant gain learning rule will converge in distribution to the REE if the E-stability condition is satisfied and the gain parameter is sufficiently small. Note beliefs under the constant gain learning rule will not converge pointwise to the RE belief because asymptotically agents will discount past data and still revise their beliefs responding to forecast errors caused by random innovations.

4 Understanding the Learning Model

This section firstly provides intuitions on the role of subjective beliefs in determining house prices in the model. It then examines the condition governing the convergence of the learning process and shows that a higher leveraged economy is more prone to self-reinforcing fluctuations. It closes with informal discussions of related papers.

4.1 Dependence of House Prices on Subjective Beliefs

Log-linearizing borrowers' collateral demand (10) under symmetric equilibrium yields[14]

$$\widehat{H}_t^B = \frac{\left(\widehat{q}_t - (1-\tau)E_{t-1}^{\mathcal{P}}\widehat{q}_t\right) - \left(\widehat{q}_t - \frac{1-\tau}{R}E_t^{\mathcal{P}}\widehat{q}_{t+1}\right)}{1-\frac{1-\tau}{R}} + \widehat{H}_{t-1}^B - \frac{\frac{1-\tau}{R}\widehat{A}_t}{\left(1-\frac{1-\tau}{R}\right)} \tag{34}$$

$\widehat{H}_t^B$ depends on expectations of two successive periods: $E_{t-1}^{\mathcal{P}}\widehat{q}_t$ and $E_t^{\mathcal{P}}\widehat{q}_{t+1}$. The former arises from debt repayments and the latter from down-payment.

Recall agents perceive prices to evolve according to (17), while their beliefs are updated following (30) and (31). The state variables of the learning algorithm are $x_t = (1 \quad \widehat{H}_{t-1}^B)'$. Agents' conditional expectations are $E_{t-1}^{\mathcal{P}}\widehat{q}_t = \zeta_{t-1}'x_{t-1}$ and $E_t^{\mathcal{P}}\widehat{q}_{t+1} = \zeta_t'x_t$ where $\zeta_t \equiv (\zeta_t^m \ \zeta_t^p)'$. Substituting the conditional expectations into the log-linearized version of equation (11) and (34) under the symmetric equilibrium, I get the actual law of motion (ALM) for collateral prices under learning

$$\widehat{q}_t = T_1(\zeta_{t-1}^m, \zeta_t^m, \zeta_t^p) + T_2(\zeta_{t-1}^p, \zeta_t^p)\widehat{H}_{t-1}^B + T_3(\zeta_t^p)\widehat{A}_t \tag{35}$$

where $T_2(\zeta_{t-1}^p, \zeta_t^p) = \frac{\left(\frac{1}{R}\zeta_t^p + \frac{1}{\eta}\frac{R-1}{R}\right)\left(1-\frac{1-\tau}{1-\frac{1-\tau}{R}}\zeta_{t-1}^p\right)}{1-\frac{\frac{1-\tau}{R}}{1-\frac{1-\tau}{R}}\zeta_t^p}$ and $\frac{1}{\eta}$ is the steady state elasticity of user cost of collateral with respect to borrowers' collateral holdings.[15] The analytical expression for $T_1(\zeta_{t-1}^m, \zeta_t^m, \zeta_t^p)$ and $T_3(\zeta_t^p)$ is presented in Online Appendix A.

The T-map maps agents' subjective beliefs to the parameters in the ALM for house prices under learning. The Minimum State Variables RE belief is the fixed point of the T-map satisfying $T_1(\overline{\zeta}^m, \overline{\zeta}^p) = \overline{\zeta}^m = 0$ and $T_2(\overline{\zeta}^m, \overline{\zeta}^p) = \overline{\zeta}^p$.[16]

Subjective beliefs ζ^p at period $t-1$ and t appear three times in the T_2-map and their effects on house prices under learning are presented below. Online Appendix A.1 keeps track of the three belief terms in deriving the T_2-map.

First, the T_2-map depends negatively on past beliefs ζ_{t-1}^p via debt repayment (see equation (34)). An increase in ζ_{t-1}^p raises borrowers' debt repayment (given borrowers' collateral holdings $\widehat{H}_{t-1}^B$) which in turn reduces their collateral demand and hence impacts negatively on house prices. Second, ζ_t^p associated with $\frac{1}{R}$ in the T_2-map comes from the expected housing re-sale price in equation (11). A higher ζ_t^p impacts positively on house prices because agents have more optimistic forecast functions. Third, ζ_t^p associated with $\frac{\frac{1-\tau}{R}}{1-\frac{1-\tau}{R}}$ in the T_2-map comes from the down-payment in the borrowers' housing demand equation (34).

[14] This equation is useful for understanding the impulse response functions and discussed further in section 5.1.

[15] The elasticity is defined as

$$\frac{1}{\eta} \equiv \frac{d\log u^e(H_t^B)}{d\log H_t^B}|_{H_t^B=H^B} = -\frac{d\log G'(H_t^L)}{d\log H_t^L}|_{H_t^L=H^L} \times \frac{H^B}{\overline{H}-H^B}$$

It is the product of the financial intermediaries' marginal product of collateral and the ratio of the households' collateral holdings to the financial intermediaries' at the steady state.

[16] The MSV RE solution for borrowers' collateral holdings and collateral prices are an AR(1) process and ARMA(1,1) process, respectively and suppressed here.

A higher ζ_t^p or more optimistic belief relaxes the credit limit and hence raises borrowers' housing demand/holding. Increased borrowers' demand influences house prices positively in two ways. On one hand, recall agents use borrowers' house holding to forecast future house prices, the rising borrowers' house holding boosts house price forecasts and prices. On the other hand, the increased borrowers' house holding raises the lenders' marginal productivity, user cost of collateral and house prices.

4.2 Expectational Stability of the REE

As the RE belief is the fixed point of the T-map, so it is the rest point of the learning process. This section examines the Expectational-Stability condition for the REE, which also governs the convergence of the learning process towards the REE. This can be analyzed by applying standard techniques elaborated in Evans and Honkapohja (2001). Denote Θ as the set of admissible parameters in the RE and the learning model.

Definition 3

The admissible parameter space $\Theta \equiv \{(\eta, R, \tau) | \eta > 0, R > 1, 0 \leq \tau < 1\}$.

This implies that the steady state leverage ratio or loan-to-value ratio is in the interval $(0, \frac{1}{R}]$, the elasticity $\frac{1}{\eta}$ is positive, and the (net) interest rate is positive.

Local stability of the Minimum State Variable REE is determined by the stability of the following associated ordinary differential equations

$$\begin{aligned} \frac{d\zeta^m}{d\tau} &= T_1(\zeta^m, \zeta^p) - \zeta^m \\ \frac{d\zeta^p}{d\tau} &= T_2(\zeta^p, \zeta^p) - \zeta^p \end{aligned}$$

The following condition establishes a sufficient condition for the E-stability of the Minimum State Variable RE equilibrium.

Proposition 4

The MSV RE equilibrium for the economy is Expectationally-stable (E-stable) for all admissible parameters in Θ.

Proof. see Online Appendix D. ■

The following illustration may help to understand the E-stability condition. Fixing agents' beliefs about ζ^m at the RE value 0 and the price elasticity ζ^p above the RE value $\overline{\zeta}^p$, so agents have an excessively optimistic forecast function. Recall the three effects discussed in section 4.1. On one hand, a high ζ^p influences negatively on house prices because agents hold excessive debt repayment. On the other hand, a high ζ^p impacts positively on house prices via two ways: excessively optimistic forecast function for housing re-sale price and excessively optimistic expectations about collateral values and credit limits relative to RE.

The E-stability result says exactly that the negative effect of the excessive debt repayment will dominate for all admissible parameterizations. This in turn leads to a low realization of the price elasticity which pushes agents' belief downward. Therefore, the asymptotic local

stability of the REE is achieved. Roughly speaking, given that the E-stability condition is satisfied and the parameter estimates are around the neighborhood of the RE value, we have $\beta_t \to \bar{\beta}$ and $\nu_t \to \infty$ almost surely.[17]

The above proposition also implies that parameter estimates coming from a constant gain learning algorithm (32)-(33) will converge in distribution to the REE as long as the gain parameter is sufficiently small.[18]

4.3 Dependence of Belief Dynamics on Loan-to-Value Ratio

House price changes display strong positive serial correlation at short time horizon, such as one year, as shown by Case and Shiller (1989), and Glaeser and Gyourko (2006). However, standard full-information RE models, e.g., Liu, Wang, Zha (2013), typically can not generate the degree of persistence of house price changes as in the data without relying on very persistent exogenous shocks. This section shows that the learning model can internally display strong persistence in belief and price changes if the Loan-to-Value ratio of the economy is sufficiently large. Put differently, a higher leveraged economy is more prone to self-reinforcing fluctuations.

Following Adam, Marcet and Nicolini (2012), one way to capture the strong persistence in the change of agents' beliefs is by momentum defined below. Momentum in belief adjustments is the key property for their asset pricing model replicating a number of equity pricing facts in the U.S. data, such as the return volatility, the persistence and volatility of price dividend ratio, etc.

Recall ζ_t^p is agents' belief about the price collateral holdings elasticity at period t, and $\overline{\zeta}^p$ the corresponding value at the RE level.

Definition 5 *Momentum*

Momentum is defined as:
(1) if $\zeta_t^p > \zeta_{t-1}^p$ and $\zeta_t^p \leq \overline{\zeta}^p$, then $\zeta_{t+1}^p > \zeta_t^p$.
(2) if $\zeta_t^p < \zeta_{t-1}^p$ and $\zeta_t^p \geq \overline{\zeta}^p$, then $\zeta_{t+1}^p < \zeta_t^p$.

The definition says following. Suppose agents' belief or parameter estimate is adjusted upward (downward), i.e., $\zeta_{t-1}^p < \zeta_t^p$ ($\zeta_{t-1}^p > \zeta_t^p$), but still not exceed (not below) the RE level, i.e., $\zeta_t^p \leq \overline{\zeta}^p$ ($\zeta_t^p \geq \overline{\zeta}^p$), this will be followed by further upward (downward) belief adjustment, i.e., $\zeta_{t+1}^p > \zeta_t^p$ ($\zeta_{t+1}^p < \zeta_t^p$). Roughly speaking, agents' optimism (pessimism) is followed by further optimism (pessimism).

To study the internal dynamics of the learning model, a deterministic model is examined by assuming $\widehat{A}_t = 0$ for all t. I further consider a simplified perceived law of motion without learning about $\overline{\zeta}^m$ or the steady state, that is, $\widehat{q}_t = \zeta_{t-1}^p \widehat{H}_{t-1}^B + \omega_t$. Recall the T-map mapping from the subjective belief to the parameter in the actual law of motion is $T_2(\zeta_{t-1}^p, \zeta_t^p) =$

[17] Once the convergence of agents' estimates in the collateral price process is achieved, agents' belief about the parameter estimates in the borrowers' collateral holding equation will also converge to the RE value.

[18] The convergence properties of learning models under the constant-gain learning algorithm are discussed in details in Evans and Honkapohja (2001).

$\frac{\left(\frac{1}{R}\zeta_t^p+\frac{1}{\eta}\frac{R-1}{R}\right)\left(1-\frac{1-\tau}{1-\frac{1-\tau}{R}}\zeta_{t-1}^p\right)}{1-\frac{\frac{1-\tau}{R}}{1-\frac{1-\tau}{R}}\zeta_t^p}$, which also determines critically the dynamics of the model with learning about $\overline{\zeta}^m$. Below I characterize a sufficient condition for the learning model to display momentum in belief adjustments.

Substituting the actual law of motion for house prices, i.e., $q_t = T_2(\zeta_{t-1}^p, \zeta_t^p)\widehat{H}_t^B$, into agents' belief updating equations (30)-(31) or (32)-(33) yields

$$\begin{aligned}\zeta_{t+1}^p &= \zeta_t^p + g_t S_{t+1}^{-1}\widehat{H}_t^B(\widehat{q}_t - \widehat{H}_t^B\zeta_t^p) \\ &= \zeta_t^p + g_t S_{t+1}^{-1}\left(\widehat{H}_t^B\right)^2(T_2(\zeta_{t-1}^p, \zeta_t^p) - \zeta_t^p) \end{aligned} \tag{36}$$

where g_t can be the decreasing gain sequence or constant gain and is positive. Assuming $\widehat{H}_t^B \neq 0$ below. Equation (36) says that agents will revise their belief about the price elasticity upward (downward) if the realized price elasticity is higher (lower) than their subjective estimate.

Performing the first-order Taylor approximation of the T_2-map around the REE belief yields

$$T_2(\zeta_{t-1}^p, \zeta_t^p) \simeq \overline{\zeta}^p + \left(-\frac{\partial T_2}{\partial \zeta_{t-1}^p}|_{\overline{\zeta}^p}\right)\left(\overline{\zeta}^p - \zeta_{t-1}^p\right) - \left(\frac{\partial T_2}{\partial \zeta_t^p}|_{\overline{\zeta}^p}\right)\left(\overline{\zeta}^p - \zeta_t^p\right) \tag{37}$$

As explained in section 4.1, we know $\frac{\partial T_2}{\partial \zeta_{t-1}^p}|_{\overline{\zeta}^p} < 0$ and $\frac{\partial T_2}{\partial \zeta_t^p}|_{\overline{\zeta}^p} > 0$. A past belief ζ_{t-1}^p which is lower than the RE value implies a lower debt repayment (given collateral holdings) relative to that under RE and contributes positively to the realized price elasticity. A current belief ζ_t^p which is lower than the RE value contributes negatively to the price elasticity.

To illustrate, consider a scenario in which agents' belief arrives at the RE value from below, i.e., $\zeta_{t-1}^p < \zeta_t^p = \overline{\zeta}^p$. According to (37), the realized price elasticity will be larger than the RE value, i.e., $T_2(\zeta_{t-1}^p, \zeta_t^p) > \overline{\zeta}^p$. The past belief lower than RE value implies a lower debt repayment and helps to generate a high price elasticity. Furthermore, belief updating rule (36) implies agents will revise their belief further upward, i.e., $\zeta_{t+1}^p > \zeta_t^p$. So there is a tendency that agents' belief will overshoot the RE value when arriving at the RE value. [19] This is true for any admissible parameterization of the model.

However, a stronger condition is needed for the learning model to display momentum in belief adjustments when agents' belief is updated upward but still below the RE value, that is, $\zeta_{t-1}^p < \zeta_t^p < \overline{\zeta}^p$. A sufficient condition[20] to ensure momentum is

$$-\frac{\partial T_2}{\partial \zeta_{t-1}^p}|_{\overline{\zeta}^p} \geq \frac{\partial T_2}{\partial \zeta_t^p}|_{\overline{\zeta}^p} \tag{38}$$

The reason is as follows. Consider agents' belief is updated upward but still below the RE value, that is, $\zeta_{t-1}^p < \zeta_t^p < \overline{\zeta}^p$. If the above condition holds, (37) implies that the price

[19] The magnitude of this further upward belief adjustment will depend on the size of the gain parameter. Also note the property that agents' beliefs hover around the RE value during the learning transition does not conflict with the convergence of agents' belief under the RLS learning to the RE value.

[20] Note the set of parameters satisfying the momentum condition $\frac{\partial T_2}{\partial \zeta_{t-1}^p}|_{\overline{\zeta}^p} + \frac{\partial T_2}{\partial \zeta_t^p}|_{\overline{\zeta}^p} \leq 0$ is a subset of the set of parameters satisfying the E-stability condition $\frac{\partial T_2}{\partial \zeta_{t-1}^p}|_{\overline{\zeta}^p} + \frac{\partial T_2}{\partial \zeta_t^p}|_{\overline{\zeta}^p} < 1$.

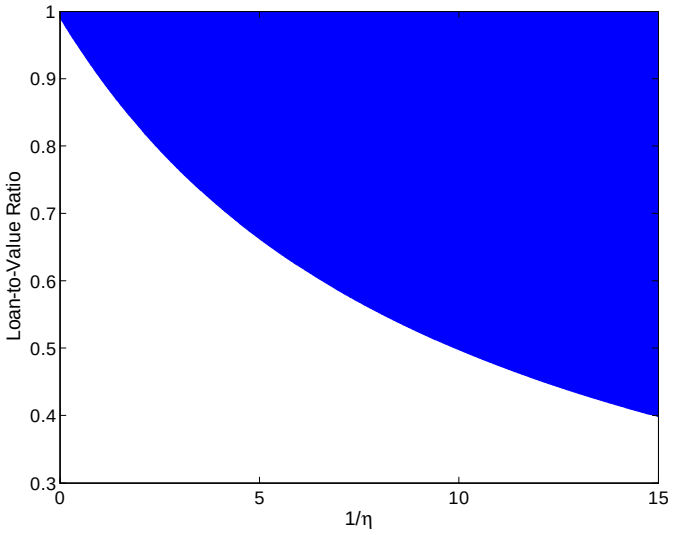

Figure 3: Threshold function $\frac{1-\tau}{R}(\frac{1}{\eta})$ and parameter combinations generating momentum

collateral holdings elasticity in the ALM will be higher than the RE value, i.e., $T_2(\zeta_{t-1}^p, \zeta_t^p) > \overline{\zeta}^p$. Using the realized price elasticity, agents will update their belief further upward according to (36).

Note if τ were 1 or equivalently the leverage ratio were zero, momentum would not arise in the learning model. This is because past beliefs would not appear in the T_2-map. (38) would not be met because its left hand side would be zero and its right hand side positive. There would not be an overshoot of agents' belief because without a shock it would stay at the RE value when arriving there. As we increase the leverage ratio above some threshold, (38) will be met so that the positive effect of a lower past belief or debt repayment will dominate and the realized price elasticity will be sufficiently high and reinforce the initial optimism.

The intuitions are summarized as follows. Increasingly optimistic price beliefs endogenously enhance borrowing capacity, so that the drag of debt repayments based on less optimistic past beliefs on house prices becomes relatively smaller. The positive effect of the relative reduction of the debt burden on house prices will be sufficiently large when the leverage ratio is sufficiently large, so that realized house prices will be sufficiently large and can reinforce agents' optimism. Similarly for increasingly pessimistic beliefs. This is shown by the following proposition which provides an explicit characterization of the momentum condition (38).

Proposition 6

A sufficient condition guaranteeing momentum in beliefs (around the neighborhood of the REE belief) in the learning model is that parameter combinations of (η, R, τ) satisfy $\frac{1-\tau}{R} \geq \frac{1}{g(R)\frac{1}{\eta}+1}$ where $g(R) = R(\sqrt{(R-1) + \frac{(R-1)^2}{4}} + \frac{R-1}{2})$.

Proof. see Online Appendix E. ■

As an example, I set the gross quarterly interest rate R to 1.0088, which is the steady state value of the interest rate I choose in the quantitative exercise later. The threshold steady state loan-to-value ratio as a function of $\frac{1}{\eta}$, i.e, $\frac{1-\tau}{R} = \frac{1}{g(R)\frac{1}{\eta}+1}$, is plotted, which is decreasing in the elasticity $\frac{1}{\eta}$. The shaded area of figure 3, which is the area above the threshold function, summarizes the parameter combinations $(\frac{1-\tau}{R}, \frac{1}{\eta})$ under which there is momentum in beliefs in the learning model.[21] As can be seen from this figure, momentum[22] will arise in the learning model when the elasticity of the user cost with respect to borrowers' collateral holdings is relatively large or the steady state leverage ratio is relatively large. Given the elasticity $\frac{1}{\eta}$, the leveraged economy with a sufficiently high steady state leverage ratio can display momentum in belief and price changes.

Online Appendix F provides a discussion of the robustness of the qualitative learning dynamics with respect to an alternative specification of the collateral constraint that the maximum loan borrowers can get is a fraction of the current instead of expected collateral values.

4.4 Discussions

4.4.1 Relation to Adam, Kuang, and Marcet (2011, henceforth AKM)

AKM develop an open economy asset pricing model with a housing collateral constraint and learning which quantitatively accounts for the heterogeneous G7 house prices and current account dynamics over 2001-2008. In the AKM model, the price boom-bust is the consequence of the dynamic interaction of only house prices and price beliefs. Feedback from credit expansion/contraction to house prices, which is believed to have played a critical role during the recent U.S. housing cycle, has been shut down in the AKM analysis.[23] For example, studying detailed zip code level data, Mian and Sufi (2009) suggest that "there may be a feedback mechanism between credit growth and house price growth" and "the evidence cautions against treating house prices movements in the last decade as independent from the expansion and collapse of subprime mortgage securitization." The current model features dynamic interaction of house prices, price beliefs *and* credit limits and can capture this important feedback.

Both AKM and the current model generate quantitatively significant differences from the RE version of the models. For both models, the critical property is the dependence of house prices on belief changes and hence the possibility of endogenously persistent belief and price

[21] The parameter combinations generating momentum in beliefs are not sensitive to a wide range of the steady state value of the interest rate R chosen here.

[22] The parameterizations in the quantitative exercise later do not fall in the shaded area but very close to the border of the threshold function. The persistence in agents' beliefs and in collateral price changes can still arise when the learning friction interacts with interest rate reductions.

[23] In AKM, price fluctuations affect the collateral values of domestic households, borrowing, and current account dynamics. Relaxation and tightening of credit limits have an effect on the borrowers' housing demand but no impact on the house price dynamics in the approximate solution. The latter is because the marginal product of houses $(\xi_t G'(H_t))$ is kept constant in the quantitative analysis and hence the collateral demand function is horizontal. No feedback from credit limits to house prices is evident by inspecting the key equations of the learning model in AKM, i.e., the belief updating equation (33) and the actual law of motion for house prices (35).

changes. The improvement in the fit of the model here arises from the intrinsic property of a collateral-constrained economy that prices are directly influenced by past beliefs via debt repayment, contrast to from learning about the permanent component of house price growth in AKM.

Different from AKM, the paper also analyzes the role of leverage ratio, convergence of the learning process, and studies the dynamics of household debt and aggregate consumption quantitatively.

4.4.2 Relation to some asset pricing models with adaptive learning

Timmermann (1996) and Carceles-Poveda and Giannitsarou (2008, henceforth CG) study asset pricing with adaptive learning in an endowment economy and a production economy without collateral constraints, respectively. They find a limited role of adaptive learning in asset pricing when agents learn about the parameters linking asset prices to fundamentals or dividends. As analyzed in section 4.3, the asset pricing equation in a collateral-constrained economy differs critically from those in the two papers. Asset prices in the model are directly influenced by past beliefs and hence by the change of agents' price beliefs. This opens the possibility for the learning model to display strong persistence in belief changes and larger fluctuations of prices and quantities even if a similar belief specification as them is considered, i.e., agents learn about the parameters linking house prices and fundamentals or dividends. In contrast, in Timmermann (1996) and CG, when agents' beliefs arrive at the RE beliefs, they will stay there without further realizations of shocks.

CG follow the Euler-equation (EE) learning approach, replacing expectation terms in agents' first-order conditions by those formed by adaptive learning. Preston (2005) shows that the decision rule under EE learning approach leads to suboptimal decisions in a New Keynesian model. In standard real business cycle models, Eusepi and Preston (2011) suggest that such learning approach is unlikely to be helpful in explaining quantitative features of macroeconomic dynamics. The suboptimality is generally true in many standard models but disappears in some specific settings, for example, if agents' preferences are risk-neutral as in the current model or in the model of Adam and Marcet (2011). Agents' optimal decisions are fully characterized by Euler equations, which is the same as under the EE learning. The more significant role of learning in asset pricing in the current model relative to Timmermann (1996) and CG arises not from the internal rationality approach but from leveraging and borrowing constraints particularly in a relatively high leveraging regime.[24]

[24]The optimal decisions under internal rationality will be different from decisions by agents under the EE learning if the risk-neutrality assumption is replaced by risk-aversion. The former requires that agents forecast house prices up to the indefinite future. Risk-aversion is quantitatively less important relative to learning as asset prices are mainly driven by expectations, as can be seen the stock pricing in model of Adam, Marcet, and Nicolini (2012). Solving a model with internal rationality and risk-averse agents requires more sophisticated numerical methods as agents make their forecasts based on the subjective density instead of the point subjective belief and up to infinite horizon. The Adam, Beutel and Marcet (2014) model of stock market booms and busts is such an example.

5 Quantitative Results and Further Mechanism

The learning model is estimated to the U.S. economy showing that the learning model can quantitatively account for the recent house prices boom and bust and the associated household debt and aggregate consumption dynamics. Around the year 2001, the U.S. real interest rate dropped considerably and stayed low for an extended period of time, before rising again around the year 2006. The average of 1-year ahead ex-ante real mortgage interest rates[25] from 1997Q1 to 2000Q4 was 3.51%, while the average of real interest rates between 2001Q1-2005Q4 was 2.28%.

The following experiment is conducted. Initially the economy is assumed to be at the steady state and agents' beliefs at 2000Q4 are set to the RE value.[26] The low real interest rates after 2000Q4 and the subsequent increase are captured in the following stylized way. The annualized real interest rate at the steady state is set to 3.51%. I let the interest rate fall from 2001Q1, stay unchanged at 2.28% until 2005Q4, and then go back to the steady state. The model is used to generate real house prices, consumption and debt/GDP ratio during 2001Q1-2008Q4. Following Campbell (1994), I set the steady state consumption-GDP ratio to 0.745.

Denote by ck the product of the productivity gap $\frac{(a+\bar{e})-G'}{(a+\bar{e})}$ and borrowers' production share $\frac{(a+\bar{e})H^B}{Y}$ in aggregate output. The gain parameter g, the elasticity $\frac{1}{\eta}$, the parameter τ, and the parameter ck, are chosen to minimize the absolute distance between the learning model generated and actual house prices, consumption and debt/GDP ratio as follows

$$\sum_{t=2001Q1}^{2008Q4}\left(\frac{|\widehat{q}_t-\widehat{\mathbf{q}}_t|}{std(\widehat{\mathbf{q}}_t)}+\frac{|\widehat{c}_t-\widehat{\mathbf{c}}_t|}{std(\widehat{\mathbf{c}}_t)}+\frac{|\widehat{Debt/GDP}_t-\widehat{\mathbf{Debt/GDP}}_t|}{std(\widehat{\mathbf{Debt/GDP}}_t)}\right)$$

where boldface letters denote actual data and std stands for standard deviation.

The minimization yields that $g=0.065, \frac{1}{\eta}=2.46, \tau=0.45$, and $ck=0.43$. This choice of τ implies that the steady state loan-to-value ratio is 0.54.[27] The value of ck implies roughly, say both the productivity gap and borrowers' production share are $\frac{2}{3}$.[28]The choice of the parameters yields impulse response functions of the learning model broadly consistent with other studies.[29] The parameterization of the RE models is the same as the estimated learning model.

[25]The mortgage rate used is the "one-year adjustable rate mortgage average in the United States" from Freddie Mac (seriesID: MORTGAGE1US). The ex-ante real interest rate is calculated as the mortgage rate minus the median expected 1 year ahead CPI inflation rate from the survey of professional forecasters.

[26]An analogy can be made between the perceived law of motion used by the agents in the model and learning house price-to-rent ratio because the regressor $\widehat{H}_t^B$ is a constant multiple of the (log-linearized) dividend of houses. The assumption of RE belief as the initial belief is based on the observation that the house price-to-rent ratio is relatively stable say from 1980s to 2000. So agents are assumed to have learned to form RE.

[27]This is consistent with the estimate of the household loan-to-value ratio by Iacoviello (2005) with mean 0.55 and standard deviation 0.09.

[28]The productivity gap of $\frac{2}{3}$ is also considered by Cordoba and Ripoll (2004).

[29]Iacoviello and Neri (2010) find that 1% *positive* i.i.d. monetary policy shock leads to a decrease of house prices by about 0.65% and hump-shaped response of consumption with the trough 0.5% below the steady state.

Recall the interest rate at period t in my model is $R_t = \frac{A_t}{\beta^L}$. To get low interest rates during 2001-2005, I assume lenders' discount factors β^L shift upward exogenously during 2001-2005 and back to their old value during 2006-2008.[30]

Under RE, two alternatives are considered: the interest rate movement is either unanticipated or anticipated. Model predictions for the two alternatives are provided below. "RE-I" model refers to the RE model with unexpected interest rate reductions, while "RE-II" model stands for the RE model with anticipated interest rates movement. Note the performance of the learning model will be the same under either of the two assumptions because expectations about future interest rates do not enter the system of equations governing the model economy.

5.1 Response to 1% unexpected interest rate reduction

RE dynamics.—Figure 4 depicts the responses to an unexpected interest rate reduction. In the impact period, real house price under RE rises by about 1.2%, while consumption and debt/GDP ratio rise by about 0.6% and 2%,[31] respectively. However, they do not rise further after the interest rate reduction disappears. Consumption decays exponentially, while the house price drops substantially due to the disappearance of the interest rate reduction and then converges persistently to the steady state.

Under rational expectations, borrowers' collateral demand increases following an unexpected interest rate reduction. In the impact period, collateral is transferred from lenders to borrowers. Due to the fixed supply of collateral and the concave technology of lenders, user cost of collateral rises above its steady state value. Since borrowers' current investment in collateral holding raises their ability to borrow in the next period, there will be persistence in their collateral holdings. The user cost of collateral stays above the steady state for many periods. Under RE, house prices are the discounted sum of current and future user cost. The persistence in user cost reinforces the effect on house prices and collateral values, which leads to a larger effect on collateral transfers and aggregate activities.

After the disappearance of the interest rate reduction, the user cost rises above the steady state, which chokes off further rises in borrowers' housing demand. House prices and borrowers' collateral holdings will revert immediately towards the steady state. Prices and quantities converge persistently and monotonically to the steady state.

Learning dynamics.—The response of the learning model is simulated by setting the initial belief to the RE value. Learning generates additional propagation of the interest rate reduction due to belief revisions and the dynamic interaction between price beliefs, credit limit and price realizations. The peak responses of house prices, consumption and debt/GDP ratio are 1.2%, 0.73%, and 2.3%, respectively. The learning model also generates positive persistence in forecast errors,[32] as can be seen from the lower right panel.

Under learning, the impact responses are the same as those under RE because agents have correct forecast functions initially. After the disappearance of the interest rate reduction, a positive surprise in house prices induces an upward belief revision. Agents partially interpret

[30] Admittedly, this is a short-cut, but necessary way, to model the interest rate reduction in my context.

[31] Note the debt/GDP ratio here is percentage changes from the steady state value.

[32] The forecast error is defined as $\widehat{q}_t - E_{t-1}\widehat{q}_t$.

the surprise in house price forecasts as a permanent change in the parameters governing the house price process. They become more optimistic about future house prices due to both more optimistic beliefs and rising amounts of collateral holdings by borrowers. Credit limit is relaxed based on the optimism. As can be seen from equation (34), borrowers' collateral holdings can rise if $\frac{1-\tau}{R} E_t^{\mathcal{P}} \widehat{q}_{t+1} > (1-\tau) E_{t-1}^{\mathcal{P}} \widehat{q}_t$. With a larger borrowing capacity, borrowers can repay the debt and increase investment in collateral holdings. Since aggregate consumption is a constant fraction of borrowers' collateral holdings, consumption follows closely borrowers' collateral holdings.[33] The debt/GDP ratio rises due to rising collateral values.

In the period 2, house prices under learning are much higher relative to RE mainly due to more optimistic expectation about future prices. The temporary decline in house prices is due to the return of the interest rate to the steady state. Nevertheless, the realized price is still higher than agents' price forecast. So the realized price reinforces agents' optimism, which leads to further optimism when the price realization is used for belief updating. The positive effects of more optimistic beliefs and expectations can temporarily dominate the negative effect of debt repayment and lead to a rise in house prices. The dynamic feedback between agents' beliefs and actual prices through the relaxation of credit limits generates prolonged periods of expansion of prices and quantities.

The reversal of prices and quantities relates to the convergence of the learning process in section 4.2. At some point, the debt repayment becomes excessive such that its negative effect dominates and the realized house prices fall short of expectations. This sets a self-reinforcing decline in motion. According to (32) and (33), agents' beliefs are revised downward and they become pessimistic. Credit limits are tightened based on the pessimism. Borrowers' housing demand falls, so does the realization of house prices. The realized house prices reinforce agents' initial pessimism, which generates further declines in prices and quantities. Eventually prices and quantities converge to the steady state.

5.2 Boom and bust in house prices, debt and aggregate consumption dynamics

Figure 5 contrasts predictions of the learning model and of the "RE-I model" with actual data. Under RE, prices and quantities increase above their steady state values following the real interest rate reduction. House prices continue to increase due to the persistence in the user cost and the persistently low interest rates. They peak at about 14.4% above the steady state. After the disappearance of the interest rate reduction, house price starts to revert to their steady state. The RE model under-predicts considerably the levels of prices and quantities.[34]

[33] See Online Appendix A for the analytical expressions for log-linearized consumption and debt/GDP ratio.

[34] Given the pattern of the interest rates I consider, the response of house prices in the "RE-I" model will be larger if the elasticity $\frac{1}{\eta}$ is larger. The improvement of the performance of the RE model with a larger $\frac{1}{\eta}$ or a larger leverage ratio is limited when the leverage is not very large or close to 1 and the response of consumption or the collateral holding transfer is not counterfactually large. Regardless of the value of these two parameters, the REE house prices will revert when the interest rate starts to revert. So the RE model cannot match the turning point of house prices.

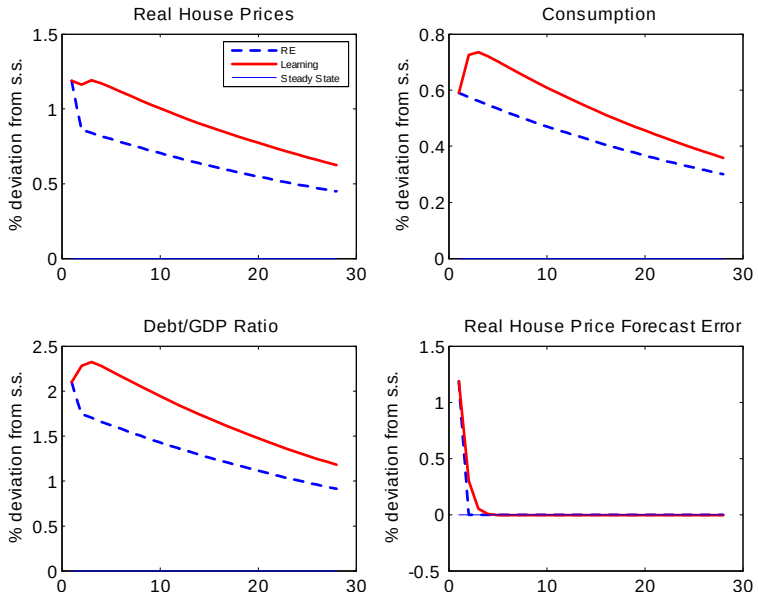

Figure 4: Response to 1% unexpected negative shock to interest rates

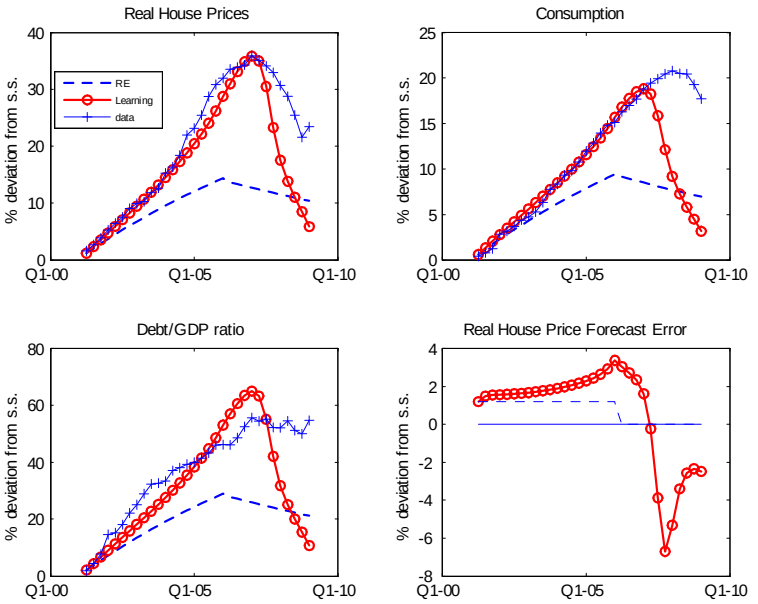

Figure 5: Model Predictions of the RE-I Model, Learning Model and Actual Data

The learning model predicts house prices, debt/GDP ratio and consumption rather well, in particular during the price boom years. Following the real interest rate reduction, real house prices under learning increase at a faster pace than under RE. The learning model generates large additional amplification of prices and quantities relative to the RE version of the model. The peak of the predicted house prices under learning is about 35.9% at 2006Q4, which is about 2.5 times the peak response of under RE. The house price boom arises mainly from more optimistic expectation about future prices due to both more optimistic beliefs and the rising amount of collateral held by households. The rising household credit market debt/GDP ratio is due to both the house price boom and the rising amount of collateral held by households. The learning model also generates a consumption boom due to shifts of collateral to more productive households. The peak response of consumption in the learning model is 18.8%, which is twice as large as that in the RE model.

House prices in the "RE-I model" start to revert once the interest rate rises, while the learning model matches rather well the turning point of house prices in the data. House prices in the learning model rise further for a few quarters as in the data even after the rise of the interest rates. This is due to belief revisions and the interaction of beliefs and price realizations.

The forecast errors of the RE model are constant during 2001Q1-2005Q4 and then become zero afterwards. They are completely driven by and correspond to the pattern of exogenous shifts in interest rates. In contrast, the learning model generates internal and positive persistence in forecast errors. Gelain, Lansing, and Mendicino (2013) derived house price forecasts from the future markets for the Case-Shiller house price index (where only the data from 2006 onwards are available) and showed that "the future market tends to overpredict future house prices when prices are falling." The learning model generates exactly this pattern of forecast errors as one can see from the lower right panel of figure 5 that the real house price forecast error becomes negative from 2006 onwards.

The model overpredicts the bust of house prices and the decline in the debt/GDP ratio and consumption. Various policies responding to the financial crisis and the Great Recession since the year 2007, such as liquidity provision to the private sector, may have contributed to the discrepancy between model predicted and actual prices. Associated with the price bust was slow changing debt/GDP ratio in the data. One reason for the gap between model predicted and actual debt/GDP ratio may be that borrowers cannot be forced to repay the mortgages. Adding such asymmetric feature of the mortgage contracts as in Justiniano, Primiceri and Tambalotti (2014) may help to explain the debt/GDP ratio during the price reversal. Incorporating these factors is beyond the scope of the present paper and left for future research.

In the learning model, agents evaluate the payoffs of different strategies using the subjective probability measure. Online Appendix G shows that the collateral constraint is indeed binding during the housing cycle over the year 2001-2008 in the learning model.

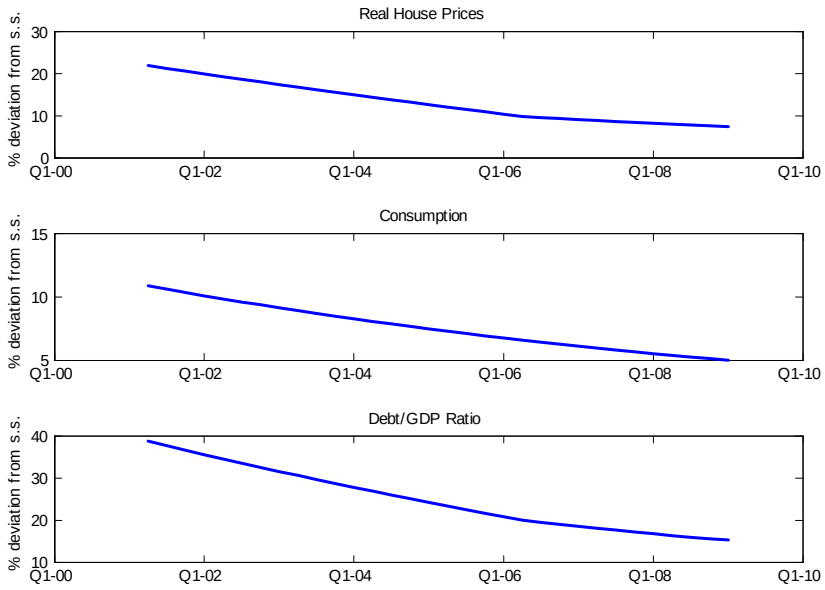

Figure 6: Predictions of the RE-II Model with Anticipated Interest Rate Movement

5.3 RE price dynamics with anticipated interest rate movement

Figure 6 displays the "RE-II" model dynamics,[35] i.e., when the low interest rates during 2001Q1-2005Q4 are anticipated by the agents. Except for the initial period, agents understand the effects of such structural change and could perfectly foresee the entire path of prices and quantities given that there is no remaining uncertainty after the initial real rate reduction. The real house prices jump immediately upward and then converge to the steady state. This is inconsistent with the pattern of prices and quantities observed in the data. In particular, the model does not generate persistent increases in house prices.

5.4 Dependence of the effect of interest rates changes on loan-to-Value (LTV) ratio

The section performs counterfactual analysis by reducing the LTV ratio and keeping other parameters unchanged. Figure 7 displays the counterfactual simulation result under RE and Learning model with LTV ratio of 40% (indicated by the line with diamonds and with cross, respectively) together with the data (the line with '+') and the result under the benchmark RE and Learning model with LTV ratio 54% as in figure 5 (the line with 'o' and the dashed line, respectively).

The effect of interest rates changes depends sensitively on the LTV ratio and the learning model with this lower LTV ratio generates counterfactually much lower rise in house prices, consumption and debt/GDP ratio. This suggests that if the LTV ratio were lower, much of the house price boom could have been avoided.

[35] For simulating the model in such senario, I firstly solve the law of motion for prices and quantities during 2006Q1-2008Q4. Then with them I recursively solve backward the policy function until 2001Q1.

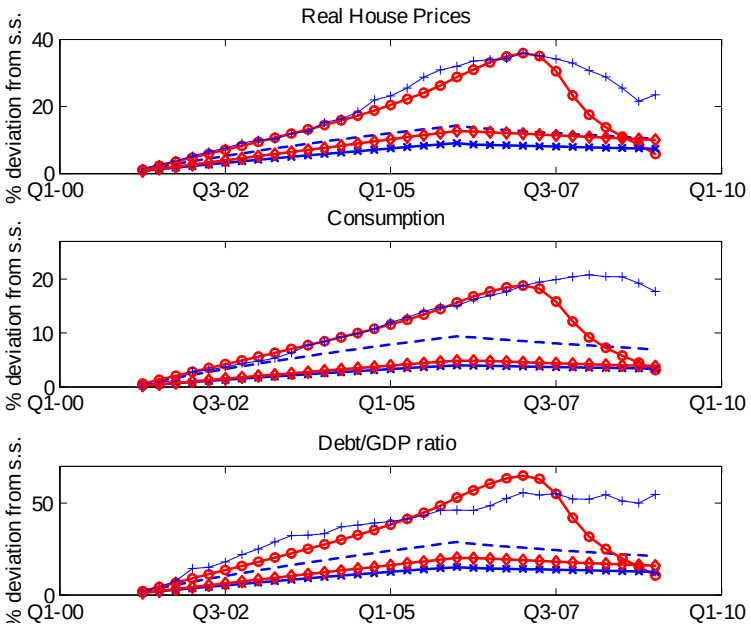

Figure 7: Data and Results under Benchmark RE, Learning and Counterfactual

I use the standardized difference between the peak response in the learning model and that under RE as a crude measure of the degree of amplification.[36] Table 1 reports the amplification of house prices, debt/GDP ratio and consumption in the learning model relative to the RE version of the model when the LTV ratio is varied in the interval $[40\%, 54\%]$. Interestingly, the amplification of the learning model is non-linear and increasing in the LTV ratio. For example, the learning model generates 150%, 125%, and 100% larger peak response of house prices, debt/GDP ratio, and consumption relative the RE model respectively when the LTV ratio is 54%, while the counterpart is 41%, 33%, and 22% when the LTV ratio is 40%.

Table 1:Dependence of Amplification on LTV Ratio

	Loan-to-Value Ratio					
	40%	43%	46%	49%	52%	54%
House prices	41%	45%	51%	61%	80%	150%
Debt/GDP ratio	33%	37%	42%	50%	67%	125%
Consumption	22%	25%	30%	38%	52%	100%

The result is in line with Proposition 6 which shows that a higher leveraged regime is more prone to self-reinforcing fluctuations. The weaker relationship between real interest rates and house prices under a lower LTV ratio regime may be a reason why changes of interest rates or monetary policy has a more muted effect on house prices in some earlier historical episodes featuring more stringent financing constraints than today.[37] It is poten-

[36] More precisely, amplification is calculated by the peak response under learning minus the peak response under RE and then divided by the latter.

[37] For example, in the published discussion of Adam, Kuang, and Marcet (2011), Robert Gordon questioned the mechanism of the AKM model based on the observation of a weaker relationship between interest rate changes and house prices in those earlier episodes.

tially interesting to explore this non-linear dependence on the leverage ratio to understand economic volatilities of aggregate variables across regimes with different leverage ratios or in cross-country comparisons.

6 Conclusion

The paper presents a model of housing and credit cycles which is consistent with the evidence on economic agents' expectations and can quantitatively account for the 2001-2008 U.S. boom and bust in house prices and associated household debt and aggregate consumption dynamics. The trigger of the price boom is the persistent fall in real interest rates after the year 2000. The response of house prices and quantities are drastically amplified due to the comovement and mutual reinforcement between agents' price beliefs and house price realizations via credit expansion/contraction. Positive (negative) development or surprise in house prices fuels optimism (pessimism) and credit expansion (contraction), which in turn boost (dampen) housing demand and house prices and reinforce agents' optimism (pessimism).

The model also uncovers an important dependence of the fluctuations of house prices and quantities on the leverage ratio in a collateral-constrained economy, which is missing in the RE analysis. It suggests that allowing for imperfect knowledge of agents, a higher leveraged economy is more prone to self-reinforcing fluctuations. In particular, the amplification of the response of house prices and quantities to the interest rates changes in the learning model relative to RE is non-linear and increasing in the leverage ratio of the economy. This gives an additional rationale for reasonable capital requirement regulation to avoid an extremely high leverage ratio regime which is prone to self-reinforcing fluctuations and associated with high economic volatilities.

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Online Appendix for "A Model of Housing and Credit Cycles with Imperfect Market Knowledge" (Pei Kuang)

A Steady State, Log-linearization and the Actual Law of Motion under Learning

The steady state of the interest rate, user cost of collateral, collateral prices, lenders' collateral holdings, borrowers' collateral holdings, borrowing, and borrowers' consumption are $R = \frac{1}{\beta^L}$, $u = \frac{a}{(1-\tau)}$, $q = \frac{aR}{(R-1)(1-\tau)}$, $H^L = G'^{-1}(\frac{aR}{1-\tau})$, $H^B = \overline{H} - H^L$, $b^B = (1-\tau)qH^B/R$ and $c^B = \overline{e}H^B$. Recall the model equations

$$H_t^B = \frac{a + q_t - (1-\tau)E_{t-1}^{\mathcal{P}} q_t}{q_t - \frac{1}{R_t}(1-\tau)E_t^{\mathcal{P}} q_{t+1}} H_{t-1}^B \tag{39}$$

$$q_t - \frac{1}{R_t} E_t^{\mathcal{P}} q_{t+1} = \frac{G'(H_t^L)}{R_t} \tag{40}$$

Equation (39) implies

$$\begin{aligned} \left[q_t - \frac{1}{R_t}(1-\tau)E_t^{\mathcal{P}} q_{t+1}\right] H_t^B &= \left[a + q_t - (1-\tau)E_{t-1}^{\mathcal{P}} q_t\right] H_{t-1}^B \\ \left[q_t R_t - (1-\tau)E_t^{\mathcal{P}} q_{t+1}\right] H_t^B &= \left[a + q_t - (1-\tau)E_{t-1}^{\mathcal{P}} q_t\right] H_{t-1}^B R_t \end{aligned}$$

Note $\frac{a+\tau q}{q} = 1 - \frac{1-\tau}{R}$. Log-linearizing this equation yields

$$\begin{aligned} & q\left[R - (1-\tau)\right] H^B \widehat{H}_t^B + H^B \left[Rq\left(\widehat{q}_t + \widehat{R}_t\right) - (1-\tau)qE_t^{\mathcal{P}}\widehat{q}_{t+1}\right] \\ = \; & (a+\tau q) RH^B \left(\widehat{R}_t + \widehat{H}_{t-1}^B\right) + RH^B \left(q\widehat{q}_t - (1-\tau)qE_{t-1}^{\mathcal{P}}\widehat{q}_t\right) \end{aligned}$$

Further simplification yields

$$\begin{aligned} & \left(1 - \frac{1-\tau}{R}\right)\widehat{H}_t^B + \left[\left(\widehat{q}_t + \widehat{R}_t\right) - \frac{1-\tau}{R}E_t^{\mathcal{P}}\widehat{q}_{t+1}\right] \\ = \; & \left(1 - \frac{1-\tau}{R}\right)\left(\widehat{R}_t + \widehat{H}_{t-1}^B\right) + \left(\widehat{q}_t - (1-\tau)E_{t-1}^{\mathcal{P}}\widehat{q}_t\right) \end{aligned}$$

Recall $R_t = \frac{A_t}{\beta^L}$, so $\widehat{R}_t = \widehat{A}_t$. This implies further

$$\widehat{H}_t^B = \frac{1}{\left(1 - \frac{1-\tau}{R}\right)}\left[\left(\widehat{q}_t - (1-\tau)E_{t-1}^{\mathcal{P}}\widehat{q}_t\right) - \left(\widehat{q}_t - \frac{1-\tau}{R}E_t^{\mathcal{P}}\widehat{q}_{t+1}\right)\right] + \widehat{H}_{t-1}^B - \frac{\frac{1-\tau}{R}}{\left(1 - \frac{1-\tau}{R}\right)}\widehat{A}_t \tag{41}$$

Now start with equation (40)

$$\begin{aligned} q_t - \frac{1}{R_t} E_t^{\mathcal{P}} q_{t+1} &= \frac{G^{'}(H_t^L)}{R_t} \\ &\iff R_t q_t - E_t^{\mathcal{P}} q_{t+1} = G^{'}(H_t^L) \end{aligned} \tag{42}$$

Recall the definition of $\frac{1}{\eta} = -\frac{G^{''}(H^L)H^L}{G^{'}(H^L)}\frac{H^B}{H^L}$. Note log-linearizing $H_t^B + H_t^L = \overline{H}$ implies $\widehat{H}_t^L = -\frac{H^B}{H^L}\widehat{H}_t^B$.

Log-linearization of equation (42) leads to

$$\begin{aligned} Rq\left(\widehat{q}_t + \widehat{R}_t\right) - qE_t^{\mathcal{P}}\widehat{q}_{t+1} &= G^{''}(H^L)H^L\widehat{H}_t^L \\ &= -\frac{G^{''}(H^L)H^L}{G^{'}(H^L)}\frac{H^B}{H^L}G^{'}(H^L)\widehat{H}_t^B \\ &= \frac{1}{\eta}G^{'}(H^L)\widehat{H}_t^B \\ &= \frac{1}{\eta}q\left(R-1\right)\widehat{H}_t^B \end{aligned}$$

In the last equation, the steady state relationship, $G^{'}\left(H^L\right) = q(R-1)$, is used. Simplification further yields

$$\frac{1}{\eta}\widehat{H}_t^B = \frac{R}{R-1}\left(\widehat{q}_t - \frac{1}{R}E_t^{\mathcal{P}}\widehat{q}_{t+1} + \widehat{R}_t\right)$$

Given that $\widehat{R}_t = \widehat{A}_t$, the above equation leads to

$$\frac{1}{\eta}\widehat{H}_t^B = \frac{R}{R-1}\left(\widehat{q}_t - \frac{1}{R}E_t^{\mathcal{P}}\widehat{q}_{t+1} + \widehat{A}_t\right) \tag{43}$$

Below the actual law of motion (ALM) for collateral prices is derived. Suppose the perceived law of motion for collateral prices is $\widehat{q}_t = \zeta_t^m + \zeta_t^p\widehat{H}_{t-1}^B + \zeta_t^s\widehat{A}_t$. Conditional expectations at period t and $t-1$ for collateral prices are $E_t^{\mathcal{P}}\widehat{q}_{t+1} = \zeta_t^m + \zeta_t^p\widehat{H}_t^B$ and $E_{t-1}^{\mathcal{P}}\widehat{q}_t = \zeta_{t-1}^m + \zeta_{t-1}^p\widehat{H}_{t-1}^B$.

Equation (41) can be simplified to

$$\widehat{H}_t^B = \frac{1}{\left(1-\frac{1-\tau}{R}\right)}\left[\frac{1-\tau}{R}E_t^{\mathcal{P}}\widehat{q}_{t+1} - (1-\tau)E_{t-1}^{\mathcal{P}}\widehat{q}_t\right] + \widehat{H}_{t-1}^B - \frac{\frac{1-\tau}{R}}{\left(1-\frac{1-\tau}{R}\right)}\widehat{A}_t \tag{44}$$

Combining (43) and (44) by eliminating $\widehat{H}_t^B$, I get

$$\widehat{q}_t = \xi_1 E_t^{\mathcal{P}}\widehat{q}_{t+1} - \xi_2 E_{t-1}^{\mathcal{P}}\widehat{q}_t + \xi_3\widehat{H}_{t-1}^B + \xi_4\widehat{A}_t$$

where $\xi_1 = \frac{1}{R} + \frac{1}{\eta}\frac{R-1}{R}\frac{1}{1-\frac{1}{R}(1-\tau)}\frac{1-\tau}{R}$, $\xi_2 = (1-\tau)\frac{1}{\eta}\frac{R-1}{R}\frac{1}{1-\frac{1}{R}(1-\tau)}$, $\xi_3 = \frac{1}{\eta}\frac{R-1}{R}$ and $\xi_4 = -(1 + \frac{\frac{1-\tau}{R}}{1-\frac{1}{R}(1-\tau)}\frac{1}{\eta}\frac{R-1}{R})$.

Substituting conditional expectations into the last equation, I get

$$\widehat{q}_t = \xi_1 \left(\zeta_t^m + \zeta_t^p \widehat{H}_t^B\right) - \xi_2 \left(\zeta_{t-1}^m + \zeta_{t-1}^p \widehat{H}_{t-1}^B\right) + \xi_3 \widehat{H}_{t-1}^B + \xi_4 \widehat{A}_t \tag{45}$$

Substituting expectations into (43), I get

$$\frac{1}{\eta}\widehat{H}_t^B = \frac{R}{R-1}\left(\widehat{q}_t - \frac{1}{R}\left(\zeta_t^m + \zeta_t^p \widehat{H}_t^B\right) + \widehat{A}_t\right)$$

or alternatively

$$\widehat{H}_t^B = \frac{\widehat{q}_t + \widehat{A}_t - \frac{1}{R}\zeta_t^m}{\frac{1}{\eta}\frac{R-1}{R} + \frac{1}{R}\zeta_t^p} \tag{46}$$

Combining (46) and (45) by eliminating $\widehat{H}_t^B$, I get the ALM for collateral prices under learning

$$\widehat{q}_t = T_1(\zeta_{t-1}^m, \zeta_t^m, \zeta_t^p) + T_2(\zeta_{t-1}^p, \zeta_t^p)\widehat{H}_{t-1}^B + T_3(\zeta_t^p)\widehat{A}_t$$

where $T_1(\zeta_{t-1}^m, \zeta_t^m, \zeta_t^p) = \frac{(\xi_1\zeta_t^m - \xi_2\zeta_{t-1}^m)(1+\frac{\zeta_t^p}{\xi_3 R}) - \frac{\xi_1}{\xi_3}\frac{\zeta_t^m\zeta_t^p}{R}}{1+\frac{\zeta_t^p}{\xi_3 R} - \xi_1\zeta_t^p\frac{1}{\xi_3}}$, $T_2(\zeta_{t-1}^p, \zeta_t^p) = \frac{\xi_3 - \xi_2\zeta_{t-1}^p}{1-\frac{\xi_1\zeta_t^p}{\xi_3+\zeta_t^p\frac{1}{R}}}$ and $T_3(\zeta_t^p) = \frac{(\xi_4 + \frac{\xi_1\zeta_t^p}{\xi_3+\frac{1}{R}\zeta_t^p})}{1-\frac{\xi_1\zeta_t^p}{\xi_3+\frac{1}{R}\zeta_t^p}}$. Those expressions are used for proving the propositions and computations.

A.1 An alternative expression of the T_2-map

In the paper, an alternative expression of the T_2-map, i.e., $T_2(\zeta_{t-1}^p, \zeta_t^p) = \frac{\left(\frac{1}{R}\zeta_t^p + \frac{1}{\eta}\frac{R-1}{R}\right)\left(1-\frac{1-\tau}{1-\frac{1-\tau}{R}}\zeta_{t-1}^p\right)}{1-\frac{\frac{1-\tau}{R}}{1-\frac{1-\tau}{R}}\zeta_t^p}$, is presented and used to illustrate some intuitions. Its derivation is provided here. I am only interested in deriving the T_2-map and keeping track of agents' beliefs, so I consider a simpler case without learning about $\overline{\zeta}^m$ and without shock $\widehat{A}_t$ for the simplicity of expositions. The T_2-map in the full model with learning about the steady state and with stochastic innovations will be the same as that in the simpler case.

Recall our model equations are (43) and (44) which contain three expectation terms. Substituting conditional expectations into equation (44), we get

$$\begin{aligned}
\widehat{H}_t^B &= \frac{1}{\left(1-\frac{1-\tau}{R}\right)}\left[\frac{1-\tau}{R}E_t^{\mathcal{P}}\widehat{q}_{t+1} - (1-\tau)E_{t-1}^{\mathcal{P}}\widehat{q}_t\right] + \widehat{H}_{t-1}^B \\
&= \frac{1}{\left(1-\frac{1-\tau}{R}\right)}\left[\frac{1-\tau}{R}\zeta_t^p\widehat{H}_t^B - (1-\tau)\zeta_{t-1}^p\widehat{H}_{t-1}^B\right] + \widehat{H}_{t-1}^B
\end{aligned}$$

Note in the above equation ζ_t^p comes from the down-payment and ζ_{t-1}^p debt repayment. So

$$\widehat{H}_t^B = \frac{1 - \frac{(1-\tau)}{1-\frac{1-\tau}{R}}\zeta_{t-1}^p}{1-\frac{\frac{1-\tau}{R}}{1-\frac{1-\tau}{R}}\zeta_t^p}\widehat{H}_{t-1}^B \tag{47}$$

Substituting conditional expectations into equation (43)

$$\begin{aligned}\widehat{q}_t &= \frac{1}{R}E_t^{\mathcal{P}}\widehat{q}_{t+1} + \frac{1}{\eta}\frac{R-1}{R}\widehat{H}_t^B \\ &= \frac{1}{R}\zeta_t^p\widehat{H}_t^B + \frac{1}{\eta}\frac{R-1}{R}\widehat{H}_t^B \\ &= \left(\frac{1}{R}\zeta_t^p + \frac{1}{\eta}\frac{R-1}{R}\right)\widehat{H}_t^B \end{aligned} \tag{48}$$

Note ζ_t^p in the above equation arise from the housing re-sale price/value. Combining equation (47) and (48) yields the T_2-map used in the paper

$$\widehat{q}_t = \left(\frac{1}{R}\zeta_t^p + \frac{1}{\eta}\frac{R-1}{R}\right)\frac{1-\frac{(1-\tau)}{1-\frac{1-\tau}{R}}\zeta_{t-1}^p}{1-\frac{\frac{1-\tau}{R}}{1-\frac{1-\tau}{R}}\zeta_t^p}\widehat{H}_{t-1}^B$$

From this equation, the three belief terms can be easily tracked. It can be shown that it is identical to the T_2-map derived earlier, i.e., $T_2(\zeta_{t-1}^p, \zeta_t^p) = \frac{\xi_3 - \xi_2\zeta_{t-1}^p}{1-\frac{\xi_1\zeta_t^p}{\xi_3+\zeta_t^p\frac{1}{R}}}$.

A.2 Consumption and Debt/GDP ratio

Denote by Y the steady state value of aggregate output. Log-linearizing aggregate production yields

$$\widehat{y}_t = \frac{(a+\overline{e}) - G'}{(a+\overline{e})}\frac{(a+\overline{e})H^B}{Y}\widehat{H}_{t-1}^B$$

It equals to the product of the productivity gap $\frac{(a+\overline{e})-G'}{(a+\overline{e})}$ between borrowers and lenders, the production share of borrowers $\frac{(a+\overline{e})H^B}{Y}$ and the redistribution of collateral. Aggregate consumption $\widehat{c}_t$ will be the same as aggregate output because of zero net investment in housing.

Denote by $\overline{C}$ and $\overline{GDP}$ aggregate consumption and GDP at the steady state, respectively. The log-linearized GDP is

$$\widehat{GDP}_t = \frac{\overline{C}}{\overline{GDP}}\widehat{c}_t \tag{49}$$

and the debt/GDP ratio is

$$\begin{aligned}(\widehat{Debt/GDP})_t &= \widehat{b}_t^B - \widehat{GDP}_t & (50)\\ &= E_t^{\mathcal{P}}\widehat{q}_{t+1} + H_t^B - \widehat{R}_t - \frac{\overline{C}}{\overline{GDP}}\widehat{y}_t & (51)\end{aligned}$$

where $\widehat{b}_t^B$ can be calculated by the collateral constraint.

B Derivation of the Bayesian Posterior Mean[38]

I assume the prior distribution of unknown parameters, i.e., the parameters linking prices and fundamentals (ζ^m, ζ^p) and the residual precision $K \equiv \frac{1}{\sigma_\varepsilon^2}$, is a Normal-Gamma distribution as following

$$\begin{array}{rcl} K & \sim & G(\gamma_0, d_0^{-2}) \\ (\zeta^m, \zeta^p)' \mid K & = k \sim & N((\theta_0^m, \theta_0^p)', (\nu_0 k)^{-1}) \end{array}$$

The prior distribution of K is a gamma distribution and the conditional prior of $\zeta \equiv (\zeta^m, \zeta^p)$ given K is a multivariate normal distribution.

I drop the terms which do not involve (θ, k) by using the proportionality symbol. The conditional probability of the collateral price is a normal distribution with following conditional probability density function

$$p(y_t|\theta, k) \propto k^{\frac{1}{2}} exp\{-\frac{k}{2}(y_t - x_t'\theta)'(y_t - x_t'\theta)\}$$

The prior density of the parameters is following

$$p(\theta, h) \propto k^{\gamma_{t-1}-1} exp\{-d_{t-1}^{-2}k\} k^{\frac{1}{2}} exp\{-\frac{k}{2}(\theta - \theta_{t-1})'\nu_{t-1}(\theta - \theta_{t-1})\}$$

I proceed to show that the posterior distribution of the parameters is as follows

$$\begin{array}{rcl} \theta|K & = & k \sim N(\theta_t, (\nu_t k)^{-1}) \\ K & \sim & G(\gamma_t, d_t^{-2}) \end{array}$$

with probability density function

$$p(\theta, k|y_t) \propto k^{\gamma_t - 1} exp\{-d_t^{-2}k\} k^{\frac{1}{2}} exp\{-\frac{k}{2}(\theta - \theta_t)'\nu_t(\theta - \theta_t)\}$$

where

$$\begin{array}{rcl} \theta_t & = & \theta_{t-1} + (x_t x_t' + \nu_{t-1})^{-1} x_t (y_t - x_t'\theta_{t-1}) \\ \nu_t & = & \nu_{t-1} + x_t x_t' \\ \gamma_t & = & \gamma_{t-1} + \frac{1}{2} \\ d_t^{-2} & = & d_{t-1}^{-2} + \frac{1}{2}(y_t - x_t'\theta_{t-1})'(x_t x_t' + \nu_{t-1})^{-1}\nu_{t-1}(y_t - x_t'\theta_{t-1}) \end{array}$$

The above equations can be derived using Bayes' law. The derivations are standard so only some critical intermediate steps are presented here. The posterior density

$$p(\theta, k|y_t) \propto p(y_t|\theta, k)p(\theta, k)$$

[38] The derivation follows DeGroot (1974).

It can be derived from the right hand side that the posterior mean of the parameters is

$$\begin{aligned}
& \theta_t \\
= \; & (x_t x_t' + \nu_{t-1})^{-1}(\nu_{t-1}\theta_{t-1} + x_t y_t) \\
= \; & (x_t x_t' + \nu_{t-1})^{-1}\nu_{t-1}\theta_{t-1} + (x_t x_t' + \nu_{t-1})^{-1} x_t y_t \\
= \; & \theta_{t-1} - (x_t x_t' + \nu_{t-1})^{-1} x_t x_t' \theta_{t-1} + (x_t x_t' + \nu_{t-1})^{-1} x_t y_t \\
= \; & \theta_{t-1} + (x_t x_t' + \nu_{t-1})^{-1} x_t (y_t - x_t'\theta_{t-1})
\end{aligned}$$

Note

$$\begin{aligned}
& (y_t - x_t'\theta)'(y_t - x_t'\theta) + (\theta - \theta_{t-1})'\nu_{t-1}(\theta - \theta_{t-1}) \\
= \; & y_t'y_t - 2\theta' x_t y_t + \theta' x_t x_t'\theta + \theta'\nu_{t-1}\theta - 2\theta'\nu_{t-1}\theta_{t-1} + \theta_{t-1}'\nu_t\theta_{t-1} \\
= \; & \theta'(x_t x_t' + \nu_{t-1})\theta - 2\theta'(x_t y_t + \nu_{t-1}\theta_{t-1}) + y_t'y_t + \theta_{t-1}'\nu_{t-1}\theta_{t-1} \\
= \; & \left(\theta - (x_t x_t' + \nu_{t-1})^{-1}(\nu_{t-1}\theta_{t-1} + x_t y_t)\right)' (x_t x_t' + \nu_{t-1}) \left(\theta - (x_t x_t' + \nu_{t-1})^{-1}(\nu_{t-1}\theta_{t-1} + x_t y_t)\right) \\
& \left[y_t'y_t + \theta_{t-1}'\nu_{t-1}\theta_{t-1} - (\nu_{t-1}\theta_{t-1} + x_t y_t)'(x_t x_t' + \nu_{t-1})^{-1}(\nu_{t-1}\theta_{t-1} + x_t y_t)\right]
\end{aligned}$$

C Deriving the Constant-Gain Learning Algorithm from Bayesian Updating[39]

Agents perceive the following random walk model of coefficient variation

$$\phi_t = \phi_{t-1} + \iota_t \qquad E\iota_t\iota_t' = R_{1t} \tag{52}$$

$$y_t = \bar{\theta}_{t-1}' x_t + \varsigma_t \qquad E\varsigma_t\varsigma_t' = R_{2t} \tag{53}$$

Define $P_{t-1} = E[(\phi_{t-1} - \theta_{t-1})(\phi_{t-1} - \theta_{t-1})]$. The prior belief about $\bar{\theta}_0$ are $N(\theta_0, P_{0|0})$. The posterior of θ_t can be represented by the basic Kalman filter, which takes the form of following recursions[40]

$$\theta_t = \theta_{t-1} + L_t[y_t - x_t'\theta_{t-1}] \tag{54}$$

$$L_t = \frac{P_{t|t-1}x_t}{R_{2t} + x_t' P_{t|t-1} x_t} \tag{55}$$

$$P_{t+1|t} = P_{t|t-1} - \frac{P_{t|t-1}x_t x_t' P_{t|t-1}}{R_{2t} + x_t' P_{t|t-1} x_t} + R_{1t+1} \tag{56}$$

Furthermore, agents are assumed to perceive $R_{1t} = \frac{g}{1-g}P_{t-1|t-1}$ and $R_{2t} = \frac{1}{g}$. Note $P_{t|t-1} = P_{t-1|t-1} + R_{1t} = \frac{1}{1-g}P_{t-1|t-1}$. Equations $(55) - (56)$ become

$$\theta_t = \theta_{t-1} + L_t[y_t - x_t'\hat{\theta}_{t-1}] \tag{57}$$

$$L_t = \frac{P_{t-1|t-1}x_t}{\frac{1-g}{g} + x_t' P_{t-1|t-1} x_t} \tag{58}$$

[39] The derivations follow Ljung (1991) and Sargent (1999) except that both of them use inconsistent notations.

[40] Note for the model considered here I have $\theta_{t|t-1} = \theta_{t-1|t-1}$, so I suppress the conditioned information set and use θ_t for both.

And equation (54) becomes

$$P_{t+1|t} - R_{1t+1} = P_{t|t-1} - \frac{P_{t|t-1}x_t x_t' P_{t|t-1}}{R_{2t} + x_t' P_{t|t-1} x_t} \tag{59}$$

$$P_{t|t} = \frac{1}{1-g}\left[P_{t-1|t-1} - \frac{P_{t-1|t-1}x_t x_t' P_{t-1|t-1}}{x_t' P_{t-1|t-1}x_t + \frac{1-g}{g}}\right] \tag{60}$$

The constant gain learning algorithm is following[41]

$$\theta_t = \theta_{t-1} + gR_t^{-1}x_t(y_t - \theta_{t-1}' x_t) \tag{61}$$

$$R_t = R_{t-1} + g\left(x_t x_t' - R_{t-1}\right) \tag{62}$$

Below I show the above two formulations are equivalent. Use $R_t^{-1} \equiv P_{t|t}$, equation (62) yields

$$R_t^{-1} = \left((1-g)R_{t-1} + g x_t x_t'^{-1}\right) \tag{63}$$

$$= \frac{1}{1-g}R_{t-1}^{-1} - \frac{1}{1-g}R_{t-1}^{-1}x_t\left[x_t'\frac{1}{1-g}R_{t-1}^{-1}x_t + \frac{1}{g}\right]^{-1} x_t'\frac{1}{1-g}R_{t-1}^{-1} \tag{64}$$

$$= \frac{1}{1-g}\left[P_{t-1|t-1} - \frac{P_{t-1|t-1}x_t x_t' P_{t-1|t-1}}{x_t' P_{t-1|t-1}x_t + \frac{1-g}{g}}\right] \tag{65}$$

From equation (63) to equation (64), the matrix inversion formula is used and stated in lemma 1 below. Specifically, it is applied with $A = (1-g)R_{t-1}$, $B = x_t$, $C = g$, $D = x_t'$.

Now I proceed to show the equivalence between equation (57) and (61). It suffices to show that $gR_t^{-1}x_t = L_t$.

$$gR_t^{-1}x_t = gP_{t|t}x_t \tag{66}$$

$$= \frac{g}{1-g}\left[P_{t-1|t-1} - \frac{P_{t-1|t-1}x_t x_t' P_{t-1|t-1}}{\frac{1-g}{g} + x_t' P_{t-1|t-1}x_t}\right]x_t \tag{67}$$

$$= \frac{P_{t-1|t-1}x_t}{\frac{1-g}{g} + x_t' P_{t-1|t-1}x_t} \tag{68}$$

$$= L_t \tag{69}$$

From equation (66) to (67), equation (60) is used.

Lemma 1. Let A, B, C and D be matrices of compatible dimensions, so that the product BCD and the sum A+BCD exist. Then

$$[A + BCD]^{-1} = A^{-1} - A^{-1}B[DA^{-1}B + C^{-1}]^{-1}DA^{-1} \tag{70}$$

Proof: see Ljung and Soederstroem (1983) pp. 19. (Sketch: show the RHS of (70) multiplied by $A + BCD$ from the right is equal to identity matrix.)

[41] The learning rule using lagged data can be derived similarly as in Adam, Beutel and Marcet (2014).

D Proof of Proposition 4

Local stability of the Minimum State Variable (MSV) RE solution is determined by the stability of the following associated ordinary differential equations (ODEs)

$$\begin{aligned}\frac{d\zeta^m}{d\tau} &= T_1(\zeta^m,\zeta^p)-\zeta^m \\ \frac{d\zeta^p}{d\tau} &= T_2(\zeta^p,\zeta^p)-\zeta^p\end{aligned}$$

where $T_1(\zeta^m,\zeta^p)=\frac{(\xi_1\zeta^m-\xi_2\zeta^m)(1+\frac{\zeta^p}{\xi_3 R})-\frac{\xi_1}{\xi_3}\frac{\zeta^m\zeta^p}{R}}{1+\frac{\zeta^p}{\xi_3 R}-\xi_1\zeta^p\frac{1}{\xi_3}}$, $T_2(\zeta^p,\zeta^p)=\frac{\xi_3-\xi_2\zeta^p}{1-\frac{\xi_1\zeta^p}{\xi_3+\zeta^p\frac{1}{R}}}$, $\xi_1=\frac{1}{R}+\frac{1}{\eta}\frac{R-1}{R}\frac{1}{1-\frac{1}{R}(1-\tau)}\frac{1-\tau}{R}$, $\xi_2=(1-\tau)\frac{1}{\eta}\frac{R-1}{R}\frac{1}{1-\frac{1}{R}(1-\tau)}$, $\xi_3=\frac{1}{\eta}\frac{R-1}{R}$.

Solving the fixed point of the T-map, I get $\overline{\zeta}^m=0$, and $\overline{\zeta}^p=\frac{(1-\frac{1}{R}(1-\tau))\frac{1}{\eta}}{1+(1-\tau)(\frac{1}{\eta}-\frac{1}{R})}$.

The E-stability condition requires that the eigenvalues of the Jacobian of the right hand side of the above ODEs are negative. Since ζ^m does not show up in the ODE for ζ^p, the eigenvalues will be on the diagonal of the Jacobian matrix and only two partial derivatives, i.e., $\frac{\partial T_1(\zeta^m,\zeta^p)}{\partial\zeta^m}|_{\zeta^m=\overline{\zeta}^m,\zeta^p=\overline{\zeta}^p}$ and $\frac{\partial T_2(\zeta^p,\zeta^p)}{\partial\zeta^p}|_{\zeta^p=\overline{\zeta}^p}$, matter for the E-stability.

Fixing the arguments of the T_2-map at ζ^p yields

$$\begin{aligned}T_2(\zeta^p,\zeta^p) &= \frac{\xi_3-\xi_2\zeta^p}{1-\frac{\xi_1\zeta^p}{\xi_3+\zeta^p\frac{1}{R}}} \\ &= \frac{(\xi_3+\frac{\zeta^p}{R})(\xi_3-\xi_2\zeta^p)}{\xi_3+(\frac{1}{R}-\xi_1)\zeta^p} \\ &= \frac{\xi_3^2+(\frac{1}{R}-\xi_2)\zeta^p\xi_3-\frac{\xi_2}{R}(\zeta^p)^2}{\xi_3+(\frac{1}{R}-\xi_1)\zeta^p}\end{aligned}$$

The derivative of T_2 with respect to ζ^p evaluated at the RE value is

$$\begin{aligned}\frac{\partial T_2(\zeta^p,\zeta^p)}{\partial\zeta^p}|_{\zeta^p=\overline{\zeta}^p} &= \frac{(\frac{1}{R}-\xi_2)\xi_3-\overline{\zeta}^p\frac{2}{R}\xi_2}{\xi_3+(\frac{1}{R}-\xi_1)\overline{\zeta}^p} \\ &\quad -(\frac{1}{R}-\xi_1)\frac{(\xi_3^2+(\frac{1}{R}-\xi_2)\zeta^p\xi_3-\frac{\xi_2}{R}(\zeta^p)^2)}{\left(\xi_3+(\frac{1}{R}-\xi_1)\overline{\zeta}^p\right)^2} \\ &= \frac{(\frac{1}{R}-\xi_2)\xi_3-\overline{\zeta}^p\frac{2}{R}\xi_2}{\xi_3+(\frac{1}{R}-\xi_1)\overline{\zeta}^p}-(\frac{1}{R}-\xi_1)\frac{\overline{\zeta}^p}{\xi_3+(\frac{1}{R}-\xi_1)\overline{\zeta}^p} \\ &= \frac{(\frac{1}{R}-\xi_2)\xi_3-\overline{\zeta}^p(\frac{2}{R}\xi_2+\frac{1}{R}-\xi_1)}{\xi_3+(\frac{1}{R}-\xi_1)\overline{\zeta}^p} \qquad (71)\end{aligned}$$

Note $\overline{\zeta}^p$ is the fixed point of the T_2-map, i.e., $\overline{\zeta}^p=\frac{\xi_3^2+(\frac{1}{R}-\xi_2)\overline{\zeta}^p\xi_3-\frac{\xi_2}{R}(\overline{\zeta}^p)^2}{\xi_3+(\frac{1}{R}-\xi_1)\overline{\zeta}^p}$, is used when deriving the second equality above. Substituting for ξ_1,ξ_3, and $\overline{\zeta}^p$, I show that the denominator of the above derivative is positive.

$$
\begin{aligned}
& \xi_3 + \overline{\zeta}^p(\frac{1}{R} - \xi_1) \\
= & \frac{1}{\eta}\frac{R-1}{R} + \frac{\overline{\zeta}^p}{R}\left(1 - R\left(\frac{1}{R} + \frac{1}{\eta}\frac{R-1}{R}\frac{1}{1-\frac{1}{R}(1-\tau)}\frac{1-\tau}{R}\right)\right) \\
= & \frac{1}{\eta}\frac{R-1}{R} - \frac{(1-\frac{1}{R}(1-\tau))\frac{1}{\eta}}{1+(1-\tau)(\frac{1}{\eta}-\frac{1}{R})}\frac{1}{R}\left(\frac{1}{\eta}\frac{R-1}{R}\frac{1}{1-\frac{1}{R}(1-\tau)}(1-\tau)\right) \\
= & \frac{1}{\eta}\frac{R-1}{R}\frac{1}{R}\left(R - \frac{\frac{1}{\eta}(1-\tau)}{1+(1-\tau)(\frac{1}{\eta}-\frac{1}{R})}\right) \\
= & \frac{1}{\eta}\frac{R-1}{R}\frac{1}{R}\left(R - \frac{\frac{1}{\eta}(1-\tau)}{1-\frac{1-\tau}{R}+\frac{1-\tau}{\eta}}\right) \\
> & 0
\end{aligned} \tag{72}
$$

I proceed to show that the derivative (71) is smaller than 1. Given (72), note

$$\frac{\partial T_2(\zeta^p, \zeta^p)}{\partial \zeta^p}|_{\zeta^p=\overline{\zeta}^p} < 1$$

is equivalent to

$$(\frac{1}{R} - \xi_2)\xi_3 - \overline{\zeta}^p(\frac{2}{R}\xi_2 + \frac{1}{R} - \xi_1) < \xi_3 + (\frac{1}{R} - \xi_1)\overline{\zeta}^p$$

Rearranging the above inequality yields

$$(\frac{1}{R} - \xi_2 - 1)\xi_3 < 2\overline{\zeta}^p(\frac{\xi_2}{R} + \frac{1}{R} - \xi_1)$$

Plugging $\xi's$, it can be shown that the left hand side of the above inequality is negative because $\xi_2 > 0, \frac{1}{R} - 1 < 0$, and $\xi_3 > 0$. It can be shown that the right hand side is exactly zero because $\xi_1 = \frac{1}{R} + \frac{\xi_2}{R}$.

Now I turn to the first derivative. Recall $T_1(\zeta^m, \zeta^p) = \frac{\xi_1\zeta^m - \xi_2\zeta^m(1+\frac{\zeta^p}{\xi_3 R})}{1+\frac{\zeta^p}{\xi_3 R} - \xi_1\zeta^p\frac{1}{\xi_3}}$. Taking the derivative of T_1 with respect to ζ^m and evaluate it at the RE belief yields

$$
\begin{aligned}
\frac{\partial T_1(\zeta^m, \zeta^p)}{\partial \zeta^m}|_{\zeta^m=\overline{\zeta}^m, \zeta^p=\overline{\zeta}^p} &= \frac{\xi_1 - \xi_2(1+\frac{\overline{\zeta}^p}{\xi_3 R})}{1+\frac{\overline{\zeta}^p}{\xi_3 R} - \xi_1\overline{\zeta}^p\frac{1}{\xi_3}} \\
&= \frac{(\xi_1-\xi_2)\xi_3 R - \xi_2\overline{\zeta}^p}{\xi_3 R + \overline{\zeta}^p(1-\xi_1 R)}
\end{aligned}
$$

I then show the above derivative is smaller than 1. Given (72), it is equivalent to show

$$(\xi_1 - \xi_2)\xi_3 R - \xi_2\overline{\zeta}^p < \xi_3 R + \overline{\zeta}^p(1 - \xi_1 R)$$

Rearranging the above inequality yields

$$(\xi_1 - \xi_2 - 1)\xi_3 R < (\xi_2 + 1 - \xi_1 R)\overline{\zeta}^p \tag{73}$$

Plugging in the parameters, it can be shown firstly the left hand side of the above inequality is negative because

$$\begin{aligned} \xi_1 - \xi_2 - 1 &= \frac{1}{R} + \frac{\xi_2}{R} - \xi_2 - 1 \\ &= \left(\frac{1}{R} - 1\right)(1 + \xi_2) \\ &< 0 \end{aligned}$$

$\xi_3 > 0$, and $R > 0$. It can also be shown that the right hand side of (73) is zero because $\xi_1 R = 1 + \xi_2$.

E Proof of Proposition 6

Define $s(\zeta_t^p) = \frac{\xi_1 \zeta_t^p}{\xi_3 + \zeta_t^p \frac{1}{R}}$. Recall the T_2-map is

$$\begin{aligned} T_2(\zeta_{t-1}^p, \zeta_t^p) &= \frac{\xi_3 - \xi_2 \zeta_{t-1}^p}{1 - \frac{\xi_1 \zeta_t^p}{\xi_3 + \zeta_t^p \frac{1}{R}}} \\ &= \frac{\xi_3 - \xi_2 \zeta_{t-1}^p}{1 - s(\zeta_t^p)} \\ &\simeq \overline{\zeta}^p + \left(-\frac{\partial T_2}{\partial \zeta_{t-1}^p}|_{\zeta_{t-1}^p = \overline{\zeta}^p, \zeta_t^p = \overline{\zeta}^p}\right)\left(\overline{\zeta}^p - \zeta_{t-1}^p\right) - \left(\frac{\partial T_2}{\partial \zeta_t^p}|_{\zeta_{t-1}^p = \overline{\zeta}^p, \zeta_t^p = \overline{\zeta}^p}\right)\left(\overline{\zeta}^p - \zeta_t^p\right) \end{aligned}$$

where $-\frac{\partial T_2}{\partial \zeta_{t-1}^p}|_{\overline{\zeta}^p}$ and $\frac{\partial T_2}{\partial \zeta_t^p}|_{\overline{\zeta}^p}$ stand for $-\frac{\partial T_2}{\partial \zeta_{t-1}^p}|_{\zeta_{t-1}^p = \overline{\zeta}^p, \zeta_t^p = \overline{\zeta}^p}$ and $\frac{\partial T_2}{\partial \zeta_t^p}|_{\zeta_{t-1}^p = \overline{\zeta}^p, \zeta_t^p = \overline{\zeta}^p}$, respectively.

A sufficient condition to guarantee momentum in belief is

$$-\frac{\partial T_2}{\partial \zeta_{t-1}^p}|_{\overline{\zeta}^p} \geq \frac{\partial T_2}{\partial \zeta_t^p}|_{\overline{\zeta}^p} \tag{74}$$

Suppose $\zeta_{t-1}^p < \zeta_t^p \leq \overline{\zeta}^p$, then $\overline{\zeta}^p - \zeta_{t-1}^p > \overline{\zeta}^p - \zeta_t^p \geq 0$. Given that the momentum condition (74) holds, we have $\left(-\frac{\partial T_2}{\partial \zeta_{t-1}^p}|_{\zeta_{t-1}^p = \overline{\zeta}^p}\right)\left(\zeta_{t-1}^p - \overline{\zeta}^p\right) > \left(\frac{\partial T_2}{\partial \zeta_t^p}|_{\zeta_t^p = \overline{\zeta}^p}\right)\left(\overline{\zeta}^p - \zeta_t^p\right)$ and hence $T_2(\zeta_{t-1}^p, \zeta_t^p) > \overline{\zeta}^p$. Using the belief updating rule, agents will update their belief upward, i.e., $\zeta_{t+1}^p > \zeta_t^p$. Similarly, if $\zeta_{t-1}^p > \zeta_t^p \geq \overline{\zeta}^p$, we have $T_2(\zeta_{t-1}^p, \zeta_t^p) < \overline{\zeta}^p$ and $\zeta_{t+1}^p < \zeta_t^p$.

I proceed to calculate the two derivatives. We have

$$-\frac{\partial T_2}{\partial \zeta_{t-1}^p}|_{\overline{\zeta}^p} = \frac{\xi_2}{1 - s(\overline{\zeta}^p)}$$

and

$$\begin{aligned}\frac{\partial T_2}{\partial \zeta_t^p}|_{\overline{\zeta}^p} &= \frac{\xi_3 - \xi_2\overline{\zeta}^p}{1-s(\overline{\zeta}^p)}\frac{s'(\overline{\zeta}^p)}{1-s(\overline{\zeta}^p)} \\ &= \overline{\zeta}^p\frac{s'(\overline{\zeta}^p)}{1-s(\overline{\zeta}^p)}\end{aligned}$$

$\overline{\zeta}^p$ is the fixed point of the T_2-map, i.e., $\frac{\xi_3-\xi_2\overline{\zeta}^p}{1-s(\overline{\zeta}^p)} = \overline{\zeta}^p$, is used in the last equality. So the condition (74) is equivalent to $\frac{-\xi_2}{1-s(\overline{\zeta}^p)} \geq \overline{\zeta}^p\frac{s'(\overline{\zeta}^p)}{1-s(\overline{\zeta}^p)}$ or

$$\xi_2 \geq \overline{\zeta}^p s'(\overline{\zeta}^p) \tag{75}$$

Calculating the derivative $s'(\overline{\zeta}^p)$ as follows

$$\begin{aligned}s'(\overline{\zeta}^p) &= \frac{\xi_1}{\xi_3 + \overline{\zeta}^p\frac{1}{R}} - \frac{\xi_1\overline{\zeta}^p}{\xi_3+\overline{\zeta}^p\frac{1}{R}}\frac{\frac{1}{R}}{\xi_3+\overline{\zeta}^p\frac{1}{R}} \\ &= \frac{s(\overline{\zeta}^p)}{\overline{\zeta}^p} - \frac{s(\overline{\zeta}^p)\frac{1}{R}}{\xi_3+\overline{\zeta}^p\frac{1}{R}}\end{aligned}$$

So

$$\begin{aligned}\overline{\zeta}^p s'(\overline{\zeta}^p) &= s(\overline{\zeta}^p) - \frac{\overline{\zeta}^p s(\overline{\zeta}^p)\frac{1}{R}}{\xi_3+\overline{\zeta}^p\frac{1}{R}} \\ &= \frac{s(\overline{\zeta}^p)\xi_3}{\xi_3+\overline{\zeta}^p\frac{1}{R}} \\ &= \frac{s(\overline{\zeta}^p)(R-1)}{(R-1)+\eta\overline{\zeta}^p}\end{aligned} \tag{76}$$

The last equality comes from plugging in the expression for ξ_3. Below we use again that $\overline{\zeta}^p$ is the fixed point of the T_2-map, i.e., $\frac{\xi_3-\xi_2\overline{\zeta}^p}{1-s(\overline{\zeta}^p)} = \overline{\zeta}^p$, which implies that

$$s(\overline{\zeta}^p) = \xi_2 + 1 - \frac{\xi_3}{\overline{\zeta}^p} \tag{77}$$

The inequality (75)

$$
\begin{aligned}
\xi_2 \quad &\geq \quad \overline{\zeta}^p s'(\overline{\zeta}^p) \\
&\iff \xi_2 \geq \frac{s(\overline{\zeta}^p)(R-1)}{(R-1)+\eta\overline{\zeta}^p} && \text{(using (76))} \\
&\iff \xi_2 \geq \left(\xi_2 + 1 - \frac{\xi_3}{\overline{\zeta}^p}\right)\frac{(R-1)}{(R-1)+\eta\overline{\zeta}^p} && \text{(using (77))} \\
&\iff \xi_2 \geq \xi_2\frac{(R-1)}{(R-1)+\eta\overline{\zeta}^p} + \left(1 - \frac{\xi_3}{\overline{\zeta}^p}\right)\frac{(R-1)}{(R-1)+\eta\overline{\zeta}^p} \\
&\iff \xi_2\frac{\eta\overline{\zeta}^p}{(R-1)+\eta\overline{\zeta}^p} \geq \left(1 - \frac{\xi_3}{\overline{\zeta}^p}\right)\frac{(R-1)}{(R-1)+\eta\overline{\zeta}^p} \\
&\iff \xi_2\eta\overline{\zeta}^p \geq \left(1 - \frac{\xi_3}{\overline{\zeta}^p}\right)(R-1) \\
&\iff (1-\tau)\frac{1}{\eta}\frac{R-1}{R}\frac{1}{1-\frac{1}{R}(1-\tau)}\eta\frac{(1-\frac{1}{R}(1-\tau))\frac{1}{\eta}}{1+(1-\tau)(\frac{1}{\eta}-\frac{1}{R})} \\
&\quad \geq \quad \left(1 - \frac{\frac{1}{\eta}\frac{R-1}{R}}{\frac{(1-\frac{1}{R}(1-\tau))\frac{1}{\eta}}{1+(1-\tau)(\frac{1}{\eta}-\frac{1}{R})}}\right)(R-1) \qquad \text{(using expressions for } \xi_2, \overline{\zeta}^p \text{ and } \xi_3) \\
&\iff \frac{(1-\tau)\frac{1}{\eta}\frac{R-1}{R}}{1+(1-\tau)(\frac{1}{\eta}-\frac{1}{R})} \geq \left(1 - \frac{\frac{R-1}{R}\left(1+(1-\tau)(\frac{1}{\eta}-\frac{1}{R})\right)}{(1-\frac{1}{R}(1-\tau))}\right)(R-1) \\
&\iff \frac{(1-\tau)}{R\eta+(1-\tau)(R-\eta)} \geq \frac{(1-\frac{1}{R}(1-\tau)) - \frac{R-1}{R}\left(1+(1-\tau)(\frac{1}{\eta}-\frac{1}{R})\right)}{(1-\frac{1}{R}(1-\tau))} \qquad (78)
\end{aligned}
$$

The right hand side of the above inequality is

$$
\begin{aligned}
RHS \quad &= \quad \frac{(1-\frac{1}{R}(1-\tau)) - \frac{R-1}{R}\left(1+(1-\tau)(\frac{1}{\eta}-\frac{1}{R})\right)}{(1-\frac{1}{R}(1-\tau))} \\
&= \quad \frac{1-\frac{1}{R}+\frac{1}{R}\tau-1+\frac{1}{R}-\frac{R-1}{R}(1-\tau)(\frac{1}{\eta}-\frac{1}{R})}{\frac{1}{R}(R-(1-\tau))} \\
&= \quad \frac{1-\frac{1}{R}+\frac{1}{R}\tau-1+\frac{1}{R}-\frac{R-1}{R}(1-\tau)(\frac{1}{\eta}-\frac{1}{R})}{\frac{1}{R}(R-(1-\tau))} \\
&= \quad \frac{\eta R\tau-(R-1)(1-\tau)(R-\eta)}{\eta R(R-(1-\tau))} \qquad (79)
\end{aligned}
$$

So with equation (79) the inequality (78) becomes

$$
\begin{aligned}
\frac{(1-\tau)}{R\eta+(1-\tau)(R-\eta)} &\geq \frac{\eta R\tau-(R-1)(1-\tau)(R-\eta)}{\eta R(R-(1-\tau))} \\
&\iff \frac{(1-\tau)\eta R^2-\eta R(1-\tau)^2}{R\eta+(1-\tau)(R-\eta)} \geq \eta R\tau-(R-1)(1-\tau)(R-\eta) \\
&\iff (1-\tau)\eta R^2-\eta R(1-\tau)^2 \geq R^2\eta^2\tau+\eta R\tau(1-\tau)(R-\eta) \\
&\quad -R\eta(R-1)(1-\tau)(R-\eta)-(R-1)(1-\tau)^2(R-\eta)^2 \\
&\iff (1-\tau)\eta R^2-\eta R(1-\tau)^2 \geq R^2\eta^2-R^2\eta^2(1-\tau) \\
&\quad +\eta R(1-\tau)(R-\eta)-\eta R(1-\tau)^2(R-\eta) \\
&\quad -R\eta(R-1)(1-\tau)(R-\eta)-(R-1)(1-\tau)^2(R-\eta)^2
\end{aligned}
$$

$\iff$

$$
\begin{aligned}
&\left((R-\eta)^2(R-1)-\eta R+\eta R(R-\eta)\right)(1-\tau)^2 \\
&+\left(\eta R^2+R^2\eta^2-\eta R^2+\eta^2 R+(R-1)(R-\eta)\eta R\right)(1-\tau) \\
&-R^2\eta^2 \\
\geq\ & 0 \qquad (80)
\end{aligned}
$$

I proceed to simplify the inequality (80). Note the coefficient on $(1-\tau)^2$ in inequality (80) is

$$
\begin{aligned}
(R-\eta)^2(R-1)-\eta R+\eta R(R-\eta) &= \left(R^2-2\eta R+\eta^2\right)R-\left(R^2-2\eta R+\eta^2\right)-\eta R+\eta R^2-\eta^2 R \\
&= R^3-\eta R^2+\eta R-R^2-\eta^2 \\
&= R^2(R-1)-\eta R(R-1)-\eta^2
\end{aligned}
$$

The coefficient on $(1-\tau)$ in inequality (80) is

$$
\begin{aligned}
\eta R^2+R^2\eta^2-\eta R^2+\eta^2 R+(R-1)(R-\eta)\eta R &= R^2\eta(R-1)-\eta^2 R(R-1)+R^2\eta^2+\eta^2 R \\
&= R\eta(R(R-1)+2\eta)
\end{aligned}
$$

So the inequality (80) is equivalent to

$$
\begin{aligned}
0 \quad &\leq \quad (R^2(R-1)-\eta R(R-1)-\eta^2)(1-\tau)^2+R\eta(R(R-1)+2\eta)(1-\tau)-R^2\eta^2 \\
&\iff 0 \leq -\eta^2\left((1-\tau)^2-2R(1-\tau)+R^2\right) \\
&\quad +\left(R^2(R-1)(1-\tau)\right)-R(R-1)(1-\tau)^2)\eta+R^2(R-1)(1-\tau)^2 \\
&\iff 0 \leq -\eta^2(R-(1-\tau))^2+R(R-1)(1-\tau)(R-(1-\tau))\eta+R^2(R-1)(1-\tau)^2 \\
&\iff (\eta(R-(1-\tau))-\frac{R(R-1)(1-\tau)}{2})^2 \leq R^2(1-\tau)^2\left((R-1)+\frac{(R-1)^2}{4}\right) \qquad (81)
\end{aligned}
$$

Case 1: assuming $\eta(R-(1-\tau)) > \frac{R(R-1)(1-\tau)}{2}$ or equivalently

$$\frac{2(R-(1-\tau))}{R(R-1)(1-\tau)} > \frac{1}{\eta} \tag{82}$$

(81) implies

$$\begin{aligned}
\eta(R-(1-\tau)) - \frac{R(R-1)(1-\tau)}{2} &\leq R(1-\tau)\sqrt{(R-1)+\frac{(R-1)^2}{4}} \\
&\iff \frac{1}{\eta} \geq \frac{R-(1-\tau)}{\frac{R(R-1)(1-\tau)}{2} + R(1-\tau)\sqrt{(R-1)+\frac{(R-1)^2}{4}}} \\
&\iff \frac{1}{\eta} \geq \frac{\frac{R}{1-\tau}-1}{g(R)} \qquad (83) \\
\text{where } g(R) &= R\sqrt{(R-1)+\frac{(R-1)^2}{4}} + \frac{R(R-1)}{2}
\end{aligned}$$

Combining (82) and (83), I get

$$\frac{2(R-(1-\tau))}{R(R-1)(1-\tau)} > \frac{1}{\eta} \geq \frac{\frac{R}{1-\tau}-1}{g(R)} \tag{84}$$

It can be shown that

$$\begin{aligned}
\frac{2(R-(1-\tau))}{R(R-1)(1-\tau)} &> \frac{\frac{R}{1-\tau}-1}{R\sqrt{(R-1)+\frac{(R-1)^2}{4}}+\frac{R(R-1)}{2}} \\
&\iff \frac{\frac{R}{(1-\tau)}-1}{\frac{R(R-1)}{2}} > \frac{\frac{R}{1-\tau}-1}{R\sqrt{(R-1)+\frac{(R-1)^2}{4}}+\frac{R(R-1)}{2}}
\end{aligned}$$

because the second inequality is true.

Case 2: assuming $\eta(R-(1-\tau)) \leq \frac{R(R-1)(1-\tau)}{2}$ or equivalently $\frac{1}{\eta} \geq \frac{(R-(1-\tau))}{\frac{R(R-1)(1-\tau)}{2}}$,(81) implies

$$\frac{R(R-1)(1-\tau)}{2} - \eta(R-(1-\tau)) \leq R(1-\tau)\sqrt{(R-1)+\frac{(R-1)^2}{4}}$$

Note $\frac{R(R-1)(1-\tau)}{2} < R(1-\tau)\sqrt{(R-1)+\frac{(R-1)^2}{4}}$, so the above inequality holds. I get

$$\frac{1}{\eta} \geq \frac{(R-(1-\tau))}{\frac{R(R-1)(1-\tau)}{2}} \tag{85}$$

Combining (84) and (85), I get the momentum condition $\frac{1-\tau}{R} \geq \frac{1}{g(R)\frac{1}{\eta}+1}$ where $g(R) = R(\sqrt{(R-1)+\frac{(R-1)^2}{4}} + \frac{R-1}{2})$.

F Robustness of the Qualitative Model Dynamics to an Alternative Specification of the Collateral Constraint

Some papers, say Boz and Mendoza (2013) use an alternative specification of collateral constraints that household borrowing is limited by a fraction of current collateral values rather than the expected liquidation value of collateral. This section shows that the qualitative dynamics, belief changes have a critical effect on collateral prices and the possibility of momentum dynamics, continues to hold in the model with this alternative specification of the collateral constraint.

Consider the following collateral constraint

$$b_t^B \leq (1-\tau)\frac{q_t}{R}H_t^B$$

The debt repayment is now $(1-\tau)q_{t-1}H_{t-1}^B$ and still depends directly on agents' price elasticity belief at $t-1$,i.e., ζ_{t-1}^p. Borrowers' collateral demand becomes

$$H_t^B = \frac{a+q_t-(1-\tau)q_{t-1}}{q_t-\frac{1}{R}(1-\tau)q_t}H_{t-1}^B$$

Consider the deterministic case and without learning about the steady state. The collateral demand equation is

$$H_t^B = \frac{a+q_t-(1-\tau)q_{t-1}}{q_t-\frac{(1-\tau)}{R}q_t}H_{t-1}^B$$

The model equations with the alternative specification of the collateral constraint contain

$$\widehat{H}_t^B = \frac{1}{\left(1-\frac{1-\tau}{R}\right)}\left[\frac{1-\tau}{R}\widehat{q}_t-(1-\tau)\widehat{q}_{t-1}\right]+\widehat{H}_{t-1}^B \tag{86}$$

$$\widehat{q}_t = \frac{1}{R}E_t^{\mathcal{P}}\widehat{q}_{t+1}+\frac{1}{\eta}\frac{R-1}{R}\widehat{H}_t^B \tag{87}$$

Agents' conditional expectations are $E_t^{\mathcal{P}}\widehat{q}_{t+1}=\zeta_t^p\widehat{H}_t^B$. Substituting the conditional expectations into the model equations, we get

$$\begin{aligned}\widehat{q}_t &= \frac{1}{R}E_t^{\mathcal{P}}\widehat{q}_{t+1}+\frac{1}{\eta}\frac{R-1}{R}\widehat{H}_t^B \\ &= \frac{1}{R}\zeta_t^p\widehat{H}_t^B+\frac{1}{\eta}\frac{R-1}{R}\widehat{H}_t^B \\ &= \left(\frac{1}{R}\zeta_t^p+\frac{1}{\eta}\frac{R-1}{R}\right)\widehat{H}_t^B \end{aligned}\tag{88}$$

So we have

$$\widehat{q}_{t-1}=\left(\frac{1}{R}\zeta_{t-1}^p+\frac{1}{\eta}\frac{R-1}{R}\right)\widehat{H}_{t-1}^B \tag{89}$$

Combining equations (86), and (89), we get

$$\begin{aligned}\widehat{H}_t^B &= \frac{1}{\left(1-\frac{1-\tau}{R}\right)}\left[\frac{1-\tau}{R}\widehat{q}_t-(1-\tau)\widehat{q}_{t-1}\right]+\widehat{H}_{t-1}^B \\ &= \frac{\frac{1-\tau}{R}}{1-\frac{1-\tau}{R}}\left(\frac{1}{R}\zeta_t^p+\frac{1}{\eta}\frac{R-1}{R}\right)\widehat{H}_t^B-\frac{1-\tau}{\left(1-\frac{1-\tau}{R}\right)}\left(\frac{1}{R}\zeta_{t-1}^p+\frac{1}{\eta}\frac{R-1}{R}\right)\widehat{H}_{t-1}^B+\widehat{H}_{t-1}^B\end{aligned}$$

So

$$\widehat{H}_t^B=\frac{1-\frac{1-\tau}{\left(1-\frac{1-\tau}{R}\right)}\left(\frac{1}{R}\zeta_{t-1}^p+\frac{1}{\eta}\frac{R-1}{R}\right)}{1-\frac{\frac{1-\tau}{R}}{1-\frac{1-\tau}{R}}\left(\frac{1}{R}\zeta_t^p+\frac{1}{\eta}\frac{R-1}{R}\right)}\widehat{H}_{t-1}^B \tag{90}$$

Equations (88) and (90) yield the ALM for collateral prices under learning

$$\begin{aligned}\widehat{q}_t &= \left(\frac{1}{R}\zeta_t^p+\frac{1}{\eta}\frac{R-1}{R}\right)\widehat{H}_t^B \\ &= T_2\left(\zeta_{t-1}^p,\zeta_t^p\right)\widehat{H}_{t-1}^B\end{aligned}$$

where

$$T_2\left(\zeta_{t-1}^p,\zeta_t^p\right)=\left(\frac{1}{R}\zeta_t^p+\frac{1}{\eta}\frac{R-1}{R}\right)\frac{1-\frac{1-\tau}{\left(1-\frac{1-\tau}{R}\right)}\left(\frac{1}{R}\zeta_{t-1}^p+\frac{1}{\eta}\frac{R-1}{R}\right)}{1-\frac{\frac{1-\tau}{R}}{1-\frac{1-\tau}{R}}\left(\frac{1}{R}\zeta_t^p+\frac{1}{\eta}\frac{R-1}{R}\right)}$$

The actual price elasticity under learning, i.e., the T_2-map, depends positively on current beliefs and negatively on past beliefs. So the qualitative learning dynamics under this alternative collateral constraint is similar to the one analyzed in the text.

G Check if the Collateral Constraint is Binding

Similar to the Kiyotaki and Moore (1997) paper, two assumptions: (1) $\beta^B<\frac{\beta^L}{A}$ (2)$\frac{\overline{e}}{1-\frac{1}{R}(1-\tau)}\frac{(R-1)(1-\tau)}{aR}>\frac{1}{\beta^B}-1$, have been made to ensure that the return to investment is higher than that to consumption and saving in a neighborhood of the steady state, so that the collateral constraint is always binding under RE. In the learning model agents evaluate the payoffs of different strategies using the subjective probability measure and there may be substantial deviations of beliefs and expectations. It is unclear that if the collateral constraint remains binding in this case without a formal check. This section confirms that the collateral constraint is indeed binding during the housing cycle over the year 2001-2008 in the learning model.

Consider a marginal unit of tradable consumption at date t where t runs from 2001Q1 to 2008Q4. The borrower could consume it and get utility 1. Alternatively he could invest it in collateral holding or save it and then invest. The payoff sequences for period $t, t+1, t+2, t+3, ...$ would be as follows.

$$\begin{aligned}
&\text{invest} \qquad 0,\ \frac{\overline{e}}{d_t},\ \frac{a+q_{t+1}-(1-\tau)E_t^{\mathcal{P}}q_{t+1}}{d_t}\frac{\overline{e}}{d_{t+1}},\\
&\frac{a+q_{t+1}-(1-\tau)E_t^{\mathcal{P}}q_{t+1}}{d_t}\frac{a+q_{t+2}-(1-\tau)E_{t+1}^{\mathcal{P}}q_{t+2}}{d_{t+1}}\frac{\overline{e}}{d_{t+2}},... \qquad (91)\\
&\text{save} \qquad 0,\ 0,\ R_t\frac{\overline{e}}{d_{t+1}},\\
&R_t\frac{a+q_{t+2}-(1-\tau)E_{t+1}^{\mathcal{P}}q_{t+2}}{d_{t+1}}\frac{\overline{e}}{d_{t+2}},.... \qquad (92)\\
&\text{consume} \quad 1,\ 0,\ 0,\ 0,... \qquad (93)
\end{aligned}$$

In period t, the borrower invests 1 unit and can get $\frac{1}{d_t}$ unit of collateral with borrowing. In period t+1, he gets the production $\frac{a+\overline{e}}{d_t}$. The non-tradable part $\frac{\overline{e}}{d_t}$ will be consumed and the tradable part $\frac{a}{d_t}$ will be re-invested. There will be capital gain $\left(q_{t+1}-(1-\tau)E_t^{\mathcal{P}}q_{t+1}\right)\frac{1}{d_t}$. In period t+1, the sum of the tradable part and the capital gain, i.e., $\frac{a+q_{t+1}-(1-\tau)E_t^{\mathcal{P}}q_{t+1}}{d_t}$, will be reinvested, so $\frac{a+q_{t+1}-(1-\tau)E_t^{\mathcal{P}}q_{t+1}}{d_t}\frac{1}{d_{t+1}}$ will be acquired in period $t+1$. In period $t+2$, $\frac{a+q_{t+1}-(1-\tau)E_t^{\mathcal{P}}q_{t+1}}{d_t}\frac{\overline{e}}{d_{t+1}}$ will be consumed; and so on.

In period t, if the borrower saves the 1 unit, then he will get R_t in period $t+1$. He can acquire $\frac{R_t}{d_{t+1}}$ at the same period. In period $t+2$, he consumes $R_t\frac{\overline{e}}{d_{t+1}}$ and invests $R_t\frac{a+q_{t+2}-(1-\tau)E_{t+1}^{\mathcal{P}}q_{t+2}}{d_{t+1}}$; and so on.

Denote P_t^I, P_t^S the payoff of the investment and saving strategy at period t, respectively. Inv_t and Sav_t stand for the discounted sum of period payoffs for the investment and saving strategy, respectively. So

$$Inv_t = \sum_{i=t}^{\infty}\left(\beta^B\right)^i P_t^I \qquad (94)$$

$$Sav_t = \sum_{i=t}^{\infty}\left(\beta^B\right)^i P_t^S \qquad (95)$$

The steady state value of Inv_t is

$$\begin{aligned}
Inv &= 0+\beta^B\frac{\overline{e}}{d}+\left(\beta^B\right)^2\frac{a+\tau q}{d}\frac{\overline{e}}{d}+\left(\beta^B\right)^3\left(\frac{a+\tau q}{d}\right)^2\frac{\overline{e}}{d}+...\\
&= \beta^B\frac{\overline{e}}{d}\left(1+\beta^B+\left(\beta^B\right)^2+...\right)\\
&= \beta^B\frac{\overline{e}}{d}\frac{1}{1-\beta^B}
\end{aligned}$$

and the steady state value of Sav_t is

$$\begin{aligned}
Sav &= \left(\beta^B\right)^2 R\frac{\overline{e}}{d}+\left(\beta^B\right)^3 R\frac{\overline{e}}{d}+\left(\beta^B\right)^4 R\frac{\overline{e}}{d}+...\\
&= \left(\beta^B\right)^2 R\frac{\overline{e}}{d}\frac{1}{1-\beta^B}
\end{aligned}$$

To ensure a higher payoff for the investment strategy than that for saving at the steady state, i.e., $Inv > Sav$, we need $1 > \beta^B R$ or $\beta^L > \beta^B$. And

$$\beta^B \frac{\overline{e}}{d} \frac{1}{1-\beta^B} > 1 \tag{96}$$

is sufficient to ensure that the payoff for investment is higher than that for consuming it immediately. Substituting the steady state values, I get the inequality (96) is equivalent to

$$\frac{\overline{e}}{1-\frac{1}{R}(1-\tau)} \frac{(R-1)(1-\tau)}{aR} > \frac{1}{\beta^B} - 1$$

Note Inv_t and Sav_t depend on the realization of future variables such as future prices. We are going to compare the conditional expectation of the payoffs of the three strategies at period t, $E_t^{\mathcal{P}} Inv_t$ and $E_t^{\mathcal{P}} Sav_t$. After log-linearization, $E_t^{\mathcal{P}} \widehat{Inv}_t$ and $E_t^{\mathcal{P}} \widehat{Sav}_t$ are identical to the log-linearization version of the discounted sum of the following payoff sequences[42], respectively

$$\text{invest} \qquad 0, \ \frac{\overline{e}}{d_t}, \ \frac{a+\tau E_t^{\mathcal{P}} q_{t+1}}{d_t} \frac{\overline{e}}{E_t^{\mathcal{P}} d_{t+1}}, \ \frac{a+\tau E_t^{\mathcal{P}} q_{t+1}}{d_t}, ... \tag{97}$$

$$\text{save} \qquad 0, \quad 0, \quad R_t \frac{\overline{e}}{E_t^{\mathcal{P}} d_{t+1}}, \qquad R_t \frac{a+\tau E_t^{\mathcal{P}} q_{t+2}}{E_t^{\mathcal{P}} d_{t+1}} \frac{\overline{e}}{E_t^{\mathcal{P}} d_{t+2}}, ... \tag{98}$$

where I replaced $q_{t+1} - (1-\tau) E_t^{\mathcal{P}} q_{t+1}, q_{t+2} - (1-\tau) E_{t+1}^{\mathcal{P}} q_{t+2} ...$ in (91) and (92) by $\tau E_t^{\mathcal{P}} q_{t+1}, \tau E_t^{\mathcal{P}} q_{t+2} ...$ in (97) and (98). I also replaced future down-payments $d_{t+1}, d_{t+2}, ...$ by $E_t^{\mathcal{P}} d_{t+1}, E_t^{\mathcal{P}} d_{t+2}, ...$

Note at the steady state $\frac{q}{d} = \frac{1}{1-\frac{1-\tau}{R}}$. Log-linearizing the discounted sum of the payoff sequences (97) and (98) yields

$$\begin{aligned}
Inv * E_t^{\mathcal{P}} \widehat{Inv}_t &= \beta^B \frac{\overline{e}}{d} \left(-\widehat{d}_t \right) + \left(\beta^B \right)^2 \frac{\overline{e}}{d} \left(-\widehat{d}_t + \frac{\tau}{1-\frac{1-\tau}{R}} E_t^{\mathcal{P}} \widehat{q}_{t+1} - E_t^{\mathcal{P}} \widehat{d}_{t+1} \right) \\
&\quad + \left(\beta^B \right)^3 \frac{\overline{e}}{d} \left(-\widehat{d}_t + \frac{\tau}{1-\frac{1-\tau}{R}} \left(E_t^{\mathcal{P}} \widehat{q}_{t+1} + E_t^{\mathcal{P}} \widehat{q}_{t+2} \right) - E_t^{\mathcal{P}} \widehat{d}_{t+1} - E_t^{\mathcal{P}} \widehat{d}_{t+2} \right) \\
&\quad + \left(\beta^B \right)^4 \frac{\overline{e}}{d} [-\widehat{d}_t - E_t^{\mathcal{P}} \widehat{d}_{t+1} - E_t^{\mathcal{P}} \widehat{d}_{t+2} - E_t^{\mathcal{P}} \widehat{d}_{t+3} \\
&\quad + \frac{\tau}{1-\frac{1-\tau}{R}} \left(E_t^{\mathcal{P}} \widehat{q}_{t+1} + E_t^{\mathcal{P}} \widehat{q}_{t+2} + E_t^{\mathcal{P}} \widehat{q}_{t+3} \right)] \\
&\quad + ... \\
&= -\beta^B \frac{\overline{e}}{d} \frac{1}{1-\beta^B} [\widehat{d}_t - \beta^B \frac{\tau}{1-\frac{1-\tau}{R}} E_t^{\mathcal{P}} \widehat{q}_{t+1} + \sum_{i=1}^{\infty} \left(\beta^B \right)^i E_t^{\mathcal{P}} \widehat{d}_{t+i} \\
&\quad - \beta^B \frac{\tau}{1-\frac{1-\tau}{R}} \sum_{i=1}^{\infty} \left(\beta^B \right)^i E_t^{\mathcal{P}} \widehat{q}_{t+i+1}]
\end{aligned} \tag{99}$$

[42] Note this does not mean that the conditional expectation of the payoff sequence (91) and (92) are (97) and (98), respectively.

and

$$\begin{aligned} Sav * E_t^{\mathcal{P}} \widehat{Sav}_t &= \left(\beta^B\right)^2 R\frac{\overline{e}}{d}\left(\widehat{R}_t - E_t^{\mathcal{P}}\widehat{d}_{t+1}\right) \\ &+ \left(\beta^B\right)^3 R\frac{\overline{e}}{d}\left(\widehat{R}_t - E_t^{\mathcal{P}}\widehat{d}_{t+1} - E_t^{\mathcal{P}}\widehat{d}_{t+2} + \frac{\tau q}{d}E_t^{\mathcal{P}}\widehat{q}_{t+2}\right) \\ &+ \left(\beta^B\right)^4 R\frac{\overline{e}}{d}[\widehat{R}_t - E_t^{\mathcal{P}}\widehat{d}_{t+1} - E_t^{\mathcal{P}}\widehat{d}_{t+2} - E_t^{\mathcal{P}}\widehat{d}_{t+3} \\ &+ \frac{\tau q}{d}E_t^{\mathcal{P}}\widehat{q}_{t+2} + \frac{\tau q}{d}E_t^{\mathcal{P}}\widehat{q}_{t+3}] + ... \\ &= \left(\beta^B\right)^2 R\frac{\overline{e}}{d}\frac{1}{1-\beta^B} * \\ &\left(\widehat{R}_t - \sum_{i=1}^{\infty}\left(\beta^B\right)^{i-1} E_t^{\mathcal{P}}\widehat{d}_{t+i} + \frac{\tau q}{d}\sum_{i=1}^{\infty}\left(\beta^B\right)^i E_t^{\mathcal{P}}\widehat{q}_{t+i+1}\right) \end{aligned} \tag{100}$$

Log-linearizing the down-payment yields

$$\widehat{d}_t = \frac{1}{1-\frac{1-\tau}{R}}\left(\widehat{q}_t - \frac{1-\tau}{R}\left(E_t^{\mathcal{P}}\widehat{q}_{t+1} - \widehat{R}_t\right)\right)$$

and for $i \geq 1$

$$E_t^{\mathcal{P}}\widehat{d}_{t+i} = \frac{1}{1-\frac{1-\tau}{R}}\left(E_t^{\mathcal{P}}\widehat{q}_{t+i} - \frac{1-\tau}{R}\left(E_t^{\mathcal{P}}\widehat{q}_{t+i+1} - E_t^{\mathcal{P}}\widehat{R}_{t+i}\right)\right)$$

Define $\varkappa_t \equiv (\varkappa_t^m, \varkappa_t^p)'$ where $\varkappa_t^m$ and $\varkappa_t^p$ are parameter estimates in the collateral holding equation. β_t^r stands for agents' belief for interest rates at period t and is assumed to be updated via steady state learning for simplicity. $\varkappa_t$ and β_t^r are updated recursively as follows

$$\begin{aligned} \varkappa_t &= \varkappa_{t-1} + \gamma\left(S_t^H\right)^{-1}\begin{pmatrix} 1 \\ \widehat{H}_{t-2}^B \end{pmatrix}\left(\widehat{H}_{t-1}^B - \begin{pmatrix} 1 & \widehat{H}_{t-2}^B \end{pmatrix}\varkappa_{t-1}\right) \\ S_t^H &= S_{t-1}^H + \gamma\left(\begin{pmatrix} 1 \\ \widehat{H}_{t-2}^B \end{pmatrix}\begin{pmatrix} 1 & \widehat{H}_{t-2}^B \end{pmatrix} - S_{t-1}^H\right) \\ \beta_t^r &= \beta_{t-1}^r + \gamma\left(\widehat{R}_{t-1} - \beta_{t-1}^r\right) \end{aligned}$$

where γ is the gain parameter.

We derive the collateral price forecast when $i \geq 1$

$$\begin{aligned} E_t^{\mathcal{P}}\widehat{q}_{t+i} &= \phi_t^m + \phi_t^p E_t^{\mathcal{P}}\widehat{H}_{t+i}^B \\ &= \phi_t^m + \phi_t^p\left(\varkappa_t^m\frac{1-\left(\varkappa_t^p\right)^i}{1-\varkappa_t^p} + \left(\varkappa_t^p\right)^{i-1}\ \widehat{H}_t^B\right) \end{aligned}$$

Substituting the conditional expectations and the down-payments into (99) and (100), we get

$$
\begin{aligned}
Inv * E_t^{\mathcal{P}} \widehat{Inv}_t &= -\beta^B \frac{\overline{e}}{d} \frac{1}{1-\beta^B} [\widehat{d}_t - \beta^B \frac{\tau}{1-\frac{1-\tau}{R}} E_t^{\mathcal{P}} \widehat{q}_{t+1} + \sum_{i=1}^{\infty} \left(\beta^B\right)^i E_t^{\mathcal{P}} \widehat{d}_{t+i} \\
&\quad -\beta^B \frac{\tau}{1-\frac{1-\tau}{R}} \sum_{i=1}^{\infty} \left(\beta^B\right)^i E_t^{\mathcal{P}} \widehat{q}_{t+i+1}] \\
&= -\beta^B \frac{\overline{e}}{d} \frac{1}{1-\beta^B} [\widehat{d}_t - \beta^B \frac{\tau}{1-\frac{1-\tau}{R}} E_t^{\mathcal{P}} \widehat{q}_{t+1} \\
&\quad + \sum_{i=1}^{\infty} \left(\beta^B\right)^i \left(\frac{1}{1-\frac{1-\tau}{R}} \left(E_t^{\mathcal{P}} \widehat{q}_{t+i} - \frac{1-\tau}{R} \left(E_t^{\mathcal{P}} \widehat{q}_{t+i+1} - E_t^{\mathcal{P}} \widehat{R}_{t+i} \right) \right) \right) \\
&\quad -\beta^B \frac{\tau}{1-\frac{1-\tau}{R}} \sum_{i=1}^{\infty} \left(\beta^B\right)^i E_t^{\mathcal{P}} \widehat{q}_{t+i+1}] \\
&= -\beta^B \frac{\overline{e}}{d} \frac{1}{1-\beta^B} [\widehat{d}_t + \frac{\beta^B (1-\tau)}{1-\frac{1-\tau}{R}} E_t^{\mathcal{P}} \widehat{q}_{t+1} \\
&\quad + \frac{\left(\beta^B - \frac{1}{R}\right)(1-\tau)}{1-\frac{1-\tau}{R}} \sum_{i=1}^{\infty} \left(\beta^B\right)^i E_t^{\mathcal{P}} \widehat{q}_{t+i+1} \\
&\quad + \frac{\frac{1-\tau}{R}}{1-\frac{1-\tau}{R}} \sum_{i=1}^{\infty} \left(\beta^B\right)^i E_t^{\mathcal{P}} \widehat{R}_{t+i}] \qquad (101)
\end{aligned}
$$

where the sum $\sum_{i=1}^{\infty} \left(\beta^B\right)^i E_t^{\mathcal{P}} \widehat{q}_{t+i+1}$ can be calculated as follows

$$
\begin{aligned}
\sum_{i=1}^{\infty} \left(\beta^B\right)^i E_t^{\mathcal{P}} \widehat{q}_{t+i+1} &= \sum_{i=1}^{\infty} \left(\beta^B\right)^i \left(\phi_t^m + \phi_t^p \left(\varkappa_t^m \frac{1-\left(\varkappa_t^p\right)^i}{1-\varkappa_t^p} + \left(\varkappa_t^p\right)^i \ \widehat{H}_t^B \right) \right) \\
&= \sum_{i=1}^{\infty} \left(\beta^B\right)^i \left(\phi_t^m + \frac{\phi_t^p \varkappa_t^m}{1-\varkappa_t^p} \right) + \sum_{i=1}^{\infty} \left(\beta^B \varkappa_t^p\right)^i \left(\widehat{H}_t^B - \frac{\varkappa_t^m \varkappa_t^p \phi_t^p}{1-\varkappa_t^p} \right)
\end{aligned}
$$

and the interest rate forecast

$$
E_t^{\mathcal{P}} \widehat{R}_{t+i} = \beta_t^r
$$

Comparing (99) with (100), we get

$$
\begin{aligned}
Sav * E_t^{\mathcal{P}} \widehat{Sav}_t &= \left(\beta^B\right)^2 R \frac{\overline{e}}{d} \frac{1}{1-\beta^B} \left(\widehat{R}_t - \sum_{i=1}^{\infty} \left(\beta^B\right)^{i-1} E_t^{\mathcal{P}} \widehat{d}_{t+i} + \frac{\tau q}{d} \sum_{i=1}^{\infty} \left(\beta^B\right)^i E_t^{\mathcal{P}} \widehat{q}_{t+i+1} \right) \\
&= \left(\beta^B\right)^2 R \frac{\overline{e}}{d} \frac{1}{1-\beta^B} \widehat{R}_t \\
&\quad + R \left(Inv * E_t^{\mathcal{P}} \widehat{Inv}_t + \beta^B \frac{\overline{e}}{d} \frac{1}{1-\beta^B} \widehat{d}_t - \left(\beta^B\right)^2 \frac{\overline{e}}{d} \frac{1}{1-\beta^B} E_t^{\mathcal{P}} \widehat{q}_{t+1} \right)
\end{aligned}
$$

Denote Δ_t^1 and Δ_t^2 the difference of the payoff between the investment strategy and the saving and the consumption strategy, respectively, so

$$\Delta_t^1 = E_t^{\mathcal{P}} \widehat{Inv}_t + \log(Inv) - \left(E_t^{\mathcal{P}} \widehat{Sav}_t + \log(Sav) \right)$$

and

$$\Delta_t^2 = E_t^{\mathcal{P}} \widehat{Inv}_t + \log(Inv) - 1$$

The parameterization in the quantitative section is used. In addition, we need to set a few more parameters. A different and smaller gain parameter $\gamma = 0.03$ is considered for updating $\varkappa_t$ and β_t^r rather than 0.065 used in updating parameters in the collateral price equation. This is because γ =0.065 will lead to $\varkappa_t^p$ larger than 1, which we want to avoid. The discount factor β^B is set to 0.97. Numerical calculation shows that Δ_t^1 and Δ_t^2 are positive over the housing cycle in the learning model over the year 2001-2008. [43]

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[43] For some alternative parameterizations, Δ_t^1 and/or Δ_t^2 can be negative for some periods. However, both Δ_t^1 and Δ_t^2 are positive for all t and for a large set of plausible and alternative parameterization, such as different gain parameter and/or different borrowers' discount factor.

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