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Abstract

We consider 2-factorizations of the complete tripartite graph $K(n;3)$ where each 2-factor consists of cycles with even length. An additional requirement is related to pairs of edges being consecutive in an arbitrary cycle. For all pairs of edges we require that the three nodes incident to these two edges must be from different partite sets of the tripartite graph. We show that such a 2-factorizations cannot exist if $n = 2$ or n is odd. Furthermore, we show how to construct such a 2-factorization for all even n with $n \notin \{2, 26, 34, 58, 74\}$.

Keywords: Graph decomposition; complete tripartite graph; 2-factorization

1 Introduction

Decompositions of graphs have received a great deal of attention. In particular, this is true for edge-disjoint decompositions of complete multipartite graphs into cycles, see Lindner and Rodger [9] for a survey.

There are several papers considering special cycle structures. Billington [1] studies decompositions of complete tripartite graphs into cycles of lengths 3 and 4 while Cavenagh and Billington [4] consider decompositions of complete multipartite graphs into cycles with even length. Cavenagh [3] shows that each complete tripartite graph G can be partitioned into cycles of length k if the number of edges of G is a multiple of k. More generally, Billington and Cavenagh [2] prove that the edges of each complete tripartite graph can be partitioned into cycles of arbitrary lengths. Hilton and Rodger [5], Laskar [6], Laskar and Auerbach [7], and Leach and Rodger [8] investigate decompositions of complete multipartite graphs into Hamilton cycles.

In this paper we study special structures of 2-factorizations of the complete tripartite graph where all three partite sets have identical size. The graph having $3n$ nodes is denoted by $K(n;3)$ in the following. The three subsets of nodes forming the partite sets of the tripartite graph are represented by V_0, V_1, and V_2. Recall that a 2-factor of a graph $G = (V, E)$ is a subset E' of edges such that each node $v \in V$ is incident to exactly two edges $e_0, e_1 \in E'$, $e_0 \neq e_1$. A 2-factorization of a graph is a partition of the edges into edge-disjoint 2-factors. Note that each 2-factor consists of a collection of node-disjoint cycles.

We consider 2-factors meeting two additional requirements:

(i) Each cycle contained in the 2-factor must have even length.

(ii) In the 2-factor each node $i \in V_s$, $s \in \{0, 1, 2\}$, must be adjacent to exactly one node $j \in V_{(s-1) \bmod 3}$ and to exactly one node $j' \in V_{(s+1) \bmod 3}$.

A 2-factor fulfilling (i) and (ii) is referred to as circular even 2-factor (ce2f) below. Accordingly, a circular even 2-factorization (ce2F) of a graph G is defined as a decomposition of E into edge-disjoint ce2fs.

In the following section we study for which numbers $n \in \mathbb{N}$ a ce2F of $K(n;3)$ exists. We answer this question for all n with $n \notin \{26, 34, 58, 74\}$.

2 Results

It is easy to see that no ce2F of $K(n;3)$ exists if n is odd. Since then the number of nodes $3n$ is odd, there must be at least one cycle having odd length in each 2-factor. Therefore, we restrict ourselves to n being even in the following.

2.1 Case $\mathbf{n = 4k}$ with $\mathbf{k \in \mathbb{N}}$

In this subsection we provide a construction scheme for a ce2F of the graph $K(n;3)$ with $n = 4k$ for all $k \in \mathbb{N}$. Note that the subgraph induced by two of the three partite sets is the complete bipartite graph $K_{n,n}$ with $2n$ nodes. It is well-known that for this graph 1-factorizations exist, e.g. the bipartite 1-factorization $F^{bip} = \left\{F_0^{bip}, \ldots, F_{n-1}^{bip}\right\}$ with

$$F_l^{bip} = \{[m, n + (m + l) \bmod n] \mid m = 0, \ldots, n-1\} \text{ for } l = 0, \ldots, n-1.$$

The basic idea of our construction scheme is to combine certain 1-factors of F^{bip} between pairs of partite sets in each ce2f. First, we adress the question how to combine 1-factors of F^{bip} in order to obtain a ce2f of $K(n;3)$. For two different partite sets $V_s, V_{s'}$ and $l \in \{0, \ldots, n-1\}$ we denote by $F_l^{bip}(s, s')$ the 1-factor F_l^{bip} between the partite sets V_s and $V_{s'}$.

Lemma 1. *Three 1-factors $F_{l_0}^{bip}(0,1)$, $F_{l_1}^{bip}(1,2)$, and $F_{l_2}^{bip}(2,0)$ with $l_0, l_1, l_2 \in \{0, \ldots, n-1\}$ form a ce2f of $K(n;3)$ if $l_0 + l_1 + l_2$ is odd.*

Proof. Obviously, $F_{l_0}^{bip}(0,1)$, $F_{l_1}^{bip}(1,2)$, and $F_{l_2}^{bip}(2,0)$ form together a 2-factor of $K(n;3)$ since exactly two 1-factors are chosen for each group. Moreover, the 2-factor fulfills condition (ii), i.e. it is circular. Let us consider an arbitrary path from a node $i \in V_s$ to another node $i' \in V_s$ induced by $F_{l_0}^{bip}(0,1)$, $F_{l_1}^{bip}(1,2)$, and $F_{l_2}^{bip}(2,0)$. Obviously, the length of this path is a multiple of 3. Now consider an arbitrary path having length 3. Note that there are exactly two such paths for each $i \in V_s$ and both connect i with another node $i' \in V_s$. Let i and i' be the start and end node of such a path. Since n is even and $l_0 + l_1 + l_2$ is odd, $|i - i'|$ is odd, as well. This implies that an even number of paths of length 3 must be concatenated to construct a circle and, hence, circles have even length. □

In the following we will show that a ce2F of the graph $K(n;3)$ with $n = 4k$ for all $k \in \mathbb{N}$ exists. According to Lemma 1 combining 1-factors $F_{l_0}^{bip}(0,1)$, $F_{l_1}^{bip}(1,2)$, and $F_{l_2}^{bip}(2,0)$ with indices $l_0, l_1, l_2 \in \{0, \ldots, n-1\}$ yields a ce2f if

- exactly one index is odd, and the other two are even, or
- all three indices are odd.

According to Lemma 1 the specific combination of 1-factors can be chosen arbitrarily. However, we focus on a specific arrangement in the following.
At first we combine the odd 1-factors F_l^{bip} for $l = 4\lambda + 1$ ($\lambda = 0, \ldots, \frac{n}{4} - 1$) with the two even 1-factors F_{l-1}^{bip} and F_{l+1}^{bip}. As an example Figure 1 illustrates the case $l = 1$ for the graph $K(4;3)$, where the 1-factors $F_0^{bip}(0,1)$, $F_1^{bip}(1,2)$ and $F_2^{bip}(2,0)$ are shown. Here, the node on top of the column representing a part has the smallest index. We apply each combination three times by rotating the specific 1-factors such that each 1-factor is applied to each pair of partite sets exactly once. This means that each combination is arranged in three ce2fs: $F_{l-1}^{bip}(0,1)$, $F_l^{bip}(1,2)$, $F_{l+1}^{bip}(2,0)$ in the first ce2f, $F_{l-1}^{bip}(1,2)$, $F_l^{bip}(2,0)$, $F_{l+1}^{bip}(0,1)$ in the second ce2f, and $F_{l-1}^{bip}(2,0)$, $F_l^{bip}(0,1)$, $F_{l+1}^{bip}(1,2)$ in the third ce2f. By arranging this configuration for all $\lambda = 0, \ldots, \frac{n}{4} - 1$ we arrange edges corresponding to $\frac{3n}{4}$ 1-factors between each pair of partite sets in $\frac{3n}{4}$ ce2fs. Note that we cover F_l^{bip} for each even index $l \in \{0, \ldots, n-1\}$.
It remains to arrange edges according to F_l^{bip} for all odd $l = 4\mu + 3$ ($\mu = 0, \ldots, \frac{n}{4} - 1$) in the $\frac{n}{4}$ remaining ce2fs. This can be done by arranging F_l^{bip} between each pair of partite sets in

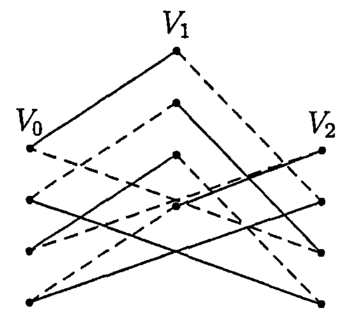

Figure 1: $F_0^{bip}(0,1)$, $F_1^{bip}(1,2)$, and $F_2^{bip}(2,0)$ for $K(4;3)$

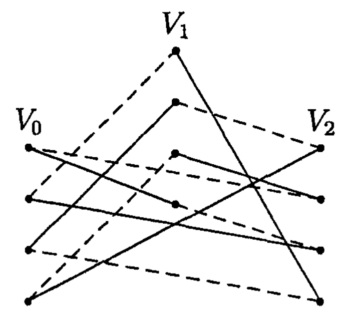

Figure 2: $F_3^{bip}(0,1)$, $F_3^{bip}(1,2)$, and $F_3^{bip}(2,0)$ for $K(4;3)$

a single ce2f, i.e. by choosing $F_l^{bip}(0,1)$, $F_l^{bip}(1,2)$, and $F_l^{bip}(2,0)$. Clearly, the prerequisite of Lemma 1 is fulfilled and, hence, we obtain a ce2f. As an example Figure 2 illustrates the case $l = 3$ for the graph $K(4;3)$, where the 1-factors $F_3^{bip}(0,1)$, $F_3^{bip}(1,2)$ and $F_3^{bip}(2,0)$ are shown. The preceding construction implies

Theorem 1. *For the graph $K(n;3)$ with $n = 4k$ a ce2F exists for all $k \in \mathbb{N}$.*

2.2 Case $\mathbf{n} \notin \{26, 34, 58, 74\}$

In this subsection we provide a construction scheme for a ce2F of the graph $K(n;3)$ for all even $n \notin \{2, 26, 34, 58, 74\}$. If n is a multiple of 4, the construction method from the previous subsection can be applied. By using an integer programming formulation it can be shown that for $n = 2$ no ce2F exists. On the other hand, using Cplex we found ce2Fs F^6 and F^{10} for $K(6;3)$ and $K(10;3)$. These ce2Fs are given in the appendix.
At first we will show that from ce2Fs for a graph $K(l;3)$ with $l \in \mathbb{N}$ ce2Fs for the larger graphs $K(kl;3)$ can be constructed for all $k \in \mathbb{N}$.

Lemma 2. *If a ce2F of $K(l;3)$ for some integer $l \in \mathbb{N}$ exists, then also a ce2F of $K(kl;3)$ for all $k \in \mathbb{N}$ exists.*

Proof. The basic idea of this constructive proof is to cluster nodes and to use the structure known for the graph induced by the cluster. This technique has been applied in several works concerning decomposition of graphs, see e.g. Laskar [6] or Laskar and Auerbach [7].
Suppose we have a ce2F F of $K(l;3)$ for some $l \in \mathbb{N}$. Then we construct a ce2F F' of $K(kl;3)$ for all $k \in \mathbb{N}$ as follows:

1. In the graph $K(kl;3)$ partition the nodes of each partite set V_s, $s \in \{0,1,2\}$, into l subparts $V_{s,m}$ for $m = 0,\ldots,l-1$, having size k each.

2. Consider $K(l;3)$ as the complete tripartite graph induced by the $3l$ subparts. Each 2-factor in the 2-factorization F of $K(l;3)$ is expanded to k 2-factors in the 2-factorization F' of $K(kl;3)$. Note that for each pair of subparts (belonging to different partite sets) all edges between the nodes in these subparts can be represented by the complete bipartite graph $K_{k,k}$. We may employ F^{bip} to decompose them into k 1-factors. For this purpose we assign an arbitrary orientation to each circle from the 2-factors in F. Then, we combine the 1-factors F^{bip}_{λ} for $\lambda = 0,\ldots,k-1$ between each pair of subparts being consecutive in a circle.

It is easy to see that F' is a 2-factorization of $K(kl;3)$. Moreover, we have a circular structure due to the circular structure of F. It is obvious that circles of even length are constructed since each underlying circle in F is of even length q and circles in F' can only have lenghts being multiples of q. $\square$

Thus, according to Lemma 2, we can construct ce2Fs of $K(6k;3)$ and $K(10k;3)$ for all $k \in \mathbb{N}$ using the ce2Fs F^6 and F^{10} as building blocks.
In a similar way the following lemma provides a basis for a more powerful recursive construction.

Lemma 3. *If a ce2F of $K(l;3)$ for some $l \in \mathbb{N}$ exists, then also a ce2F of $K(kl+4m;3)$ for each pair of integers k,m with $4m \geq kl \geq 3m$ exists.*

Proof. The basic idea of this constructive proof is to divide the nodes into two parts inducing subgraphs $G^{kl} := K(kl;3)$ and $G^{4m} := K(4m;3)$. Suppose we have a ce2F F of $K(l;3)$ for some $l \in \mathbb{N}$. Then we construct a ce2F of $K(kl+4m;3)$ using structural elements of F and known ce2Fs of G^{4m} (cf. Subsection 2.1) as follows:

1. According to Lemma 2 from F we can construct a ce2F F' of G^{kl} for all $k \in \mathbb{N}$. We arrange F' of G^{kl} as part of the first kl ce2fs of $K(kl+4m;3)$.

2. We complete the first kl ce2fs of $K(kl+4m;3)$ by ce2fs of G^{4m}. These ce2fs of G^{4m} are arranged by combining 1-factors according to F^{bip} of $K_{4m,4m}$ as in the construction in Section 2.1. More specifically, we choose all $2m$ even 1-factors $F^{bip}_{2\lambda}$ for $\lambda = 0,\ldots,2m-1$ and the $kl-2m$ odd 1-factors $F^{bip}_{4m-kl+1+2\lambda}$ for $\lambda = 0,\ldots,kl-2m-1$. Since $kl \geq 3m$, the number of used odd 1-factors is at least m and, therefore, at least half the number of used even 1-factors. Regarding Lemma 1, we can arrange $3m$ ce2fs of G^{4m} by combining two even 1-factors with one odd 1-factor in each ce2f. Furthermore, we can arrange $kl-3m$ ce2fs of G^{4m} using the remaining odd 1-factors (only a single 1-factor in each ce2f, cf. Subsection 2.1).

Up to now, we have kl complete ce2fs of $K(kl+4m;3)$. It remains to arrange all edges of G^{4m} according to $F^{bip}_{2\lambda+1}$ and $F^{bip}_{4m-1-2\lambda}$ for all $\lambda = 0,\ldots,\frac{4m-kl}{2}-1$ (these edges have been left out in Step 2) and all edges between G^{kl} and G^{4m} in the remaining $4m$ ce2fs of $K(kl+4m;3)$. In the following we represent node i by $(s(i),i')$ where $s(i)$ denotes the partite set of i and $i' := i \bmod n$ is the index of i within its partite set. W.l.o.g. we assume that the nodes of G^{4m} are numbered such that they satisfy $0 \leq i' \leq 4m-1$.

3. For $\lambda = 0, \ldots, 4m-1$ in the ce2f with index $kl+\lambda$ for $q = 0, 1, 2$ and $i' = 4m, \ldots, kl+4m-1$ we arrange edges

$$[(q, i'), ((q+1) \bmod 3, (i' - 4m + \lambda) \bmod 4m)] \quad \text{and}$$
$$[(q, i'), ((q-1) \bmod 3, (i' - 4m + \lambda) \bmod 4m)].$$

In Step 3 we exclusively arrange circles of length 6. In the ce2f with index $\mu \in \{kl, \ldots, kl + 4m-1\}$ of $K(kl+4m; 3)$ the nodes $(q, (\mu+i') \bmod 4m)$ for $q = 0, 1, 2$ and $i' = 0, \ldots, 4m-kl-1$ are not involved yet.

4. For $\mu = kl, \ldots, kl+4m-1$ in the ce2f with index μ for $q = 0, 1, 2$ and $i' = 0, \ldots, 4m-kl-1$ we arrange edges

$$[(q, (\mu + i') \bmod 4m), ((q+1) \bmod 3, (\mu + 4m - kl - 1 - i') \bmod 4m)] \quad \text{and}$$
$$[(q, (\mu + i') \bmod 4m), ((q-1) \bmod 3, (\mu + 4m - kl - 1 - i') \bmod 4m)].$$

In Step 4 we, again, exclusively arrange circles of length 6. Edges originating from Step 4 form ce2fs corresponding to $F^{bip}_{2\lambda+1}$ and $F^{bip}_{4m-1-2\lambda}$ for $\lambda = 0, \ldots, \frac{4m-kl}{2} - 1$. Summarizing, we obtain a ce2F of $K(kl + 4m; 3)$. $\square$

In the following we want to apply Lemma 3 with $k = 1$. Thus, we study which even numbers $n \in \mathbb{N}$ can be written in the form $n = l + 4m$ with $l, m \in \mathbb{N}$ and $4m \geq l \geq 3m$. We get

$$(l, m) = \begin{cases} \left(\frac{n}{2}, \frac{n}{8}\right) & \text{for } n \bmod 8 = 0, \quad n \geq 8, \\ \left(\frac{n-6}{2}, \frac{n+6}{8}\right) & \text{for } n \bmod 8 = 2, \quad n \geq 42, \\ \left(\frac{n-4}{2}, \frac{n+4}{8}\right) & \text{for } n \bmod 8 = 4, \quad n \geq 28, \\ \left(\frac{n-2}{2}, \frac{n+2}{8}\right) & \text{for } n \bmod 8 = 6, \quad n \geq 14. \end{cases} \tag{1}$$

Thus, each even number $n \geq 36$ can be written in the form $n = l + 4m$ with $l, m \in \mathbb{N}$ and $4m \geq l \geq 3m$.

n	existence	argument
2	no	IP
6	yes	IP
10	yes	IP
14	yes	Lemma 3 ($k = 1, l = 6, m = 2$)
18	yes	Lemma 2 ($k = 3, l = 6$)
22	yes	Lemma 3 ($k = 1, l = 10, m = 3$)
26	?	—
30	yes	Lemma 2 ($k = 5, l = 6$)
34	?	—

Table 1: Existence results for a ce2F of $K(n; 3)$

Theorem 2. *For the graph $K(n; 3)$ a ce2F exists for all even $n \notin \{2, 26, 34, 58, 74\}$.*

l	D_m	m	$l+4m$	argument
26	{7,8}	7	54	Lemma 2 ($k=9, l=6$)
		8	58	—
34	{9,10,11}	9	70	Lemma 3 ($k=1, l=30, m=10$)
		10	74	—
		11	78	Lemma 2 ($k=13, l=6$)
58	{15,16,17,18,19}	15	118	Lemma 3 ($k=1, l=54, m=16$)
		16	122	Lemma 3 ($k=1, l=54, m=17$)
		17	126	Lemma 3 ($k=1, l=54, m=18$)
		18	130	Lemma 3 ($k=1, l=62, m=17$)
		19	134	Lemma 3 ($k=1, l=62, m=18$)
74	{19,20,21,22,23,24}	19	150	Lemma 3 ($k=1, l=70, m=20$)
		20	154	Lemma 3 ($k=1, l=70, m=21$)
		21	158	Lemma 3 ($k=1, l=70, m=22$)
		22	162	Lemma 3 ($k=1, l=70, m=23$)
		23	166	Lemma 3 ($k=1, l=78, m=22$)
		24	170	Lemma 3 ($k=1, l=78, m=23$)

Table 2: Numbers $l+4m$ depending on $l \in \{26, 34, 58, 74\}$

Proof. Due to Theorem 1 we already know a ce2F of $K(n;3)$ for all n which are a multiple of 4. For all other even $n < 36$ in Table 1 we summarize the results whether a ce2F is known for the graph $K(n;3)$ or not. Only for $n \in \{2, 26, 34\}$ no ce2F is known. Note that for $n = 26, 34$ no numbers k, l, m exist which satisfy the conditions of Lemma 3.
Due to (1) each even $n \geq 36$ can be represented as $n = l + 4m$ with $4m \geq l \geq 3m$. Thus, if a ce2F for the specific graph $K(l;3)$ is known, according to Lemma 3 also a ce2F of $K(n;3)$ exists.
This construction covers all values n which do not depend on a (direct or indirect) representation with $l = 26$ or $l = 34$. The first part of Table 2 lists all numbers which directly depend on $l = 26$ or $l = 34$. The second column gives the feasible domain D_m consisting of all values $m \in \mathbb{N}$ with $4m \geq l \geq 3m$ for the specific value of l. For each element of D_m the third column contains the value $l + 4m$. If another argument not based on the specific value of l for the existence of a ce2F of $K(l+4m;3)$ is known, it is stated in the fourth column.
The only numbers which directly depend on $l_1 = 26$ or $l_2 = 34$ and for which no other argument is known are $l_1 + 4 \cdot 8 = 58$ and $l_2 + 4 \cdot 10 = 74$. These numbers are treated in the same way in the second part of Table 2. It can be seen that all numbers which directly depend on $l_3 = 58$ or $l_4 = 74$ can be covered by other arguments. Thus, our construction only fails for all even $n \in \{2, 26, 34, 58, 74\}$. While for $n = 2$ it is proven that no ce2F exists, the other four cases remain open. $\square$

We can conclude that a ce2F of the graph $K(n;3)$ for all even $n \notin \{2, 26, 34, 58, 74\}$ does exist. For $n = 2$ or n odd no ce2F can exist. Cases $n \in \{26, 34, 58, 74\}$ remain open so far. However, we strongly conjecture ce2Fs to exist for these cases.

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A Ce2Fs of $\mathrm{K}(6;3)$ and $\mathrm{K}(10;3)$

In the following we represent the nodes $i = 0, \ldots, n-1$ of the graph $K(\frac{n}{3};3)$ by pairs $(s(i), i')$ where $i' = i \bmod \frac{n}{3}$. In the following a ce2F $F^6 = \{F^6_f \mid 0 \leq f \leq 5\}$ of $K(6;3)$ and a ce2F $F^{10} = \{F^{10}_f \mid 0 \leq f \leq 9\}$ of $K(10;3)$ are shown. Each ce2f is given as collection of circles.

$$
\begin{aligned}
F^6_0 &= \{[(0,0),(1,3),(2,0),(0,3),(1,0),(2,3)],\\
&\quad [(0,1),(2,4),(1,1),(0,4),(2,1),(1,4)],\\
&\quad [(0,2),(1,5),(2,2),(0,5),(1,2),(2,5)]\}\\
F^6_1 &= \{[(0,0),(1,1),(2,2),(0,3),(1,4),(2,5)],\\
&\quad [(0,1),(1,2),(2,3),(0,4),(1,5),(2,0)],\\
&\quad [(0,2),(1,3),(2,4),(0,5),(1,0),(2,1)]\}\\
F^6_2 &= \{[(0,0),(2,2),(1,4),(0,2),(2,0),(1,2),\\
&\quad (0,3),(2,1),(1,1),(0,5),(2,3),(1,3),\\
&\quad (0,4),(2,4),(1,5),(0,1),(2,5),(1,0)],\}\\
F^6_3 &= \{[(0,0),(1,4),(2,0),(0,4),(1,2),(2,1)],\\
&\quad [(0,1),(2,2),(1,3),(0,5),(2,5),(1,5),\\
&\quad (0,3),(2,3),(1,1),(0,2),(2,4),(1,0)]\}\\
F^6_4 &= \{[(0,0),(2,0),(1,1),(0,3),(2,4),(1,2),\\
&\quad (0,2),(2,2),(1,0),(0,4),(2,5),(1,3),\\
&\quad (0,1),(2,3),(1,4),(0,5),(2,1),(1,5)]\}\\
F^6_5 &= \{[(0,0),(1,2),(2,2),(0,4),(1,4),(2,4)],\\
&\quad [(0,1),(1,1),(2,5),(0,3),(1,3),(2,1)],\\
&\quad [(0,2),(2,3),(1,5),(0,5),(2,0),(1,0)]\}
\end{aligned}
$$

$$
\begin{aligned}
F_0^{10} &= \{[(0,0),(1,7),(2,3),(0,4),(1,3),(2,1),(0,9),(1,9),(2,4),(0,3),(1,8),(2,0)],\\
&\quad [(0,1),(1,6),(2,7),(0,7),(1,0),(2,8),(0,5),(1,4),(2,6),(0,8),(1,5),(2,2),\\
&\quad (0,2),(1,2),(2,9),(0,6),(1,1),(2,5)]\}\\
F_1^{10} &= \{[(0,0),(1,3),(2,0),(0,6),(1,5),(2,5),(0,4),(1,2),(2,4),(0,8),\\
&\quad (1,0),(2,6),(0,7),(1,7),(2,9),(0,5),(1,9),(2,3),(0,1),(1,4),\\
&\quad (2,7),(0,3),(1,1),(2,1),(0,2),(1,8),(2,2),(0,9),(1,6),(2,8)]\}\\
F_2^{10} &= \{[(0,0),(1,8),(2,7),(0,9),(1,1),(2,3)],[(0,3),(1,2),(2,1),(0,6),(1,7),(2,5)],\\
&\quad [(0,1),(1,3),(2,8),(0,2),(1,5),(2,9),(0,7),(1,4),(2,0),(0,8),(1,9),(2,6)],\\
&\quad [(0,4),(1,0),(2,4),(0,5),(1,6),(2,2)]\}\\
F_3^{10} &= \{[(0,0),(1,6),(2,9),(0,2),(1,0),(2,0),(0,5),(1,2),(2,7),(0,4),\\
&\quad (1,9),(2,1),(0,3),(1,3),(2,5),(0,7),(1,1),(2,2),(0,6),(1,4),\\
&\quad (2,4),(0,1),(1,7),(2,8),(0,9),(1,5),(2,3),(0,8),(1,8),(2,6)]\}\\
F_4^{10} &= \{[(0,0),(1,0),(2,7),(0,1),(1,9),(2,5),(0,6),(1,8),(2,1),(0,8),\\
&\quad (1,6),(2,3),(0,5),(1,5),(2,8),(0,4),(1,1),(2,4),(0,2),(1,7),\\
&\quad (2,0),(0,7),(1,3),(2,2)],[(0,3),(1,4),(2,9),(0,9),(1,2),(2,6)]\}\\
F_5^{10} &= \{[(0,0),(1,2),(2,2),(0,3),(1,5),(2,0),(0,9),(1,4),(2,8),(0,1),\\
&\quad (1,8),(2,4),(0,6),(1,3),(2,6),(0,4),(1,6),(2,5)],\\
&\quad [(0,2),(1,1),(2,7),(0,5),(1,0),(2,3),(0,2),(1,1),(2,7),(0,5),(1,0),(2,3)]\}\\
F_6^{10} &= \{[(0,0),(1,1),(2,8),(0,3),(1,0),(2,9),(0,1),(1,5),(2,7),(0,6),\\
&\quad (1,9),(2,2),(0,8),(1,2),(2,0),(0,4),(1,7),(2,6),(0,2),(1,4),\\
&\quad (2,1),(0,5),(1,8),(2,5),(0,9),(1,3),(2,3),(0,7),(1,6),(2,4)]\}\\
F_7^{10} &= \{[(0,0),(1,5),(2,1),(0,1),(1,0),(2,5),(0,5),(1,7),(2,4),(0,4),\\
&\quad (1,4),(2,2),(0,7),(1,2),(2,8),(0,8),(1,1),(2,6),(0,9),(1,8),\\
&\quad (2,3),(0,6),(1,6),(2,0),(0,2),(1,3),(2,9),(0,3),(1,9),(2,7)]\}\\
F_8^{10} &= \{[(0,0),(1,4),(2,5),(0,8),(1,3),(2,4),(0,9),(1,7),(2,7),(0,2),\\
&\quad (1,9),(2,0),(0,1),(1,2),(2,3),(0,3),(1,6),(2,1),(0,4),(1,8),\\
&\quad (2,8),(0,7),(1,5),(2,6),(0,6),(1,0),(2,2),(0,5),(1,1),(2,9)]\}\\
F_9^{10} &= \{[(0,0),(1,9),(2,8),(0,6),(1,2),(2,5),(0,2),(1,6),(2,6),(0,5),\\
&\quad (1,3),(2,7),(0,8),(1,4),(2,3),(0,9),(1,0),(2,1)],\\
&\quad [(0,1),(1,1),(2,0),(0,3),(1,7),(2,2)],[(0,4),(1,5),(2,4),(0,7),(1,8),(2,9)]\}
\end{aligned}
$$

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