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WARP Decompositions

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Abstract

The relation between Sen's decomposition of the weak axiom of revealed preferences (WARP), and WARP directional decomposition introduced in Nosratabadi (2017) is investigated. It is argued that the latter generalizes the former. A pairwise transitive WARP decomposition consistent with Sen's property α is given. It is also argued that the notion of *referential DM* can not be built upon Sen's properties.

JEL Classification: D11, D81.

Keywords: The Weak Axiom of Revealed Preferences, WARP Decomposition.

1 Introduction

The *weak axiom of revealed preferences* introduced in Samuelson (1938) is the central notion in the theory of revealed preference. Implications of WARP in social choice has also been investigated extensively. Sen's decomposition of WARP (Sen (1971)) is a well-known result. A new decomposition theorem has recently been provided in Nosratabadi (2017). The purpose of this paper is to investigate the connection between these WARP decompositions. It is shown that Sen's decompositions can be, indeed, strengthened using the directional rationalities in Nosratabadi (2017). Sen's property α resembles the *top-down rationality* in Nosratabadi (2017). Noting that α is indeed stronger than TDR it is shown that WARP is decomposable to TDR and β . On the other hand, Sen's property β is reminiscent of the *bottom-up rationality* in Nosratabadi (2017). However, it is shown that there is no logical relation between the two notion. Finally it is shown that a pairwise transitive WARP decomposition consistent with Sen's property α is also achievable.

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2 Preliminaries

Let X be a finite set. For $A \subseteq X$ let $\mathcal{P}(A) = 2^A$. Also let

$$\bar{\mathcal{P}}(A) := \mathcal{P}(X) \setminus A.$$

Next let

$$\mathcal{P}^{\geq k}(S) = \{A \in \mathcal{P}(S) : |A| \geq k\}.$$

A choice correspondence on X is a function $\mathbf{c} : \mathcal{P}^{\geq 2}(X) \rightarrow \mathcal{P}(X)$ such that $\emptyset \neq \mathbf{c}(A) \subseteq A$. A binary relation, R , on X is a subset of $X \times X$. R is said to be complete if for all $x, y \in X$ either xRy or yRx is true. R is said to be transitive if for x, y, z in X xRy and yRz implies xRz . Define

$$x \succsim^p y \iff x \in \mathbf{c}\{x, y\}.$$

2.1 The WARP

Following the notion for WARP in [Samuelson \(1938\)](#), [Houthakker \(1950\)](#) proposes the following statement for WARP which is simply based on observed choice data.

Houthakker's WARP: If $x, y \in A \cap B$, $x \in \mathbf{c}(A)$, and $y \in \mathbf{c}(B)$, then $x \in \mathbf{c}(B)$.

[Sen \(1971\)](#) provide a decomposition of WARP into the following properties.

Sen's Property α : If $x \in B \subseteq A$ and $x \in \mathbf{c}(A)$, then $x \in \mathbf{c}(B)$.

Sen's Property β : If $x, y \in \mathbf{c}(A)$, $A \subseteq B$ and $y \in \mathbf{c}(B)$, then $x \in \mathbf{c}(B)$.

[Nosratabadi \(2017\)](#) introduces a new decompositions of WARP in a directional sense, separating the horizontal rationalities from vertical ones. Following the notions in this paper, next I introduce this decomposition. The directional rationalities in this paper are defined in order to capture the referential effects. Here we look at a more general case where they are independent of references. To do this let

$$S^- = \{S \setminus \{x\} : x \in S\}$$

Definition 1. (Beating) Let $S \subseteq X$. For $x, y \in S$ we say x *beats* y in S whenever $x \in \mathbf{c}(S)$ and $y \notin \mathbf{c}(S)$.

Definition 2. (Dominance) Let $S \subseteq X$. We say x *dominates* y in S and we write

$$x \blacktriangleright_S y,$$

if there exists $\bar{A} \in S^-$ such that x beats y in \bar{A} , and there does not exist $A \in S^-$ such that y beats x in A .

Top-Down Rationality - TDR: We say S satisfies TDR if

$$x \in \mathbf{c}(S) \implies x \text{ is a maximal element of } \blacktriangleright_S.$$

We say a choice correspondence \mathbf{c} satisfies TDR on S if all $A \in \mathcal{P}^{\geq 3}(S)$ satisfy TDR. If $S = X$, then we simply say \mathbf{c} satisfies TDR.

Bottom-Up Rationality - BUR: We say S satisfies BUR if

$$x \text{ is a maximal element of } \blacktriangleright_S \implies x \in \mathbf{c}(S).$$

We say a choice correspondence \mathbf{c} satisfies BUR on S if all $A \in \mathcal{P}^{\geq 3}(S)$ satisfy BUR. If $S = X$, then we simply say \mathbf{c} satisfies BUR.

Pairwise Transitivity - PT: We say a choice correspondence \mathbf{c} satisfies PT on S if for all $x, y, z \in S$

$$x \succsim^p y \text{ and } y \succsim^p z \text{ implies } x \succsim^p z.$$

If $S = X$, then we simply say \mathbf{c} satisfies PT. Following [Nosratabadi \(2017\)](#) a “classical DM” (CDM) is defined as a choice correspondence that satisfies TDR, BUT, and PT. The next proposition, the proof of which is already provided in the literature, shows the equivalence of these WARP variations.

Proposition 1. *\mathbf{c} satisfies Houthakker’s WARP, if and only if \mathbf{c} satisfies Sen’s properties α and β , if and only if \mathbf{c} is a CDM.*

3 WARP Decompositions

In this section I investigate how Sen’s decomposition is related to that of in [Nosratabadi \(2017\)](#).

3.1 α and TDR

Property α resembles the TDR as it forces a consistency from a larger menu to a smaller one. The next proposition shows that, indeed, α is stronger than TDR.

Proposition 2. α implies TDR.

Proof. Assume that \mathbf{c} satisfies α . Take $S \subseteq X$ with $|S| \geq 3$. We need to show S satisfies TDR. Take $x \in \mathbf{c}(S)$. Then property α implies that $x \in \mathbf{c}(A)$ for all $A \in S^-$ such that $x \in A$. Therefore x is not dominated by any elements of S and therefore is a maximal element of S . \blacktriangleright \blacksquare

To see the converse of the statement in Proposition 2 is not true consider the following example.

Example 1.

$$\begin{aligned} & \{(\textcircled{x}), (\textcircled{y}), z, t\} \\ & \{(\textcircled{x}), y, z\}, \{x, (\textcircled{y}), t\}, \{(\textcircled{x}), z, r\}, \{(\textcircled{y}), z, r\} \\ & \{(\textcircled{x}), (\textcircled{y})\}, \{(\textcircled{y}), z\}, \{(\textcircled{x}), z\}, \{(\textcircled{x}), t\}, \{(\textcircled{y}), t\}, \{(\textcircled{z}), t\} \end{aligned}$$

The choice behavior above satisfies TDR, however it violates Sen's property α . For example, $y \in \{x, y, z\} \subset \{x, y, z, t\}$ and $y \in \mathbf{c}\{x, y, z, t\}$, however $y \notin \mathbf{c}\{x, y, z\}$.

Since TDR is weaker than α , one might conjecture that Sen's decomposition could be strengthened by replacing property α with TDR. This, indeed, turn out to be true and is presented in the following theorem.

Theorem 1. \mathbf{c} satisfies WARP if and only if it satisfies TDR and β .

Proof. (\implies): For the proof of $\text{WARP} \implies \text{TDR}$ see [Nosratabadi \(2017\)](#) Section 9.1.1 and for the proof of $\text{WARP} \implies \beta$ see [Sen \(1971\)](#).

(\impliedby): Assume \mathbf{c} satisfies TDR and β . Take $x, y \in A \cap B$ with $x \in \mathbf{c}(A)$ and $y \in \mathbf{c}(B)$. We must show $x \in \mathbf{c}(B)$. First note that TDR implies $x \sim^p y$ and therefore $\mathbf{c}\{x, y\} = \{x, y\}$. Next β implies $x \in \mathbf{c}(B)$. \blacksquare

Next I show the independence of these properties.

Example 2. Consider the following behaviors:

(i)

$$\begin{aligned} & \{\textcircled{x}, y, z\} \\ & \{\textcircled{x}, \textcircled{y}\}, \{\textcircled{x}, z\}, \{\textcircled{y}, z\} \end{aligned}$$

This behavior satisfies TDR. However it violates β . It obviously violates WARP.

(ii)

$$\begin{aligned} & \{\textcircled{x}, y, z\} \\ & \{x, \textcircled{y}\}, \{\textcircled{y}, z\}, \{\textcircled{x}, z\} \end{aligned}$$

This choice function satisfies β as no indifference is observed. However it violates TDR. It obviously violates WARP.

3.2 β and BUR

By the nature of the two rationales β and BUR one might think there is a logical relation between the two. However the following example shows such resemblance between the two concepts does not imply any logical relation.

Example 3. Consider the following examples:

(i)

$$\begin{aligned} & \{x, \textcircled{y}, \textcircled{z}\} \\ & \{\textcircled{x}, y\}, \{\textcircled{x}, z\}, \{\textcircled{y}, \textcircled{z}\} \end{aligned}$$

This behavior satisfies β . $y, z \in \mathbf{c}\{y, z\}$. Also $y, z \in \mathbf{c}\{x, y, z\}$. However it does not satisfy BUR. In particular, x is not dominated by either y or z , and therefore, is the maximal element of $\blacktriangleright_{\{x, y, z\}}$. But $x \notin \mathbf{c}\{x, y, z\}$.

(ii)

$$\begin{aligned} & \{\textcircled{x}, y, z\} \\ & \{\textcircled{x}, y\}, \{\textcircled{y}, z\}, \{\textcircled{x}, \textcircled{z}\} \end{aligned}$$

This behavior satisfies BUR. x is the maximal element of $\blacktriangleright_{\{x, y, z\}}$ and $x \in \mathbf{c}\{x, y, z\}$. However it violates property β . $z, x \in \mathbf{c}\{x, z\}$, $\{x, z\} \subset \{x, y, z\}$, $\{x, z\} \in \mathbf{c}\{x, y, z\}$, however $z \notin \mathbf{c}\{x, y, z\}$.

3.3 Relation to PT

Since Sen's properties decompose WARP they should imply PT. However, neither α nor β imply PT.

Example 4. Consider the following behaviors:

(i)

$$\begin{aligned} & \{(\textcircled{x}), y, z\} \\ & \{(\textcircled{x}), y\}, \{(\textcircled{y}), z\}, \{(\textcircled{x}), (\textcircled{z})\} \end{aligned}$$

This behavior satisfies property α , however it violates PT.

(ii)

$$\begin{aligned} & \{(\textcircled{x}), y, (\textcircled{z})\} \\ & \{(\textcircled{x}), y\}, \{(\textcircled{y}), z\}, \{(\textcircled{x}), (\textcircled{z})\} \end{aligned}$$

This behavior satisfies Property β , however it violates PT.

Theorem 1 provides a new WARP decomposition that contains property β . What about property α ? This next result shows that a pairwise transitive decomposition consistent with α is achievable. To do this let me introduce a new property β' .

Property β' : Let A_x be an arbitrary subset of S that contains x . We say \mathbf{c} satisfies β' if and only if

$$x \in A_x \text{ for all } A_x \in \bar{\mathcal{P}}(S) \text{ implies } x \in \mathbf{c}(S).$$

Theorem 2. \mathbf{c} satisfies WARP if and only if it satisfies α , PT, and β'

Proof. (\implies): We only need to show WARP implies β' . To do this assume $x \in \mathbf{c}(A_x)$ for all $A_x \in \bar{\mathcal{P}}(A)$. Next assume $y \in \mathbf{c}(A)$ then, since $x, y \in A_x$ for some A_x then WARP implies $x \in \mathbf{c}(A)$.

(\impliedby): Assume \mathbf{c} satisfies α , PT, and β' . Take $x, y \in A \cap B$ with $x \in \mathbf{c}(A)$ and $y \in \mathbf{c}(B)$. First note that α implies $x \in \mathbf{c}\{x, y\}$, and therefore $x \succsim^p y$. It also implies that $y \in \mathbf{c}\{y, z\}$ for all $z \in B$ and therefore $y \succsim^p z$ for all $z \in B$. The last two assertion along with PT imply $x \succsim^p z$ for all $z \in B$ and therefore $x \in \mathbf{c}\{x, z\}$ for all $z \in B$. Next We show that $x \in \mathbf{c}(B)$ by strong induction on $|B|$. For $B=3$ the result goes through since x is chosen on all doubletons containing elementals of B and \mathbf{c} satisfies β' . Next assume the statement is true for all B with cardinality less than or equal to $k - 1$ and assume $|B| = k$. Then β' implies that $x \in \mathbf{c}(A_x)$ for all $A_x \in \bar{\mathcal{P}}(B)$ and therefore we conclude $x \in \mathbf{c}(B)$. ■

Next example shows the independence of these three properties. e.

Example 5. Consider the following behaviors:

(i)

$$\begin{aligned} & \{\textcircled{x}, y, z\} \\ & \{\textcircled{x}, \textcircled{y}\}, \{\textcircled{x}, z\}, \{\textcircled{y}, z\} \end{aligned}$$

This behavior satisfies α and PT . However it violates β' . It obviously violates WARP.

(ii)

$$\begin{aligned} & \{\textcircled{x}, y, \textcircled{z}\} \\ & \{\textcircled{x}, y\}, \{\textcircled{y}, z\}, \{\textcircled{x}, z\} \end{aligned}$$

This behavior satisfies β' and PT . However it violates α . It obviously violates WARP.

(iii)

$$\begin{aligned} & \{x, \textcircled{y}, z\} \\ & \{x, \textcircled{y}\}, \{\textcircled{y}, \textcircled{z}\}, \{\textcircled{x}, z\} \end{aligned}$$

This behavior satisfies β' and α . However it violates PT . It obviously violates WARP.

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