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Rational Filters

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Abstract

It has been widely documented that reference points influence the choice. If references affect choice by attracting attention towards an alternative, what can be said about the joint effect of the references? Assuming that references form preferences, or are *rational filters*, this paper extracts a reference-dependent choice model with joint referential effects from WARP. Assume that a DM has an “unfiltered” pairwise preference which is inferred from her choice over doubletons. The DM’s uses her rational filters consecutively on the pairwise preference in order to finalize her choice.

1 Introduction

Reference-dependent choice model have received quiet an attention in the literature as they help explaining behavioral anomalies. One explanation for such effects is through relaxing *independence of irrelevant alternatives*. That is a third alternative, a reference point, can filter the naive pairwise preference between two alternative. This is indeed the essence of the rational shortlist method introduced in [Manzini and Mariotti \(2007\)](#), where a DM uses preference relations consecutively to rationalize her choice. To clarify, consider the following scenario where a DM is choosing from Brands A-E with the following pairwise preference:

$$[A \sim^p B \sim^p C] \succ^p D \sim^p E$$

Next assume from the respective of pairwisely dominated Brand D the follwoign rational filtering is observed:

$$A \succ^D B \succ^D C.$$

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and that Brand E induces the following rational filtering on the most favorite alternatives:

$$B \sim^E C \succ^E A.$$

The choice under classical rationality assumption, induced by *the weak axiom of revealed preferences*, should be consistent with the pairwise choice and therefore consists of all elements A, B, C . However, under presence of the rational filter D choice will be filtered to only A . On the other hand, and in the presence of E the choice should be refined to B, C . What would be the “joint” effect of filters D, E ? In particular, if the filters work sequentially then the choice of C is not rationalizable, since after surviving the pairwise comparison, and E filtering, C is ranked lower than B under D .

The rational filters in [Manzini and Mariotti \(2007\)](#) are exogenously given. The purpose of this paper is to produce an axiomatic foundations for decision making where such sequential rational filtering is *endogenously* produced. The approach taken in this paper is akin to the one in [Nosratabadi \(2017\)](#). I show that how an extension of the model in that paper can help to introduce the notion of join referential effect. Such phenomenon is absent in the results in [Nosratabadi \(2017\)](#). The referential effects in this work are completely separable. This is formalized in the main results of ht paper: *the choice in any arbitrary set is characterized with the alternative that has the highest number of reference preferences that puts it at top*. Obviously, from such a view, the only important information that a DM receives from reference preferences are the top class and ignores the information about the ranking that goes beyond the top class.

1.1 Related Literature

The approaches that have been taken can are divided in two categories. rational, and behavioral. In the latter category the models developed in [Masatlioglu, Nakajima, and Ozbay \(2012\)](#) and [Ok, Ortovela, and Riella \(2015\)](#) capture the concept using the behavioral notion of “inattention”. In the former category the results in [Cherepanov, Feddersen, and Sandroni \(2013\)](#) and [Nosratabadi \(2017\)](#) provided explanation where rationals (reference preference) are formed, respectively, exogenously and endogenously.

2 Preliminaries

Let X be a finite set. X is the set of all “relevant” alternatives for the DM. Therefore, it contains not only the concrete options available to the DM, but also, for example, alternatives that she has chosen before, or phantom alternatives that are not available to choose but

presented to her (e.g., items that are out of stock, or shows that are sold out). In terms of the nature of the elements, X might be alternatives available for grocery shopping, different colleges to attend, various policies to be followed by the policy maker, etc. Let 2^X be the power set of X . Also let

$$\mathfrak{X}^k := \{A \subseteq X : |A| = k\};$$

that is the set of all subsets of X with cardinality equal to k , and

$$\mathfrak{X}^{\geq k} := \{A \subseteq X : |A| \geq k\};$$

that is the set of all subsets of X with cardinality of at least k . In order to simplify the domain of the discussion on choice I only consider the sets that have at least two elements, as the choice over the empty set and the singletons are trivially interpreted. Therefore let $\mathfrak{X} := \mathfrak{X}^{\geq 2}$. A *choice correspondence* on X is a function $\mathbf{c} : \mathfrak{X} \rightarrow 2^X$ such that for all $A \in \mathfrak{X}$ we have $\mathbf{c}(A) \subseteq A$. \mathbf{c} is called a *non-empty valued choice correspondence* if $\mathbf{c}(A) \neq \emptyset$ for all $A \subseteq X$. We make the common notational abuses:

$$\mathbf{c}\{x, y, z\} := \mathbf{c}(\{x, y, z\}) \quad \text{and} \quad \mathbf{c}\{x, y\} := \mathbf{c}(\{x, y\}),$$

for all $x, y, z \in X$.

Let $S \subseteq X$. Unless otherwise stated, whenever used throughout this paper let $S \in \mathfrak{X}^{\geq 3}$; that is let S have at least three elements. Similar to X , for S let

$$\mathfrak{S}^k := \{A \subseteq S : |A| = k\} \quad \text{and} \quad \mathfrak{S}^{\geq k} := \{A \subseteq S : |A| \geq k\}.$$

For $x \in S$ let $S - x := S \setminus \{x\}$; that is the set which is derived by removing x from S .

A binary relation R on X is a subset of $X \times X$. Let \mathcal{R} be the asymmetric relation derived from R ; that is

$$x\mathcal{R}y \iff xRy \text{ and } \neg(yRx).$$

A *cycle* of order k in R is a set $\{x_1, x_2, \dots, x_k\}$ with $x_i \in X$ such that

$$x_1 \mathcal{R} x_2 \mathcal{R} \dots \mathcal{R} x_k \mathcal{R} x_1.$$

R is said to be *acyclic* if it does not possess any cycle of any order. A *preference relation* on X is a binary relation which is transitive and complete. For a binary relation R on X , and

$S \subseteq X$, x is called a *maximum* element of R on S if

$$xRy : \forall y \in S.$$

Let

$$\operatorname{argmax}_S R := \{x \in S : x \text{ is a maximum for } R \text{ on } S\};$$

x is called a *maximal* element of R on S if there does not exist $y \in A$ such that yRx , where \mathcal{R} is the asymmetric relation derived from R .

A *cover* for $S \subseteq X$ is a family of sets, $\{A_i\}_{i=1}^n$ such that $A_i \subseteq S$ for all i and

$$S = \bigcup_{i=1}^n A_i.$$

For a choice correspondence \mathbf{c} define the relation \succsim^p on X by

$$x \succsim^p y \iff x \in \mathbf{c}\{x, y\}.$$

Let \succ^p and \sim^p be asymmetric and symmetric parts of \succsim^p . Note that \succsim^p matches the notion of revealed preference in the sense of Samuelson (1938). We call \succsim^p the *pairwise revealed preference* throughout this paper. We next define the key notion of references.

In order to extend the model in Nosratabadi (2017) to the case where reference has correlated effect, it turns out, an expansion of the consideration sets with which a DM makes here choice does the job. To do this let

$$S^- = \{S - x : x \in S\}.$$

Note that here a DM considers her respective choice in *all* of the first-order diminished sets. Next following the same idea introduced in Nosratabadi (2017) I introduce the axioms to be used in the decomposition theorem by introducing the concepts of beating and dominance.

Definition 1. (Beating) Let $S \subseteq X$. For $x, y \in S$ we say x *beats* y in S whenever $x \in \mathbf{c}(S)$ and $y \notin \mathbf{c}(S)$.

Definition 2. (Dominance) Let $S \subseteq X$. We say x *dominates* y in S and we write

$$x \blacktriangleright_S y,$$

if there exists $\bar{A} \in S^-$ such that x beats y in \bar{A} , and there does not exist $A \in S^-$ such that y beats x in A .

3 Choice Axioms

I start this section with a formal statement of WARP.

Axiom 0. (Weak Axiom of Revealed Preferences - WARP): We say a choice correspondence \mathbf{c} satisfies WARP if for $S_1, S_2 \subseteq X$ such that $x, y \in S_1 \cap S_2$ we have

$$x \in \mathbf{c}(S_1) \text{ and } y \in \mathbf{c}(S_2) \text{ implies } x \in \mathbf{c}(S_2).$$

Axiom 1. (Top-Down Rationality - TDR) We say S satisfies TDR if

$$x \in \mathbf{c}(S) \implies x \text{ is a maximal element of } \blacktriangleright_S.$$

We say a choice correspondence \mathbf{c} satisfies TDR on S if all $A \in \mathfrak{G}^{\geq 3}$ satisfy TDR. If $S = X$, then we simply say \mathbf{c} satisfies TDR.

Axiom 2. (Bottom-Up Rationality - BUR) We say S satisfies BUR if

$$x \text{ is a maximal element of } \blacktriangleright_S \implies x \in \mathbf{c}(S).$$

We say a choice correspondence \mathbf{c} satisfies BUR on S if all $A \in \mathfrak{G}^{\geq 3}$ satisfy BUR. If $S = X$, then we simply say \mathbf{c} satisfies BUR.

Axiom 3. (Pairwise Transitivity - PT) We say a choice correspondence \mathbf{c} satisfies PT on S if for all $x, y, z \in S$

$$x \succsim^p y \text{ and } y \succsim^p z \text{ implies } x \succsim^p z.$$

If $S = X$, then we simply say \mathbf{c} satisfies PT.

Note that PT implies transitivity of both \succ^p and \sim^p . To conclude this section note that from the structure of my axioms it follows that if \mathbf{c} satisfies TDR (resp. BUR, PT), then it satisfies TDR (resp. BUR, PT) on all $S \subseteq X$; that is these rationales are induced from larger menus to smaller ones.

It turns out the same decomposition theorem works in this extended version of referential revealed preference theory. This theorem is presented here.

Theorem 1. (Decomposition Theorem) *Let \mathbf{c} be a choice correspondence. Then the following are equivalent:*

- (i) \mathbf{c} is non-empty valued and satisfies WARP
- (ii) $\mathbf{c}\{x, y\} \neq \emptyset$ for all $x, y \in X$ and \mathbf{c} satisfies TDR, BUR, and PT.

4 Classical and Referential Decision Makers

Following [Nosratabadi \(2017\)](#) let me use the notation $\mathcal{I}^p(S) =: \operatorname{argmax}_S \succsim^p$. $\mathcal{I}^p(S)$ is, therefore, the set of best alternatives in S from the perspective of the pairwise revealed preference. S is called *fully indecisive* if $S \subseteq \mathcal{I}^p(S)$.

Definition 3. (Referential DM: RDM) Let

$$\mathfrak{A} = \{A \in \mathfrak{G}^3 \text{ such that } A \text{ is not fully indecisive}\}.$$

A choice correspondence \mathbf{c} is called a *RDM* on S if

- (i) \mathbf{c} satisfies TDR and PT on S
- (ii) \mathbf{c} satisfies BUR for all $A \in \mathfrak{G}^{\geq 3} \setminus \mathfrak{A}$.

If $S = X$, then we simply say \mathbf{c} is a RDM.

The basic properties of a RDM follows directly from the results in [Nosratabadi \(2017\)](#). In particular, references are indeed elements of a lower indifference class and change the relative importance of two more preferable alternatives.¹ Also the decision maker still chooses an alternative from the most preferable indifference class. That is $\mathbf{c}(S) \subseteq \mathcal{I}^p(S)$ for all $S \subseteq X$.²

5 Reference Transitivity

Definition 4. (References) For a choice correspondence \mathbf{c} and $S \subseteq X$ we say r is a *reference* in S if there exists two distinct elements $x, y \in S$, both different from r such that

$$\mathbf{c}\{x, y, r\} \subset \mathbf{c}\{x, y\}.$$

We say r is a *maximal* reference if there exists $x, y \in \mathcal{I}^p(S)$ such that

$$\mathbf{c}\{x, y, r\} \subset \mathbf{c}\{x, y\}.$$

Definition 5. (Reference Relation) Let $S \subseteq X$ and $r \in \mathfrak{R}(S)$. For two distinct element $x, y \in S$, both different from r , define

$$x \succsim^r y \iff x \in \mathbf{c}\{x, y, r\}.$$

Also let \succ^r and \sim^r be the asymmetric and symmetric parts of \succsim^r .

¹See Proposition ??? in [Nosratabadi \(2017\)](#)

²See Proposition ??? in [Nosratabadi \(2017\)](#).

As shown in [Nosratabadi \(2017\)](#) if \mathbf{c} is non-empty valued then \succsim^r defines a complete and acyclic binary relation on $\mathcal{I}^p(S)$.³ In order to make the choice of a DM more structured I strengthen the latter result by enforcing transitivity of reference as an axiom.

Axiom 4. (Reference Transitivity - RT): \succsim^r defines a transitive binary relation on $\mathcal{I}^p(S)$.

6 Main Results

The referential effects produced in [Nosratabadi \(2017\)](#) are completely separable. In particular, the from Theorem ??? it follows that a choice in a set is characterized by the element(s) that have the “most number” of references. That is the effect of references from the respective of the DM are completely separable. Let me consider a example here. Suppose that the menu in a restaurant included the item $\{x, y, z, r_1, r_2\}$. DM’s preferences on the menu is defined by

$$x \sim^p y \sim^p z \succ^p r_1 \sim^p r_2.$$

Next assume that r_1, r_2 are maximal referneces that produce the following reference preference relation over the most favorable alternatives; that is $\mathcal{I}^p(S) = \{x, y, z\}$:

$$r_1 : x \succ^{r_1} y \succ^{r_1} z,$$

$$r_2 : y \sim^{r_2} z \succ^{r_2} x.$$

That is from the perspective of r_1 DM likes x the best, and from the perspective of r_2 does y, z . If DM completely separates the referential effects then the choice in the menu would consist of $\{x, y, z\} = \mathcal{I}^p(S)$. That is DM’s acts as her behavior was satisfying WARP. On the other hand it is viable to think that a DM considers the *join* effects of the references. That is after adopting to a reference point r_1 she further “filters” her choice by adopting to r_2 and vice versa.⁴ In this example under r_1 the unique most preferable alternative is x and applying r_2 does not change the choice. However, y, z are most preference under r_2 but $y \succ^{r_1} z$. If referential effects are joint then the choice of z is not desirable from the DM.

To make this formal, let $S \subseteq X$ be a set that has at most two maximal reference. Let

³See Proposition ??? and Theorem ??? in [Nosratabadi \(2017\)](#)

⁴This indeed is generalizing the notion of sequential rationality introduced in [Manzini and Mariotti \(2007\)](#)

$M_1(S) = \operatorname{argmax}_{\mathcal{I}^p(S)} \succsim^{r_1}$. For a reference r_1 , define

$$r_{12}(S) = \operatorname{argmax}_{M_1(S)} \succsim^{r_2},$$

and respectively,

$$r_{21}(S) = \operatorname{argmax}_{M_2} \succsim^{r_1}.$$

Also define

$$\mathfrak{r}(S) = r_{12}(S) \cup r_{21}(S).$$

Note that if there are no maximal references in S then $\mathfrak{r}(S) = \mathcal{I}^p(S)$. Also Let $\operatorname{argmax}_{\mathcal{I}^p(S)} \succsim^{r_1 r_2} = \operatorname{argmax}_{\mathcal{I}^p(S)} \succsim^{r_1} \cap \operatorname{argmax}_{\mathcal{I}^p(S)} \succsim^{r_2}$. The next theorem formalized the notion of joint referential effect.

Theorem 2. *Let $S \subseteq X$ have at most two maximal references. Then*

$$\mathfrak{c}(S) = \mathfrak{r}(S).$$

The results in Theorem 2 shows the endogenous formation of the notion of *rational short-list model* introduced in [Manzini and Mariotti \(2007\)](#). The two preferences are exogenously given in their work. From the point of view of joint referential effects introduce in this paper, a rational referential DM has a “naive” preference which is free of referential effects. Such preference however get updated adopting to reference points one after the other. The order does not matter here and the results are consisted of both directions of the reference influence.

7 Appendix

7.1 Proof of Decomposition Theorem

Since under WARP there are no reference, therefore $\mathfrak{B}(S)$ in [Nosratabadi \(2017\)](#) is indeed S^- and the proof coincidences with the proof of Theorem 1 in [Nosratabadi \(2017\)](#).

7.2 Proof of Theorem 2

7.2.1 The Case of No Maximal Reference

Lemma 6. *Let \mathfrak{c} be a RDM. If $x \in \mathfrak{c}\{x, y, z\}$ for all pair of distinct elements $y, z \in S$, both different from x , then $x \in \mathfrak{c}(S)$.*

Proof. We prove this by induction on $|S|$. For $|S| = 3$ there is nothing to prove. Assume that the statement is true for all the sets with cardinality k . Let $|S| = k + 1$ and take $x \in S$ be such that $x \in \mathbf{c}\{x, y, z\}$ for all two distinct elements $y, z \in S$, both different from x . Let

$${}^xS^- = \{A \in S^- : x \in A\}.$$

Obviously ${}^xS^- \neq \emptyset$. Take $A \in {}^xS^-$. Note that since

$$x \in \mathbf{c}\{x, y, z\},$$

for two distinct elements $y, z \in S$ and, since $A \subseteq S$, we conclude that

$$x \in \mathbf{c}\{x, y, z\},$$

for two distinct elements $y, z \in A$, both different from x . Finally induction assumption implies that $x \in \mathbf{c}(A)$. Since A was an arbitrary element of ${}^xS^-$ we conclude x is chosen in all elements of ${}^xS^-$. This means x is not dominated by any element of S . Therefore BUR implies that $x \in \mathbf{c}(S)$. ■

Lemma 7. *If $\mathfrak{R}_M(S) = \emptyset$ then $\mathbf{c}(S) = \mathcal{I}^p(S)$.*

Proof. Assume $\mathfrak{R}_M(S) = \emptyset$. We need to show $\mathbf{c}(S) = \mathcal{I}^p(S)$. First note that in the proof of Lemma ??? (\Rightarrow) we only used TDR. Since an RDM satisfies TDR we conclude that from Lemma ??? (\Rightarrow) that $\mathbf{c}(S) \subseteq \mathcal{I}^p(S)$. So we only need to prove $\mathcal{I}^p(S) \subseteq \mathbf{c}(S)$. Take $x \in \mathcal{I}^p(S)$ and consider the set $\{x, y, z\}$ for two distinct elements $y, z \in S$, both different from x .

Step 1: $\mathfrak{R}\{x, y, z\} = \emptyset$.

Proof. By contradiction assume $\mathfrak{R}\{x, y, z\} \neq \emptyset$. First note that by Proposition ??ii it has to be the case that there is only one reference in $\{x, y, z\}$. Second by Corollary ?? and the fact that $x \in \mathcal{I}^p(S)$ we conclude $x \notin \mathfrak{R}(S)$. Wlog, assume z is the reference in $\{x, y, z\}$. Then it has to be the case that

$$\mathbf{c}\{x, y, z\} \subset \mathbf{c}\{x, y\},$$

which implies $\mathbf{c}\{x, y\} = \{x, y\}$, which in turn implies $y \succ^p x$. Since $x \in \mathcal{I}^p(S)$ PT implies $y \in \mathcal{I}^p(S)$. This means $z \in \mathfrak{R}_M(S)$ which is not possible. This completes the proof of Step 1. □

Step 2: $x \in \mathbf{c}\{x, y, z\}$.

Proof. From Step 1 we have $\mathfrak{R}\{x, y, z\} = \emptyset$. Therefore

$$\mathfrak{B}\{x, y, z\} = \{\{x, y\}, \{y, z\}, \{x, z\}\}.$$

We first argue that y or z can not be the single choice in $\mathbf{c}\{x, y, z\}$. To do this, and wlog, assume $\mathbf{c}\{x, y, z\} = \{y\}$. Then TDR implies $y \succsim^p x$ which in turn means $x, y \in \mathcal{I}^p(S)$. Then we conclude that

$$\mathbf{c}\{x, y, z\} \subset \mathbf{c}\{x, y\},$$

which means z is a maximal reference in S which is impossible. Second assume $\{y, z\} \subseteq \mathbf{c}\{x, y, z\}$. We show that $x \in \mathbf{c}\{x, y, z\}$. From TDR we conclude $y, z \succsim^p x$ and, since $x \in \mathcal{I}^p(S)$, it follows that $x, y, z \in \mathcal{I}^p(S)$ which means $\{x, y, z\}$ is fully indecisive. Using Proposition ??i we conclude $\mathbf{c}\{x, y, z\} = \{x, y, z\}$ which obviously means $x \in \mathbf{c}\{x, y, z\}$. Since $\mathbf{c}\{x, y, z\} \neq \emptyset$ the proof of this Step 2 is complete. \square

To finish the proof note that by Lemma 6.ii we conclude $x \in \mathbf{c}(S)$. This means $\mathcal{I}^p(S) \subseteq \mathbf{c}(S)$. Therefore $\mathbf{c}(S) = \mathcal{I}^p(S)$. \blacksquare

7.3 The Case of Single Maximal Reference

Lemma 8. *If $\mathfrak{R}_M(S) = \{r\}$ then $\mathbf{c}(S) = \operatorname{argmax}_{\mathcal{I}^p(S)} \succsim^r$.*

Proof. We prove this by induction on $|S|$. For $k = 3$ assume $S = \{x_1, x_2, r\}$. Obviously $x \in \mathbf{c}\{x_1, x_2, r\}$ if and only if $x \in \operatorname{argmax}_{\mathcal{I}^p(S)} \succsim^r$. Now assume for $|S| = k$ the statement is true and let $|S| = k + 1$. First note that $\mathbf{c}(S) \subseteq \mathcal{I}^p(S)$. Take $x \in \mathbf{c}(S)$. First note that by Theorem 7 we conclude that $\mathbf{c}(S - r) = \mathcal{I}^p(S)$. Assume there exists $y \in S$ and $A \in S^-$ such that y beats x in A . Therefore $r \in A$ and by induction assumption $y \succ^r x$. Note that $y \in \mathcal{I}^p(S)$. This means y beats x in all elements of S^- except $S - r$. Since $y \in \mathbf{c}(S - r)$ then we conclude x does not beat y in any elements of S^- and therefore $y \blacktriangleright_S x$. This contradicts the assumption that $x \in \mathbf{c}(S)$ and \mathbf{c} satisfies TDR. This establishes that $\mathbf{c}(S) \subseteq \operatorname{argmax}_{\mathcal{I}^p(S)} \succsim^r$.

To show the other inclusion take $x \in \operatorname{argmax}_{\mathcal{I}^p(S)} \succsim^r$. Note that by induction assumption and since $x \in \operatorname{argmax}_{\mathcal{I}^p(S^-)} \succsim^r$ for all $t \neq r$ we conclude $x \in \mathbf{c}(S - r)$. Also since $x \in \mathcal{I}^p(S)$ we conclude from Theorem 7 that $x \in \mathbf{c}(S - r)$. Now BUR implies $x \in \mathbf{c}(S)$. This establishes that $\operatorname{argmax}_{\mathcal{I}^p(S)} \succsim^r \subseteq \mathbf{c}(S)$. Therefore the proof is complete. \blacksquare

7.3.1 The Case of Double Maximal References

Lemma 9. (WARP Lemma) Let \succsim be a preference relation on $S \subseteq X$. For $A \subseteq B \subseteq S$ we have

$$\operatorname{argmax}_B \succsim \cap A = \operatorname{argmax}_A \succsim,$$

whenever $\operatorname{argmax}_B \succsim \cap A \neq \emptyset$.

Proof. Take $x \in \operatorname{argmax}_B \succsim \cap A$. Obviously $x \in A$. Also $x \succsim y$ for all $y \in B \supseteq A$ and therefore $x \in \operatorname{argmax}_A \succsim$. To see the other inclusion take $x \in \operatorname{argmax}_A \succsim$. Since $\operatorname{argmax}_B \succsim \cap A \neq \emptyset$ also take $y \in \operatorname{argmax}_B \succsim \cap A$. Since $y \in A$ and $x \in \operatorname{argmax}_A \succsim$ we conclude $x \succsim y$. Since $y \in \operatorname{argmax}_B \succsim$ it follows that $y \succsim t$ for all $t \in B$. Therefore transitivity of \succsim implies $x \succsim t$ for all $t \in B$, which in turn means $x \in \operatorname{argmax}_B \succsim$. Finally $x \in A$ and we conclude $x \in \operatorname{argmax}_B \succsim \cap A$. \blacksquare

Lemma 10. The following are true:

(i) $x \in \mathfrak{r}(S)$ then $x \in \operatorname{argmax}_{\mathcal{I}^p(S)} \succsim^{r_i}$ for some $i \in \{1, 2\}$.

(ii) For $x \in \mathfrak{r}(S)$ if $y \succ^{r_1} x$ then $x \succ^{r_2} y$.

(iii) For $x, y \in r_{12}(S)$ we have $x \sim^{r_i} y$ for all $i \in \{1, 2\}$.

Proof. (i) Wlog, assume that $x \in r_{12}(S)$. Then

$$x \in \operatorname{argmax}_{M_1(S)} \succsim^{r_2}$$

this means $x \in M_1(S) = \operatorname{argmax}_{\mathcal{I}^p(S)} \succsim^{r_1}$

(ii) Since $y \succ^{r_1} x$ then we conclude that $x \notin M_1(S)$ and therefore $x \notin r_{12}(S)$. This means $x \in r_{21}(S) = \operatorname{argmax}_{M_2(S)} \succsim^{r_1}$. Since $y \succ^{r_1} x$ then we conclude $y \notin M_2(S)$, which by transitivity of \succsim^{r_2} and the fact that $x \in M_2(S)$ implies $x \succ^{r_2} y$.

(iii)

$$x, y \in r_{12}(S) = \operatorname{argmax}_{M_1(S)} \succsim^{r_2}$$

This, first means $x, y \in M_1(S)$ and therefore $x \sim^{r_1} y$, second, $x \sim^{r_2} y$. \blacksquare

Lemma 11. *The followings are true:*

i. *If $x_i \in \mathbf{r}(S)$ then $x_i \in \mathbf{r}(S - x_t)$ for $t \neq i$.*

ii. *$\operatorname{argmax}_{\mathcal{I}^p(S)} \succsim^{r_1 r_2} \neq \emptyset$ if and only if $\mathbf{r}(S) = \operatorname{argmax}_{\mathcal{I}^p(S)} \succsim^{r_1 r_2}$*

Proof. i. Take $x_i \in \mathbf{r}(S)$ and $t \neq i$. Wlog assume $x_i \in r_{12}(S)$, so $x_i \in \operatorname{argmax}_{\mathcal{I}^p(S)} \succsim^{r_1}$.

First note that by WA Lemma we conclude

$$x_i \in \operatorname{argmax}_{\mathcal{I}^p(S)} \succsim^{r_1} \cap \mathcal{I}^p(S - x_t) = \operatorname{argmax}_{\mathcal{I}^p(S-x_t)} \succsim^{r_1}.$$

Let $N(S) = \operatorname{argmax}_{\mathcal{I}^p(S)} \succsim^{r_1}$ and $N(S - x_t) = \operatorname{argmax}_{\mathcal{I}^p(S-x_t)} \succsim^{r_1}$. Note that $x_i \in N(S-x_t) \cap N(S)$.

This latter fact means $N(S - x_t) \subseteq N(S)$. To see that note take $x_j \in N(S - x_t)$. This means $x_j \succsim^{r_1} x$. Since $x_i \in N(S)$ and $x \in S$ from transitivity of \succsim^{r_1} it follows that $x_j \in N(S)$. Next using WA Lemma one more time we conclude

$$x_i \in \operatorname{argmax}_{N_1(S)} \succsim^{r_2} \cap N_1(S - x_t) = \operatorname{argmax}_{N_1(S-x_t)} \succsim^{r_2} = r_{12}(S - x_t) \subseteq \mathbf{r}(S - x_t)$$

ii. (\Leftarrow): Since $\mathbf{r}(S) \neq \emptyset$ then this is obvious.

(\Rightarrow): Now assume $\operatorname{argmax}_{\mathcal{I}^p(S)} \succsim^{r_1 r_2} \neq \emptyset$ and take $x_i \in \operatorname{argmax}_{\mathcal{I}^p(S)} \succsim^{r_1 r_2}$. Then we have $x_i \in M_1(S)$ and since $M_1(S) \subseteq \mathcal{I}^p(S)$ by WA Lemma

$$x_i \in \operatorname{argmax}_{\mathcal{I}^p(S)} \succsim^{r_2} \cap M_1(S) = \operatorname{argmax}_{M_1(S)} \succsim^{r_2} = r_{12}(S) \subseteq \mathbf{r}(S).$$

This establishes that $\operatorname{argmax}_{\mathcal{I}^p(S)} \succsim^{r_1 r_2} \subseteq \mathbf{r}(S)$. Now take $x_i \in \mathbf{r}(S)$. By contradiction

assume $x_i \notin \operatorname{argmax}_{\mathcal{I}^p(S)} \succsim^{r_1 r_2}$. Since this latter set is non-empty take $x_j \in \operatorname{argmax}_{\mathcal{I}^p(S)} \succsim^{r_1 r_2}$.

Wlog, it follows that $x_j \succ^{r_1} x_i$. This means $x_i \notin M_1(S)$ which in turn means $x_i \notin r_{12}(S)$. Also since $x_j \in M_2(S)$ and $x_j \succ^{r_1} x_i$ it follows that $x \notin r_{21}(S)$ and therefore $x_i \notin \mathbf{r}(S)$. This is a contradiction. This establishes that $x_i \in \operatorname{argmax}_{\mathcal{I}^p(S)} \succsim^{r_1 r_2}$ and

therefore $\mathbf{r}(S) = \operatorname{argmax}_{\mathcal{I}^p(S)} \succsim^{r_1 r_2}$

■

Theorem 2. *Let $S \subseteq X$ have at most two maximal references. Then*

$$\mathbf{c}(S) = \mathbf{r}(S)$$

Proof. Let $S = \{x_1, x_2, \dots, x_n, r_1, r_2\}$. We prove the result by induction on n .

Induction Base: (\Rightarrow) For $k = 2$ assume $S = \{x_1, x_2, r_1, r_2\}$ and assume wlog $x_1 \in \mathbf{c}(S)$. If $x_1 \succ^{r_i} x_2$ for all $i \in \{1, 2\}$ then $x \in \operatorname{argmax}_{\mathcal{I}^p(S)} \succ^{r_i}$ for all $i \in \{1, 2\}$ and therefore $x \in \mathbf{r}(S)$. So assume there exists $i \in \{1, 2\}$ such that $x_2 \succ^{r_i} x_1$. Then TDR implies $x_1 \succ^{r_j} x_2$ for $j \neq i$ which implies $x_1 \in r_{ji}(S) \subseteq r_i r_j(S)$.

(\Leftarrow) Now assume $x_1 \in \mathbf{r}(S)$. We show that x_2 does not dominate x_1 in S (that is we show that $x_2 \blacktriangleright_S x_1$ is not true). For this assume, wlog, $x_2 \succ^{r_1} x_1$, then $x_1 \notin r_{12}(S)$ which implies $x_1 \in r_{21}(S)$. This means $x_1 \succ^{r_2} x_2$. Therefore BUR implies $x_1 \in \mathbf{c}(S)$.

Induction Assumption: Assume that for S with $n - 1$ non-maximal reference elements the statement is true and assume S has n non-maximal reference elements.

(\Rightarrow): Let $x_i \in \mathbf{c}(S)$.

Claim: $x_i \in \operatorname{argmax}_{\mathcal{I}^p(S)} \succ^{r_i}$ for some $i \in \{1, 2\}$.

Proof. By contradiction assume that $x_i \notin \operatorname{argmax}_{\mathcal{I}^p(S)} \succ^{r_t}$ for all $t \in \{1, 2\}$. Take $x_j \in \mathbf{r}(S)$. Then Lemma 11.ii and for all $t \notin \{i, j\}$ we conclude from induction assumption that

$$x_i \in \mathbf{c}(S - x_t).$$

Next note that since $x_j \in \mathbf{r}(S)$ then Lemma 10.ii, and wlog, $x_j \in \operatorname{argmax}_{\mathcal{I}^p(S)} \succ^{r_1}$. Since $x_i \notin \operatorname{argmax}_{\mathcal{I}^p(S)} \succ^{r_1}$ Then Lemma 8 implies that $\mathbf{c}(S - r_2) \cap \{x_i, x_j\} = \{x_j\}$. This means that x_j beats x_i in $S - r_2$. By the argument before we also conclude that x_i does not beat x_j in all $A \in S^-$ and therefore $x_j \blacktriangleright_S x_i$. This contradicts $x_i \in \mathbf{c}(S)$ and the fact that \mathbf{c} satisfies TDR. \square

To complete the proof we consider two cases:

Case I: If $x_i \in \operatorname{argmax}_{\mathcal{I}^p(S)} \succ^{r_1 r_2}$ by Lemma 11.ii then $x_i \in \mathbf{r}(S)$ and the proof is complete.

Case II: Wlog, assume that $x_i \in \operatorname{argmax}_{\mathcal{I}^p(S)} \succ^{r_1}$ and $x_i \notin \operatorname{argmax}_{\mathcal{I}^p(S)} \succ^{r_2}$. Note that the latter, by Lemma 8 implies that $x_i \notin \mathbf{c}(S - r_1)$. Obviously $x_i \notin r_{21}(S)$. By contradiction assume $x_i \notin r_{12}(S)$. Take $x_j \in \operatorname{argmax}_{\mathcal{I}^p(S)} \succ^{r_1}$ such that $x_j \in r_{12}(S)$. Note that since $r_{12}(S) \neq \emptyset$

such x_j exists. Next note that it has to be the case that $x_j \succ^{r_2} x_i$. Now consider the set $S - x_t$ for $t \notin \{i, j\}$. By Lemma 11.i $x_j \in \mathbf{r}(S - x_t)$. Also $x_i \notin \mathbf{r}(S - x_t)$ which by induction hypothesis implies $x_i \notin \mathbf{c}(S - x_t)$. This means x_j beats x_i in $S - x_t$. Since $x_j, x_i \in \mathbf{c}(S - r_2)$ and $x_i \notin \mathbf{c}(S - r_1)$ it follows that x_i does not beat x_j in any elements of S^- which in turn means $x_j \blacktriangleright_S x_i$. This contradicts the fact that $x_i \in \mathbf{c}(S)$ and \mathbf{c} satisfies TDR.

(\Leftarrow): Now assume that $x_i \in \mathbf{r}(S)$. First note that by Lemma 11.i $x_i \in \mathbf{r}(S - x_t)$ and by induction assumption $x_i \in \mathbf{c}(S - x_t)$ for all $t \neq i$. Wlog, assume $x_i \in \operatorname{argmax}_{\mathcal{I}^p(S)} \succ^{r_1}$, which implies $x_i \in \mathbf{c}(S - r_2)$. If $x_i \in \operatorname{argmax}_{\mathcal{I}^p(S)} \succ^{r_2}$ then $x_i \in \mathbf{c}(S - r_1)$ and BUR implies $x_i \in \mathbf{c}(S)$. So take x_j such that $x_j \succ^{r_2} x_i$. Therefore Lemma 8 implies that x_j beats x_i in $S - r_1$. Since $x_i \in \mathbf{r}(S)$ this implies $x_i \in \operatorname{argmax}_{\mathcal{I}^p(S)} \succ^{r_1}$. Next by Lemma 10.iii $x_i \succ^{r_1} x_j$ and therefore x_i beats x_j in $S - r_2$. Now from BUR it follows that $x_i \in \mathbf{c}(S)$. \blacksquare

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