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Small Informational Size and Interim Cores of Large Quasilinear Economies*

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Abstract. We provide nonemptiness results of approximate *interim* cores with endogenous communication in large quasilinear economies, where every agent's informational size is small. We offer results for both replica and more general sequences of economies.

Keywords: Asymmetric Information; Large Quasilinear Economies; Informational Smallness; Approximate Interim Cores.

JEL Classification: C71, C72, D51, D82.

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1 Introduction

In this paper we consider nonemptiness of the core in *interim* exchange economies with asymmetric information when agents' informational size is small (McLean and Postlewaite (2002)). Within this model of informational size, previous results are available for the *ex ante* core (McLean and Postlewaite (2003, 2005)), but existence results for the *interim* core are missing.¹ If coalitions are formed at the *interim* stage, we need to specify the information that agents in a coalition are allowed to use in constructing an objection. Depending on the extent of communication within the coalition, several *interim* core concepts have been proposed. The literature began with concepts where communication is exogenous, such as the coarse core and the fine core in Wilson (1978), but more recently, *interim* cores with endogenous communication have been made available; see Forges, Minelli, and Vohra (2002) or Forges and Serrano (2013) for surveys.

The current paper concentrates on a class of *interim* cores with endogenous information transmission, namely, the core with respect to (w.r.t.) equilibrium blocking, as termed in Kamishiro and Serrano (2011). Indeed, Serrano and Vohra (2007) show that several core notions previously proposed in the literature, such as the credible core in Dutta and Vohra (2005) or the virtual-utility core in Myerson (2007), can be unified into the notion of core w.r.t. equilibrium blocking (see Kamishiro and Serrano (2011) for a further elaboration).

In general, when incentive constraints are incorporated into core notions, it is difficult to obtain nonemptiness of *interim* cores of exchange economies, given the results in Vohra (1999) and Forges, Mertens, and Vohra (2002), which show that the incentive-compatible coarse core, a superset of all cores w.r.t equilibrium blocking, might be empty. This suggests the consideration of approximate cores, where the blocking constraints are somewhat relaxed.

Going in that direction, this paper considers large economies, and restricts attention to quasilinear preferences. Assumptions made on the information structure will in general matter for the results one can obtain. For instance, Kamishiro, Serrano, and Wooders (2015) consider the case where there is a small number of informed agents and an unbounded number of uninformed agents. In fact, the bulk of their argument

¹However, Kamishiro and Serrano (2011) study the convergence question in this model in their Section 5.2.

is built around economies with exclusive information, in which there are finitely-many monopolists of different pieces of information. The paper shows that, in large quasilinear economies, the “small group effectiveness” condition suffices to obtain nonemptiness of approximate *interim* cores w.r.t. equilibrium blocking. In contrast, it also shows that the exact incentive-compatible coarse core might still be empty.

The current paper analyzes the opposite case, in the sense that all agents can have private information and all of them are *informationally small*, in the sense of McLean and Postlewaite (2002). Roughly speaking, agents are informationally small if, with high probability, agents are not able to change the estimate that others have of the likelihood of the state by misreporting their signals. In such scenarios, we show nonemptiness of the approximate *interim* cores w.r.t. equilibrium blocking, without relying on “small group effectiveness.” Theorem 3.2 shows the result for replica sequences of economies, in which the number of copies of each agent type is the same, and Theorem 3.3 extends it to more general sequences.

In order to obtain nonemptiness of approximate *interim* cores, we mainly rely on the results of Dutta and Vohra (2005) and Kamishiro and Serrano (2011). These results establish that having an *ex post* core allocation that is incentive compatible is sufficient in quasilinear economies to show that the core w.r.t. equilibrium blocking be nonempty. This is the reason to restrict attention to quasilinear preferences, and we do not know whether the argument extends to more general settings. In addition, a variant of a lemma in McLean and Postlewaite (2002), adapted to our model, will be key to complete our proof, by allowing us to construct an incentive-compatible rule that lies in the approximate *ex post* core.

This brief paper proceeds as follows: while section 2 provides the basic framework and definitions, and briefly reviews the notions of informational size in McLean and Postlewaite (2002), section 3 offers nonemptiness results of the approximate *interim* core.

2 The Model

We consider sequences of economies with the property that all agents are informationally small in the sense of McLean and Postlewaite (2002). These authors consider a model in which the agents’ utility functions depend on an underlying but unobserved state of nature, and in which each agent receives a private signal that is correlated with the state.

2.1 Notation and Definitions

We begin with a few preliminaries in order to define a pre-economy and an economy. Let $N = \{1, \dots, n\}$ denote a fixed finite set of agent types. We denote a (finite) state space as $\Theta = \{\theta_1, \dots, \theta_m\}$. In this model, nature chooses an element $\theta \in \Theta$. The state of nature is unobservable, but each agent type i receives a “signal” that is correlated with nature’s choice of θ . Let T_1, \dots, T_n be finite sets, where T_i represents the set of possible signals that agent type $i \in N$ might receive and let $T = \prod_i T_i$. We take as given a probability distribution, denoted by P , on $\Theta \times T$. We assume that for every θ, θ' with $\theta \neq \theta'$, there exists a $t \in T$ such that $P(t | \theta) \neq P(t | \theta')$.

The consumption set of each agent type, denoted by X_i , is $R_+^{L-1} \times R$ and for each $\theta \in \Theta$, $e_i \in X_i$ denotes the (state-independent) endowment of agent type i in state $\theta \in \Theta$. The preferences of agent type i are represented by a utility function $u_i : R_+^{L-1} \times R \times \Theta \rightarrow R$ where $u_i(x_i, \theta) = v_i(x_i^{-L}, \theta) + x_i^L$ is the utility function of agent type i in state θ . We note that in this model agents’ utility functions do not depend on T , but do depend on the state of nature.

The collection $\mathcal{E} = (\{u_i, X_i, e_i, T_i\}_{i \in N}, \Theta, P)$ will be called a *private information pre-economy*. And for each state $\theta \in \Theta$, the collection $\{u_i(\cdot, \theta), e_i\}_{i \in N}$ defines an associated (complete information) pre-economy.

A *profile* on N is a function f from N to Z_+ , the nonnegative integers; we interpret $f(i)$ as the number of agents with the same type. Given the set of agent types N and a profile f , we denote the agent set as \mathbf{N} , i.e., $\mathbf{N} = \{(i, j) : i = 1, \dots, n; j = 1, \dots, f(i)\}$. Let $\|f\|$ denote the number of agents in a group represented by a profile f , i.e., $\|f\| = \sum_{i \in N} f(i)$.

We begin by considering exact replicas; that is, sequences of economies where the r^{th} economy has r agents of each type. This yields simpler mathematical expressions and more transparent arguments. However, we are able to extend our results for the case of more general sequences, as we explain below. That is, we begin by considering only profiles f of agent types such that $f(i) = f(i')$ for all $i, i' \in N$.

Given a pre-economy \mathcal{E} and a profile f , let

$$\mathcal{E}^f = (\{u_{(i,j)}, X_{(i,j)}, e_{(i,j)}, T_{(i,j)}\}_{i \in N, j=1, \dots, r}, \Theta, P^f)$$

be a private information economy, with $\|f\| (= r \times n)$ agents, for some integer r , satisfying the following conditions. (Hereafter we denote $T^f := \prod_{i \in N, j=1, \dots, r} T_{(i,j)}$.)

- (1) $e_{(i,j)} = e_i$ and $X_{(i,j)} = X_i$ for all $i \in N$ and all $j = 1, \dots, r$,
- (2) $u_{(i,j)}(x, \theta) = u_i(x, \theta)$ for all $x \in R_+^{L-1} \times R$, $i \in N$ and $j = 1, \dots, r$.
- (3) $T_{(i,j)} = T_i$ for all $i \in N$ and $j = 1, \dots, r$.
- (4) P^f is a probability distribution on $\Theta \times T^f$ satisfying the following²:
 - (a) For each $j = 1, 2, \dots, r$, and each $(\theta, t_1, \dots, t_n) \in \Theta \times T$,

$$P(\theta, t_{(1,j)}, t_{(2,j)}, \dots, t_{(n,j)}) = P(\theta, t_1, t_2, \dots, t_n);$$

- (b) for each θ , the probability distributions over

$$(T_{(1,1)}, T_{(2,1)}, \dots, T_{(n,1)}), \dots, (T_{(1,r)}, T_{(2,r)}, \dots, T_{(n,r)})$$

are independent given θ .

Thus, \mathcal{E}^f is a private information economy with $\|f\|$ agents, containing r copies of each agent type $i \in N$. With the standard nonredundancy assumption on the probability distributions, by the law of large numbers, it follows that, if r is sufficiently large for all i , and if one could pool all signals, then one could assign a probability to each state that is arbitrarily close to 1. We note that no agent's information is redundant in this process: regardless of the number of copies, each agent still has information that cannot be inferred from the aggregate information of other agents.

2.2 Core Notions

Let \mathcal{E}^f be a private information economy. For each $S \subseteq \mathbf{N}$, S -feasible allocations for the economy are of the form $z : T_S \rightarrow \mathcal{A}_S$, where $\mathcal{A}_S = \{(\xi_i)_{i \in S} \mid \xi_i \in R_+^{L-1} \times R, \sum_{i \in S} \xi_i \leq \sum_{i \in S} e_i\}$.

For ease of notation, we often denote agent (i, j) simply by k . An S -feasible allocation $(z_k)_{k \in S}$ is *incentive compatible* if

$$\begin{aligned} \sum_{\theta \in \Theta} \sum_{t_{S \setminus \{k\}} \in T_{S \setminus \{k\}}} u_k(z_k(t_{S \setminus \{k\}}, t_k), \theta) P^f(\theta, t_{S \setminus \{k\}} \mid t_k) \\ \geq \sum_{\theta \in \Theta} \sum_{t_{S \setminus \{k\}} \in T_{S \setminus \{k\}}} u_k(z_k(t_{S \setminus \{k\}}, t'_k), \theta) P^f(\theta, t_{S \setminus \{k\}} \mid t_k) \end{aligned}$$

for each $t_k, t'_k \in T_k$ and $k \in S$.

²These sequences are called conditionally independent replicas in McLean and Postlewaite (2002).

An incentive compatible allocation z for \mathbf{N} is in the *core with respect to* (*w.r.t.*) *equilibrium blocking* of some class of coalitional communication mechanisms whenever there does not exist an equilibrium rejection of z , i.e., a random blocking plan against z , for a communication mechanism in that class; see Kamishiro and Serrano (2011) for details.

Particular cases of this definition are the following:

- The *incentive-compatible coarse core* (Vohra (1999)) corresponds to the core of no communication mechanism.
- The *credible core* (Dutta and Vohra (2005)) corresponds to the core of deterministic mechanisms.
- The *virtual-utility core* (Myerson (2007)) corresponds to the core of measurable random mechanisms.
- The *randomized mediated core* (Serrano and Vohra (2007)) corresponds to the core of unrestricted random mechanisms.

The larger the class of communication mechanisms considered, the smaller the corresponding core. It follows that the core notions just described are therefore nested, the latter one being the smallest.

Coalition S is said to have an *ex post* objection to a feasible allocation z for \mathbf{N} if there exist $y_S \in \mathcal{A}_S$ and $t^f \in T^f$ such that

$$\sum_{\theta \in \Theta} P^f(\theta | t^f) u_k(y_k^S, \theta) > \sum_{\theta \in \Theta} P^f(\theta | t^f) u_k(z_k(t^f), \theta)$$

for all $k \in S$.³ The *ex post* core is the set of allocations to which there is no *ex post* objection.

The inclusion of the *ex post* core allocations satisfying incentive compatibility in the core w.r.t. equilibrium blocking of unrestricted random communication mechanisms was shown in Kamishiro and Serrano (2011, Proposition 3.1), which follows an argument in Dutta and Vohra (2005, Proposition 5.1) for the credible core.

³Our use of the term *ex post* refers to events that occur after the realization of the signal profile t , but before the realization of the state θ (allocations can depend on agents' types but not on θ , which is assumed to be unobservable).

Given $\varepsilon > 0$, coalition S is said to have an ε -ex post objection to z if there exist $y_S \in \mathcal{A}_S$ and $t^f \in T^f$ such that

$$\sum_{\theta \in \Theta} P^f(\theta | t^f) u_k(y_k^S, \theta) > \sum_{\theta \in \Theta} P^f(\theta | t^f) u_k(z_k(t^f), \theta) + \varepsilon$$

for all $k \in S$. The ε -ex post core (or approximate ex post core) is the set of all allocations to which no coalition has an ε -ex post objection. The definitions of approximate or ε -interim blocking and its associated cores combine the definitions of ε -blocking and *interim* cores in the obvious way, and are therefore omitted.

2.3 Definitions on Informational Size

For the private information economy \mathcal{E}^f , we define next the notions of informational size, aggregate uncertainty, and distributional variability, originally introduced in McLean and Postlewaite (2002), as follows:

1. Informational size:

Let P_Θ^f be the conditional probability distribution on Θ given $t^f \in T^f$. Any array of agents' types $t^f = (t_{-k}, t_k) \in T^f$ induces a conditional distribution on Θ and, if agent k unilaterally changes his announced signal from t_k to t'_k , this conditional distribution in general will also change. We consider agent k to be informationally small if, for each t_k , there is a small probability assigned to the event that he can induce a large change in the conditional distribution on Θ by changing his announced signal from t_k to some other t'_k . McLean and Postlewaite (2002) formalized this in the following definition.

Let

$$I_\varepsilon^k(t'_k, t_k) := \{t_{-k} \in T_{-k} \mid \|P_\Theta^f(\cdot | t_{-k}, t_k) - P_\Theta^f(\cdot | t_{-k}, t'_k)\| > \varepsilon\},$$

where $\|\cdot\|$ is the 1-norm.

The informational size of agent k is defined as

$$\nu_k^{P^f} = \max_{t_k \in T_k} \max_{t'_k \in T_k} \min\{\varepsilon \geq 0 \mid \Pr\{\tilde{t}_{-k} \in I_\varepsilon^k(t'_k, t_k) \mid \tilde{t}_k = t_k\} \leq \varepsilon\}.$$

Loosely speaking, we will say that agent k is *informationally small* with respect to P^f if his informational size $\nu_k^{P^f}$ is small. That is, if agent k receives signal t_k

but reports $t'_k \neq t_k$, then the effect of this misreport is a change in the conditional distribution on θ from $P_{\Theta}^f(\cdot | t_{-k}, t_k)$ to $P_{\Theta}^f(\cdot | t_{-k}, t'_k)$. If $t_{-k} \in I_{\varepsilon}^k(t'_k, t_k)$ then this change is large in the sense that $\|P_{\Theta}^f(\cdot | t_{-k}, t_k) - P_{\Theta}^f(\cdot | t_{-k}, t'_k)\| > \varepsilon$. Therefore, $\Pr\{\tilde{t}_{-k} \in I_{\varepsilon}^k(t'_k, t_k) | \tilde{t}_k = t_k\}$ is the probability that k can have a large influence on the conditional distribution on Θ by reporting t'_k instead of t_k when his observed signal is t_k . An agent is informationally small if for each of his possible signals t_k , he assigns small probability to the event that he can have a large influence on the distribution $P_{\Theta}^f(\cdot | t_{-k}, t_k)$, given his observed signal.

2. Negligible aggregate uncertainty:

Next, aggregate uncertainty can be quantified. Let

$$\mu_k^{P^f} := \max_{t_k \in T_k} \min\{\varepsilon | \Pr\{\|P_{\Theta}^f(\cdot | t^f) - I_{\theta}\| > \varepsilon \text{ for all } \theta \in \Theta | \tilde{t}_k = t_k\} \leq \varepsilon\},$$

where I_{θ} denotes the measure that puts all its mass on state θ .

If $\mu_k^{P^f}$ is small for each k , then we will say that P^f exhibits negligible aggregate uncertainty. In this case, each agent knows that, conditional on his own signal, the aggregate information of all agents will, with high probability, provide a good prediction of the true state.

3. Distributional variability:

To define the measure of variability, we first define a metric d on the simplex Δ_{Θ} as follows: for each $\alpha, \beta \in \Delta_{\Theta}$, let

$$d(\alpha, \beta) := \left\| \frac{\alpha}{\|\alpha\|_2} - \frac{\beta}{\|\beta\|_2} \right\|_2,$$

where $\|\cdot\|_2$ denotes the 2-norm. Hence, $d(\alpha, \beta)$ measures the Euclidean distance between the normalizations of α and β . We define

$$\Lambda_k^{P^f} = \min_{t_k \in T_k} \min_{t'_k \in T_k \setminus \{t_k\}} d(P_{\Theta}^f(\cdot | t_k), P_{\Theta}^f(\cdot | t'_k))^2.$$

This is a measure of the variability of the conditional distribution $P_{\Theta}^f(\cdot | t_k)$ as a function of t_k .

3 The Main Result

Next, we present an important lemma for our main result:

Lemma 3.1. *For each $\eta > 0$ and each (complete information) feasible allocation $\mathcal{A} = (\zeta_i(\theta_1), \dots, \zeta_i(\theta_m))_{i \in N}$ in each state in a pre-economy, there exists a number \hat{r} such that: for all $r > \hat{r}$ and every profile f satisfying $f(i) = r$ for all i there exist an incentive compatible allocation $z(\cdot)$ for \mathcal{E}^f and a collection B_1, \dots, B_m of disjoint subsets of T^f such that*

- (i) $\Pr\{\tilde{t}^f \in \cup_{h=1}^m B_h\} \geq 1 - \eta$,
- (ii) $\Pr\{\tilde{\theta} = \theta_h \mid \tilde{t}^f = t^f\} \geq 1 - \eta$ for all $h = 1, \dots, m$ and all $t^f \in B_h$,
- (iii) for all agents (i, j) in the profile f and all $t^f \in B_h$,

$$u_{(i,j)}(z_{(i,j)}(t^f); \theta_h) \geq u_{(i,j)}(\zeta_i(\theta_h), \theta_h) - \eta.$$

- (iv) for all agents (i, j) in the profile f and all $t^f \in T^f \setminus [\cup_{h=1}^m B_h]$,

$$z_{(i,j)}(t^f) = \hat{x}_{(i,j)}(t^f),$$

where $(\hat{x}_{(i,j)}(t^f))_{(i,j)}$ is an ex post core allocation in signal profile t^f with agents' profile f .

Proof. By the Claim in McLean and Postlewaite (2002, page 2449), for every $\lambda > 0$, there exists an integer \hat{r} such that for every $r > \hat{r}$ and every profile f satisfying $f(i) = r$ for all i , $\nu_{i,j}^{P^f} \leq \lambda$ and $\mu_{i,j}^{P^f} \leq \lambda$.

For a conditionally independent sequence, $P_{\Theta}^f(\cdot | t_{i,j}) = P_{\Theta}(\cdot | t_i)$ for every profile f and all $t_{i,j} \in T_i$. In particular, $P_{\Theta}^f(\cdot | t_{i,j})$ is independent of r and it follows that $\Lambda_{i,j}^{P^f} = \Lambda_i^P$ for all r and j . Furthermore, $\Lambda_i^P > 0$.

Let $\mathcal{A} = (\zeta_i(\theta_1), \dots, \zeta_i(\theta_m))_{i \in N}$ be a (complete information) feasible allocation in each state $\theta_h \in \Theta$, and $\eta > 0$. Let

$$K_1 := \max_{\theta} \max_i v_i \left(\sum_{j=1}^n e_j^{-L}, \theta \right) + \sum_{j=1}^n e_j^L.$$

Choose δ so that

$$0 < \delta < \min \left\{ \frac{c(\eta, \mathcal{A})}{20\sqrt{m}K_1}, \frac{\eta}{4}, \frac{1}{6} \right\},$$

where $c(\eta, \mathcal{A})$ is the number given in Lemma A.1 in McLean and Postlewaite (2002) introduced below (we use this lemma later; we omit to write down the definition of the function c since we will only directly use the lemma.)

Define $\hat{\mu}^{P^r} = \max_{i,j} \mu_{i,j}^{P^r}$, $\hat{\nu}^{P^r} = \max_{i,j} \nu_{i,j}^{P^r}$, $\Lambda^{P^r} = \min_{i,j} \Lambda_{i,j}^{P^r}$, and $\Lambda^P = \min_i \Lambda_i^P$. Applying Steps 1 and 2 in McLean and Postlewaite (2002, page 2449), there exists an \hat{r} such that for every $r > \hat{r}$ and every profile f satisfying $f(i) = r$, $\hat{\nu}^{P^f} \leq \delta \Lambda^P = \delta \Lambda^{P^f}$, $\hat{\mu}^{P^f} \leq \delta \Lambda^P = \delta \Lambda^{P^f}$.

For each $h = 1, \dots, m$, let

$$A_h^f = \{t^f \in T^f \mid \|P_{\Theta}^f(\cdot \mid t^f) - I_{\theta_h}\| \leq \hat{\mu}^{P^f}\}$$

and

$$B_h^f = \{t^f \in T^f \mid \|P_{\Theta}^f(\cdot \mid t^f) - I_{\theta_h}\| \leq \hat{\mu}^{P^f} + \hat{\nu}^{P^f}\},$$

where we recall that $\|\cdot\|$ is the 1-norm. Let

$$A_0^f = T^f \setminus [\cup_h A_h^f] \quad \text{and} \quad B_0^f = T^f \setminus [\cup_h B_h^f].$$

We rely on the following:

Lemma (McLean and Postlewaite (2002, Lemma A.1)). Let $\mathcal{A} = (\zeta_i(\theta_1), \dots, \zeta_i(\theta_m))_{i \in N}$ be a (complete information) feasible allocation in each state $\theta_h \in \Theta$. For each $\eta \geq 0$, there exists a collection $\{\{y_i(t_i, \theta_h)\}_{(t_i, \theta_h) \in T_i \times \Theta}\}_{i \in N}$ satisfying:

- (i) $y_i(t_i, \theta_h) \in R_+^L$ and $\sum_{i \in N} (y_i(t_i, \theta_h) - e_i) \leq 0$ for all $t_i \in T_i$ and all $\theta_h \in \Theta$,
- (ii) $u_i(\zeta_i(\theta_h); \theta_h) \geq u_i(y_i(t_i, \theta_h)) \geq u_i(\zeta_i(\theta_h); \theta_h) - \eta$ for all $t_i \in T_i$ and all $\theta_h \in \Theta$,
- (iii) for each $t_i, t'_i \in T_i$ with $t_i \neq t'_i$,

$$\sum_{\theta_h} [u_i(y_i(t_i, \theta_h), \theta_h) - u_i(y_i(t'_i, \theta_h), \theta_h)] P(\theta_h \mid t_i) \geq \frac{c(\eta, \mathcal{A})}{2\sqrt{m}} \min_i \Lambda_i^P.$$

Applying this lemma, there exists a collection $(\{y_i(t_i, \theta_h)\}_{(t_i, \theta_h) \in T_i \times \Theta})_{i \in N}$ satisfying the above three conditions. In the private information economy, Let $z(\cdot)$ be the allocation defined as

$$z_{i,j}(t^f) := \begin{cases} y_i(t_i, \theta_h) & \text{if } t^f \in B_h^f (h = 1, \dots, m) \text{ and } t_{i,j}^f = t_i, \\ \hat{x}_{i,j}(t^f) & \text{if } t^f \in B_0^f, \end{cases}$$

where $(\hat{x}_{(i,j)}(t^f))_{(i,j)}$ is an *ex post* core allocation in signal profile t^f when agents' profile is f . The proof of the lemma can now be completed using exactly the same arguments as in Kamishiro (2011). \blacksquare

From this lemma, we can obtain the following nonemptiness result of the approximate core.

Theorem 3.2. *For every $\varepsilon > 0$, there exists a number \hat{r} such that: for all $r > \hat{r}$ and every profile f satisfying $f(i) = r$ for all i , any ε -interim core w.r.t. equilibrium blocking of a private information economy \mathcal{E}^f is nonempty.*

Proof: Given $\varepsilon > 0$, let

$$K_1 := \max_{\theta} \max_i v_i \left(\sum_{i=1}^n e_i^{-L}, \theta \right) + \sum_{i=1}^n e_i^L \quad \text{and} \quad \eta := \frac{\varepsilon}{1 + K_1}.$$

We pick a Walrasian equilibrium allocation for all the economies in a replication sequence, which, in the case of replicas, is simply an allocation for the basic economy $(\zeta_i(\theta_1), \dots, \zeta_i(\theta_m))_{i \in N}$ with one agent of each characteristic. Given $\eta > 0$, we are able to choose \hat{r} in accordance with the above lemma, such that for all $r > \hat{r}$ and every profile f satisfying $f(i) = r$ for every i there exists an incentive compatible allocation $z(\cdot)$ for \mathcal{E}^f satisfying the conditions in the lemma.

From Dutta and Vohra (2005, Proposition 5.1) and Kamishiro and Serrano (2011, Proposition 3.1) it suffices to show that this allocation $z(\cdot)$ is not *ex post* ε -blocked by any coalition S . If $t^f \in T^f \setminus [\cup_{h=1}^m B_h]$, then there is no such blocking from condition (iv) in Lemma 3.1. So suppose that there exist $(y_k^S)_{k \in S} \in \mathcal{A}_S$ and $t^f \in B_h$ satisfying

$$\sum_{k \in S} \sum_{\theta \in \Theta} u_k(y_k^S, \theta) P^f(\theta | t^f) > \sum_{k \in S} \sum_{\theta \in \Theta} u_k(z_k(t^f), \theta) P^f(\theta | t^f) + \varepsilon \cdot |S|.$$

We will show that this leads to a contradiction.

For each $t^f \in B_h$,

$$\begin{aligned} \sum_{k \in S} \sum_{\theta \in \Theta} u_k(z_k(t^f), \theta) P^f(\theta | t^f) &\geq (1 - \eta) \sum_{k \in S} u_k(z_k(t^f), \theta_h) \\ &\geq (1 - \eta) \sum_{k \in S} [u_k(\zeta_k(\theta_h), \theta_h) - \eta] \\ &> (1 - \eta) \sum_{k \in S} u_k(\zeta_k(\theta_h), \theta_h) - \eta \cdot |S|. \end{aligned}$$

On the other hand,

$$\sum_{k \in S} \sum_{\theta} u_k(y_k^S, \theta) P^f(\theta | t^f) \leq (1 - \eta) \sum_{k \in S} u_k(y_k^S, \theta_h) + \eta \cdot K_1 \cdot |S|.$$

These imply that

$$\begin{aligned}
(1 - \eta) \sum_{k \in S} u_k(y_k^S, \theta_h) + \eta \cdot K_1 \cdot |S| &\geq \sum_{k \in S} \sum_{\theta} u_k(y_k^S, \theta) P^f(\theta | t^f) \\
&> \sum_{k \in S} \sum_{\theta \in \Theta} u_k(z_k(t^f), \theta) P^f(\theta | t^f) + \varepsilon \cdot |S| \\
&> (1 - \eta) \sum_{k \in S} u_k(\zeta_k(\theta_h), \theta_h) - \eta \cdot |S| + \varepsilon \cdot |S|.
\end{aligned}$$

Thus, we conclude that

$$\sum_{k \in S} u_k(y_k^S, \theta_h) > \sum_{k \in S} u_k(\zeta_k(\theta_h), \theta_h)$$

contradicting the assumption that $(\zeta_k(\theta_h))_{k \in N}$ is a Walrasian allocation of the associated economy in state θ_h . Therefore, the allocation $z(\cdot)$ is not *ex post* ε -blocked by any coalition, i.e., it therefore must belong to the *ex post* ε -core of the $r \times n$ agents' economy. ■

Finally, we can extend our result to general nonreplica sequences, as follows:

Theorem 3.3. *Given $g = (g_1, \dots, g_n) \in Z_+^n$, for every $\varepsilon > 0$, there exists a number \hat{r} such that: for all $r > \hat{r}$ and every profile f satisfying $f(i) = rg_i$ for all i , any ε -interim core w.r.t. equilibrium blocking of a private information economy \mathcal{E}^f is nonempty.*

This theorem clearly holds since we can regard an economy consisting of $g(i)$ copies of each characteristic i as another original pre-economy.

Kovalenkov and Wooders (2003) define the ε_1 -remainder ε_2 -core in a complete information setting. Following their definition, the ε_1 -remainder ε_2 -core in our model is defined as an incentive compatible allocation in the ε_2 core of a sub-profile containing all but a fraction ε_1 of the agents. We can therefore state the following:

Corollary: For every $\varepsilon = \min\{\varepsilon_1, \varepsilon_2\} > 0$, there exists a number \hat{B} such that: for all $B > \hat{B}$ and every profile f satisfying $\|f\| \geq B$, any ε_1 -remainder ε_2 -interim core w.r.t. equilibrium blocking of a private information economy \mathcal{E}^f is nonempty.

In closing, a word about the technical differences with McLean and Postlewaite (2002) will be instructive. McLean and Postlewaite (2002) also show that, if the economy is sufficiently replicated, all agents are informationally small in the sense that the conditional distribution on the state of nature does not vary significantly in any agent's signal if

other agents' signals are known. Moreover, they construct a mechanism that is incentive compatible, *ex post* individually rational, and nearly *ex post* Pareto efficient.

We have constructed an incentive compatible mechanism that is slightly different than the one considered by McLean and Postlewaite (2002) in Step 3 of their proof of Theorem 2. When the received signals are too weak to allow estimation of the true state with high probability, our mechanism assigns each agent an *ex post* core allocation. In contrast, McLean and Postlewaite (2002) constructs an incentive compatible mechanism that satisfies (i)–(iii) in the lemma above (for large r) but, when the received signals cannot estimate the true state with high probability, their mechanism assigns each agent their endowment. That is, the McLean and Postlewaite mechanism allocates each agent their endowment when, for some sets B_h , $t^f \in T^f \setminus [\cup_{h=1}^m B_h]$, where B_h is the set of signal profiles such that the true state is estimated as θ_h with probability almost one.

As a consequence of their treatment of the case where signals are too weak, in general the McLean and Postlewaite (2002) mechanism does not yield an (approximate) *ex post* core allocation. If one tries to modify their rule directly so that it would yield an *ex post* core allocation for $t^f \in T^f \setminus [\cup_{h=1}^m B_h]$, then a key inequality of their proof that shows incentive compatibility breaks down. In order to overcome this problem, we have presented a variant of an argument in Kamishiro (2011), who takes a collection B_1, \dots, B_m of T^f somewhat different from that in McLean and Postlewaite (2002). With Kamishiro's approach, for sufficiently large r , there exists an incentive compatible rule that yields an *ex post* core allocation when $t^f \in T^f \setminus [\cup_{h=1}^m B_h]$. (In order for the argument to go through, r needs to be taken larger than in McLean and Postlewaite, 2002). To underscore the difference of our result here with Kamishiro (2011), we remark that his paper fixes the set of agents and assumes they have very accurate signals in order to establish nonemptiness of the exact coarse core, while we increase the number of agents and prove nonemptiness of approximate cores. With such a change, a result similar to McLean and Postlewaite's Step 3 in their proof of Theorem 2 has been obtained. (See also Lemma 4.5 in Kamishiro (2011)).

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