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Working Paper

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Working Paper, No. 2017-07

Provided in Cooperation with:
Department of Economics, Rutgers University

Suggested Citation: Nosratabadi, Hassan (2017) : Referential revealed preference theory, Working Paper, No. 2017-07, Rutgers University, Department of Economics, New Brunswick, NJ

This Version is available at:
https://hdl.handle.net/10419/200263

Referential Revealed Preference Theory

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This Draft: October 16, 2017

Abstract

References can influence choice. One of the well-studied cases is when a *decoy* is added to the menu. However small in magnitude, decoy effect violates the *weak axiom of revealed preferences* (WARP). In order to explain the small deviation from the classical revealed preference theory, I decompose WARP into independent axioms in order to only remove the ones which are inconsistent with the this effect. This *minimal* deviation produces the referential revealed preference theory that keeps the strong predictive power offered in the classical theory while explaining the referential effect.

JEL Classification: D11, D81.
Keywords: Reference-Dependent Choice, Reference Preferences, Decoy Effect, Attraction Effect, WARP Decomposition.

1 Introduction

Experiments have documented that references can influence choice. In that, the presence of a third alternative can affect the relative ranking of a pair of alternatives. Among the different forms, the *decoy effect* is widely studied.[1] A decoy is an alternative which is obviously dominated by the *target* alternative, and, therefore, *attracts* the decision maker (DM) toward the target relative to a *competitor*. The following table presents the individual level experimental results from Huber, Payne, and Puto (1982).[2]

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[1]This effects is also referred to as *asymmetrically dominated effect* and *attraction effect* in the experimental literature.

[2]The experimental evidence for decoy effect is vast. Also see Doyl, O'Conner, Reynolds, and Bottomly (1999), and Tversky and Kahneman (1991).

	Choice After Addition of Decoy			
Pairwise Choice	Target	Competitor	Decoy	Total
Target	242 (44%)	40 (7%)[a]	8 (1%)	290 (53%)
Competitor	69 (12%)[b]	190 (34%)	3 (1%)	262 (47%)
Total	311 (56%)	230 (42%)	11 (2%)	552 (100%)

Table 1: Individual Choice Reversal Due to Addition of a Decoy

Note: Subjects can only choose one. Out of 109 choice reversals (cells a and b) observed in the experiment 63% were to the target and 37% to the competitor. The difference is statistically significant at $p \leq 0.05$. The share of the target increased by 3% after addition to the decoy.
Source: Huber et al. (1982).

The results of this experiment, arguably, violate the *weak axiom of revealed preference* (WARP). To see this, let us introduce the main components of the classical revealed preference theory. Let X be the set of all alternatives and $\boldsymbol{c}$ be a choice correspondence.

Revealed Preference ($\succsim^*$): x is revealed to be *at least as good as* y if there exists $S \subseteq X$ such that $x, y \in S$ and $x \in \boldsymbol{c}(S)$.

WARP: $\boldsymbol{c}$ satisfies WARP if and only if for all $A, B \subseteq X$ and all $x, y \in A \cap B$, $x \in \boldsymbol{c}(A)$ and $y \in \boldsymbol{c}(B)$ imply $x \in \boldsymbol{c}(B)$.

Under $\succsim^*$, a switch from the competitor to the target (or vice versa) in the presence of a decoy is an indication of indifference between the two. Note that a single such observation is not necessarily a violation of WARP as the indifferent DM randomizes between the two alternatives in the two menus. However, if such randomization is being used, the difference between the distribution of the choice over the target and the competitor should not be statistically significant within a plausible number of subjects. The increase in the share of the target due to adding the decoy, therefore, suggests that some DMs are not randomizing in both menus. One can hypothesize that the decoy is influencing the choice by breaking the indifference.[3] In that, some subjects are indifferent between the target and the competitor when faced a choice between only the two, but inclined toward the target *in the presence*

[3]Using the notion of $\succsim^*$, this phenomenon can not be referred to as *preference reversal*. Also the behavioral literature on this suggest that x is indeed preferred to y and the choice of y in the menu xyr is caused by a failure of paying complete *attention* to the most preference alternatives. For the theories on this interpretation see Masatlioglu, Nakajima, and Ozbay (2012) and Ok, Ortovela, and Riella (2015). Finally, it seems equally valid to assume that the DM prefers x over y in the menu xy and becomes indifferent when the decoy is presented in the menu. This later interpretation is not in line with the experimental arguments of the decoy effect and, therefore, is not discussed in this paper.

of the decoy. Under this hypothesis, the behavior of such DMs reveals preferences of the following form:

$$\text{target} \sim^{p} \text{competitor} \succ^{p} \textit{decoy},$$

$$\text{target} \succ^{\text{decoy}} \text{competitor},$$

where $\succsim^{p}$ and $\succsim^{\text{decoy}}$ represent *revealed pairwise preference* and *revealed reference relation*, respectively. A choice correspondence of the form presented in Figure 1 is consistent with such interpretation (circles represent choice):

Figure 1: Attraction Effect

$$\{\textcircled{x}, y, r_x\}$$
$$\{\textcircled{x}, \textcircled{y}\}, \{\textcircled{x}, r_x\}, \{\textcircled{y}, r_x\}$$

where x, y, and r_x are the target, competitor, and the decoy, respectively. This choice correspondence violates WARP; however, in a *weak sense*:[4] r is only *refining* $\succsim^{p}$, not *reversing* it. Since the violation is small in magnitude, and in order to keep the predictive power offered via WARP to the extent possible, I provide an axiomatic approach to decision making which *minimally* distances itself from WARP, only to explain behaviors of the type in this Figure 1. A natural way to address this minimal deviation is to decompose WARP into independent subrationalities in order to only remove the inconsistent parts. Keeping this motivation in mind, Figure 2 shows the same behavior of interest in a larger choice problem. *What WARP rationalities are being violated here?* Obviously, the choice still satisfies *pairwise transitivity* (PT), so the cause of this behavioral anomaly can not be associated with PT. Next we look at the way choice is formed in the triplets which is referred to as referential level in figure. First, consider the blue part which is a typical case of decoy effect. x, indeed, is one of the best alternative considering pairwise revealed preference. Therefore, there is a rational force that works its way from the referential level down to the pairwise level. To formalized this *top-down rationality* (TDR), we say an alternative a is *dominated* by b if, when considering the choice on one level lower, there is a submenu where b is chosen *over* a (b is chosen and a is not), and in no submenu a is chosen over b. Then TDR requires the choice in a set to be undominated by any other alternatives. This is true for x in the

[4]This weak violation is also confirmed by the experimental results. Obviously, and from the decision theoretical standpoint, evidence within the subjects are the ones we need to concentrate on. As shown in Table 1 the increase of the share of the target in Huber et al. (1982) is 3%. Between subjects increases are also small. In Huber et al. (1982) the distortion of WARP due to decoy effect between subjects ranges from 2% to 16% with the average of 9.2% across different treatments. Doyl et al. (1999)'s result on decoy effect also shows a range between 17% and 23%. For phantom decoys the reported effect in this paper is 16%.

blue part of Figure 2. x is not dominated in the menu xyz by either y or r, therefore this behavior is in line with TDR.

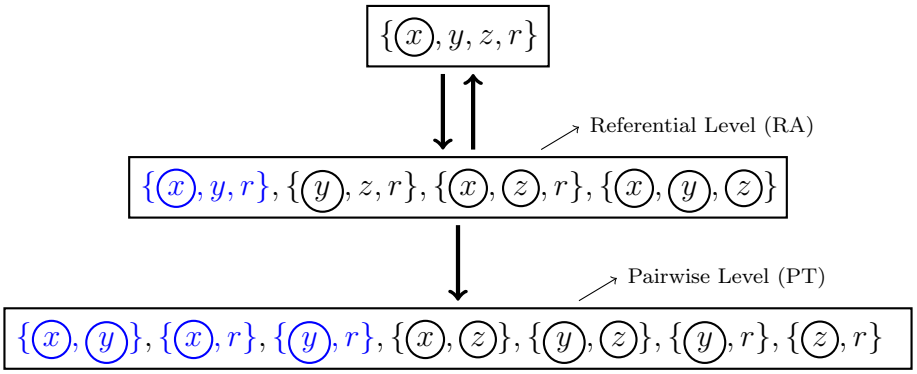

Figure 2: Referential DM - The Case of Decoy Effect

What other rationality is forced upon the WARP-minded DM? In order to answer this let us look at the version of example in Figure 2 that is consistent with the classical theory. This is shown in Figure 3. PT and TDR are still satisfied in this example. WARP, in addition, forces a rationality which works its way from the pairwise level up to the referential level. Using the notion of dominance, this *bottom-up rationality* requires the undominated alternatives to survive in the larger menu. The blue part in Figure 3, for example, shows that y is, indeed, not dominated by either x or r and therefore should be in the choice of the menu xyr. This bottom-up force does not exist in the case of decoy effect: *y does not survive in the menu xyr in the presence of decoy effect.* Therefore, BUR between the pairwise and referential level is eliminated in Figure 2.

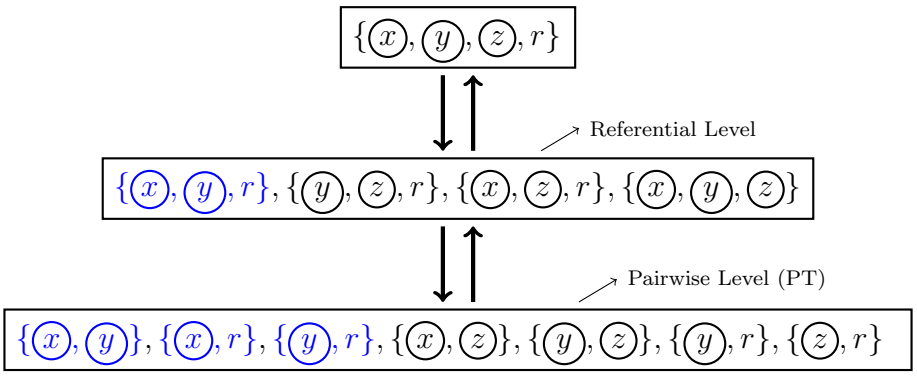

Figure 3: Classical DM - WARP Decomposition

PT carries a *horizontal* sense of rationality. On the other hand, TDR and BUR, working in opposite direction, capture a *vertical* sense of rationality. Intuitively, as a result, one might conjecture that they are independent. This, indeed, is be true. Furthermore, Theorem 1

shows that WARP is decomposable to these three independent rationalities.[5]

WARP decomposition helps to achieve the ambition of introducing a minimal deviation from the classical theory. That is we can extend the classical theory to include references if and only if we relax BUR on the referential level. Proposition 1 formalizes this result. The notion of *referential decision maker* is, naturally, defined as a DM who only violates the bottom-up rationality in a triplet via a reference point, and embodies the *classical decision maker* who fully submits WARP. As shown in Figure 2, referential revealed preference theory extends the classical theory to the case where the choice behavior of the DM is explicitly pinned down by her action over doubletons and triplets. This is because TDR and BUR are assumed to hold on levels above the referential level. In particular, and from the observations over doubletons and tripletons, the DM reveals a pairwise preference, $\succsim^p$, references, and reference relations. References are revealed wherever indifference is broken. A reference relation, $\succsim^r$, is also revealed by observing the relative ranking r induces on the referential level. In Figure 2, adopting to the reference point, r, the pairwise indifference, $x \sim^p y$, evolves to $x \succ^r y$. Theorem 2 shows that each reference relation is acyclic (abbreviated by RA in Figure 2), and therefore, attains a maximum. The main result of the paper, stated in Theorem 3, provides a full characterization of the choice of the referential DM. This result states that the choice in any arbitrary set consists of those element(s) that posses the maximum number of reference(s) declaring them as the most preferred alternative. Let me explore the implication of the main result in our example in Figure 2. The DM has the pairwise preference relation:

$$x \sim^p y \sim^p z \succ^p r.$$

Reference point r, however, revealed the following preference:

$$x \succ^r y \ , \ y \succ^r z \ , \ x \sim^r z.$$

It is revealed that $x \succ^r y$, and $y \succ^r z$. However, r fails to break the indifference between x and z; that is $x \sim^r z$. Even though $\succ^r$ is not a transitive reference preference, it does not produce any cycles, and therefore attains its maximum, x. This matches choice in the menu $xyzr$ since y, z, and r are all dominated alternatives. (y is dominated by x, z is dominated by y, and r is dominated by x.)

The notion of top-down and bottom-up rationalities is reminiscent in presentation to properties α and β in Sen (1971) decomposition of WARP. Section 7.2 argues that a referential

[5]The decomposition theorem, in fact, makes the nature of the *fundamental therm of revealed preferences* clear: As long as TDR and BUR are satisfied in all levels the choice in every level completely submits to the level before, and inductively, to the pairwise level.

DM does not satisfy either properties α or β, and, therefore, can not be built upon these properties.[6]

One approach to explain behavioral anomalies is to assume that a DM uses two preference in order to makes her choice. This is introduced in Manzini and Mariotti (2007) and is referred to as *rational short-list* method where the DM makes a short list via the first preferences and finalizes her choice via the second preference on that short list. Section 7.1 that shows that the referential revealed preference theory endogenizes the notion of rational short-list method. In that, the DM has a pairwise preference that is updated via a reference point.

1.1 Related Literature

WARP anomalies are not restricted to the mentioned effects and the body of the theoretical literature that addresses them is vast. These works include, but are not limited to, Shafir, Simonson, and Tversky (1993), Tversky and Simonson (1993), Masatlioglu and Ok (2005), Kalai, Rubinstein, and Spiegler (2002), Rubinstein and Salant (2006), Manzini and Mariotti (2007), Salant and Rubinstein (2008), Kamenica (2008), Lombardi (2009), de Clippel and Eliaz (2012), Masatlioglu et al. (2012), Bordalo, Gennaioli, and Shleifer (2013), Cherepanov, Feddersen, and Sandroni (2013), Apesteguia and Ballester (2013), Ok et al. (2015), Lleras, Masatlioglu, Nakajima, and Ozbay (2017), and Gerasimou (2016). My approach is distinguished from the existing literature via decomposition theorem that enables a minimal deviation from WARP. Therefore, just like the classical theory, choice is completely characterized in the referential revealed preference theory. In what follows I explore the relation of my results to those in the literature that carry the closest relations.

Manzini and Mariotti (2007) (and in a similar manner Apesteguia and Ballester (2013)) takes an interesting approach where a DM's choice behavior is rationalized using a *sequential procedure* with two preferences. These authors concentrate on cyclical choice, however, and, therefore, only consider choice functions. Nonetheless, the application of their model to choice correspondence can also explain the anomalies of the nature of menu effect. In this regard, I show that, interestingly, the *methodology* of sequential thinking is endogenously derived from my approach. The two preferences P_1 and P_2 in these author's work can essentially be considered as $\succsim^p$ and $\succsim^r$ for any reference r.

The notion of multiple rationales in Kalai et al. (2002) is closely connected to the reference-dependent model in Tversky and Kahneman (1991). These rationales can be thought of as reference preference revealed in my model. However, this work is an effort

[6] In a separate note I explore the way these properties relate to TDR and BUR. See Nosratabadi (2017).

to endogenize these reference preferences as opposed to their focus which is to find a minimal number of exogenously given rationales with which the choice behavior is rationalized. Similarly in Cherepanov et al. (2013) the concept of rationales are exogenously given. However, referential revealed preference theory produces results which are consistent with theirs. To see this once again, we can interpret the rationales as reference preferences in my model. Following their motivating example, assume that a decision maker who is choosing from the alternatives $\{x, y, z\}$ has the two following rationales (reference preferences): $x \succ^{r_1} y \succ^{r_1} z$ and $z \succ^{r_2} y \succ^{r_2} x$. As my model predicts y can not be the choice in the menu xyz as it is not a maximum element under any of the reference preferences. This matches the prediction in Cherepanov et al. (2013) as choosing y is not rationalizable with respect to any of the rationales regardless of the structure of DM's innate preference.[7]

A different approach is to introduce the behavioral notion of *constrained consideration*. Masatlioglu et al. (2012) (and similarly Lleras et al. (2017)) is a predominant paper in this category.[8] The main idea of the approach introduced in these papers is to add behavioral postulates to the analysis of choice. This is translated to an *attention filter* in Masatlioglu et al. (2012) (or the more general notion of consideration mapping in Lleras et al. (2017)) that limits the consideration set of the DM. The approach taken in Ok et al. (2015) is akin to that of the aforementioned paper. In these authors' work the consideration constraint is imposed via a *reference map* and an *attraction correspondence* that operates on the reference mapping. These features can be incorporated in my design if one thinks of a reference as the source of inattention and the set of maximum elements induced by it as the consideration set under that reference point. The idea of endogenous formation of references through weakening WARP is originated in Ok et al. (2015). Following their motivation and extending their results reference preferences are also endogenously derived in this work. Some predictions of the theory developed here, however, are different from these works. In particular, it is natural in my model that a DM who admits a pairwise preference relation (complete and transitive) is not unaware of her most preferred alternatives; that is, she is never inattentive towards a dominant alternative in favor of a dominated one.[9]

The structure of this paper is as follows: Section 2 provides notations and basic definitions to be used in this paper. Section 3 builds the axiomatic foundations for the decomposition theorem, which is stated in Section 4. In Section 5, I introduce the notions of classical and

[7]The concept of innate preference in Cherepanov et al. (2013) is a fixed preference which is only used if it matches a rationale, however, I distance myself from such concept and allow for breaking down indifference via references. This difference has not relevance in terms of theoretical ramifications of these two theories.

[8]In addition to attraction effect, Masatlioglu et al. (2012) is also concerned with cyclical choice and choosing pairwisely unchosen. Lleras et al. (2017) is concentrated on choice overload phenomenon.

[9]For a detailed discussion on this see Section 5.3.

referential DMs and explore some basic properties of the latter. The main results of the paper are presented in Section 6. Section 7 discusses main implications of the referential revealed preference theory. Finally, Section 8 provides a conclusion.

2 Preliminaries

Let X be a finite set. X is the set of all "relevant" alternatives for the DM. Therefore, it contains not only the concrete options available to the DM, but also, for example, alternatives that she has chosen before, or phantom alternatives that are not available to choose but presented to her (e.g., items that are out of stock, or shows that are sold out). In terms of the nature of the elements, X might be alternatives available for grocery shopping, different colleges to attend, various policies to be followed by the policy maker, etc. Let 2^X be the power set of X. Also let

$$\mathfrak{X}^k := \{A \subseteq X : |A| = k\};$$

that is the set of all subsets of X with cardinality equal to k, and

$$\mathfrak{X}^{\geq k} := \{A \subseteq X : |A| \geq k\};$$

that is the set of all subsets of X with cardinality of at least k. In order to simplify the domain of the discussion on choice I only consider the sets that have at least two elements, as the choice over the empty set and the singletons are trivially interpreted. A *choice correspondence* on X is a function $\boldsymbol{c} : \mathfrak{X}^{\geq 2} \to 2^X$ such that for all $A \in \mathfrak{X}^{\geq 2}$ we have $\boldsymbol{c}(A) \subseteq A$. $\boldsymbol{c}$ is called a *non-empty valued* choice correspondence if $\boldsymbol{c}(A) \neq \emptyset$ for all $A \in \mathfrak{X}^{\geq 2}$. We make the common notational abuses:

$$\boldsymbol{c}\{x, y, z\} := \boldsymbol{c}(\{x, y, z\}) \text{ and } \boldsymbol{c}\{x, y\} := \boldsymbol{c}(\{x, y\}),$$

for all $x, y, z \in X$.

For $S \subseteq X$, unless otherwise stated, whenever used throughout this paper let $S \in \mathfrak{X}^{\geq 3}$; that is let S have at least three elements. For $x \in S$ let $S - x := S \setminus \{x\}$; that is the set which is derived by removing x from S.

A binary relation R on X is a subset of $X \times X$. Let $\mathcal{R}$ be the asymmetric relation derived from R; that is

$$x \mathcal{R} y \iff xRy \text{ and } \neg(yRx).$$

A *cycle* of order k in R is a set $\{x_1, x_2, \ldots, x_k\}$ with $x_i \in X$ such that

$$x_1 \mathcal{R} x_2 \mathcal{R} \ldots \mathcal{R} x_k \mathcal{R} x_1.$$

R is said to be *acyclic* if it does not posses any cycle of any order. A *preference* relation on X is a binary relation which is transitive and complete. For a binary relation R on X, and $S \subseteq X$, x is called a *maximum* element of R on S if

$$xRy : \forall y \in S.$$

Let

$$\underset{S}{\operatorname{argmax}} R := \{x \in S : x \text{ is a maximum for } R \text{ on } S\};$$

x is called a *maximal* element of R on S if there does note exist $y \in A$ such that $y\mathcal{R}x$, where $\mathcal{R}$ is the asymmetric relation derived from R.

A *cover* for $S \subseteq X$ is a family of sets, $\{A_i\}_{i=1}^n$ such that $A_i \subseteq S$ for all i and

$$S = \bigcup_{i=1}^{n} A_i.$$

For a choice correspondence $\boldsymbol{c}$ define the relation $\succsim^p$ on X by

$$x \succsim^p y \iff x \in \boldsymbol{c}\{x, y\}.$$

Let $\succ^p$ and $\sim^p$ be asymmetric and symmetric parts of $\succsim^p$. Note that $\succsim^p$ matches the notion of revealed preference in the sense of Samuelson (1938). We call $\succsim^p$ the *pairwise revealed preference* throughout this paper. We next define the key notion of references.

Definition 1. (**References**) For a choice correspondence $\boldsymbol{c}$ and $S \subseteq X$ we say r is a *revealed reference*[10] in S if there exits two distinct elements $x, y \in S$, both different from r such that

$$\boldsymbol{c}\{x, y, r\} \subset \boldsymbol{c}\{x, y\}.^{11}.$$

Note that references are not chosen in sets where they operate; that is $r \notin \boldsymbol{c}\{x, y, r\}$ in the previous definition. Let

$$\mathfrak{R}(S) = \{r \in S : r \text{ is a reference in } S\};$$

[10]The term is originally introduced in Ok et al. (2015). For the sake of parsimony in writing, I will drop the term "revealed" for the rest of this paper.

[11]I am borrowing the definition of references from Ok et al. (2015).

that is the set of references in S.

Assume that a decision maker rationalizes her choice in a menu by her relative choice in submenus. Recall from the intuitive introductory argument that, wherever operational, reference are not elements of choice themselves and only relevant by breaking the indifference.[12] Therefore assume that submenus are derived by removing references from the original menu, one at a time, to the extent that referential effect is preserved; This creates a family of *first-order diminished* subsets that constitutes a cover for the original set. To see this in a formal setting, let

$$R_1(S) = \{S - x : x \in \mathfrak{R}(S)\}, \text{ and, } R_2(S) = \{S - x : x \in S \setminus \mathfrak{R}(S)\}.$$

R_1 is the family of first-order diminished subsets of S derived from removing references, one at a time, from S. R_2, in a similar fashion, is the family derived from removing non-reference elements. Next let

$$\mathfrak{B}(S) = \begin{cases} R_1(S) & S \text{ has at least two references} \\ R_2(S) & \text{Otherwise} \end{cases}$$

To explore the nature of $\mathfrak{B}(S)$ and why it forms a cover for S let us consider the following cases:

(i) If $\mathfrak{R}(S) = \emptyset$ then there is no reference in S and therefore $\mathfrak{B}(S) = R_2(S)$. To show that $\mathfrak{B}(S)$ is a cover take $x \in S$. Since $|S| \geq 3$ there exists $y \in S$ different from x and $S - y \in \mathfrak{B}(S)$. Obviously $x \in \boldsymbol{c}(S - y)$.

(ii) If $\mathfrak{R}(S) = \{r\}$ then $\mathfrak{B}(S) = R_2(S)$ and all elements of $\mathfrak{B}(S)$ are derived by taking an element different than r out of S; that is r is in all elements of $\mathfrak{B}(S)$. To show that $\mathfrak{B}(S)$ is a cover take $x \in S$. If $x = r$ then $x \in A$ for all $A \in \mathfrak{B}(S)$ and therefore $\mathfrak{B}(S)$ is a cover for S. So assume $x \neq r$. Since $|S| \geq 3$ it follows that there exists a non-reference element $y \in S$ and $S - y \in \mathfrak{B}(S)$. Obviously $x \in S - y$.

(iii) If $|\mathfrak{R}(S)| \geq 2$ then $\mathfrak{B}(S) = R_1(S)$. That is all elements of $\mathfrak{B}(S)$ are of the form $S - r$ for some reference r in S. To show that $\mathfrak{B}(S)$ is a cover for S take $x \in S$. If $x \in \mathfrak{R}(S)$ take a reference r different from x and note that $S - r \in \mathfrak{B}(S)$. Obviously $x \in S - r$. So assume x is not a reference in S. Take a reference r in S. It follows that $x \in S - r$.

As the argument above shows, the covers are built by removing the references from a menu, one at a time. If such procedure leads to blockage of the referential effect (that is when there

[12] This is naturally derived from my model. See Corollary 16 in Appendix.

is only one reference in the menu) the process is performed by removing non-referential elements. Obviously in the case of no references the cover contains all first-order diminished subsets.[13]

Definition 2. (**Beating**) Let $S \subseteq X$. For $x, y \in S$ we say x *beats* y in S whenever $x \in \boldsymbol{c}(S)$ and $y \notin \boldsymbol{c}(S)$.

Conceptually, "beating" captures the idea of *context-based* choice. For example assume that a DM who is indifferent between x, y choses x under the reference r_1. Then x beats y in the context $\{x, y, r_1\}$. On the other hand, and under the reference r_2, the DM might choose y in which case y beats x in the context of $\{x, y, r_2\}$. Therefore it is natural to assume that the DM is still not able to break the indifference between x, y if presented to both r_1 and r_2. Building on beating, I next a develop a stronger notion which is a criteria for the DM to judge the relative importance of two alternatives. We refer to this latter notion as *dominance*.

Definition 3. (**Dominance**) Let $S \subseteq X$. We say x *dominates* y *in* S and we write

$$x \underset{S}{\blacktriangleright} y,$$

if there exists $\bar{A} \in \mathfrak{B}(S)$ such that x beats y in $\bar{A}$, and there does not exist $A \in \mathfrak{B}(S)$ such that y beats x in A.

Note that dominance defines an asymmetric relation on S.[14] Domination is a key notion in WARP decomposition which is developed in Section 4. It delivers a criteria for a DM to eliminate an alternative. To clarify, if y is dominated by x, then it is never the case that DM choses y over x in all considered submenus. However, there is a submenu in her consideration where she choses x over y. Therefore DM can not rationalize choosing y using her relative choices on submenus as the criteria.

3 Choice Axioms

I start this section with a formal statement of WARP.

Axiom 0. (**Weak Axiom of Revealed Preferences - WARP**): We say a choice correspondence $\boldsymbol{c}$ satisfies WARP if for $S_1, S_2 \subseteq X$ such that $x, y \in S_1 \cap S_2$ we have

[13] The rationalizability motivation proposed here is particularly close to that of "divide and conquer" in Plott (1973). However, here, covers are not arbitrary and are specified via references.

[14] To see this assume that $x \underset{S}{\blacktriangleright} y$. Then for all $A \in \mathfrak{B}(S)$ we have y does not beat x in A. This means $y \underset{S}{\not\blacktriangleright} x$.

$$x \in \boldsymbol{c}(S_1) \text{ and } y \in \boldsymbol{c}(S_2) \text{ implies } x \in \boldsymbol{c}(S_2).$$

Axiom 1. (**Top-Down Rationality - TDR**) For all $S \in \mathfrak{X}^{\geq 3}$

$$x \in \boldsymbol{c}(S) \implies x \text{ is a maximal element of } \underset{S}{\blacktriangleright}.$$

Axiom 2. (**Bottom-Up Rationality - BUR**) For all $S \in \mathfrak{X}^{\geq 3}$

$$x \text{ is a maximal element of } \underset{S}{\blacktriangleright} \implies x \in \boldsymbol{c}(S).$$

Following the introductory discussion, TDR and BUR capture the notion of independence of irrelevant alternatives forced upon the DM via WARP. TDR requires the choice of the DM to be rationlizable with her choice on the submenus, where rationalizability is enforced via the notion of dominance. BUR, conversely, requires the rationlizable choice from the perspective of submenus, to survive in the larger menu.

The final ingredient of WARP decomposition is pairwise transitivity which is addressed in the following axiom.

Axiom 3. (**Pairwise Transitivity - PT**) $\succsim^p$ defines a transitive relation on X; that is,

$$x \succsim^p y \text{ and } y \succsim^p z \implies x \succsim^p z.$$

Note that PT implies transitivity of both $\succ^p$ and $\sim^p$. To conclude this section note that from the structure of my axioms it follows that if $\boldsymbol{c}$ satisfies TDR (resp. BUR, PT), then it satisfies TDR (resp. BUR, PT) on all $S \subseteq X$; that is these rationales are induced from larger menus to smaller ones.

4 Decomposition Theorem

Recall from the motivation that what distinguishes the axiomatic approach in this paper from those in the existing literature, is to extract axioms from WARP. This main innovation is achieved through the following theorem.

Theorem 1. *(**Decomposition Theorem**) Let $\boldsymbol{c}$ be a choice correspondence. Then the following are equivalent:*

(i) *$\boldsymbol{c}$ is non-empty valued and satisfies WARP*

(ii) *$\boldsymbol{c}\{x, y\} \neq \emptyset$ for all $x, y \in X$ and $\boldsymbol{c}$ satisfies TDR, BUR, and PT.*

The proof of Theorem 1 is presented in Section 9.1.1. Also, for independence of the axioms see Section 9.2.

5 Referential Decision Maker

For the remainder of this paper assume $\boldsymbol{c}$ is a *non-empty valued* choice correspondence. Let $S \subseteq X$. For the sake of parsimony I use the following notation: $\mathcal{I}^p(S) =: \underset{S}{\operatorname{argmax}} \succsim^p . \mathcal{I}^p(S)$ is, therefore, the set of best alternatives in S from the perspective of the pairwise revealed preference. S is called *fully indecisive* if $S \subseteq \mathcal{I}^p(S)$.

Definition 4. (**Classical DM: CDM**) A choice correspondence $\boldsymbol{c}$ is called a *CDM* on S if it satisfies TDR, BUR, and PT on S. If $S = X$, then we simply say $\boldsymbol{c}$ is a CDM.

Note that by decomposition theorem a CDM completely admits to WARP. Therefore the fundamental theorem of revealed preferences implies that the choice for CDM is completely pinned down by her transitive choice over doubletons. In the formal sense, if $\boldsymbol{c}$ is a CDM on S then

$$\boldsymbol{c}(S) = \mathcal{I}^p(S);$$

that is a CDM's choice is not dependent on *context*. Obviously such contextlessness is violated in phenomena like attraction effect. In order to capture these contextual effects I next introduce the notion of *referential* DM.

As discussed in the introduction, the main source of the behavioral anomalies of concern in this paper is the BUR on tripletons where a reference operates. Therefore we use the following axiom:

Axiom 4. (**Weak Bottom-Up Rationality - WBUR**) Let

$$\mathfrak{A} = \{A \in \mathfrak{X}^3 \text{ such that } A \text{ is not fully indecisive }\}.$$

For all $S \in \mathfrak{X}^{\geq 3} \setminus \mathfrak{A}$

$$x \text{ is a maximal element of } \underset{S}{\blacktriangleright} \implies x \in \boldsymbol{c}(S).$$

Definition 5. (**Referential DM: RDM**) A choice correspondence $\boldsymbol{c}$ is called a *RDM* on S if it satisfies PT, TDR, and WBUR.[15,16]

[15]Note that references are defined by breaking indifference. Therefore, if $\boldsymbol{c}$ is a choice function (that in the absence of indifference), then there are no references in X and Proposition 1 implies that a RDM is a CDM; that is the classical and referential revealed preference theories coincide in the absence of pariwise indifference.

[16]The notion of *path independence* in Plott (1973) is related to CDM and RDM. We say a choice corre-

5.1 RDM's Properties

Definition 6. A choice correspondence $\boldsymbol{c}$ is called a *pure* referential DM if it is a RDM but not a CDM.

This next proposition formalizes how the *minimal* deviation from WARP introduce via WBUR can generate references.

Proposition 1. *Let $\boldsymbol{c}$ be a RDM. Then*

$$\mathfrak{R}(S) \neq \emptyset \iff \boldsymbol{c} \text{ is a pure RDM}$$

The proofs of Proposition 1 is presented in Section 9.1.2. Note that $\boldsymbol{c}$ in this proposition has all the rationalities of a CDM except, possibly, BUR on referential tripletons. Proposition 1, therefore, simply states that the emergence of references in my model is *equivalent* to violation of BUR on referential tripletons. Such relaxation, as a result, guarantees the proportionate deviation from WARP.[17]

Note that it is obvious from the definitions that if $\boldsymbol{c}$ is a RDM (resp. CDM) then $\boldsymbol{c}$ is a RDM (resp. CDM) on all $S \subseteq X$. Next I explore some basic, yet critical, properties of a RDM. Recall from my motivational examples that references lie on indifference curves below the ones where they operate. One of the benefits of my approach via decomposition theorem is that such phenomenon is naturally derived from my setup. This next proposition, for which the proof is presented in Section 9.1.3, formalizes this observation.

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spondence $\boldsymbol{c}$ satisifies path independence if for all $S \subseteq X$ and two finite covers of S, v_1 and v_2, we have

$$c\Big(\bigcup_{v\in v_1} c(v)\Big) = c\Big(\bigcup_{v\in v_2} c(v)\Big).$$

To explore this relation first note that a CDM admits to WARP and therefore satisfies path independence. On the other hand, path independence does not imply WARP. To see this, consider the following example which is introduced in Plott (1973). Let $\boldsymbol{c}\{x,y,r\} = \{x,y\}, \boldsymbol{c}\{x,y\} = \{x,y\}, \boldsymbol{c}\{x,r\} = \{x,r\}$, and $\boldsymbol{c}\{y,r\} = \{y,r\}$. This choice structure satisfies path independence (see Plott (1973) for the proof), but it obviously violates WARP. There is no logical relation between RDM and a path independent choice structure however. To see this consider the following choice correspondence: $\boldsymbol{c}\{x,y,r\} = \{x\}, \boldsymbol{c}\{x,y\} = \{x,y\}, \boldsymbol{c}\{x,r\} = \{x\}$, and $\boldsymbol{c}\{y,r\} = \{y\}$. This is typical case of attraction effect and therefore consistent with RDM. However this choice correspondence does not satisfy path independence. To see this consider the two following covers: $v_1 = \big\{\{x,y,r\}\big\}$ and $v_2 = \big\{\{x,r\},\{y,r\}\big\}$. Then $\boldsymbol{c}\big(\bigcup_{v\in v_1} v\big) = \boldsymbol{c}\{x,y,r\} = \{x\}$ and $\boldsymbol{c}\big(\bigcup_{v\in v_2} v\big) = \boldsymbol{c}\big(\boldsymbol{c}\{x,r\}\cup\boldsymbol{c}\{y,r\}\big) = \boldsymbol{c}\{x,y\} = \{x,y\}$. To see that path independence does not imply RDM consider, again, the example from Plott (1973). Since $x \sim^p y \sim^p r$ we conclude that $\{x,y,r\}$ is fully indecisive. If $\boldsymbol{c}$ was a RDM then we would have $\boldsymbol{c}\{x,y,r\} = \{x,y,r\}$ which is not possible.

[17] Relaxing BUR on fully indecisive tripletons, in fact, captures another behavioral anomaly referred to as *compromise effect* in the literature. This effect pertains to references that act on the element from the same indifference curve. However consistent with my framework, explaining this effect is beyond the scope of this work.

Proposition 2. *Let $\boldsymbol{c}$ be a RDM. Also let $r \in \mathfrak{R}(S)$ and $x, y \in S$ such that*

$$\boldsymbol{c}\{x, y, r\} \subset \boldsymbol{c}\{x, y\}.$$

Then $x \sim^p y \succ^p r$.

Having defined both notions of CDM and RDM, an important question arises: *How is the choice behavior of RDM related to that of CDM?* As discussed before CDM's choice is independent of context and summarized by $\boldsymbol{c}(S) = \mathcal{I}^p(S)$. For RDM, on the other hand, such contextlessness is violated. Nonetheless, RDM still satisfies TDR. This leads to a natural conjecture: *RDM's choice must still be rationalizable by $\succsim^p$*. This is in fact true and formalized in the next proposition.

Proposition 3. *Let $\boldsymbol{c}$ be a RDM and $S \subseteq X$. Then*

$$\boldsymbol{c}(S) \subseteq \mathcal{I}^p(S).$$

The proofs of Proposition 3 is presented in Section 9.1.3. The intuition is quite simple. As long as DM is "capable" to reveal her rational preference in all pairs of alternatives, that is if $\succsim^p$ is a complete and transitive relation, then she is completely "aware" of all available options and the manner in which she ranks them. Therefore, there is no reason to ignore dominant alternatives in favor of dominated ones.[18]

5.2 Reference Relations

After observing the pairwise revealed preference ($\succsim^p$) from doubletons, the next step is to observe where indifferences are broken as a result of adding a third alternative. This revelation of references is then followed by observation in regards to the manner references rank other elements, in particular those in a highest indifference curve, giving birth to *revealed reference relations*.[19] This notion is introduced in the following definition.

Definition 7. (**Reference Relation**) Let $S \subseteq X$ and $r \in \mathfrak{R}(S)$. For two distinct element $x, y \in S$, both different from r, define

$$x \succsim^r y \iff x \in \boldsymbol{c}\{x, y, r\}.$$

Also let $\succ^r$ and $\sim^r$ be the asymmetric and symmetric parts of $\succsim^r$.

[18] This intuitive result, indeed, is key to the distinction between predictions of my rational approach and the existing behavioral treatments in the literature. I elaborate on this point in Section 5.3.

[19] As in the case of references, I drop the word "revealed" from the rest of this paper.

This next proposition explores the nature of the reference relations.

Proposition 4. *Let $\boldsymbol{c}$ be a RDM, $S \subseteq X$, and $r \in \mathfrak{R}(S)$. Then*

(i) $x \succsim^r y \implies x \succsim^p y$.

(ii) $\succsim^r$ *defines a complete binary relation on* $\mathcal{I}^p(S)$.

The proof of Proposition 4 is presented in Section 9.1.4 . Proposition 4.i speaks to the evolution of the pairwise revealed preference by adapting to a reference point r. To see this let us look at an example that shows the reverse direction may not hold. Let $S = \{x, y, r\}$ such that $\boldsymbol{c}\{x, y\} = \{x, y\}, \boldsymbol{c}\{x, r\} = \{x\}$, $\boldsymbol{c}\{y, r\} = \{y\}$, and $\boldsymbol{c}\{x, y, r\} = \{x\}$. This is typical case of attraction effect (or statues quo bias). $\boldsymbol{c}$ is a RDM on S, and $y \succsim^p x$, but $x \succ^r y$. Proposition 4.ii, on the other hand, guarantees one of the two essential characteristics of reference relations: *completeness*.[20] As I show in Section 6 reference relations are also *acyclic*, completing their characterization.[21]

5.3 Maximal References

Definition 8. We say $r \in S$ is a *maximal* reference if there exists $x, y \in \mathcal{I}^p(S)$ such that

$$\boldsymbol{c}\{x, y, r\} \subset \boldsymbol{c}\{x, y.\}$$

We also use the following notation:

$$\mathfrak{R}_M(S) := \{r \in S \text{ such that } r \text{ is a maximal reference in } S\}.$$

Definition 9. For a maximal reference r and $x \in S$ we say r *refers* x in S (or equivalently, r is a x-*maximal* reference) if

$$x \in \underset{\mathcal{I}^p(S)}{\operatorname{argmax}} \succsim^r .$$

We also use the the notation:

$$\mathfrak{R}^x_M(S) := \{r \in \mathfrak{R}_M(S) : r \text{ is a } \ x \text{ -maximal reference}\}.$$

We call $\mathfrak{R}^x_M(S)$ the *maximal reference set* of x in S.

[20] As discussed before for any arbitrary sets S, a RDM's behavior satisfies $\boldsymbol{c}(S) \subseteq \mathcal{I}^p(S)$ and, therefore, $\mathcal{I}^p(S)$ is the domain of relevance for $\succsim^r$. This intuition is in fact true; that is $\underset{S}{\operatorname{argmax}} \succsim^r = \underset{\mathcal{I}^p(S)}{\operatorname{argmax}} \succsim^r$. For the proof of this latter statement see footnote 33 in the Appendix.

[21] $\succsim^r$ are assumed to be transitive in Tversky and Kahneman (1991). However, for the purpose of maximization the weaker notion of acyclicity of references is sufficient.

Note that

$$\mathfrak{R}(S) \supseteq \mathfrak{R}_{M}(S) = \bigcup_{x \in \mathcal{I}^{p}(S)} \mathfrak{R}_{M}^{x}(S).$$

The difference between the maximal and non-maximal references is illustrated in Figure 4. In this figure, brackets represent indifference classes (indifference curves) and the most preferred class, $\mathcal{I}^{p}(S)$, is the one of the left containing x_1, x_2, and x_3. Recall from the motivational examples (and also from Proposition 2) that references are from lower indifference curves and therefore all referential effects (presented by both dashed and solid arrows) are from a class on the right to one on the left. x_7 is a reference that breaks down the indifference between x_4 and x_5 in favor of x_4, however, it does not affect the indifference between the elements of the most preferred class and therefore not a maximal reference (these are shown by dashed arrows). x_6, and x_4, on the other hand, are maximal references since they affect the indifference in $\mathcal{I}^{p}(S)$ (these are shown by solid arrows). In particular, $\{x_2\} = \underset{\mathcal{I}^{p}(S)}{\operatorname{argmax}} \succsim^{x_6}$, and $\{x_1, x_3\} = \underset{\mathcal{I}^{p}(S)}{\operatorname{argmax}} \succsim^{x_4}$. As shown in this figure, references can operate in more than one class. For example, x_6 breaks the indifference in both first and second class. Also note that even though x_5 is dominated from the preceptive of pairwise comparison, it is not a reference since there is no observation of indifference break-down for this element. Finally, and obviously from my setup, elements of $\mathcal{I}^{p}(S)$ do not induce referential effects. Note that the notion of referring is only defined for maximal references. In Figure 4, x_6 refers to x_2 and x_4 refers to x_1, and x_3.

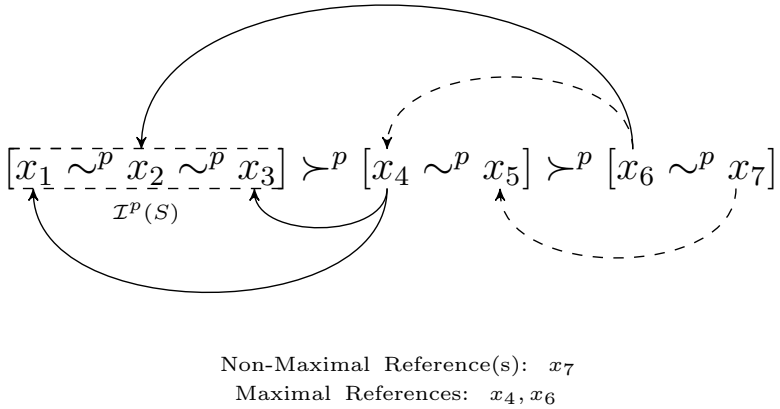


Figure 4: Maximal and Non-Maximal References

A natural question arises here: *why is the notion of referring only defined for maximal references?* This is purposeful, and indeed, speaks to the key difference that distinguishes the rational treatment in referential revealed preference theory from the behavioral approaches

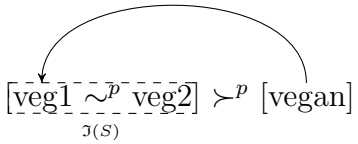
taken in the existing literature.[22] As I show in Section 6 (Theorem 3) the maximal references are the sole determinants of the RDM's behavior that goes beyond WARP. Indeed, recall from Proposition 3 that RDM's behavior satisfies $\boldsymbol{c}(S) \subseteq \mathcal{I}^p(S)$. As a result only those references that effect the relative ranking of elements of $\mathcal{I}^p(S)$ play a role in the characterization of choice.

Let me consider an example. Assume that a DM is choosing from a menu in a restaurant. The menu consists of five options: i. beef ribs with a side of lentil soup, ii. pork ribs with side that does not contain lentil, nor beans, iii. a vegetarian dish (veg1) containing both lentil and beans, iv. a vegetarian dish (veg2) that does not contain lentil, nor beans, and v. vegan dish that contains beans. Assume that price is of no concern for the DM. DM is an absolute meat lover who, even though does not mind vegetarian dishes, will not choose them over meat options. Assume the following pairwise preference:

$$[\text{beef ribs} \sim^p \text{pork ribs}] \succ^p [\text{veg1} \sim^p \text{veg 2}] \succ^p [\text{vegan}].$$

DM associates the variety of lentil and beans in the menu as a sign of chef's specialty and therefore veg1 and vegan dish acts as references. Now consider the two following scenarios:

Assume, in the first scenario, that the restaurant is out of ribs (both pork and beef). Then veg1 and veg2 are the most favored alternatives from the perspective of pairwise comparison. In this scenario, offering beans, vegan dish acts as a reference that effect the most favored class and therefore a maximal reference under which DM chooses veg1.

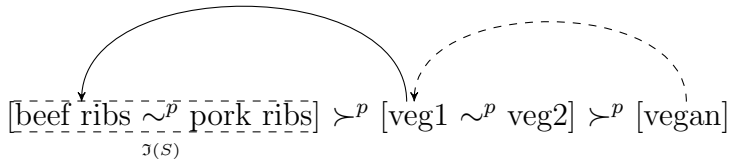


Scenario 1

In the second scenario, all options in the menu are available. In this case, offering lentil, veg1 also acts as a reference and the following referential effects are observed:

Veg1 is a maximal reference (solid arrow) that favors beef ribs over pork ribs. However vegan dish is not a maximal reference anymore (dashed arrow) as it fails to effect the indifference between the most preferred options. The choice of veg1 in this scenario is consistent if an arbitrary notion of "inattention" (or attraction) is employed. In words, a DM's might use the vegan dish as a reference and become inattentive towards the dominant options of

[22] See, for example, Masatlioglu et al. (2012), Ok et al. (2015), and Lleras et al. (2017). Masatlioglu et al. (2012), Lleras et al. (2017) only consider choice functions. Here I consider the implications of their approach on choice correspondence.



Scenario 2

meat, and therefore choose the dominated alternative veg1. However, such prediction is not consistent with the referential revealed preference theory; that is WARP deviations are only caused by maximal references and, therefore, the consistent choice in this scenario is beef ribs. In words, a RDM who admits to a complete and transitive preference relation on the entire set of alternative is not "irrational" in her inattention.[23]

6 Main Results

Now I can proceed to the main results of the paper. Recall form Proposition 4.ii that $\succsim^r$ is a complete relation on $\mathcal{I}^p(S)$. The first result completes the characterization of reference relations asserting that they are also acyclical.[24]

Theorem 2. *(**Acyclicity of Reference Relations**) Let $\boldsymbol{c}$ be a RDM and $r \in \mathfrak{R}(S)$. $\succsim^r$ defines an acyclic relation on $\mathcal{I}^p(S)$.*

The proof of Theorem 2 is presented in Section 9.1.6. The final theorem in this paper is a characterization of choice on an arbitrary set with arbitrary number of references.

Theorem 3. *Let $\boldsymbol{c}$ be a RDM and $S \subseteq X$. Then $x \in \boldsymbol{c}(S)$ if and only if $x \in \mathcal{I}^p(S)$ and for all $t \in \mathcal{I}^p(S)$, the number of x-maximal references in S is greater than or equal to the number of t-maximal references in S.*

Proof. We need to show that

$$x \in \boldsymbol{c}(S) \iff x \in \mathcal{I}^p(S), \text{ and } |\mathfrak{R}^x_{_M}(S)| \geq |\mathfrak{R}^t_{_M}(S)| \text{ for all } t \in \mathcal{I}^p(S).$$

[23] To explore this distinction in more detail, let me employ the jargon of inattention and say veg1 attracts attention to beef ribs, and vegan dish attracts attention to veg1. Then DM chooses to be *rational* in her inattention by only using those references that affect her most preferred alternatives. The aforementioned interpretation is, in a sense, in line with the talking point of the *rational inattention* models introduced in Sims (2003). These models, however, are indistinguishable from the processing capacity constrain (Shannon entropy) under which the choice is made. This constraint is lacking in the current work, and therefore, to avoid confusion, I keep the jargon of this work akin to that of classical revealed preference theory.

[24] Reference acyclicity (RA) also appears in Ok et al. (2015). Two things are important to note here. The notion in these authors' paper refers to the manner references operates on each other and, therefore, different from the acyclic "revealed binary relation" induced here. Indeed, the two notions are not logically nested. Second, RA is an axiom in Ok et al. (2015) and a result in this paper.

I start with following lemmas the proof of which, in order to keep the flow of the argument, are presented in Section 9.1.7.

Lemma 10. *Let $\boldsymbol{c}$ be a RDM and $S \subseteq X$ that possesses at least two references. There exists $\bar{r} \in \mathfrak{R}(S)$ such that $x \in \boldsymbol{c}(S - \bar{r})$.*

Lemma 11. *Let $\boldsymbol{c}$ be a RDM and $S \subseteq X$ that possesses at least two maximal references. For any reference r in S assume*

$$t^* \in \boldsymbol{c}(S - r) \iff |\mathfrak{R}^{t^*}_{M}(S - r)| \geq |\mathfrak{R}^{t}_{M}(S - r)|, \text{ for all } t \in \mathcal{I}^p(S - r.)$$

Also assume that $x \in \boldsymbol{c}(S)$ and $x, y \in \boldsymbol{c}(S - \bar{r})$ for a reference $\bar{r}$ in S. If $\bar{r}$ refers to y then it refers to x.

We now, start the proof of the theorem by considering two cases.

Case 1: $\mathfrak{R}_{M}(S) = \emptyset$.

Note that in this case all elements in $\mathcal{I}^p(S)$ posses zero maximal references. Take $x \in \boldsymbol{c}(S)$. From Proposition 3 we have $x \in \mathcal{I}^p(S)$. Also, in S, the number of x-maximal references (zero) is greater than or equal to the number of t-maximal references (zero), for all $t \in \mathcal{I}^p(S)$ which completes $\Rightarrow$ direction of the proof. Next assume $x \in \mathcal{I}^p(S)$ (and note that it is true in case that, in S, the number of x-maximal references is greater than or equal to the number of t-maximal references, for all $t \in \mathcal{I}^p(S)$). It directly follows from Lemma 17 that $x \in \boldsymbol{c}(S)$. This completes the proof of $\Leftarrow$ and, hence, the proof of the statement for Case 1.

Case 2: $\mathfrak{R}_{M}(S) \neq \emptyset$

Assume S at least has one maximal reference. We prove the statement by induction on $|\mathfrak{R}_{M}(S)|$. Note that since the set of maximal references in S is non-empty we conclude the set of references in S is also non-empty.

Induction Base: Let $\mathfrak{R}_{M}(S) = \{r\}$.

($\Rightarrow$): Let $x \in \boldsymbol{c}(S)$. Since r is the unique maximal reference in S Lemma 20 implies $x \in \underset{\mathcal{I}^p(S)}{\operatorname{argmax}} \succsim^{r}$. First note $x \in \mathcal{I}^p(S)$. Also, r, as the single maximal reference in S, refers

to x and the result follows.

($\Leftarrow$): Now assume $x \in \mathcal{I}^p(S)$ and

$$|\mathfrak{R}^x_{_M}(S)| \geq |\mathfrak{R}^t_{_M}(S)| \text{ for all } t \in \mathcal{I}^p(S).$$

We must show $x \in \boldsymbol{c}(S)$. Note that since r is a maximal reference in S Corollary 21 implies that $\underset{\mathcal{I}^p(S)}{\operatorname{argmax}} \succsim^r \neq \emptyset$. So take $z \in \underset{\mathcal{I}^p(S)}{\operatorname{argmax}} \succsim^r$. This means r refers to z. Since the number of x-maximal references in S is greater or equal to the number of z-maximal references, and S only possesses a single maximal reference, r, we conclude r refers to x. Therefore $x \in \underset{\mathcal{I}^p(S)}{\operatorname{argmax}} \succsim^r$. Note that $\boldsymbol{c}$ satisfies the conditions in Lemma 20 and we conclude from $\Leftarrow$ of that theorem that $x \in \boldsymbol{c}(S)$.

Induction Hypothesis: Assume that the statement is true for $A \subseteq X$ with $|\mathfrak{R}_{_M}(A)| = k$ and let $|\mathfrak{R}(S)| = k+1$. Note that we have $|\mathfrak{R}(S)| \geq |\mathfrak{R}_{_M}(S)| \geq 2$ in our induction. Therefore we have $\mathfrak{B}(S) = R_1(S)$. That is all elements of $\mathfrak{B}(S)$ are of the form $S - r$ for some $r \in \mathfrak{R}(S)$.

($\Rightarrow$): Let $x \in \boldsymbol{c}(S)$. Note that Lemma 15 implies $x \in \mathcal{I}^p(S)$. From Lemma 10 we conclude $x \in \boldsymbol{c}(S - \bar{r})$ for some $\bar{r} \in \mathfrak{R}(S)$.

Recall that we must show that, in S, the number of x-maximal references is greater than or equal to the number of t-maximal references, for all $t \in \mathcal{I}^p(S)$. Consider the set $S - \bar{r}$. Note that $\mathcal{I}^p(S - \bar{r}) = \mathcal{I}^p(S)$. Take $y \in \mathcal{I}^p(S) = \mathcal{I}^p(S - \bar{r}) \subseteq S - \bar{r}$. We consider two cases here:

Case 1: $y \notin \boldsymbol{c}(S - \bar{r})$.

Since $x \in \boldsymbol{c}(S - \bar{r})$ and $y \notin \boldsymbol{c}(S - \bar{r})$ then induction hypothesis implies that

$$|\mathfrak{R}^x_{_M}(S - \bar{r})| > |\mathfrak{R}^y_{_M}(S - \bar{r})|,$$

and therefore adding $\bar{r}$ to the set $S - \bar{r}$ does not increase the number of y-maximal references over the number of x-maximal references; that is,

$$|\mathfrak{R}^x_{_M}(S)| \geq |\mathfrak{R}^y_{_M}(S)|,$$

and the proof in this case is complete.

Case 2: $y \in \boldsymbol{c}(S - \bar{r})$.

Note that since $x, y \in \boldsymbol{c}(S - \bar{r})$ induction hypothesis implies

$$|\mathfrak{R}^x_{_M}(S - \bar{r})| = |\mathfrak{R}^y_{_M}(S - \bar{r})|.$$

Note that all the conditions of Lemma 19 are met (considering induction hypothesis) and therefore we conclude if $\bar{r}$ refers to y then it also refers to x. This means adding $\bar{r}$ to the set $S - \bar{r}$ does not increase the number of y-maximal references over the number of x-maximal references; that is,

$$|\mathfrak{R}^x_{_M}(S)| \geq |\mathfrak{R}^y_{_M}(S)|.$$

This complete the proof in this case and, in turn, the proof of $\Rightarrow$ of the theorem.

($\Leftarrow$): Take $x \in \mathcal{I}^p(S)$ and assume that

$$|\mathfrak{R}^x_{_M}(S) \geq |\mathfrak{R}^t_{_M}(S)|, \text{ for all } t \in \mathcal{I}^p(S). \tag{1}$$

We must show $x \in c(S)$. First note in our induction there are at least two references in S and therefore $|S| \geq 4$[25] and, therefore, S satisfies BUR. s a result in order to show $x \in \boldsymbol{c}(S)$ we prove that there does not exist $y \in S$ such that $y \underset{S}{\blacktriangleright} x$; that is we show x is a maximal element of $\underset{S}{\blacktriangleright}$ on S. To do this first recall that all elements of $\mathfrak{B}(S)$ are of the form $S - r$ for some reference r is S. Therefore assume that there exist $y \in S$ and $r^* \in \mathfrak{R}(S)$ such that y beats x in $S - r^*$. In particular $y \in \boldsymbol{c}(S - r^*)$ and therefore by Proposition 3 $y \in \mathcal{I}^p(S - r^*) = \mathcal{I}^p(S)$. Note that since in our induction $|\mathfrak{R}_{_M}(S)| \geq 2$ it follows that there exists a maximal reference in $S - r^*$. Therefore induction hypothesis applies to this set and we conclude

$$|\mathfrak{R}^y_{_M}(S - r^*)| > |\mathfrak{R}^x_{_M}(S - r^*)|.$$

First note that this along with (1) implies $|\mathfrak{R}^x_{_M}(S)| = |\mathfrak{R}^y_{_M}(S)|$. Second note that since dropping r^* lowers the number of maximal references of x relative to y it has to be the case that r^* refers to x and r^* does not refer to y. Since the number of maximal references that

[25] To see this note that since r is a maximal reference in S there exists two distinct elements, t_1, t_2 in $\mathcal{I}^p(S)$, both different from r. Next since $|\mathfrak{R}(S)| \geq 2$ there exists $s \neq r$ in S. Since s is a reference in S it follows from Proposition 6. iv that $s \notin \mathcal{I}^p(S)$. This in turn means $t_1, t_2 \succ^p s$ and there for t_1, t_2 are different from s. Therefore we conclude t_1, t_2, r, s are four distinct elements. This means $|S| \geq 4$.

refer to x in S is equal to the number of maximal references that refer to y there must exist a maximal reference r^{**} such that r^{**} refers to y, and r^{**} does not refer to x. To complete the proof of this direction consider the set $S - r^{**}$. Since x, y have the equal number of maximal references in S dropping r^{**} from S should lower the number of y-maximal references below the number of x-maximal references; that is

$$|\mathfrak{R}^x_M(S - r^{**})| > |\mathfrak{R}^y_M(S - r^{**})|$$

.

By induction assumption this means x beats y in $S - r^{**}$. Therefore x is not dominated by y and BUR implies that $x \in \boldsymbol{c}(S)$. This completes this direction of the proof. ■

7 Discussion

7.1 Interpretation of the Main Results

Some implication of the final theorem are important to note. First, if there are no maximal references in the set S then the result formulated in Theorem 3 is simplified to

$$\boldsymbol{c}(S) = \mathcal{I}^p(S);$$

that is, in the absence of maximal references the referential revealed preference theory coincides with the classical revealed preference theory. As we discussed in Section 5.3 this is a key distinction between the rational treatment in this paper and the behavioral ones in the existing literature.

Second, assume that r is the unique maximal reference in the set S. Then from Theorem 3 the choice is characterized by those elements of $\mathcal{I}^p(S)$ to which r refers. That is:

$$\boldsymbol{c}(S) = \operatorname*{argmax}_{\mathcal{I}^p(S)} \succsim^r = \operatorname*{argmax}_{\operatorname{argmax}_S \succsim^p} \succsim^r .$$

That is the choice in the set S is rationalized by, first, applying $\succsim^p$ on S, and second, applying $\succsim^r$ on the resulting set. This is exactly the *rational shortlist method (RSM)* in Manzini and Mariotti (2007) applied to choice correspondence; that is, the *methodology* of sequential rationalizability arises, endogenously, in the referential revealed preference theory.

Third, it is implied from the result in Theorem 3 that opposing references *nullify* the referential effect. To clarify, assume that a DM who is indifferent between two items x, y when confronted with the menu xy, prefers x over y when a reference r_1 is introduced into

the menu, and y over x when an opposing reference r_2 is added. Therefore r_1, and r_2 are opposing maximal references that refer to x and y, respectively, and Theorem 3 predicts that DM's choice on the menu $xyzr_1r_2$ should be xy; that is, she acts as if there were no references in the menu. This intuitive prediction is supported by the results in Teppan and Felfering (2012).

7.2 Sen's Decomposition and RDM

In terms of presentation Sen's Dcomposition (Sen (1971)) is reminiscent to the decomposition provided in this paper. In this section I provide an example to show that the notion of RDM can not be constructed via Sen's Decomposition. In order to do this let me introduced Sen's decomposition:

Sen's Property α: If $x \in B \subseteq A$ and $x \in \boldsymbol{c}(A)$, then $x \in \boldsymbol{c}(B)$.
Sen's Property β: If $x, y \in \boldsymbol{c}(A)$, $A \subseteq B$ and $y \in \boldsymbol{c}(B)$, then $x \in \boldsymbol{c}(B)$.

Sen (1971) proves that WARP is decomposable to properties α and β. The following example shows that a RDM need not satisfy either of these properties.

Example 1. Consider the following choice correspondence where circles represent choice.

$$\{\textcircled{x}, \textcircled{y}, r_1, r_2\}$$
$$\{\textcircled{x}, y, r_1\}, \{x, \textcircled{y}, r_2\}, \{\textcircled{x}, r_1, r_2\}, \{\textcircled{y}, r_1, r_2\}$$
$$\{\textcircled{x}, \textcircled{y}\}, \{\textcircled{x}, r_1\}, \{\textcircled{y}, r_1\}, \{\textcircled{x}, r_2\}\ \{\textcircled{y}, r_2\}, \{\textcircled{r_1}, r_2\}$$

This choice correspondence satisfies TDR and PT, and BUR above the referential level, and therefore is a RDM. However, it violates β. The violation of β is obviously a result of referential effect where indifference is getting resolved via a reference. For example, $y \in \boldsymbol{c}\{x, y, \}$ and $\{x, y\} \subseteq \{x, y, r_1\}$, but $y \notin \boldsymbol{c}\{x, y, r_1\}$. It also violates α since one would expect the opposing referential effects to *cancel out* each other as results in Teppan and Felfering (2012) suggests. For example, since r_1 breaks the indifference in favor of x and r_2 does in favor of y the referential effect disappears when both decoys are introduced in the menu. Under α such nullification is impossible.

7.3 Context-Based Decision Making and Welfare Analysis

The referential revealed preference theory, as one probably expects, produces testable predictions. One major precondition is the notion of "best in one attribute" vs. "better in more

attributes". To see that assume that a DM who is indifferent between the four alternatives x_1, x_2, x_3, x_4 also faces some less favorite alternatives r_1, r_2, r_3 that have features she likes. Therefore she has the following pairwise preference relation:

$$\left[x_1 \sim^p x_2 \sim^p x_3 \sim^p x_4\right] \succ^p r_1 \succ^p r_2 \succ^p r_3$$

In particular, assume that, r_1, r_2, and r_3 present favorite color, style, and the discount offered, respectively. Obviously the prediction here is that she will choose an alternative from $\{x_1, x_2, x_3, x_4\}$. Assume from each reference point the following ranking is observed:

$$r_1 \text{ (Color)} : x_1 \succ^{r_1} x_2 \succ^{r_1} x_3 \succ^{r_1} x_4$$

$$r_1 \text{ (Style)} : x_2 \succ^{r_2} x_3 \succ^{r_2} x_4 \succ^{r_2} x_1,$$

$$r_1 \text{ (Dicsount)} : x_3 \succ^{r_3} x_2 \succ^{r_3} x_4 \succ^{r_1} x_1.$$

Therefore x_1, x_2, and x_3 all have a reference that puts them at the most preferred alternative. x_4 does note have such support from any reference. From the point of view the theory in this paper, a switch from x_4 to any alternatives in the set $\{x_1, x_2, x_3\}$ is welfare improving. However when only compared to x_1, x_4 is indeed better in both style and the offered discount, and x_1 is only favorite in color. That is in the absence of alternatives x_2, x_3, switching from x_1 to x_4, indeed, increases welfare.

The notion of revealed preferences developed in Bernheim and Rangel (2009) has some relations to the referential revealed preference in terms of welfare analysis. However, the theory in this paper is also concerned with keeping the predictive power of the classical theory to the extent possible. If I interpret the "ancillary condition" in these authors' paper as the referential effect then we can explore the relation. To make this formal, assume that the ancillary condition are captured by the referential effect introduce via a references that breaks the indifference between any arbitrary pair x, y. Next define a variation of Bernheim and Rangel (2009)'s *strict unambiguous preference relation*, P^*, in the following manner:

$$xP^*y \text{ if and only if for all } \{x, y, r_i\} \text{ we have } y \notin \boldsymbol{c}\{x, y, r_i\}.$$

Under such case, and for all $S \subseteq X$ with $x, y \in S$ there are no maximal reference that refer to y (since $x \succ^{r_i} y$ for all i), and therefore Theorem 3 implies for all $S \subseteq X$, with $x, y \in S$ we have

$$xP^*y \implies y \notin \boldsymbol{c}(S).$$

That is if x is strictly unambiguously chosen over y in via all references then x is strictly unambiguously chosen over y in all possible set of alternatives that contain both x, y. However such prediction is not necessarily true for the *weak unambiguous choice* relation, R'. Using the same interpretation this a variation of this revealed preference relation could be defined as:

$$xR'y \text{ if and only if for all } \{x, y, r_i\},\ y \in \boldsymbol{c}\{x, y, r_i\} \text{ implies } x \in \boldsymbol{c}\{x, y, r_i\}.$$

Now consider the following example, where a DM has the follwoign pairwise preference:

$$x \sim^p y \sim^p z \succ^p r_1 \succ^p r_2,$$

with the following acyclic reference relations:

$$r_1 : x \sim^{r_1} y, y \succ^{r_1} z, z \succ^{r_1} x,$$

$$r_2 : x \sim^{r_2} y, y \succ^{r_2} z, z \succ^{r_2} x.$$

It is true that $x, y \in \boldsymbol{c}\{x, y, r_i\}$ for $i \in \{1, 2\}$. However, Theorem 3 implies

$$\boldsymbol{c}\{x, y, z, r_1, r_2\} = \{y\};$$

that is we can not conclude x is "weakly unambiguously chosen over y" in all possible scenarios, if it indeed does so via any possible references. Therefore, for $S \subseteq X$ with $x, y \in S$ and $xR'y$ then:

$$y \in \boldsymbol{c}(S) \not\implies x \in \boldsymbol{c}(S).$$

7.4 Referential Revealed Preference Theory And Other Behavioral Anomalies

In this section I discuss how the theory developed in this paper can be used in order to address other behavioral anomalies in the literature. Another behavioral anomaly which has been experimentally documented is the tendency to retain status quo. Assume that a DM, to start, is indifferent between alternatives x and y. In period 1, she is given one of these alternatives, and then makes a choice from the two in period 2. Figure 5 depicts the choice scenario in

period 2. The main idea in status quo bias is that the initial *endowments* produce biases that break indifference.[26] Assume that in period 1 the decision maker is given the alternative x. A risk averse DM who assigns higher weights to losses than gains will favor x in period 2 to retain her status quo. Let $\succsim^p$ denote the preference derived from the pairwise comparison in period 1 (the one that matches the revealed preference theory) and (a, t_i) denote the alternative a in time i. Then $(x, t_2) \sim^p (y, t_2)$, but $(x, t_2) \succ^{(x,t_1)} (y, t_2)$. Obviously both (x, t_2) and (y, t_2) dominate (x, t_1) in period 2 as the consumption in period 1 is not available anymore. The same argument works in the reverse direction when the initial endowment is y.[27] Therefore, under such interpretation, referential revealed preference theory also explain this effect. The significance of this effect has also been vastly documented in the literature. See, for example, Samuelson and Zeckhauser (1988), Tversky and Kahneman (1991), Knetsch and Sinden (1984), and Knetsch (1989).

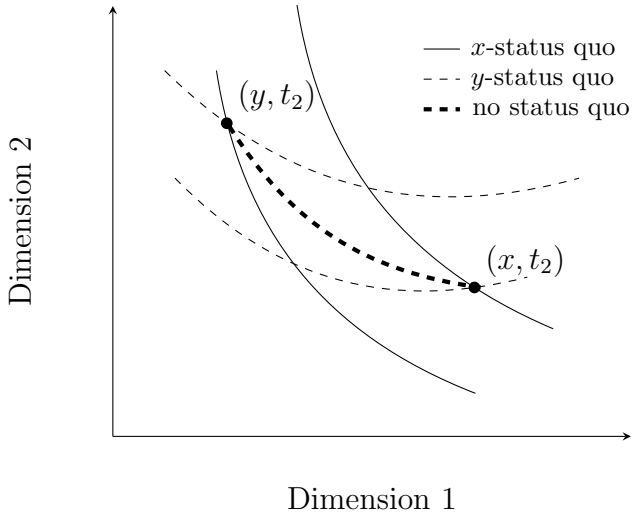


Figure 5: Status-Que Bias (or Endowment Effect)

As mentioned in Footnote 17 relaxing BUR on fully indecisive triplets will explain the compromise effect. In order to be explain choosing pairwise unchosen phenomenon one could, along with BUR, relax TDR from the referential level as well. The framework here, because of PT, is not consistent with the cyclical choice. However, one could think of an extension of the results for the case when weaker notions of transitivity are assumed at the pairwise level, eg. pseudotransitivity, or acyclicity.

[26] This effect and its twin, *endowment effect*, are essentially the same. For a detailed discussion on this see Tversky and Kahneman (1991).

[27] Our approach to status quo bias is similar to that of *phantom* alternatives. A phantom alternative is an alternative which, for example because of being out of stock, is in the consideration set of the DM but can not be chosen. The significance of phantom alternatives are documented in the experimental works. See for example Doyl et al. (1999) for the case of phantom decoys.

8 Conclusion

In this paper, I develop "revealed preference"-type theory (in the sense of Samuelson (1938)) that is consistent with the decoy effect. The main innovation of this paper is that I search for a minimal deviation from WARP (that is one which is proportionate to the extend WARP is violated in the data). This is done using a decomposition theorem of WARP. I extract the axiomatic approach from WARP. Next I track those WARP-rationals which are the sole reason of the inability of the classical theory in explaining behavioral anomalies such as attraction effect. Removing these rationals from the classical DM, naturally, gives birth to the notion of the referential DM. I show that the rational treatment in this paper preserves the predictive power of the classical theory to the extent possible; that is, RDM's choice behavior is completely characterized by the observations on doubletons and triplets (as compared to the classical revealed preference theory where such characterization is made with only observations on doubletons.) In addition, the methodology of sequential rationalizability in Manzini and Mariotti (2007) arises, endogenously, as an untapped potential in WARP coming to effect in the referential revealed preference theory.

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9 Appendix

9.1 Proofs

Throughout this section, and in order to avoid confusion, I preserve the symbol ■ for Halmos Q.E.D. sign for the proofs of the main statements, and, wherever applied, □ for subproofs. Let $\mathcal{F}$ be any of the operators $\mathfrak{R}, R_1, \mathfrak{B}$, or $\mathcal{I}^p$. I also make the following notational abuse throughout the arguments in this section.

$$\mathcal{F}(\{x, y, z\}) := \mathcal{F}\{x, y, z\}$$

For $S \subseteq X$, let

$$S^- = \{A - x : x \in A\}.$$

Recall that $R_1(S)$ is the collection of of subsets of S derived from taking references out of S, one at a time, and $R_2(S)$ is the collection of all subsets of S derived from taking non-reference elements out. S^-, on the other hand, is the collection of all subsets of S derived from removing any arbitrary element. Note that if $\mathfrak{R}(S) = \emptyset$ then $\mathfrak{B}(S) = R_2(S) = S^-$.

9.1.1 Proof of Decomposition Theorem

The following results are key to decomposition theorem.

Proposition 5. *Let $\boldsymbol{c}$ be a choice correspondence that satisfies BUR and PT, and also assume $S \subseteq X$ such that $\boldsymbol{c}\{x, y\} \neq \emptyset$ for all $x, y \in S$. Then*

(i) $\mathfrak{R}(S) = \emptyset$

(ii) $\mathfrak{B}(S) = S^-$.

Proof. (i) By contradiction assume there exists $r \in \mathfrak{R}(S)$. By definition there exists two distinct elements, $x, y \in S$, both different from r such that

$$\boldsymbol{c}\{x, y, r\} \subset \boldsymbol{c}\{x, y\}.$$

Obviously $r \notin \boldsymbol{c}\{x, y, r\}$. Since $\mathfrak{B}\{x, y, r\}$ is a cover for $\{x, y, r\}$ we conclude that $\{x, r\}$ or $\{y, r\}$ is in $\mathfrak{B}\{x, y, r\}$. Since $r \notin \boldsymbol{c}\{x, y, r\}$ BUR implies that there exists $t \in \{x, y\}$ such that $t \succ^p r$. Wlog, assume $t = x$.

Claim: $y \in \boldsymbol{c}\{x, y\}$.

Since $\boldsymbol{c}\{x, y\} \neq \emptyset$ assume $x \in \boldsymbol{c}\{x, y\}$. Note that since $x \in \boldsymbol{c}\{x, r\}$ BUR implies $x \in \boldsymbol{c}\{x, y, r\}$. From $\boldsymbol{c}\{x, y, r\} \subset \boldsymbol{c}\{x, y\}$ it follows that $\boldsymbol{c}\{x, y\} = \{x, y\}$ and therefore $y \in \boldsymbol{c}\{x, y\}$.

To finish the proof note that since $y \succsim^p x$ and $x \succ^p r$ PT implies that $y \succsim^p r$ which in turn by BUR implies $y \in \boldsymbol{c}\{x, y, r\}$. That is $\boldsymbol{c}\{x, y, r\} = \boldsymbol{c}\{x, y\} = \{x, y\}$ which is a contradiction.

(ii) Since $\mathfrak{R}(S) = \emptyset$ it follows from the definition of that $\mathfrak{B}(S) = R_2(S) = S^-$.

■

Lemma 12. *(**Quick Sort**) Assume that a choice correspondence $\boldsymbol{c}$ satisfies PT, and let $S \subseteq X$ such that $\boldsymbol{c}\{x, y\} \neq \emptyset$ for all $x, y \in S$. Then there exists $x \in S$ such that $x \in \boldsymbol{c}\{x, y\}$ for all $y \in S$.*

Proof. Let $|S| = k$. We need to prove that there exists $x \in S$ such that $x \succsim^p y$ for all $y \in S$. First note that since $\succsim^p$ is transitive then it follows that it is also acyclic. Now by contradiction assume that for all $x \in S$ there exists $y \in S$ such that $y \succ^p x$. Take $x_1 \in S$. Then there must exist $x_2 \in S$ such that $x_2 \succ^p x_1$. Next there must exist $x_3 \in S$ such that $x_3 \succ^p x_2 \succ^p x_1$. By repeating this procedure for k times we conclude that

$$x_{k+1} \succ^p x_k \succ^p \ldots \succ^p x_1,$$

where $x_{k+1} \in \{x_1, x_2, \ldots, x_k\}$. This produces a cycle in $\succsim^p$ which is a contradiction. ■

Lemma 13. *Assume that a choice correspondence* $\boldsymbol{c}$ *satisfies TDR, BUR, and PT, and also assume* $S \subseteq X$ *such that* $\boldsymbol{c}\{x,y\} \neq \emptyset$ *for all* $x, y \in S$*. Then* $\boldsymbol{c}(S) \neq \emptyset$*, and*

$$x \in \boldsymbol{c}(S) \quad \textit{if and only if} \quad x \in \boldsymbol{c}\{x,y\} \quad \textit{for all} \quad y \in S.^{28}$$

Proof. Take $S \subseteq X$. First note that since $\boldsymbol{c}$ satisfies BUR, TDR, and PT Proposition 5.ii implies $\mathfrak{B}(S) = S^{-}$.

($\Leftarrow$): We prove this by induction on $|S|$. For $k = 2$ the statement is obvious. For $|k| = 3$ let $S = \{x, y, z\}$. Then $x \in \{x, y\}$ and $x \in \{x, z\}$, by BUR, implies that $x \in \boldsymbol{c}(S)$. Now assume that for any set with cardinality k the claim is true. For S with caridnality $k+1$, let ${}^{x}S^{-} = \{A \in S^{-} : x \in A\}$. Take $A \in {}^{x}S^{-}$. Next assume $x \in \boldsymbol{c}\{x, y\}$ for all $y \in S$. This means $x \in c\{x, y\}$ for all $y \in A$. Now by induction hypothesis we conclude $x \in \boldsymbol{c}(A)$. Since ${}^{x}S^{-}$ contains all elements of S^{-} that contain x, and x is chosen in all such elements, we conclude x is not dominated by any element in S and from BUR it follows that $x \in \boldsymbol{c}(S)$.

$\boldsymbol{c}$ is non-empty valued: Let $S \subseteq X$.

Proof. First note since $\boldsymbol{c}$ satisfies PT then it follows from Quick Sort Lemma that there exists $x \in S$ such that $x \in \boldsymbol{c}\{x, y\}$ for all $y \in S$. Therefore by $\Leftarrow$ we conclude $x \in \boldsymbol{c}(S)$; that is $\boldsymbol{c}(S) \neq \emptyset$.

($\Rightarrow$): We prove the statement by induction on $|S|$.

Induction Base: For $k = 3$ let $S = \{x, y, z\}$ and assume $x \in \boldsymbol{c}(S)$. TDR implies that there does not exists $t \in S$ such that $t \underset{S}{\blacktriangleright} x$. Since $\mathfrak{B}(S) = S^{-}$ it follows that $x \in \boldsymbol{c}\{x, y\}$ and $x \in \boldsymbol{c}\{x, z\}$.

Induction Hypothesis: Let $|S| = k + 1$ and assume the claim in true for all sets with cardinality k (in particular for $A \in S^{-}$). Take $x \in \boldsymbol{c}(S)$ and an element $y \in S$ different from x. We must show $x \in \boldsymbol{c}\{x, y\}$. Let

$${}^{xy}S^{-} = \{A \in S^{-} : x, y \in A\}.$$

[28] This Lemma is exatcly the axiom refered to as *always chosen* in Manzini and Mariotti (2007) which, in turn, captures the *expansion* axiom in these author's paper.

First note that since $|S| \geq 4$ in our induction it follows that ${}^{xy}S^{-} \neq \emptyset$.[29] If there exists $A \in {}^{xy}S^{-}$ such that $x \in \boldsymbol{c}(A)$ then by induction hypothesis we have $x \in \boldsymbol{c}\{x, y\}$ and the proof is complete. So suppose for all $A \in {}^{xy}S^{-}$ we have $x \notin \boldsymbol{c}(A)$.

Claim: $y \notin c(A)$ for all $A \in {}^{xy}S^{-}$.

Proof. Take $A \in^{xy} S^{-}$. By contradiction assume $y \in \boldsymbol{c}(A)$. Note that since $x \notin \boldsymbol{c}(A)$ this means y beats x in A. Next since x is never chosen in the elements of ${}^{xy}S^{-}$, it follows that x does not beat y in all such elements. Therefore $y \underset{S}{\blacktriangleright} x$ which is a contradiction since $x \in \boldsymbol{c}(S)$ and $\boldsymbol{c}$ satisfies TDR.

Now Take $A_1 \in {}^{xy}S^{-}$. By the previous claim $x, y \notin \boldsymbol{c}(A)$. Since $|A_1| \geq 3$, and $\boldsymbol{c}(A_1) \neq \emptyset$ it follows that there exists an element $z \in A_1$, different from x, y, such that $z \in \boldsymbol{c}(A_1)$. Therefore we conclude z beats x and y in A_1. The latter fact by induction hypothesis implies $z \in \boldsymbol{c}\{y, z\}$. From the former, since $x \in \boldsymbol{c}(S)$ and by TDR, we conclude that there exists $A_2 \in S^{-}$ such that x beats z in A_2[30]. Therefore by induction hypothesis we conclude $x \in \boldsymbol{c}\{x, z\}$. So we have $z \in \boldsymbol{c}\{y, z\}$ and $x \in \boldsymbol{c}\{x, z\}$ which in turn mean $z \succsim^{p} y$ and $x \succsim^{p} z$. Now PT implies $x \succsim^{p} y$ which in turn means $x \in \boldsymbol{c}\{x, y\}$. ■

Theorem 1. ***(Decomposition Theorem)*** *Let $\boldsymbol{c}$ be a choice correspondence. Then the following are equivalent.*

(i) *$\boldsymbol{c}$ is non-empty valued and satisfies WARP*

(ii) *$\boldsymbol{c}\{x, y\} \neq \emptyset$ for all $x, y \in X$ and $\boldsymbol{c}$ satisfies TDR, BUR, and PT.*

Proof. $(ii) \Rightarrow (i)$: First note that from Lemma 13 we conclude $\boldsymbol{c}$ is non-empty valued. To prove that TDR, BUR, and PT imply WARP take $S_1, S_2 \subseteq X$ and $x, y, \in S_1 \cap S_2$, $x \in \boldsymbol{c}(S_1)$, and $y \in \boldsymbol{c}(S_2)$. We must show $x \in \boldsymbol{c}(S_2)$. Applying $\Rightarrow$ in Lemma 13 to the set S_1 we conclude that $x \in c\{x, y\}$ which in turn means $x \succsim^{p} y$. Applying $\Rightarrow$ in Lemma 13 to the set S_2 we conclude that $y \in \boldsymbol{c}\{y, t\}$ for all $t \in S_2$ which in turn means $y \succsim^{p} t$ for all $t \in S_2$. Now PT implies $x \succsim^{p} t$ for all $t \in S_2$. Therefore by $\Leftarrow$ in Lemma 13 we conclude $x \in \boldsymbol{c}(S_2)$.

$(i) \Rightarrow (ii)$: Let $S \subseteq X$. First note that since $\boldsymbol{c}$ is non-empty valued obviously $\boldsymbol{c}\{x, y\} \neq \emptyset$ for all $x, y, \in S$.

[29] To see this note that there exists an element $z \in S$ different from both x, y. It follows that $S - z \in S^{-}$. Obviously $x, y \in S - z$. This means $S - z \in {}^{xy}S^{-}$.

[30] Indeed, $A_2 = S - y$.

Assume that $\boldsymbol{c}$ satisfies WARP. We first prove the following claim.

Claim: $\mathfrak{R}(S) = \emptyset$.

Proof. By contradiction assume r is a reference in S. From definition it follows that there exists two distinct elements $x, y \in S$, both different from r, such that

$$\boldsymbol{c}\{x, y, r\} \subset \boldsymbol{c}\{x, y\}.$$

Obviously $r \notin \boldsymbol{c}\{x, y, r\}$. Since $\boldsymbol{c}\{x, y, r\} \neq \emptyset$ and $\boldsymbol{c}\{x, y, r\} \subset \boldsymbol{c}\{x, y\}$ it follows that $\boldsymbol{c}\{x, y\} = \{x, y\}$. Now WARP implies $\boldsymbol{c}\{x, y, r\} = \{x, y\}$. This is a contradiction. □

Note that from the previous claim it follows that $\mathfrak{B}(S) = S^-$.

WARP ⇒ TDR: Let $x \in \boldsymbol{c}(S)$. Then WARP implies $x \in \boldsymbol{c}(A)$ for all $A \in S^-$ such that $x \in A$. Therefore, we conclude x is not beaten by any elements of S in any elements of S^- and, as a result, x is not dominated by any elements of S. This means x is a maximal element of $\underset{S}{\blacktriangleright}$.

WARP ⇒ BUR: Assume that x is the maximal element of $\underset{S}{\blacktriangleright}$. We must show $x \in \boldsymbol{c}(S)$. Take $y \in \boldsymbol{c}(S)$ different from x[31]. Let

$$^{xy}S^- = \{A \in S^- : x, y \in A\}.$$

First note that $^{xy}S^- \neq \emptyset$.[32] Take $A_1 \in^{xy} S^-$. WARP (the equivalent statement) implies $y \in \boldsymbol{c}(A_1)$. If $x \in \boldsymbol{c}(A_1)$ then by WARP $x \in \boldsymbol{c}(S)$ and the proof is complete. So assume $x \notin \boldsymbol{c}(A_1)$. This means y beats x in A_1. Since x is not dominated by y it follows that there exists $A_2 \in {}^{xy}S^-$ such that x beats y in A_2. In particular, $x \in \boldsymbol{c}(A_2)$. Now WARP implies $x \in \boldsymbol{c}(S)$.

WARP ⇒ PT: Take $x, y, z \in S$, and assume $x \succsim^p y$ and $y \succsim^p z$. These, respectively, mean $x \in \boldsymbol{c}\{x, y\}$ and $y \in \boldsymbol{c}\{y, z\}$. Consider the set $\{x, y, z\}$. Note the following cases: i. if $x \in \boldsymbol{c}\{x, y, z\}$ then WARP implies $x \in \boldsymbol{c}\{x, z\}$ and therefore $x \succsim^p z$. ii. If $y \in \boldsymbol{c}\{x, y, z\}$ WARP and $x \in \boldsymbol{c}\{x, y\}$ imply that $x \in \boldsymbol{c}\{x, y, z\}$ and by i. $x \succsim^p z$. iii. If $z \in \boldsymbol{c}\{x, y, z\}$

[31] Note that if such y does not exists, since $\boldsymbol{c}(S) \neq \emptyset$, we conclude $x \in \boldsymbol{c}(S)$ and the proof is complete.

[32] To see this note that since $|S| \geq 3$, there exists an element $z \in S$ different from x, y. Obviously $x, y \in S - z$, and therefore $^{xy}S^- \neq \emptyset$.

WARP and $y \in \boldsymbol{c}\{y,z\}$ imply that $y \in \boldsymbol{c}\{x,y,z\}$ which by ii. implies $x \succsim^p z$. Since $\boldsymbol{c}\{x,y,z\} \neq \emptyset$ we conclude $x \succsim^p z$ and the proof is complete. ■

9.1.2 Proof of Proposition 1

Proposition 1. *Let $\boldsymbol{c}$ be a RDM. Then*

$$\mathfrak{R}(S) \neq \emptyset \iff \boldsymbol{c} \text{ satisfies WBUR}$$

Proof. ($\Rightarrow$): Assume that $\boldsymbol{c}$ satisfies BUR on $\mathfrak{A}$. Note that this means $\boldsymbol{c}$ satisfies BUR. Then from Proposition 5.i it follows that $\mathfrak{R}(S) = \emptyset$.

($\Leftarrow$): We only need to show all non-fully indecisive triplets satisfy BUR. To do this Let $\{x,y,z\} \in \mathfrak{A}$. To show that $\{x,y,z\}$ satisfies BUR, wlog, assume that no elements of S dominate x. We must show $x \in \boldsymbol{c}\{x,y,z\}$. First note that since there are no references in S we have $\mathfrak{B}\{x,y,z\} = \{x,y,z\}^{-}$. Since y and z do not dominate x, it follows that $x \in \boldsymbol{c}\{x,y\}$ and $x \in \boldsymbol{c}\{x,z\}$, which respectively, mean $x \succsim^p y$ and $x \succsim^p z$. We show that $x \notin \boldsymbol{c}\{x,y,z\}$ reaches a contradiction. Assume $x \notin \boldsymbol{c}\{x,y,z\}$. Wlog, let $y \in \boldsymbol{c}\{x,y,z\}$. Then TDR implies that $y \in \boldsymbol{c}\{x,y\}$, which in turn means $y \succsim^p x$, and therefore $x \sim^p y$. Now PT implies $y \succsim^p z$; that is $y \in \boldsymbol{c}\{y,z\}$. If $z \notin \boldsymbol{c}\{x,y,z\}$ then we will have

$$\boldsymbol{c}\{x,y,z\} \subset \boldsymbol{c}\{x,y\},$$

and therefore z will be a reference in S which is not possible. So assume $z \in \boldsymbol{c}\{x,y,z\}$. Then TDR implies $z \in \boldsymbol{c}\{x,z\}$ which in turn means $z \succsim^p x$ and therefore $x \sim^p z$. From PT it follows that $x \sim^p y \sim^p z$ and therefore $\{x,y,z\}$ is fully indecisive. A contradiction. ■

9.1.3 Proof of Propositions 2 and 3

I start by the following basic observations about RDM.

Proposition 6. *Let $\boldsymbol{c}$ be a RDM. The following is true:*

(i) If S is fully indecisive then $\boldsymbol{c}(S) = S$.

(ii) $|\mathfrak{R}\{x,y,z\}| \leq 1$.

Proof. (i) We prove this by induction on $|S|$.

Induction Base: For $|S| = 3$ let $S = \{x,y,z\}$. First note that since S is fully indecisive it satisfies BUR. Take $t \in \{x,y,z\}$. First note that since $\mathfrak{B}\{x,y,z\}$ is a cover for S

and the fact that $x \sim^p y \sim^p z$ we conclude that $q \not\blacktriangleright_S t$ for all $q \in S$. Therefore, by BUR, we conclude $t \in \boldsymbol{c}(S)$. Therefore $\boldsymbol{c}\{x, y, z\} = \{x, y, z\}$.

Induction Hypothesis: Now assume that the statement is true if $|S| = k$. Let $|S| = k+1$. Take $x \in S$. Take $t \in S$. Note that for $A \in S^-$ we have A is fully indecisive and therefore by induction base $\boldsymbol{c}(A) = A$. Next note that $\mathfrak{B}(S) \subseteq S^-$. Therefore for all $A \in \mathfrak{B}(S)$ we have $\boldsymbol{c}(A) = A$. This implies there does not exists $q \in S$ such that $q \blacktriangleright_S t$. BUR implies $t \in \boldsymbol{c}(S)$. This means $\boldsymbol{c}(S) = S$.

(ii) Consider the set $\{x, y, z\}$. By definition $t \in \mathfrak{R}\{x, y, z\}$ implies $t \notin \boldsymbol{c}\{x, y, z\}$. Since $\boldsymbol{c}\{x, y, z\} \neq \emptyset$ then we conclude $|\mathfrak{R}\{x, y, z\}| < 3$. It suffice to show $|\mathfrak{R}\{x, y, z\}| \neq 2$. Assume, wlog, $\mathfrak{R}\{x, y, z\} = \{y, z\}$. First note that this means $\boldsymbol{c}\{x, y, z\} = \{x\}$. Next since y, z are references we should have

$$\{x\} = \boldsymbol{c}\{x, y, z\} \subset \boldsymbol{c}\{x, z\},$$

and

$$\{x\} = \boldsymbol{c}\{x, y, z\} \subset \boldsymbol{c}\{x, y\},$$

which, respectively, imply

$$\boldsymbol{c}\{x, z\} = \{x, z\},$$

and

$$\boldsymbol{c}\{x, y\} = \{x, y\}.$$

Next PT implies $x \sim^p y \sim^p z$ and therefore we conclude $\{x, y, z\}$ is fully indecisive and by part (i) we conclude $\boldsymbol{c}\{x, y, z\} = \{x, y, z\}$ which is a contradiction. Therefore $|\mathfrak{R}\{x, y, z\}| \leq 1$.

■

Proposition 2. *Let $\boldsymbol{c}$ be a RDM. Also let $r \in \mathfrak{R}(S)$ and $x, y \in S$ such that*

$$\boldsymbol{c}\{x, y, r\} \subset \boldsymbol{c}\{x, y\}.$$

Then $x \sim^p y \succ^p r$.

Proof. Take $r \in \mathfrak{R}(S)$ and $x, y \in S$ such that

$$\boldsymbol{c}\{x, y, r\} \subset \boldsymbol{c}\{x, y\}.$$

First note that $r \notin \boldsymbol{c}\{x,y,r\}$, and, since $\boldsymbol{c}\{x,y,r\} \neq \emptyset$, it has to be the case that $\boldsymbol{c}\{x,y\} = \{x,y\}$. Therefore $x \sim^p y$. Since r is also a reference in $\{x,y,r\}$ from Proposition 6.ii we conclude $\mathfrak{R}\{x,y,r\} = \{r\}$. This implies

$$\mathfrak{B}(S) = \big\{\{x,r\},\{y,r\}\big\}.$$

If, wlog, $r \in \boldsymbol{c}\{x,r\}$, then PT implies $x \sim^p y \sim^p r$ and therefore $\{x,y,z\}$ is fully indecisive and by Proposition 6.i we will have $\boldsymbol{c}\{x,y,z\} = \{x,y,z\}$ which is not possible. So it has to be the case that $\boldsymbol{c}\{x,r\} = \{x\}$ and $\boldsymbol{c}\{y,r\} = \{y\}$ which imply, respectively, $x \succ^p r$, and $y \succ^p r$. ■

This next statement is a direct result of the Proposition 2 and is used in the future results of the paper.

Corollary 14. *Let $\boldsymbol{c}$ be a RDM and $r \in \mathfrak{R}(S)$. Then $r \notin \mathcal{I}^p(S)$.*

Proof. Take $r \in \mathfrak{R}(S)$. From the definition of reference we conclude that there exists two distinct elements $x, y, \in S$, both different from r such that

$$\boldsymbol{c}\{x,y,r\} \subset \boldsymbol{c}\{x,y\}.$$

Now Proposition 2 implies $x \sim^p y \succ^p r$. This means $r \notin \mathcal{I}^p(S)$. ■

Proposition 3. *Let $\boldsymbol{c}$ be a RDM and $S \subseteq X$. Then*

$$\boldsymbol{c}(S) \subseteq \mathcal{I}^p(S)$$

Proof. We prove this by induction on $|S|$.

Induction Base: For $|S| = 3$ let $S = \{x,y,z\}$. Let $x \in \boldsymbol{c}\{x,y,z\}$. We need to show $x \in \mathcal{I}^p\{x,y,z\}$. If $\mathfrak{R}(S) = \emptyset$ then $\mathfrak{B}(S) = \{x,y,z\}^-$ and TDR implies that $x \in \boldsymbol{c}\{x,y\}$ and $x \in \boldsymbol{c}\{x,z\}$ which in turn implies $x \succsim^p y$ and $x \succsim^p z$. This means $x \in \mathcal{I}^p\{x,y,z\}$ and the proof is complete. So assume $\mathfrak{R}\{x,y,z\} \neq \emptyset$. Since $x \in \boldsymbol{c}\{x,y,z\}$ then the definition of reference implies $x \notin \mathfrak{R}\{x,y,z\}$. Also note that prp:properties.ii implies that there only exists one reference in $\{x,y,z\}$. Wlog, assume $\mathfrak{R}\{x,y,z\} = \{z\}$. Then Proposition 2 implies $x \sim^p y \succ^p z$ and therefore $x \in \mathcal{I}^p\{x,y,z\}$ and the proof is complete.

Induction Hypothesis: Now assume the statement is true if $|S| = k$. Let $|S| = k+1$.

Assume $x \in \boldsymbol{c}(S)$ and take an element y in S different from x. We must show $x \succsim^p y$. Let

$$^{xy}\mathfrak{B}(S) = \{A \in \mathfrak{B}(S) : x, y \in A\}.$$

Claim: $^{xy}\mathfrak{B}(S) \neq \emptyset$.

Proof. By contradiction assume $^{xy}\mathfrak{B}(S) = \emptyset$. Since $\mathfrak{B}(S)$ is a cover for S there has to be a set in it that contain x, and a set that contains y. note that it follows from the contradiction that

$$\mathfrak{B}(S) = \big\{S - x, S - y\big\}.$$

Since $|S| > 3$ and $\mathfrak{B}(S)$ only has two elements it has to be the case that $\mathfrak{B}(S) = R_1(S)$ which in turn means $\mathfrak{R}(S) = \{x, y\}$.

Next note that $x \in \boldsymbol{c}(S - y)$. To see this, by contradiction assume $x \notin \boldsymbol{c}(S - y)$. Take $z \in \boldsymbol{c}(S-y)$. Since x only appears in $S-y$ this implies that z dominates x in S and therefore $x \notin \boldsymbol{c}(S)$ which is a contradiction. Therefore assume $x \in \boldsymbol{c}(S-y)$. Note that from induction assumption we conclude that $x \in \mathcal{I}^p(S - y)$. This means $x \succsim^p t$ for all t in S different from y.

Finally since $x \in \mathfrak{R}(S)$ it follows from the definition that there exist two distinct elements $t_1, t_2 \in S$ both different from x, such that

$$\boldsymbol{c}\{t_1, t_2, x\} \subset \boldsymbol{c}\{t_1, t_2\}.$$

Now Proposition 2 implies $t_1 \sim^p t_2 \succ^p x$. Since t_1, t_2 are distinct elements, at least one of them is different from y. This contradicts our earlier observation that $x \succsim^p t$ for all t different from y. □

To finish the proof let $^{xy}\mathfrak{B}(S) \neq \emptyset$. If $A \in {}^{xy}\mathfrak{B}(S)$ exists such that $x \in \boldsymbol{c}(A)$ then by induction assumption we conclude $x \succsim^p y$ and the proof is complete. So assume for all $A \in {}^{xy}\mathfrak{B}(S)$ we have $x \notin \boldsymbol{c}(A)$. First note that this implies $y \notin \boldsymbol{c}(A)$ for all $A \in {}^{xy}\mathfrak{B}(S)$. To see this note that if y is chosen on an element of $^{xy}\mathfrak{B}(S)$, it would mean that y beats x in that set. And since x is never chosen in the elements of $^{xy}\mathfrak{B}$ x does not beat y in any of those elements. This means $y \underset{S}{\blacktriangleright} x$ which is not possible because of TDR and the fact that $x \in \boldsymbol{c}(S)$.

Therefore take $A_1 \in {}^{xy}\mathfrak{B}(S)$ and $z \in \boldsymbol{c}(A_1)$ different from x, y. Note that by induction assumption this implies $z \succsim^p y$. Next this implies that z beats x in A_1. Since $x \in \boldsymbol{c}(S)$ TDR implies that there exists $A_2 \in \mathfrak{B}(S)$ such that x beats z in A_2. This from induction

assumption on A_2 implies $x \succsim^p z$. Combining this latter fact with $z \succsim^p y$, from PT it follows that $x \succsim^p y$. This completes the proof of the proposition. ■

I finish this section by proving the following lemma and corollary that are used in the proof of the main results.

Lemma 15. *Let $\boldsymbol{c}$ be a RDM. If $x \in \boldsymbol{c}\{x, y, z\}$ for all pair of distinct elements $y, z \in S$, both different from x, then $x \in \boldsymbol{c}(S)$.*

Proof. We prove this by induction on $|S|$. For $|S| = 3$ there is nothing to prove. Assume that the statement is true for all the sets with cardinality k. Let $|S| = k+1$ and take $x \in S$ be such that $x \in \boldsymbol{c}\{x, y, z\}$ for all two distinct elements $y, z \in S$, both different from x. Let

$$^x\mathfrak{B}(S) = \{A \in \mathfrak{B}(S) : x \in A\}.$$

Note that since $\mathfrak{B}(S)$ is a cover for S we conclude $^x\mathfrak{B}(S) \neq \emptyset$. Take $A \in {}^x\mathfrak{B}(S)$. Note that since

$$x \in \boldsymbol{c}\{x, y, z\},$$

for two distinct elements $y, z \in S$ and, since $A \subseteq S$, we conclude that

$$x \in \boldsymbol{c}\{x, y, z\},$$

for two distinct elements $y, z \in A$, both different from x. Finally induction assumption implies that $x \in \boldsymbol{c}(A)$. Since A was an arbitrary element of $^x\mathfrak{B}(S)$ we conclude x is chosen in all elements of $^x\mathfrak{B}(S)$. This means x is not dominated by any element of S. Therefore BUR implies that $x \in \boldsymbol{c}(S)$. ■

Corollary 16. *Let $\boldsymbol{c}$ be a RDM. If $r \in \mathfrak{R}(S)$ then $r \notin \boldsymbol{c}(S)$.*

Proof. Take $r \in \mathfrak{R}(S)$. Then Corollary 14 implies $r \notin \mathcal{I}^p(S)$. Next Proposition 3 implies $r \notin \boldsymbol{c}(S)$. ■

9.1.4 Proof of Proposition 4

Proposition 4. *Let $\boldsymbol{c}$ be a RDM and $r \in \mathfrak{R}(S)$. Then*

(i) $x \succsim^r y \implies x \succsim^p y$.

(ii) $\succsim^r$ *defines a complete binary relation on* $\mathcal{I}^p(S)$.

Proof. (i) Assume $x \succsim^r y$. If follows from the definition that $x \in \boldsymbol{c}\{x,y,r\}$. Note that Corollary 16 implies that $x \notin \mathfrak{R}\{x,y,r\}$. If r is a reference in $\{x,y,r\}$ then from Proposition 2 we conclude $x \sim^p y$ which completes the proof. If y is a reference in $\{x,y,r\}$ again Proposition 2 implies that $x \succ^p y$ in which case the proof is also complete. So assume there are no references in $\{x,y,r\}$. Then $\mathfrak{B}(S) = \big\{\{x,y\},\{x,r\},\{y,r\}\big\}$. Since $x \in \boldsymbol{c}\{x,y,r\}$ from TDR we conclude that $x \in \boldsymbol{c}\{x,y\}$ and therefore $x \succsim^p y$.

(ii) Take $x,y \in \mathcal{I}^p(S)$. Note that since r is a reference in S it follows from Corollary 14 that $r \notin \mathcal{I}^p(S)$. Next Proposition 3 implies that $r \notin \boldsymbol{c}\{x,y,r\}$. Since $\boldsymbol{c}$ is non-empty valued we have $x \in \boldsymbol{c}\{x,y,r\}$ or $y \in \boldsymbol{c}\{x,y,r\}$, which in turn imply $x \succsim^r y$ or $y \succsim^r x$. This completes the proof.

■

9.1.5 The Case of No Maximal Reference

Lemma 17. *Let $\boldsymbol{c}$ be a RDM and $S \subseteq X$. If $\mathfrak{R}_M(S) = \emptyset$ then $\boldsymbol{c}(S) = \mathcal{I}^p(S)$.*

Proof. Assume $\mathfrak{R}_M(S) = \emptyset$. We need to show $\boldsymbol{c}(S) = \mathcal{I}^p(S)$. To do this first note that Proposition 3 implies $\boldsymbol{c}(S) \subseteq \mathcal{I}^p(S)$. So we only need to prove $\mathcal{I}^p(S) \subseteq \boldsymbol{c}(S)$. Take $x \in \mathcal{I}^p(S)$ and consider the set $\{x,y,z\}$ for two distinct elements $y,z \in S$, both different from x.

Step 1: $\mathfrak{R}\{x,y,z\} = \emptyset$.

Proof. By contradiction assume $\mathfrak{R}\{x,y,z\} \neq \emptyset$. First note that by Proposition 6.ii it has to be the case that there is only one reference in $\{x,y,z\}$. Second by Corollary 14 and the fact that $x \in \mathcal{I}^p(S)$ we conclude $x \notin \mathfrak{R}(S)$. Wlog, assume z is the reference in $\{x,y,z\}$. Then it has to be the case that

$$\boldsymbol{c}\{x,y,z\} \subset \boldsymbol{c}\{x,y\},$$

which implies $\boldsymbol{c}\{x,y\} = \{x,y\}$, which in turn implies $y \succsim^p x$. Since $x \in \mathcal{I}^p(S)$ PT implies $y \in \mathcal{I}^p(S)$. This means $z \in \mathfrak{R}_M(S)$ which is not possible. This completes the proof of Step 1. □

Step 2: $x \in \boldsymbol{c}\{x,y,z\}$.

Proof. From Step 1 we have $\mathfrak{R}\{x,y,z\} = \emptyset$. Therefore

$$\mathfrak{B}\{x,y,z\} = \big\{\{x,y\},\{y,z\},\{x,z\}\big\}.$$

We first argue that y or z can not be the single choice in $\boldsymbol{c}\{x,y,z\}$. To do this, and wlog, assume $\boldsymbol{c}\{x,y,z\}=\{y\}$. Then TDR implies $y \succsim^p x$ which in turn means $x,y \in \mathcal{I}^p(S)$. Then we conclude that

$$\boldsymbol{c}\{x,y,z\} \subset \boldsymbol{c}\{x,y\},$$

which means z is a maximal reference in S which is impossible. Second assume $\{y,z\} \subseteq \boldsymbol{c}\{x,y,z\}$. We show that $x \in \boldsymbol{c}\{x,y,z\}$. From TDR we conclude $y,z \succsim^p x$ and, since $x \in \mathcal{I}^p(S)$, it follows that $x,y,z \in \mathcal{I}^p(S)$ which means $\{x,y,z\}$ is fully indecisive. Using Proposition 6.i we conclude $\boldsymbol{c}\{x,y,z\}=\{x,y,z\}$ which obviously means $x \in \boldsymbol{c}\{x,y,z\}$. Since $\boldsymbol{c}\{x,y,z\} \neq \emptyset$ the proof of this Step 2 is complete. □

To finish the proof note that by Lemma 15.ii we conclude $x \in \boldsymbol{c}(S)$. This means $\mathcal{I}^p(S) \subseteq \boldsymbol{c}(S)$. Therefore $\boldsymbol{c}(S)=\mathcal{I}^p(S)$. ■

For the sake of avoiding repetition, I make the the following obvious observations that are used in the remaining argument in several occasions.

Let $\boldsymbol{c}$ be an RDM and $S \subseteq X$. Then

(i) By Corollary 14 for a reference $r \in \mathfrak{R}(S)$, we have $r \notin \mathcal{I}^p(S)$. Therefore dropping r from the set S does note change the maximal class of the resulting subset; that is, $\mathcal{I}^p(S-r)=\mathcal{I}^p(S)$. Note that, using the same argument we deduce that

$$\mathcal{I}^p(S)=\mathcal{I}^p(S \setminus R),$$

for all $R \subseteq \mathfrak{R}(S)$; that is the maximal class of S is the same in all subsets of S which are derived from removing any arbitrary numbers of references from S.

(ii) Maximal references in $S-r$ and S only (potentially) differ in r. That is $\mathfrak{R}_{_M}(S)=\mathfrak{R}_{_M}(S-r)$ if r is not a maximal reference in S and $\mathfrak{R}_{_M}(S)=\mathfrak{R}_{_M}(S-r)\cup\{r\}$ if r is a maximal reference in S. Similarly such preservation happens for maximal reference sets of elements; that is for $t \in \mathcal{I}^p(S)$, we have $\mathfrak{R}^t_{_M}(S)=\mathfrak{R}^t_{_M}(S-r)$ if r is not a maximal reference in S and $\mathfrak{R}^t_{_M}(S)=\mathfrak{R}^t_{_M}(S-r)\cup\{r\}$ if r is a maximal reference in S.

9.1.6 Proof of Theorem 2

Lemma 18. *Let $\boldsymbol{c}$ be a RDM and $S \subseteq X$. Also assume $\mathfrak{R}(S)=\mathfrak{R}_{_M}(S)=\{r\}$. Then*

$$\boldsymbol{c}(S) = \underset{\mathcal{I}^p(S)}{\operatorname{argmax}} \succsim^r .^{33}$$

Proof. First note that since r is a reference in S it follows that there exists two distinct elements in S both different from r and therefore $|S| \geq 3$.

($\Leftarrow$): Take $x \in \underset{\mathcal{I}^p(S)}{\operatorname{argmax}} \succsim^r$. Obviously $x \in \mathcal{I}^p(S)$. We must show $x \in \boldsymbol{c}(S)$. To do this we first make the following claim.

Claim: For two distinct elements $y, z \in S$, both different from x, we have $x \in \boldsymbol{c}\{x, y, z\}$.

Proof. Take two distinct elements $y, z \in S$, both different from x, and consider the set $\{x, y, z\}$. We consider two cases here:

Case 1: $\mathfrak{R}\{x, y, z\} = \emptyset$.

Note that $\mathfrak{R}\{x, y, z\} = \emptyset$ implies $\mathfrak{R}_M\{x, y, z\} = \emptyset$. Since $\boldsymbol{c}$ is a RDM on $\{x, y, z\}$ and there are no maximal references in $\{x, y, z\}$ Lemma 17 implies $\boldsymbol{c}\{x, y, z\} = \mathcal{I}^p\{x, y, z\}$. Since $x \in \mathcal{I}^p(S)$ we conclude $x \in \mathcal{I}^p\{x, y, z\}$ and therefore $x \in \boldsymbol{c}\{x, y, z\}$. This completes the proof of this case.

Case 2: $\mathfrak{R}\{x, y, z\} \neq \emptyset$.

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[33] As discussed before, the domain of relevance for $\succsim^r$ is $\mathcal{I}^p(S)$; that is, indeed, $\underset{S}{\operatorname{argmax}} \succsim^r = \underset{\mathcal{I}^p(S)}{\operatorname{argmax}} \succsim^r$. To see this note that obviously $\underset{S}{\operatorname{argmax}} \succsim^r \subseteq \underset{\mathcal{I}^p(S)}{\operatorname{argmax}} \succsim^r$. To see the other inclusion take $x \in \underset{\mathcal{I}^p(S)}{\operatorname{argmax}} \succsim^r$ and $y \in S$ such that $y \notin \mathcal{I}^p(S)$. Since $x \in \mathcal{I}^p(S)$ this means $x \succ^p y$. We must show $x \succsim^r y$. First note that by Corollary 14 we conclude that $r \notin \mathcal{I}^p(S)$ and since $x \in \mathcal{I}^p(S)$ we have $x \succ^p r$. Consider the set $\{x, y, r\}$. We must show $x \in \boldsymbol{c}\{x, y, r\}$. Note that $x \in \mathcal{I}^p(S)$ implies $x \in \mathcal{I}^p\{x, y, r\}$ and by Corollary 14 we conclude x is not a reference in $\{x, y, r\}$. Also note that since there is no pairwise indifference between either x, y or x, r we conclude y, r are also not references in $\{x, y, r\}$. As a result there is no reference in $\{x, y, r\}$ and we conclude

$$\mathfrak{B}\{x, y, r\} = \big\{\{x, y\}, \{y, r\}, \{x, r\}\big\}.$$

Since $x \succ^p y$ and $x \succ^p r$ it follows that y, r are dominated by x in $\{x, y, r\}$ and TDR implies $y, r \notin \boldsymbol{c}\{x, y, r\}$. Finally since $\boldsymbol{c}\{x, y, r\} \neq \emptyset$ we conclude $x \in \boldsymbol{c}\{x, y, r\}$. This in turn means $x \succsim^r y$. Therefore, $x \in \underset{S}{\operatorname{argmax}} \succsim^r$; that is $\underset{\mathcal{I}^p(S)}{\operatorname{argmax}} \succsim^r \subseteq \underset{S}{\operatorname{argmax}} \succsim^r$. From this we conclude

$$\underset{\mathcal{I}^p(S)}{\operatorname{argmax}} \succsim^r = \underset{S}{\operatorname{argmax}} \succsim^r .$$

First note that by Proposition 6.ii we conclude $|\mathfrak{R}\{x,y,z\}| = 1$. Second Corollary 14 and the fact that $x \in \mathcal{I}^p(S)$ imply $x \notin \mathfrak{R}\{x,y,z\}$. Wlog, assume $z \in \mathfrak{R}\{x,y,z\}$. It follows from the definition of references that

$$\boldsymbol{c}\{x,y,z\} \subset \boldsymbol{c}\{x,y\},$$

and by Proposition 2 we conclude $y \sim^p x$. Since $x \in \mathcal{I}^p(S)$ this implies $y \in \mathcal{I}^p(S)$. Therefore we conclude $z \in \mathfrak{R}_M(S)$ which means $z = r$. Since $x \in \underset{\mathcal{I}^p(S)}{\operatorname{argmax}} \succsim^r$ we conclude $x \succsim^r y$ which in turn implies $x \in \boldsymbol{c}\{x,y,r\}$. This completes the proof in this case and therefore the claim. □

To finish the proof of $\Leftarrow$ note that Lemma 15 implies $x \in \boldsymbol{c}(S)$. This finishes the proof of $\Leftarrow$.

($\Rightarrow$): Take $x \in \boldsymbol{c}(S)$. We must show $x \in \underset{\mathcal{I}^p(S)}{\operatorname{argmax}} \succsim^r$.

First note that since r is the only reference in S we conclude $\mathfrak{B}(S) = R_2(S)$ and that all the elements in $\mathfrak{B}(S)$ are of the form $S - t$ for some t different from r. Also note that $r \in A$ for all $A \in \mathfrak{B}(S)$.

We prove this by strong induction on $|S|$. Note that for $|S| = 3$ the statement is obvious; that is if $S = \{x,y,r\}$ then if follows from the definition of $\succsim^r$ that $x \in c(S)$ if and only if $x \succsim^r y$. We base our induction on $|S| = 4$.

Induction Base: Assume $|S| = 4$. Let $S = \{x,y,z,r\}$. Assume, wlog, that $x \in \boldsymbol{c}(S)$. We need to show $x \in \underset{\mathcal{I}^p(S)}{\operatorname{argmax}} \succsim^r$. First note that

$$\mathfrak{B}(S) = \big\{\{x,y,r\},\{x,z,r\},\{y,z,r\}\big\}.$$

Note that since r is a reference in S, Proposition 2 implies that at least two elements in the set $\{x,y,z\}$ are preferred to r under pairwise comparison. Therefore it follows from Proposition 3 that $r \notin \boldsymbol{c}\{x,y,r\}$ and $r \notin \boldsymbol{c}\{y,z,r\}$.

Next note that since $r \notin \mathcal{I}^p(S)$ we only need to show $x \succsim^r y$ and $x \succsim^r z$. To do this, and wlog, consider the set $\{x,y,r\}$. Note that if $x \notin \boldsymbol{c}\{x,y,r\}$ then it has to be the case that $\boldsymbol{c}\{x,y,r\} = \{y\}$. Since $\{x,y,r\}$ is the only element in $\mathfrak{B}(S)$ that contains both x,y this would imply that y dominates x in S and therefore $x \notin \boldsymbol{c}(S)$ which is not possible. So we have

$x \in \boldsymbol{c}\{x, y, r\}$. This by definition implies $x \succsim^r y$. Therefore $x \in \underset{S}{\text{argmax}} \succsim^r = \underset{\mathcal{I}^p(S)}{\text{argmax}} \succsim^r$. This completes the proof for induction base.

Induction Hypothesis: Assume the statement is true for all sets with cardinality less than or equal to k and let $|S| = k + 1$. Take $x \in \boldsymbol{c}(S)$. We first make the following claim.

Claim: $\succsim^r$ does not posses any cycles of the length less than or equal to $k - 1$ in $\mathcal{I}^p(S)$.

Proof. Let $t \leq k-1$. We must show that $\succsim^r$ does not possess any cycle of degree t in $\mathcal{I}^p(S)$. To do this let $A = \{x_1, x_2, \ldots, x_t\} \subseteq \mathcal{I}^p(S)$ and and assume $x_1 \succ^r x_2 \succ^r \ldots \succ^r x_t$. We show $x_1 \succsim^r x_t$. Consider the set $A \cup \{r\}$. First note that since $\boldsymbol{c}$ is a RDM on S it follows that $\boldsymbol{c}$ is also an RDM in $A \cup \{r\}$. Second, the number of elements in A, t, is less than $k - 1$ and therefore we have $|A \cup \{r\}| < |S| = k + 1$. Third, take an element $x_i \in A$. Note that since

$$x_i \succ^r x_{i+1} \text{ and } x_i \sim^p x_{i+1}$$

we conclude r is the unique maximal reference in $A \cup \{r\}$. The last three observations together imply that induction hypothesis is applicable to $A \cup \{r\}$.

To finish the poof of this claim first note that since r is a reference in $A \cup \{r\}$ by Corollary 16 we have $r \notin \boldsymbol{c}(A \cup \{r\})$. Since $\boldsymbol{c}(A \cup \{r\}) \neq \emptyset$ take $x_i \in \boldsymbol{c}(S)$. Induction hypothesis implies that $x_i \succsim^r x_j$ for all $j \neq i$. So it has to be the case that $x_i = x_1$. This means $x_1 \succsim^r x_t$. So the proof of the claim is complete. □

In order to proof the lemma by contradiction assume $y \succ^r x$, for $y \in \mathcal{I}^p(S)$. Let

$$^{xy}\mathfrak{B}(S) = \{A \in \mathfrak{B}(S) : x, y \in A\};$$

that is, the set of all elements of $\mathfrak{B}(S)$ that contain both elements x, y.

Next recall that $r \in A$ for all $A \in \mathfrak{B}(S)$ which in turn implies $r \in A$ for all $A \in {}^{xy}\mathfrak{B}(S)$. Also since $y \succ^r x$ and $y \sim^p x$ we conclude r is a maximal reference in A for all $A \in {}^{xy}\mathfrak{B}(S)$. To summarize for all $A \in {}^{xy}\mathfrak{B}(S)$ we have made the following observations: i. $x, y, r \in A$, ii. r is a maximal reference in A. Note that this means induction hypothesis is applicable to all of these sets.

Step 1: ${}^{xy}\mathfrak{B}(S) \neq \emptyset$.

Proof. First note that in our induction $|S| \geq 5$ and, therefore, there is at least an element $z \in S$ that is different from x, y, r. Since $\mathfrak{B}(S) = R_2(S)$ we conclude $S - z$ is an element of $\mathfrak{B}(S)$. Obviously $x, y \in \boldsymbol{c}(S - z)$. This implies $S - z \in {}^{xy}\mathfrak{B}(S)$ and we conclude ${}^{xy}\mathfrak{B}(S) \neq \emptyset$. □

Step 2: $x, y \notin \boldsymbol{c}(A)$ for all $A \in {}^{xy}\mathfrak{B}(S)$.

Proof. Take $A \in {}^{xy}\mathfrak{B}(S)$. Since $y \succ^r x$ we conclude $x \notin \underset{\mathcal{I}^p(A)}{\operatorname{argmax}} \succsim^r$ and therefore induction hypothesis implies that $x \notin \boldsymbol{c}(A)$. Note that since A was an arbitrary element of ${}^{xy}\mathfrak{B}(S)$ we conclude that x does not beat y in all $A \in {}^{xy}\mathfrak{B}(S)$.

Next we show that y is also not chosen in all elements of ${}^{xy}\mathfrak{B}(S)$. To do this take $A \in {}^{xy}\mathfrak{B}(S)$. If $y \in \boldsymbol{c}(A)$, since $x \notin \boldsymbol{c}(A)$, we conclude that y beats x in A. Since x never beats y it follows that $y \underset{S}{\blacktriangleright} x$ and therefore TDR implies $x \notin \boldsymbol{c}(S)$ which is not possible. Since A was arbitrary we conclude y is not chosen in all elements of ${}^{xy}\mathfrak{B}(S)$ as well. □

Now let $\mathcal{T} = \{z \in S : z \in \boldsymbol{c}(A) \text{ for some } A \in {}^{xy}\mathfrak{B}(S)\}$. First note that since ${}^{xy}\mathfrak{B}(S) \neq \emptyset$ and also that $\boldsymbol{c}$ is non-empty valued we conclude $\mathcal{T} \neq \emptyset$. Second by Step 2 it follows that $\{x, y\} \cap \mathcal{T} = \emptyset$. Finally since r is a (maximal) reference in all elements of ${}^{xy}\mathfrak{B}(S)$ Corollary 16 implies that r is never chosen in elements of ${}^{xy}\mathfrak{B}(S)$ and therefore $r \notin \mathcal{T}$. To summarize we have made the following observation: $\mathcal{T} \cap \{x, y, r\} = \emptyset$.

Step 3: $\mathcal{T} \subseteq \mathcal{I}^p(S)$.

Proof. Take $z \in \mathcal{T}$. Then there exists $A \in {}^{xy}\mathfrak{B}(S)$ such that $z \in \boldsymbol{c}(A)$. Since $x \in A$ we conclude from Proposition 3 that $z \succsim^p x$. Next $x \in \mathcal{I}^p(S)$ implies $z \in \mathcal{I}^p(S)$. □

Step 4: For all $z \in \mathcal{T}$ there exists $t \in \mathcal{I}^p(S)$ such that $t \succ^r z$.

Proof. Take $z \in \mathcal{T}$. Then there exists $A_1 \in {}^{xy}\mathfrak{B}(S)$ such that $z \in \boldsymbol{c}(A_1)$. Note that by Step 2 we have $x \notin \boldsymbol{c}(A_1)$, and therefore z beats x in A_1. Since $x \in \boldsymbol{c}(S)$, TDR implies that x is not dominated by z and therefore it has to be the case that x beats z_2 in A_2 for some $A_2 \in \mathfrak{B}(S)$.[34]

Next, since $\boldsymbol{c}$ is a RDM in A_2, and $z \notin \boldsymbol{c}(A_2)$ induction hypothesis implies $z \notin \underset{\mathcal{I}^p(A_2)}{\operatorname{argmax}} \succsim^r$ and it follows that there exists $t \in \mathcal{I}^p(A_2)$ such that $t \succ^r z$. To show that $t \in \mathcal{I}^p(S)$, note that it follows from the definition of reference relations that $t \in \boldsymbol{c}\{t, z, r\}$. Since $\boldsymbol{c}$ is a RDM on $\{t, z, r\}$ Lemma 15 implies $t \in \mathcal{I}^p\{t, z, r\}$. This means $t \succsim^p z$. Since $z \in \mathcal{I}^p(S)$ we conclude $t \in \mathcal{I}^p(S)$. □

[34] In fact, $A_2 = S - y$.

Step 5: For $z_1 \in \mathcal{T}$ there exists $z_2 \in \mathcal{T}$ such that $z_1 \succ^r z_2$.

Proof. Take $z_1 \in \mathcal{T}$. Consider the set $S - z_1$. It follows that $S - z_1 \in \mathfrak{B}(S)$. Note that $z_1 \notin \{x, y, r\}$. Therefore we conclude $S - z_1 \in {}^{xy}\mathfrak{B}(S)$. Take $z_2 \in \boldsymbol{c}(S - z_1)$.[35] Since $S - z_1 \in {}^{xy}\mathfrak{B}(S)$ it follows that $z_2 \in \mathcal{T}$.

Next we show that $z_1 \succ^r z_2$. To do this note that induction hypothesis implies $z_2 \in \underset{\mathcal{I}^p(S-z_1)}{\operatorname{argmax}} \succsim^r$ which means $z_2 \succsim^r t$ for all $t \in \mathcal{I}^p(S - z_1)$. Also by Step 4 there exists $t^* \in \mathcal{I}^p(S)$ such that $t^* \succ^r z_2$. Since $z_2 \succsim^r t$ for all $t \in \mathcal{I}^p(S - z_1)$, and also there is an element t^* in $\mathcal{I}^p(S)$ such that $t^* \succ^r z_2$, there is only one element that can be preferred to z_2 under r and that is the element which is not in $S - z_1$: z_1. This means $z_1 \succ^r z_2$. □

To finish the proof of the lemma note that since $x, y \notin \mathcal{T}$ we have $|\mathcal{T}| = t$ for $t \leq k - 1$. Take $z_1 \in \mathcal{T}$ from Step 5 there exists $z_2 \in \mathcal{T}$ such that $z_2 \succ^r z_1$. Applying Step 5 one more time we conclude that there exists $z_3 \in \mathcal{T}$ such that $z_3 \succ^r z_2$. By repeating this argument t times we conclude

$$z_1 \succ^r z_2 \succ^r \ldots \succ^r z_t \succ^r z_{t+1}$$

Since $|\mathcal{T}| = t$ this always produces a $\succsim^r$-cycle in $\mathcal{I}^p(S)$ of degree $t \leq k - 1$. This contradicts the starting claim. So proof is complete. ■

We next prove the following lemma which states this intuitive statement: *non-maximal references are irrelevant to RDM's choice behavior.*

Lemma 19. *Let $\boldsymbol{c}$ be a RDM and $S \subseteq X$. Assume $\mathfrak{R}_{_M}(S) = \{r\}$. Also Let S_r be the subset of S which is derived by removing all the non-maximal references from S. Then*

$$\boldsymbol{c}(S) = \boldsymbol{c}(S_r)$$

Proof. First note that if r is the only reference in S; that is if $|\mathfrak{R}(S)| = 1$ then $S_r = S$ and the statement is obvious. So for the rest of this argument assume $|\mathfrak{R}(S)| \geq 2$. Note that this latter fact implies $\mathfrak{B}(S) = R_1(S)$. Therefore elements of $\mathfrak{B}(S)$ are derive by taking references out of the set S. We start with the following tow claims.

Claim 1: If $x \in \boldsymbol{c}(S)$ then there exists a non-maximal reference t such that $x \in \boldsymbol{c}(S - t)$.

[35] Note that since $|S| \geq 5$ there exists an element in $S - z_1$ different from x, y, r and therefore an element z_2 exists in $\mathcal{T}$.

Proof. Take $x \in \boldsymbol{c}(S)$. First note that $\mathcal{I}^p(S-r) = \mathcal{I}^p(S)$. Since $S-r$ does note posses any maximal reference Lemma 17 implies $\boldsymbol{c}(S-r) = \mathcal{I}^p(S-r) = \mathcal{I}^p(S)$. This means x does not beat any elements of $\mathcal{I}^p(S)$ in $S-r$. Now take a reference $t_1 \neq r$ in S and consider the set $S-t_1$. If $x \in \boldsymbol{c}(S-t_1)$ there is nothing to prove. So assume $x \notin \boldsymbol{c}(S-t_1)$ and take $y \in \boldsymbol{c}(S-t_1)$. It follows that y beats x in $S-t_1$. Since $x \in \boldsymbol{c}(S)$, TDR implies that x is not dominated by y and therefore there exists a reference t_2 such that x beats y in $S-t_2$. Note that since x is beating y in $S-t_2$ we conclude $t_2 \neq r$ and therefore t_2 is a non-maximal reference. This finishes the proof of this claim. □

Claim 2: If $x \notin \boldsymbol{c}(S)$ then there exists a non-maximal reference $t \neq r$ in S such that $x \notin \boldsymbol{c}(S-t)$.

Proof. First note that $|S| \geq 4$.[36] Recall from the proof of Claim 1 that $\boldsymbol{c}(S-r) = \mathcal{I}^p(S)$. We consider two cases here.

Case 1: $x \notin \boldsymbol{c}(S-r)$.
Since $\boldsymbol{c}(S-r) = \mathcal{I}^p(S)$ this means that $x \notin \mathcal{I}^p(S)$. Now consider the set $S-t$ for a reference $t \neq r$ in S. Proposition 3 implies that $\boldsymbol{c}(S-t) \subseteq \mathcal{I}^p(S-t) = \mathcal{I}^p(S)$ and therefore we conclude $x \notin \boldsymbol{c}(S-t)$. This completes the proof in this case.

Case 2: $x \in \boldsymbol{c}(S-r)$.
First from $|S| \geq 4$ it follows that S satisfies BUR. Since $x \notin \boldsymbol{c}(S)$ we conclude that there exists $y \in S$ such that y dominates x in S. Therefore there exists a reference t in S such that $y, x \in \boldsymbol{c}(S-t)$ and y is chosen in $S-t$ and x is not chosen in $S-t$. So it only remains to prove that $t \neq r$. This follows from the fact that $x \in \boldsymbol{c}(S-r)$; that is y does not beat x in $S-r$. This completes the proof in this case. □

Now we proceed to prove the Lemma.

First we show $\boldsymbol{c}(S) \subseteq \boldsymbol{c}(S_r)$. To do this take $x \in \boldsymbol{c}(S)$. By Claim 1 there exists a non-maximal reference $t_1 \neq r$ in S such that $x \in \boldsymbol{c}(S-t)$. If $S-t = S_r$ the proof is complete. If not then there must exist another reference different from r in S and therefore $|\mathfrak{R}(S-t)| \geq 2$. Applying Claim 1 one more time we conclude there exits a non-maximal reference t_2 in S such that $x \in \boldsymbol{c}((S-t_1)-t_2)$. By repeating this algorithm for a finite number of time, and after

[36] To see this note that since r is a maximal reference in S there exists, at least two distinct elements, t_1, t_2 in $\mathcal{I}^p(S)$, both different from r. Next since $|\mathfrak{R}(S)| \geq 2$ there exists a reference $s \neq r$ in S. Since s is a reference in S it follows from Proposition 6. iv that $s \notin \mathcal{I}^p(S)$. This in turn means $t_1, t_2 \succ^p s$ and there for t_1, t_2 are different from s. Therefore we conclude t_1, t_2, r, s are four distinct elements. This means $|S| \geq 4$.

taking all non-maximal references out of S, we conclude $x \in \boldsymbol{c}(S_r)$. This means $\boldsymbol{c}(S) \subseteq \boldsymbol{c}(S_r)$.

To show $\boldsymbol{c}(S_r) \subseteq \boldsymbol{c}(S)$ we prove that $x \notin \boldsymbol{c}(S) \implies x \notin \boldsymbol{c}(S_r)$. Assume $x \notin \boldsymbol{c}(S)$. By Claim 2 there exists a non-maximal reference $t_1 \neq r$ in S such that $x \notin \boldsymbol{c}(S-t)$. If $S - t = S_r$ the proof is complete. If not then there must exist another reference different from r in S and therefore $|\mathfrak{R}(S-t)| \geq 2$. Applying Claim 2 one more time we conclude there exits a non-maximal reference t_2 in S such that $x \notin \boldsymbol{c}((S-t_1)-t_2)$. By repeating this algorithm for a finite number of time, and after taking all non-maximal references out of S, we conclude $x \notin \boldsymbol{c}(S_r)$. This implies $\boldsymbol{c}(S_r) \subseteq \boldsymbol{c}(S)$, and therefore $\boldsymbol{c}(S) = \boldsymbol{c}(S_r)$. ■

Lemma 20. *Let $\boldsymbol{c}$ be a RDM such that $\mathfrak{R}_M(S) = \{r\}$. Then*

$$c(S) = \underset{\mathcal{I}^p(S)}{\operatorname{argmax}} \succsim^r .$$

Proof. Let S_r be the subset of S which is derived by taking all the non-maximal references out of S. Note that $\mathcal{I}^p(S_r) = \mathcal{I}^p(S)$. It follows that $\boldsymbol{c}$ is a RDM on S_r and S_r only possesses one reference, r, which is maximal. Then from Lemmas 19, and 18 we conclude

$$x \in \boldsymbol{c}(S) \overset{Lem.19}{\iff} x \in \boldsymbol{c}(S_r) \overset{Lem.18}{\iff} x \in \underset{\mathcal{I}^p(S_r)}{\operatorname{argmax}} \succsim^r = \underset{\mathcal{I}^p(S)}{\operatorname{argmax}} \succsim^r .$$

Therefore the proof is complete. ■

Theorem 2. ***(Acyclicity of Reference Relations)*** *Let $\boldsymbol{c}$ be a RDM and $r \in \mathfrak{R}(S)$. $\succsim^r$ defines an acyclic relation on $\mathcal{I}^p(S)$.*

Proof. Let $t \leq |\mathcal{I}^p(S)|$. We must show that r does not possess any cycle of degree t. To do this let $A = \{x_1, x_2, \ldots, x_t\} \subseteq \mathcal{I}^p(S)$ and assume $x_1 \succ^r x_2 \succ^r \ldots \succ^r x_t$. We show $x_1 \succsim^r x_t$. Consider the set $A \cup \{r\}$. First assume that r is not a maximal reference in $A \cup \{r\}$ and consider the set $\{x_1, x_t, r\}$. Note that since $x_1 \succ^p r$ Proposition 3 implies that $r \notin \boldsymbol{c}\{x_1, x_2, r\}$. Since r is not a maximal reference in $A \cup \{r\}$ then it should be the case that $\boldsymbol{c}\{x_1, x_t, r\} = \boldsymbol{c}\{x_1, x_t\} = \{x_1, x_t\}$, which, in turn, means $x_1 \succsim^r x_t$. To finish the proof assume that r is a maximal reference in $A \cup \{r\}$. First note that $\mathcal{I}^p(A \cup \{r\}) = A$. Second, since $r \notin \boldsymbol{c}(A \cup \{r\})$ and $\boldsymbol{c}(A \cup \{r\}) \neq \emptyset$ take $x_i \in \boldsymbol{c}(A \cup \{r\})$. Then Lemma 20 implies that $x_i \in \underset{A}{\operatorname{argmax}} \succsim^r$; that is $x_i \succsim^r x_j$. Therefore $x_i = x_1$ and as a result $x_1 \succsim^r x_t$. This completes the proof. [37] ■

[37] It worths noting that $\succsim^r$, indeed, defines an acyclic relation on S. To see this assume, for $x_i \in S$, $x_1 \succ^r x_2 \succ^r \ldots \succ^r x_n$. Let $A = \{x_1, x_2, \ldots, x_n\} \cup \{r\}$. Note that $x_1 \in \underset{A}{\operatorname{argmax}} \succsim^r$ we conclude that $x_1 \in \boldsymbol{c}(A)$ which in turn, using Proposition 3 implies $x_1 \in \mathcal{I}^p(A)$. If $x_n \in \mathcal{I}^p(A)$ then Proposition 4.i implies

This next direct corollary of Theorem 2 is useful in the proof of the final theorem.

Corollary 21. *Let $\boldsymbol{c}$ be a RDM and $S \subseteq X$. For a maximal reference r is S we have* $\operatorname*{argmax}_{\mathcal{I}^p(S)} \succsim^r \neq \emptyset$.

Proof. From Proposition 4.ii and Theorem 2 we conclude $\succsim^r$ defines a complete and acylice relation on $\mathcal{I}^p(S)$. Therefore $\succsim^r$ attains a maximum on $\mathcal{I}^p(S)$ and we conclude $\operatorname*{argmax}_{\mathcal{I}^p(S)} \succsim^r \neq \emptyset$. ■

9.1.7 Proofs of Lemmas 10, 11

Lemma 10. *Let $\boldsymbol{c}$ be a RDM and $S \subseteq X$ such that $|\mathfrak{R}(S)| \geq 2$. If $x \in \boldsymbol{c}(S)$ then there exists a reference r in S such that $x \in \boldsymbol{c}(S - r)$.*

Proof. Assume $|\mathfrak{R}(S)| \geq 2$. Note that all elements of $\mathfrak{B}(S)$ are of the form $S - r$ for some references in r in S. Now take $S - r_1$ in $\mathfrak{R}(S)$. If $x \in \boldsymbol{c}(S - r_1)$ then there is nothing to prove. So take y different from x such that $y \in \boldsymbol{c}(S - r_1)$. If follows that y beats x in $S - r_1$. Since $x \in \boldsymbol{c}(S)$ and $\boldsymbol{c}$ satisfies TDR we conclude there exists $r_2 \in \mathfrak{R}(S)$ such that x beats y in $S - r_2$.[38] In particular, we have $x \in \boldsymbol{c}(S - r_2)$. ■

Lemma 11. *Let $\boldsymbol{c}$ be a RDM and $S \subseteq X$ that possesses at least two maximal references. For any reference r in S assume*

$$t^* \in \boldsymbol{c}(S - r) \iff |\mathfrak{R}^{t^*}_M(S - r)| \geq |\mathfrak{R}^t_M(S - r)|, \text{ for all } t \in \mathcal{I}^p(S - r.)$$

Also assume that $x \in \boldsymbol{c}(S)$ and $x, y \in \boldsymbol{c}(S - \bar{r})$ for a reference $\bar{r}$ in S. If $\bar{r}$ refers to y then $\bar{r}$ refers to x.

Proof. To start we partition the set $\mathcal{I}^p(S)$ into the following sets:

$$\mathcal{T}_0 = \{t \in \mathcal{I}^p(S) : t \notin \boldsymbol{c}(S - \bar{r})\},$$

$$\mathcal{T}_1 = \{t \in \boldsymbol{c}(S - \bar{r}) : \bar{r} \text{ does not refer to } t\},$$

that $x_{n-1} \in \mathcal{I}^p(A)$ which in turn, using the same proposition, implies $x_i \in \mathcal{I}^p(A)$ for all $i \in \{1, 2, \ldots, n\}$. Therefore from Theorem 2 we conclude $x_1 \succsim^r x_n$ and the proof is complete. If, otherwise, $x_n \notin \mathcal{I}^p(S)$ then we have $x_1 \succ^p x_n$ and Proposition 4.i implies that $x_n \succsim^r x_1$ can not hold and therefore $x_n \notin \boldsymbol{c}\{x_1, x_n, r\}$. Since, $r \notin \boldsymbol{c}\{x_1, x_n, r\}$ we conclude $x_1 \in \boldsymbol{c}\{x_1, x_n, r\}$ and therefore $x_1 \succsim^r x_n$.

[38] Note that such r_2 exists since $|\mathfrak{R}(S)| \geq 2$.

$$\mathcal{T}_2 = \{t \in \boldsymbol{c}(S - \bar{r}) : \bar{r} \text{ refers to } t\}.$$

Note that by Corollary 16 $r \notin \mathcal{I}^p(S)$ and, as a result, $\{\mathcal{T}_0, \mathcal{T}_1, \mathcal{T}_2\}$ partitions the set $\mathcal{I}^p(S)$.

Assume $\bar{r}$ refers to y. Note that by definition $\bar{r}$ is a maximal reference. Also note that since $y \in \boldsymbol{c}(S - \bar{r})$ it follows that $y \in \mathcal{T}_2$ and therefore $\mathcal{T}_2 \neq \emptyset$. By contradiction assume that $\bar{r}$ does not refer to x. Note that this implies $x \in \mathcal{T}_1$.

Step 1:

$$|\mathfrak{R}^z_{_M}(S)| > |\mathfrak{R}^x_{_M}(S)|, \tag{1}$$

for all $z \in \mathcal{T}_2$.

Proof. Take an arbitrary element $z \in \mathcal{T}_2$. We have x, z are chosen in $S - \bar{r}$ and therefore

$$|\mathfrak{R}^x_{_M}(S - \bar{r})| = |\mathfrak{R}^z_{_M}(S - \bar{r})|.$$

Since $\bar{r}$ refers to z and not to x, adding it to set the set $S - \bar{r}$ increases the number of z-maximal references over x-maximal references; that is

$$|\mathfrak{R}^z_{_M}(S)| > |\mathfrak{R}^x_{_M}(S)|.$$

□

Step 2: x does note beat z in A, for all $A \in \mathfrak{B}(S)$, and all $z \in \mathcal{T}_2$.

Proof. First note that since $|\mathfrak{R}(S)| \geq |\mathfrak{R}_{_M}(S)| \geq 2$ then all $A \in \mathfrak{B}(S)$ are of the form $S - r$ for some reference r in S. So take an arbitrary element $A = S - r \in \mathfrak{B}(S)$. Also take an arbitrary element $z \in \mathcal{T}_2$. From (1), and in S, the number of z-maximal references is more than the number of $x-$maximal references, and therefore dropping r will in the worse case equalize these two numbers (that worse case happens when r refers to z and not x). Therefore

$$|\mathfrak{R}^z_{_M}(S - r)| \geq |\mathfrak{R}^x_{_M}(S - r)| \tag{2}$$

To complete the proof of this step we show that $x \in \boldsymbol{c}(S - r)$ implies $z \in \boldsymbol{c}(S - r)$ and therefore x does not beat z in $S - r$. In order to do this assume $x \in \boldsymbol{c}(S - r)$. Then it follows

that

$$|\mathfrak{R}^x_{_M}(S-r)| \geq |\mathfrak{R}^t_{_M}(S-r)|, \tag{3}$$

for all $t \in \mathcal{I}^p(S-r)$. From (2) and (3) we conclude

$$|\mathfrak{R}^z_{_M}(S-r)| \geq |\mathfrak{R}^t_{_M}(S-r)|,$$

for all $t \in \mathcal{I}^p(S-r)$. This implies $z \in \boldsymbol{c}(S-r)$. Therefore we conclude x does not beat z in $S-r$. □

Step 3: There exists a x-maximal reference in S.

Proof. Since there are at least two maximal references in S we conclude that $S-\bar{r}$ has at least one maximal reference. Take $r \in \mathfrak{R}_{_M}(S-\bar{r})$, By Corollary 21 we conclude $\underset{\mathcal{I}^p(S-\bar{r})}{\operatorname{argmax}} \succsim^r \neq \emptyset$ and therefore r refers to an element of $\mathcal{I}^p(S-\bar{r})$; that is each maximal references in the set $S-\bar{r}$ refers, at least, to one element of $\mathcal{I}^p(S-\bar{r})$. Since $x \in \boldsymbol{c}(S-\bar{r})$ it follows that, in $S-\bar{r}$, the number of x-maximal references is greater than or equal to the number of t-maximal references, for all $t \in \mathcal{I}^p(S-\bar{r})$. Since there exists at least one maximal references in $S-\bar{r}$ if x has no maximal reference in $S-r$ this last assertion would be violated. Therefore there exists a x-maximal reference in $S-r$. □

As a result of Step 3 let r_x be a (maximal) reference in S that refers to x.

Step 4: $x \notin \boldsymbol{c}(S-r_x)$.

Proof. Take $z \in \mathcal{T}_2$. Recall from (1) that z has more maximal references in S than x. Since r_x refers to x dropping r_x from S results in

$$|\mathfrak{R}^z_{_M}(S-r_x)| > |\mathfrak{R}^x_{_M}(S-r_x)|.$$

It follows that $x \notin \boldsymbol{c}(S-r_x)$. □

Step 5:

$$|\mathfrak{R}^z_{_M}(S-r_x)| \geq |\mathfrak{R}^t_{_M}(S-r_x)|,$$

for all $z \in \mathcal{T}_2$, and all $t \in \mathcal{T}_0$.

Proof. Take an arbitrary element $t \in \mathcal{T}_0$ and an arbitrary element $z \in \mathcal{T}_2$. First note that since $\bar{r}$ refers to z (and r_x might or might not refer to z) then dropping $\bar{r}$ from S does not increase the number of maximal references of z relative to dropping r_x. Therefore we have

$$|\mathfrak{R}^z_{_M}(S - r_x)| \geq |\mathfrak{R}^z_{_M}(S - \bar{r})|.$$

Next recall that we have i. z is chosen in $S - \bar{r}$ and ii. t is not chosen in $S - \bar{r}$. Therefore we conclude

$$|\mathfrak{R}^z_{_M}(S - \bar{r})| > |\mathfrak{R}^t_{_M}(S - \bar{r})| \text{ (or equivalently } |\mathfrak{R}^z_{_M}(S - \bar{r})| \geq |\mathfrak{R}^t_{_M}(S - \bar{r})| + 1).$$

Finally consider the two sets $S - r_x$, and $S - \bar{r}$. Since these sets are derived by dropping only of reference from the set S it is true that the difference between the number of t-maximal references in these two sets is less then or equal to 1; that is

$$|\mathfrak{R}^t_{_M}(S - \bar{r})| + 1 \geq |\mathfrak{R}^t_{_M}(S - r_x)|.$$

Combining these assertion in order we conclude

$$|\mathfrak{R}^z_{_M}(S - r_x)| \geq |\mathfrak{R}^z_{_M}(S - \bar{r})| \geq |\mathfrak{R}^t_{_M}(S - \bar{r})| + 1 \geq |\mathfrak{R}^t_{_M}(S - r_x)|.$$

This complete the proof of this step. □

Step 6: for $z \in \mathcal{T}_2$ and $t \in \mathcal{T}_1$ we have:

$$|\mathfrak{R}^z_{_M}(S - r_x)| \geq |\mathfrak{R}^t_{_M}(S - r_x)|$$

Proof. Take an arbitrary element $z \in \mathcal{T}_2$ and an arbitrary element $t \in \mathcal{T}_1$. Note that since $\bar{r}$ refers to z (and r_x might or might not refer to z) then dropping $\bar{r}$ from S does not increase the number of maximal references of z relative to dropping r_x. Therefore we have

$$|\mathfrak{R}^z_{_M}(S - r_x)| \geq |\mathfrak{R}^z_{_M}(S - \bar{r})|.$$

Next recall z, t are chosen in $S - \bar{r}$. Therefore it follows that

$$|\mathfrak{R}^z_{_M}(S - \bar{r})| = |\mathfrak{R}^t_{_M}(S - \bar{r})|.$$

Combining these two assertions then we conclude

$$|\mathfrak{R}^z_{_M}(S-r_x)| \geq |\mathfrak{R}^z_{_M}(S-\bar{r})| = |\mathfrak{R}^t_{_M}(S-r_x)|.$$

This complete the proof of this claim. □

Step 7: There exists $z^* \in \mathcal{T}_2$ such that $z^* \in \boldsymbol{c}(S-r_x)$.

Proof. Take $z \in \boldsymbol{c}(S-r_x)$. First note that this implies Then

$$|\mathfrak{R}^z_{_M}(S-r_x)| \geq |\mathfrak{R}^t_{_M}(S-r_x)|$$

for all $t \in \mathcal{I}^p(S-r_x)$. Second Proposition 3 implies that $\boldsymbol{c}(S-r_x) \subseteq \mathcal{I}^p(S-r_x) = \mathcal{I}^p(S)$ and we conclude $z \in \mathcal{I}^p(S)$. Recall that $\{\mathcal{T}_0, \mathcal{T}_1, \mathcal{T}_2\}$ partitions the set $\mathcal{I}^p(S)$. If $z \in \mathcal{T}_2$ there is nothing to prove. So assume z in either $\mathcal{T}_0$ or $\mathcal{T}_1$; that is $z \in \mathcal{T}_0 \cup \mathcal{T}_1$. Next take $z^* \in \mathcal{T}_2$. By Steps 5, 6 we conclude

$$|\mathfrak{R}^{z^*}_{_M}(S-r_x)| \geq |\mathfrak{R}^z_{_M}(S-r_x)|,$$

Combining the last two equations we conclude

$$|\mathfrak{R}^{z^*}_{_M}(S-r_x)| \geq |\mathfrak{R}^t_{_M}(S-r_x)|,$$

for all $t \in \mathcal{I}^p(S-r_x)$. It follows that $z^* \in \boldsymbol{c}(S-r_x)$. This completes the proof of this step. □

To finish the proof of the Lemma note that by Step 7 there exists $z^* \in \mathcal{T}_2$ such that $z^* \in \boldsymbol{c}(S-r_x)$. Also from Step 4 we have $x \notin \boldsymbol{c}(S-r_x)$. This means z^* beats x in $S-r_x$. Next from Step 2 we have x does not beat z^* in all $A \in \mathfrak{B}(S)$. This implies $z^* \underset{S}{\blacktriangleright} x$. This means x is not a maximal element of $\underset{S}{\blacktriangleright}$, which in turn by TDR, implies $x \notin \boldsymbol{c}(S)$. A contradiction. ■

9.2 Independence of Axioms

In this section I show the independence of TDR, BUR, and PT using the following examples (circles represent choice). For the following three examples let $S = \{x, y, z\}$.

Example 2. Let $\boldsymbol{c}_1$ be the following choice correspondence:

$$\{\textcircled{x}, \textcircled{y}, \textcircled{z}\}$$
$$\{\textcircled{x}, y\}, \{\textcircled{y}, z\}, \{\textcircled{x}, z\}$$

There are no reference in S under $\boldsymbol{c}_1$. Therefore

$$\mathfrak{B}(S) = \big\{\{x, y\}, \{y, z\}, \{x, z\}\big\}.$$

$\boldsymbol{c}_1$, obviously, satisfies PT. Also since x is the unique maximal element of $\underset{S}{\blacktriangleright}$ and $x \in \boldsymbol{c}(S)$ we conclude that $\boldsymbol{c}_1$ also satisfies BUR. However, it does not satisfy TDR. In particular, $y \in \boldsymbol{c}(S)$ but $x \underset{S}{\blacktriangleright} y$. $\boldsymbol{c}_1$ does not satisfy WARP, because $x, z \in \{x, z\} \cap \{x, y, z\}$, $z \in \boldsymbol{c}\{x, y, z\}$, and $x \in \boldsymbol{c}\{x, z\}$, but $z \notin \boldsymbol{c}\{x, z\}$.

Example 3. Let $\boldsymbol{c}_2$ be the following choice correspondence:

$$\{\textcircled{x}, y, z\}$$
$$\{\textcircled{x}, \textcircled{y}\}, \{\textcircled{y}, z\}, \{\textcircled{x}, z\}$$

This is an example of attraction effect discussed in the paper. z is the unique reference in S under $\boldsymbol{c}_2$. Therefore

$$\mathfrak{B}(S) = \big\{\{y, z\}, \{x, z\}\big\}.$$

Obviously $\boldsymbol{c}_1$ satisfies PT. Since $\{x\} = \boldsymbol{c}(S)$ and x is not dominated by y, nor is it dominated by z, then $\boldsymbol{c}_2$ satisfies TDR. $\boldsymbol{c}_2$, however, does not satisfy BUR. To see this note that y is not dominated by x nor z and therefore is a maximal element of $\underset{S}{\blacktriangleright}$, but not chosen in S. Note that $\boldsymbol{c}_2$ is a RDM. Finally $\boldsymbol{c}_2$ does not satisfy WARP since $x, y \in \{x, y\} \cap \{x, y, z\}$, and $y \in \boldsymbol{c}\{x, y\}$, and $x \in \boldsymbol{c}\{x, y, z\}$, but $y \notin \boldsymbol{c}\{x, y, z\}$.

Example 4. Let $\boldsymbol{c}_3$ be the following choice correspondence:

$$\{\textcircled{x}, y, z\}$$
$$\{\textcircled{x}, \textcircled{y}\}, \{y, \textcircled{z}\}, \{\textcircled{x}, z\}$$

z is the unique reference in S under $\boldsymbol{c}_3$. Therefore

$$\mathfrak{B}(S) = \big\{\{y, z\}, \{x, z\}\big\}.$$

x is the unique maximal element of $\underset{S}{\blacktriangleright}$ which is chosen in S. Therefore $\boldsymbol{c}_3$ satisfies both TDR and BUR. However, it violates PT and, as a result, WARP.

The following table summarizes the aforementioned argument:

Behavior	TDR	BUR	PT	WARP
$\boldsymbol{c}_1$	×	✓	✓	×
$\boldsymbol{c}_2$	✓	×	✓	×
$\boldsymbol{c}_3$	✓	✓	×	×

9.3 RDM: Majority Rule Vs. Condorcet Criterion

The notion of RDM can be conveniently perceived as a voting situation where candidate are represented by the elements of the most preferred class, $\mathcal{I}^p(S)$, and voters are references whose rankings of the candidates are given with the associated reference preference. In the view of the result in Theorem 3, then, RDM is consistent with the majority rule; that is: *the candidate who has the majority of the votes is elected.* RDM, however, is not consistent with the Condorcet criterion. To see this let $S = \{x_1, x_2, x_3, x_4, r_1, r_2, r_3\}$, where r_i is a reference (voters), and $\{x_1, x_2, x_3, x_4\} = \mathcal{I}^p(S)$ (candidates). Assume the following preference for each voter:

$$r_1 : x_1 \succ^{r_1} x_2 \succ^{r_1} x_3 \succ^{r_1} x_4$$

$$r_2 : x_2 \succ^{r_2} x_3 \succ^{r_2} x_4 \succ^{r_2} x_1$$

$$r_3 : x_3 \succ^{r_3} x_2 \succ^{r_3} x_4 \succ^{r_3} x_1$$

From Theorem 3 it follows that $\boldsymbol{c}(S) = \{x_1, x_2, x_3\}$. However, x_1 does not satisfy the Condorcet criterion. In particular, if restricted to only x_1, x_4, the latter has more votes than the former.