ECONSTOR
Make Your Publications Visible.

A Service of
ZBW
Leibniz-Informationszentrum Wirtschaft
Leibniz Information Centre for Economics

Koch, Torben

Working Paper

Universal bounds and monotonicity properties of ratios of hermite and parabolic cylinder functions

Center for Mathematical Economics Working Papers, No. 615

Provided in Cooperation with:
Center for Mathematical Economics (IMW), Bielefeld University

Suggested Citation: Koch, Torben (2019) : Universal bounds and monotonicity properties of ratios of hermite and parabolic cylinder functions, Center for Mathematical Economics Working Papers, No. 615, Bielefeld University, Center for Mathematical Economics (IMW), Bielefeld, https://nbn-resolving.de/urn:nbn:de:0070-pub-29357054

This Version is available at:
https://hdl.handle.net/10419/201640

Standard-Nutzungsbedingungen:

Die Dokumente auf EconStor dürfen zu eigenen wissenschaftlichen Zwecken und zum Privatgebrauch gespeichert und kopiert werden.

Sie dürfen die Dokumente nicht für öffentliche oder kommerzielle Zwecke vervielfältigen, öffentlich ausstellen, öffentlich zugänglich machen, vertreiben oder anderweitig nutzen.

Sofern die Verfasser die Dokumente unter Open-Content-Lizenzen (insbesondere CC-Lizenzen) zur Verfügung gestellt haben sollten, gelten abweichend von diesen Nutzungsbedingungen die in der dort genannten Lizenz gewährten Nutzungsrechte.

Terms of use:

Documents in EconStor may be saved and copied for your personal and scholarly purposes.

You are not to copy documents for public or commercial purposes, to exhibit the documents publicly, to make them publicly available on the internet, or to distribute or otherwise use the documents in public.

If the documents have been made available under an Open Content Licence (especially Creative Commons Licences), you may exercise further usage rights as specified in the indicated licence.

WWW.ECONSTOR.EU

Mitglied der Leibniz-Gemeinschaft

Center for
Mathematical Economics
Working Papers

615

May 2019

Universal Bounds and Monotonicity Properties of Ratios of Hermite and Parabolic Cylinder Functions

Torben Koch

Center for Mathematical Economics (IMW)
Bielefeld University
Universitätsstraße 25
D-33615 Bielefeld · Germany

e-mail: imw@uni-bielefeld.de
http://www.imw.uni-bielefeld.de/wp/
ISSN: 0931-6558

UNIVERSAL BOUNDS AND MONOTONICITY PROPERTIES OF RATIOS OF HERMITE AND PARABOLIC CYLINDER FUNCTIONS

TORBEN KOCH

Abstract. We obtain so far unproved properties of a ratio involving a class of Hermite and parabolic cylinder functions. Those ratios are shown to be strictly decreasing and bounded by universal constants. Differently to usual analytic approaches, we employ simple purely probabilistic arguments to derive our results. In particular, we exploit the relation between Hermite and parabolic cylinder functions and the eigenfunctions of the infinitesimal generator of the Ornstein-Uhlenbeck process. As a byproduct, we obtain Turán type inequalities.

Key words: Hermite functions; parabolic cylinder functions; Turán type inequalities; Ornstein-Uhlenbeck process.

1. Introduction

Consider the ordinary differential equation (ODE)

$$u''(x) - 2xu'(x) + 2\nu u(x) = 0, \quad \nu < 0,\, x \in \mathbb{R}. \tag{1.1}$$

Following Section 10.2 in [14], the solutions to (1.1) are called Hermite functions and denoted by H_ν. They are closely connected to parabolic cylinder functions. In fact, letting Γ be the Euler's Gamma function, the parabolic cylinder function D_ν, introduced in [25], admits the representation (cf. Section 8.3 in [10])

$$D_\nu(x) := \frac{e^{-\frac{x^2}{4}}}{\Gamma(-\nu)} \int_0^\infty t^{-\nu-1} e^{-\frac{t^2}{2} - xt} dt, \quad x \in \mathbb{R}, \tag{1.2}$$

and satisfies

$$D_\nu(x) = 2^{-\frac{\nu}{2}} e^{-\frac{x^2}{4}} H_\nu\left(\frac{x}{\sqrt{2}}\right), \quad x \in \mathbb{R}. \tag{1.3}$$

In this paper, we study properties of the ratio $\mathcal{R}_\nu : \mathbb{R} \mapsto \mathbb{R}$, where

$$\mathcal{R}_\nu(x) := \frac{(H_{\nu-1}(x))^2}{H_\nu(x) H_{\nu-2}(x)}, \quad x \in \mathbb{R}, \tag{1.4}$$

and thanks to (1.3), our results carry over to the ratio of D_ν as well. In particular, we show that $\mathcal{R}_\nu$ is strictly decreasing, and we derive its best possible upper and lower bounds.

The ratio (1.4) is closely related to the so-called Turán types inequalities. Those inequalities have been discovered in 1941 by P. Turán (published in 1950, see [24]) for Legendre Polynomials P_n, $n \in \mathbb{N}$, and for those functions they read as

$$P_{n-1}(x) P_{n+1}(x) - P_n^2(x) < 0, \quad \text{for all } x \in (-1, 1). \tag{1.5}$$

Notice that the validity of (1.5) was first proved by G. Szegö in 1948 (see [22]). Since then, inequalities of this form have attracted a lot of attention, and have been proved to be valid for other polynomials such as Hermite (obtained from Hermite functions by taking $\nu \in \mathbb{N}$), Jacobi, Laguerre or ultraspherical polynomials (see [12, 22], among others), and for special functions as (modified) Bessel, Gamma, parabolic cylinder or hypergeometric functions (see

Date: May 21, 2019.

[2, 4, 5, 6, 7, 23], among many others). Applications of Turán type inequalities can be found in many fields, ranging from biophysics (see [3] and the references therein) to information theory (see [15]) and stochastic control (see [8, 11]).

Properties of ratios of special functions as in (1.4) have also gained interest in recent years. In [21], conjectures about the monotonicity of a ratio associated to exponential series sections are formulated. Those conjectures are then proved in [16, 17] for classical Kummer and Gauss hypergeometric functions, as well as for the so-called q-Kummer confluent hypergeometric and q-hypergeometric functions. Moreover, the monotonicity of a ratio like (1.4) associated to modified Bessel functions of the first and second kind has been proved to be valid and used for the proofs of Theorem 2.1 and Theorem 3.1 in [4]. Our focus on (1.4) is motivated by an optimal liquidation problem in a financial market (see Remark 6.8 in [8]). Lower and upper bounds for $\mathcal{R}_\nu$ have already been studied by [20], but we are able to show that our bounds are the best possible ones, and this leads to a discrepancy between the results in [20] and ours (see Remark 3.4).

In all the aforementioned references on Turán type inequalities (see [2, 4, 5, 6, 7, 12, 16, 17, 22, 23, 24]), the authors use purely analytic approaches to prove their results. Here, instead, we follow a completely different approach that uses probabilistic arguments, and leads to a simple and short proof of our results. In particular, we exploit the relation of Hermite functions to the eigenfunctions of the infinitesimal generator of the Ornstein-Uhlenbeck process.

The paper is organised as follows: in Section 2, we introduce and recall the properties of the Ornstein-Uhlenbeck process and point out its connections to Hermite and parabolic cylinder functions. Then, in Section 3, the results from Section 2 are used to prove the claimed properties of $\mathcal{R}_\nu$.

2. Ornstein-Uhlenbeck Process and Hermite Functions

Let $(\Omega, \mathcal{F}, \mathbb{F} := (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ be a filtered probability space with a filtration $\mathbb{F}$ satisfying the usual conditions, and carrying a standard one-dimensional $\mathbb{F}$-Brownian motion W.

We now introduce the so-called Ornstein-Uhlenbeck process and we will recall some of its well known properties. The Ornstein-Uhlenbeck process has been introduced for the first time in [19], and in modern stochastic analysis it is defined as the unique strong solution to the stochastic differential equation

$$dX_t^x = (\mu - X_t^x)\, dt + \sigma dW_t, \quad X_0^x = x \in \mathbb{R}, \tag{2.1}$$

for $\mu \in \mathbb{R}$ and $\sigma > 0$. For any given initial value $x \in \mathbb{R}$, the process $X^x := (X_t^x)_{t\geq 0}$ is Gaussian. In particular, equation (2.1) admits the explicit solution

$$X_t^x = xe^{-t} + \mu(1 - e^{-t}) + \int_0^t \sigma e^{s-t} dW_s, \tag{2.2}$$

and it follows from (2.2) that for any $t \geq 0$

$$X_t^x \sim \mathcal{N}\left(xe^{-t} + \mu\left(1 - e^{-t}\right), \frac{\sigma^2}{2}\left(1 - e^{-2t}\right)\right),$$

where $\mathcal{N}(\alpha, \gamma)$ denotes the Gaussian distribution function with mean $\alpha \in \mathbb{R}$ and variance $\gamma > 0$.

The infinitesimal generator associated to X^x is denoted by $\mathcal{L}$ and, for any $u : \mathbb{R} \mapsto \mathbb{R}$ s.t. $u \in C^2(\mathbb{R})$, it is defined by

$$[\mathcal{L}u](x) := \lim_{t\downarrow 0} \frac{\mathbb{E}\left(u(X_t^x)\right) - u(x)}{t}, \quad x \in \mathbb{R}. \tag{2.3}$$

In particular, by an application of Dynkin's formula (cf. Theorem 7.4.1 in [18]) and of the mean-value theorem, one obtains

$$\mathcal{L}u(x) = \frac{\sigma^2}{2}u''(x) + (\mu - x)u'(x), \quad x \in \mathbb{R}.$$

Given $\nu < 0$, it is well known that the ODE $\mathcal{L}u + \nu u = 0$ admits a strictly increasing positive fundamental solution ψ. The function ψ can be expressed in terms of the cylinder function (see, e.g., p. 280 in [13]) (up to a positive constant); that is

$$\psi(x) \propto e^{\frac{(x-\mu)^2}{2\sigma^2}} D_\nu\left(\frac{\mu - x}{\sigma}\sqrt{2}\right), \quad x \in \mathbb{R}. \tag{2.4}$$

In light of (1.3) and (2.4), from now on we thus identify the positive strictly increasing eigenvector of $\mathcal{L}$ with

$$\psi(x) = H_\nu\left(\frac{\mu - x}{\sigma}\right), \quad x \in \mathbb{R}. \tag{2.5}$$

3. The Main Result: Monotonicity of Ratios of Hermite Functions

In this section, we use the link between the Ornstein-Uhlenbeck process and the Hermite functions in order to study the monotonicity of the ratio $\mathcal{R}_\nu$ from (1.4).

Theorem 3.1. *For all $\nu < 0$, the function $\mathcal{R}_\nu$ as in (1.4) is strictly decreasing.*

Proof. The proof is organised in two steps. First, recalling ψ from (2.5), in *Step 1* we prove that the function $\Psi : \mathbb{R} \mapsto \mathbb{R}$ such that

$$\Psi(x) := \frac{\psi'(x)^2}{\psi(x)\psi''(x)}, \quad x \in \mathbb{R},$$

is strictly increasing. Then, in *Step 2* we make the conclusion for $\mathcal{R}_\nu$.

Step 1. Let $x, y \in \mathbb{R}$ be such that $y > x$, recall X^x solving (2.1), and define the first hitting time of X^x at level y by

$$\tau_y := \inf\{t \geq 0 : X_t^x \geq y\}, \quad \mathbb{P}\text{-a.s.}$$

Direct calculations on (2.5) and the identity (cf., e.g., equation (10.4.4) in [14])

$$H_\nu'(x) = 2\nu H_{\nu-1}(x), \quad x \in \mathbb{R},\ \nu < 0, \tag{3.1}$$

show that the k-th derivative of ψ, denoted by $\psi^{(k)}$, is a strictly increasing positive solution to $(\mathcal{L} + (\nu - k))\,\psi^{(k)} = 0$. Now, since $\psi^{(k)} \in C^2(\mathbb{R})$ for any $k \in \mathbb{N}_0$, Itô's formula (together with a standard localization argument) yields, after taking expectations,

$$\mathbb{E}\left[e^{(\nu-k)\tau_y}\psi^{(k)}\left(X_{\tau_y}^x\right)\right] = \psi^{(k)}(x),$$

and hence

$$\mathbb{E}\left[e^{(\nu-k)\tau_y}\right] = \frac{\psi^{(k)}(x)}{\psi^{(k)}(y)}, \tag{3.2}$$

since $\tau_y < \infty$ $\mathbb{P}$-a.s. as X^x is positively recurrent (cf. Appendix 1.24 in [9]). Now, Hölder's inequality yields

$$\mathbb{E}\left[e^{\nu\tau_y}\right]^{\frac{1}{2}}\mathbb{E}\left[e^{(\nu-2)\tau_y}\right]^{\frac{1}{2}} > \mathbb{E}\left[e^{(\nu-1)\tau_y}\right], \tag{3.3}$$

which is strict since the function $f(z) := e^{\nu z}$ is not a multiple of the function $g(z) := e^{(\nu-2)z}$, and the random variable τ_y has a distribution which is absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}_+$. From both (3.2) with $k = 0, 1, 2$ and (3.3), we find

$$\Psi(y) > \Psi(x).$$

Since $y > x$ were arbitrary, we have that $x \mapsto \Psi(x)$ is strictly increasing.

Step 2. Exploiting the identities (2.5) and (3.1), we find that for any $x \in \mathbb{R}$

$$\Psi(x) = \frac{\nu}{\nu - 1} \mathcal{R}_\nu \left(\frac{\mu - x}{\sigma} \right).$$

Therefore, because $\nu < 0$, we conclude by *Step 1* that the function $\mathcal{R}_\nu$ is strictly decreasing. □

The following corollary gives the best possible bounds for $\mathcal{R}_\nu$. These in turn imply the Turán type inequality.

Corollary 3.2. *For all $\nu < 0$, the function $\mathcal{R}_\nu$ as in (1.4) is such that*

$$1 < \mathcal{R}_\nu(x) < \frac{\nu - 1}{\nu}, \quad \text{for all } x \in \mathbb{R}. \tag{3.4}$$

In particular, the following Turán-type inequality holds:

$$H_{\nu-1}(x)^2 - H_\nu(x) H_{\nu-2}(x) > 0, \quad x \in \mathbb{R}.$$

Proof. Equations (10.6.4) and (10.6.7) in [14] provide the asymptotic behavior of $H_\nu(x)$ for both (large) positive and (large) negative values of x. In particular, it holds that

$$\lim_{x \downarrow -\infty} \mathcal{R}_\nu(x) = \frac{\nu - 1}{\nu}, \qquad \lim_{x \uparrow +\infty} \mathcal{R}_\nu(x) = 1.$$

Thus, (3.4) follows from the strict monotonicity of $x \mapsto \mathcal{R}_\nu(x)$ proved in Theorem (3.1). □

Since, by (1.3), we have $\mathcal{R}_\nu \left(\frac{x}{\sqrt{2}} \right) = \frac{D_{\nu-1}(x)^2}{D_\nu(x) D_{\nu-2}(x)}$ for any $x \in \mathbb{R}$, the next proposition easily follows from Theorem 3.1 and Corollary 3.2.

Proposition 3.3. *For all $\nu < 0$, the function $\widetilde{\mathcal{R}}_\nu : \mathbb{R} \mapsto \mathbb{R}$ defined as*

$$\widetilde{\mathcal{R}}_\nu(x) := \frac{D_{\nu-1}(x)^2}{D_\nu(x) D_{\nu-2}(x)}, \quad x \in \mathbb{R},$$

is strictly decreasing. Moreover, it holds

$$1 < \widetilde{\mathcal{R}}_\nu(x) < \frac{\nu - 1}{\nu}, \quad \text{for all } x \in \mathbb{R}.$$

Remark 3.4. *It is worth mentioning that lower and upper bounds of the function $\widetilde{\mathcal{R}}_\nu$ associated to the parabolic cylinder function $U_{-\nu-\frac{1}{2}}(x) = D_\nu(x)$ (see (19.3.1) in [1]) have also been derived in [20]. In that paper, the right-hand side of equation (28) (see also Remark 1 in [7]) yields an upper bound for $\widetilde{\mathcal{R}}_\nu$ which is strictly less than the one we have obtained in Proposition 3.3. Given that our upper bound is optimal by the proved strict monotonicity of $\widetilde{\mathcal{R}}_\nu$, it seems that there is something fishy in eq. (28) of [20]. Also, a simple numerical analysis seems to contradict the upper bound found in [20].*

Acknowledgments. Financial support by the German Research Foundation (DFG) through the Collaborative Research Centre 1283 "Taming uncertainty and profiting from randomness and low regularity in analysis, stochastics and their applications" is gratefully acknowledged by the author. I wish to thank Giorgio Ferrari for useful discussions.

References

[1] Abramowitz, M., Stegun, I.A. (1964). *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables*. Dover Book. New York.

[2] Alzer, H., Felder, G. (2009). A Turán-Type Inequality for the Gamma Function. *J. Math. Anal. Appl.* **350** 276-282.

[3] Baricz, A. (2009). On a Product of Modified Bessel Function. *Proc. Amer. Math. Soc.* **137(1)** 189-193.

[4] Baricz, A. (2010). Turán Type Inequalities for Modified Bessel Functions. *Bull. Aust. Math. Soc.* **82** 254-264.

[5] Baricz, A. (2013). On Turán Type Inequalities for Modified Bessel Functions. *Proc. Amer. Math. Soc.* **141(2)** 523-532.

[6] Baricz, A., Ismail, M.E.H. (2008). Turán Type Inequalities for Hypergeometric Functions. *Proc. Amer. Math. Soc.* **136(9)** 3223-3229.

[7] Baricz, A., Ismail, M.E.H. (2013). Turán Type Inequalities for Tricomi Confluent Hypergeometric Functions. *Constr. Approx.* **37(2)** 195-221.

[8] Becherer, D., Bilarev, T., Frentrup, P. (2017). Optimal Liquidation under Stochastic Liquidity. *Finance Stoch.* **22(1)** 39-68.

[9] Borodin, W.H., Salminen, P. (2002). *Handbook of Brownian Motion-Facts and Formulae*. 2nd Edition. Birkhäuser.

[10] Erdélyi, A., Magnus, W., Oberhettinger, F., Tricomi, F.G. (1953). *Higher Transcendental Functions*. Volume 2. Based, in Part, on Notes left by Harry Bateman. McGraw-Hill Book Co. New York.

[11] Ferrari, G., Koch, T. (2018). An Optimal Extraction Problem with Price Impact. Preprint. **ArXiv**: 1812.01270.

[12] Gasper, G. (1972). An Inequality of Turán Type for Jacobi Polynomials. *Proc. Amer. Math. Soc.* **32(2)** 435-439.

[13] Jeanblanc, M., Yor, M., Chesney, M. (2009). *Mathematical Methods for Financial Markets*. Springer.

[14] Lebedev, N. (1972). *Special Functions and their Applications*. Dover Books on Mathematics. Dover Publications.

[15] McEliece, R.J., Reznick, B., Shearer, J.B. (1981). A Turán Inequality arising in Information Theory. *SIAM. J. Math. Anal.* **12(6)** 931-934.

[16] Mehrez, K., Sitnik, S.M. (2014). Monotonicity of Ratios of q-Kummer Confluent Hypergeometric and q-Hypergeometric Functions and Associated Turán Types Inequalities. *Preprint.* **ArXiv:** 1412.1634v2.

[17] Mehrez, K., Sitnik, S.M. (2015). Proofs of Some Conjectures on Monotonicity of Ratios of Kummer, Gauss and generalized Hypergeometric Functions. *Preprint.* **ArXiv:** 1410.6120v3.

[18] Øksendal, B. (1995). *Stochastic Differential Equations: an Introduction with Applications*. 5th Edition. Springer, New York.

[19] Ornstein, L.S., Uhlenbeck, G.E. (1930). On the Theory of the Brownian Motion. *Phys. Rev.* **36** 823-841.

[20] Segura, J. (2012). On Bounds for Solutions of Monotonic First Order Difference-Differential System. *J. Inequal. Appl.* DOI: 10.1186/1029-242X-2012-65.

[21] Sitnik, S.M. (1993). Inequalities for the Exponential Remainder. *Preprint.* Institute of Automation and Control Process, Far Eastern Branch of the Russian Academy of Sciences, Vladivostok.

[22] Szegö, G. (1948). On an Inequality of P.Turán Concerning Legendre Polynomials. *Bull. Amer. Math. Soc.* **54(4)** 401-405.

[23] Thiruvenkatachar, V.R., Nanjundiah, T.S. (1951). Inequalities Concerning Bessel Functions and Orthogonal Polynomials. *Proc. Indian Acad. Math. Sci.* **33** 373-384.

[24] Turán, P. (1950). On the Zeros of the Polynomials of Legendre. *Casopis Pest. Mat. Fys.* **75(3)** 113-122.

[25] Whittaker, E.T., Watson, G.N. (1927). *A Course of Modern Analysis*. Cambridge University Press. 4th Edition.

T. Koch: Center for Mathematical Economics (IMW), Bielefeld University, Universitätsstrasse 25, 33615, Bielefeld, Germany

Email address: t.koch@uni-bielefeld.de