

Li, Chuan-Zhong

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*An explicit formula for optimal carbon taxes
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Chuan-Zhong Li

Department of Economics
Uppsala University
Box 513
751 20 Uppsala
Sweden

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AN EXPLICIT FORMULA FOR OPTIMAL CARBON TAXES
UNDER GENERAL ECONOMIC SETTINGS

CHUAN-ZHONG LI

An explicit formula for optimal carbon taxes under general economic settings*

Chuan-Zhong Li[†]

January 31, 2018

Abstract

The paper develops an explicit formula for the calculation of optimal carbon taxes in a dynamic integrated assessment framework. We attempt to generalize the Gosolov et al. (2014) theory by relaxing the restrictions with logarithmic preferences, Cobb-Douglas production and the full periodwise capital depreciation. By taking advantage of the cumulative climate response (CCR) function, we show that all that matters for the tax formula from the economic module pins down to a single economic parameter i.e. a weighted harmonic mean of the growth-adjusted consumption rate of discount. We demonstrate the theory with a stylized climate-economy model with depletable fossil resources, test the formula with the new DICE2016 model, and provide an application to the real world economy beyond any integrated modeling framework.

Keywords: Climate change, analytical integrated assessment, optimal carbon tax, harmonic mean, DICE model

JEL classification: H21, Q43 and Q54

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[†]Department of Economics, Uppsala University, SE-751 20, Uppsala, and the Beijer Institute, Royal Swedish Academy of Sciences, Stockholm, Sweden. Email: Chuanzhong.Li@nek.uu.se

1 Introduction

To aid policy discussions on global warming, economists have developed various computer-based quantitative integrated assessment models coupled with climate and economic modules (cf. Nordhaus, 2007; Stern, 2010; Nordhaus, 2014; 2017). As marked by the seminal work of Golosov et al. (2014), the recent literature has seen increased interest in analytical integrated assessment under simplifying assumptions. The striking finding in Golosov et al. (2014) is an explicit formula for optimal carbon tax (i.e. the social cost of carbon) which is proportional to GDP up to a constant factor which depends only on a few science and economic parameters. Along the same research line, Li et al. (2012), Anderson et al. (2014), Traeger (2015), and Engstrom and Gars (2016), among others, have extended the formula with more climate-related parameters such as climate sensitivity uncertainty and the risk of catastrophic damages.

These contributions, however, rely on rather similar economic assumptions with a logarithmic preference, a Cobb-Douglas production technology and a full capital depreciation of capital in a discrete time period (say 10 years). With these settings, the marginal propensity to consume i.e. the consumption-output ratio remains constant over time (cf. Sargent, 1987). For this specific case, as we will show in the present paper, the pure rate of time preference and the real interest rate would be identical and the real GDP deflated by the Weitzman (2001) ideal price index would be constant over time, which underlies the Golosov et al. constant factor result. Apparently, the analytical formula does not hold under alternative economic assumptions although computer simulations by the authors indicate that the optimal taxes based on the formula roughly agree with those calculated from numerical optimization.

This paper attempts to generalize the Golosov et al. (2014) formula by relaxing all of the restrictive economic assumptions. For the utility and production functions, we allow any regular functional forms satisfying the Inada conditions i.e. they are strictly increasing, concave and continuously differentiable and satisfy certain boundary conditions. With a continuous time optimal control model, we also relax their assumption of full capital depreciation. To focus on the economic module, however, we adopt the cumulative climate response (CCR) function developed by Matthews et al. (2009) and Matthews et al. (2012) to represent the atmospheric temperature dynamics. The CCR function describes the direct relationship between anthropogenic carbon emissions and increases in global mean temperature that can be readily estimated from historical

data. In contrast, the approach based on equilibrium climate sensitivity involves more uncertainty which is unlikely to be resolved in the near future despite the much research by scientists (cf. Roe and Baker, 2007; Weitzman, 2009b). An obvious advantage with the CCR approach lies in its simplest form that enables us to develop the closed-form solution for the tax formula (cf. Anderson et al., 2014; Engstrom and Gars, 2016). In addition to the CCR function, we also rely on the one-parameter exponential damage function introduced by Golosov et al. (2014) to approximate the quadratic damage function underlying the DICE model (Nordhaus, 2017).

With the simplified treatment on the temperature dynamics and damage functions, we derive the optimal tax formula under more general economic settings with essentially no restrictions on the utility and production functional forms. It turns out that what matters for the tax-GDP ratio from the economic module is a weighted harmonic mean of the growth-adjusted consumption rate of discount i.e. the difference between the real interest rate and GDP growth rate. This mean measure is the reciprocal of the weighted mean of the reciprocals of the growth-adjusted discount rates. For more on such harmonic mean measures, the readers are referred to de Carvalho (2016), Samuelson (2004), Andor and Dulk (2013) and d'Aspremont (2017), among others.

We demonstrate the use of the explicit formula first with a stylized climate-economy model with a depletable resource, and then move to the well-known DICE model (Nordhaus, 2017) both with and without the possibility of negative emissions. Finally, we go beyond the integrated modeling framework to apply the formula to the real world problem based on growth data from the World Bank and the suggested time-varying discount rate schemes by the UK and France governments (cf. Li and Lofgren, 2000; HM Treasury, 2003; Lebègue, 2005; Karp, 2005; Arrow et al., 2014). The results indicate that the formula involving only a few key parameters provides rather consistent optimal taxes as compared to the DICE model, and the optimal tax for the year 2015 in the real world ranges from about \$20 to \$64 (in 2011 international dollars) depending on the discount schemes and the damage coefficient assumed.

It is worth mentioning that our explicit formula attempts to provide insights and facilitate the interpretation of numerical results rather than to replace the fully-fledged numerical integrated models with more climate details. The remaining part of the paper is structured as follows: Section 2 formalizes the theoretical model and derives the explicit formula with a simplified climate module but more general economic module. In section 3, we solve our stylized growth model using the CCR function and demonstrate the use of the formula for calculating the optimal carbon taxes

with special reference to the role of the harmonic mean discount rate. Section 4 applies the formula first to the DICE 2016 model both with and without negative emissions, and then attempts to study the real world problem beyond any integrated assessment framework. In section 5, we summarize the study.

2 The theoretical model and analytical results

We consider a closed-economy world model with continuous time in infinite time horizon. Without loss of generality, we assume a constant population normalized to be unity and define the social objective as to maximize the Ramsey-Koopman intertemporal welfare

$$\int_0^{\infty} U(C(t)) e^{-\rho t} dt \quad (1)$$

where $U(\cdot)$ is a standard concave utility function with $C(t)$ representing the aggregated final-good consumption at time t and $\rho > 0$ the pure rate of time preference. The final goods sector uses capital (K), labor (N) and energy (E) to produce output. The labor is supplied inelastically and the gross output before interacting with the climate module takes the general form $F(K, N, E)$ satisfying the usual Inada conditions. With climate changes taken into account, we have the net output function at time t as

$$Y(t) = D(T(t)) F(K(t), N(t), E(t)) \quad (2)$$

where $D(T(t)) \in (0, 1)$ denotes the damage function with $T(t)$ as the increase in the atmospheric temperature time t from the pre-industrial level. For notational ease, we will often abbreviate from the time indices, and describe the capital stock by the following differential equation

$$\dot{K} = D(T) F(K, N, E) - \delta K - C, \text{ with } K(0) = K_0 \quad (3)$$

where $\delta \in (0, 1)$ is the rate of capital depreciation.

To be general on the economic settings, we do not assume any explicit functional form on the preference, gross production technology and the full capital depreciation as in Golosov et al. (2014) and Anderson et al (2014). However, we abstract from the detailed modeling on the climate module. Similar to Anderson et al. (2014), we adopt the cumulative (carbon) climate response (CCR) measure of Matthews et al. (2009) and Matthews, Solomon and Pierrehumbert (2012)

rather than the commonly-used equilibrium climate sensitivity parameter. One advantage of the CCR approach is that the response parameter can be conveniently estimated using empirical data as compared to the equilibrium climate sensitivity whole value is far less known (Roe and Baker, 2007), and the other one is that it enables us to predict temperature changes without tracing the flows of carbon between the atmosphere, surface oceans and deeper oceans. With the CCR function, the atmospheric temperature dynamics is simply described by a single differential equation

$$\dot{T}(t) = \sigma E, \text{ with } T(0) = T_0 \quad (4)$$

where $\sigma > 0$ is the CCR parameter and E is the carbon inflow into the atmosphere at any time t . This implies that

$$T(t) - T_0 = \sigma \int_0^t E(s) ds \quad (5)$$

i.e. the rise in temperature is proportional to the cumulative carbon emissions. Note that the validity of this relationship is supported by empirical data rather than any structural model relying on the equilibrium climate sensitivity. Part of the emission may be absorbed into the oceans and even to the deeper layers while some portion may return back to the atmosphere. As long as the unit stays around the Earth system rather than the outer space, it has a temperature effect as given in (5).

As for energy, we assume a single composite energy of oil and coal with a finite initial reserve $R(0)$. The resource stock follows the differential equation

$$\dot{R} = -E, \text{ with } R_0 > 0 \text{ given} \quad (6)$$

Concerning climate damage, we assume the same convenient exponential form as Golosov et al. (2014) in that

$$D(T) = e^{-\gamma T} \text{ satisfying } D'(T) = -\gamma D(T) \quad (7)$$

where γ is a damage coefficient. The dynamic optimization problem is to maximize (1) with respect to consumption C and energy use E subject to the constraints (2), (3), (4) and (6). The current value Hamiltonian is

$$H(t) = U(C) + \lambda_K (D(T) F(K, N, E) - \delta K - C) + \lambda_T E - \lambda_R E \quad (8)$$

The first-order conditions are

$$U'(C) - \lambda_K = 0 \quad (9)$$

$$\lambda_K D(T) F_E(K, N, E) + \lambda_T - \lambda_R = 0$$

both with the usual economic interpretations. The first condition implies that, along the optimal path, the marginal utility of consumption should be equal to the shadow price of capital at each point in time, and the second condition can be interpreted as that the marginal contribution per unit carbon energy use should be equal to the Hotelling rent plus the climate damage (Note that $\lambda_T < 0$).

The equations of motion for the co-state variables are given as

$$\dot{\lambda}_K - \rho\lambda_K = -\lambda_K (D(T) F_K(K, N, E) - \delta) \quad (10)$$

$$\dot{\lambda}_T - \rho\lambda_T = \lambda_K D'(T) F(K, N, E) \quad (11)$$

$$\dot{\lambda}_R - \rho\lambda_R = 0 \quad (12)$$

From (7) and (11), we can solve for the shadow price of temperature at any time t as

$$\lambda_T(t) = \gamma \int_t^\infty \lambda_k(s) Y(s) e^{-\rho(s-t)} ds \quad (13)$$

i.e. the welfare loss in present value at time t due to a marginal unit increase in temperature T .

Normalizing the measure in monetary terms, we have

$$\bar{\lambda}_T(t) = \gamma \int_t^\infty \frac{\lambda_k(s)}{\lambda_k(t)} Y(s) e^{-\rho(s-t)} ds \quad (14)$$

As the temperature $T(t)$ follows (5) and $\frac{\partial T(t)}{\partial E(t)} = \sigma$, we have now the social cost of carbon i.e. the optimal carbon tax as

$$\tau(t) = \bar{\lambda}_T(t) \frac{\partial T(t)}{\partial E(t)} = \sigma \gamma \int_t^\infty \frac{\lambda_k(s)}{\lambda_k(t)} Y(s) e^{-\rho(s-t)} ds \quad (15)$$

Except the reduced-form climate and damage effects, this is essentially the same formula as widely conceptualized in the literature i.e. the present discounted value, in monetary terms, of all future losses caused by a marginal unit of carbon stock at an "initial" date t (cf. Weitzman, 2009a; Arrow et al, 2012; Golosov et al., 2014; Nordhaus, 2017). Using a discrete time model, with a log utility function, Cobb-Douglas production function and full capital depreciation (over a basic period of 10

years), Golosov et al. (2014) show that the optimal saving rate is constant and the integrand term $\frac{\lambda_k(s)}{\lambda_k(t)} Y(s)$ collapses to $\frac{C_s^{-1} Y_s}{C_t^{-1} Y_t} Y_t = \frac{C_s^{-1} C_t}{C_t^{-1} C_s} Y_t = Y_t$. The optimal carbon tax can then be simplified to correspond to the elegant expression

$$\tau(t) = \frac{\sigma\gamma}{\rho} Y(t) \quad (16)$$

which is proportional to income in each time period. The proportionality factor remains constant over time involving no forward-looking component and its size depends on a few key parameters in their coupled climate-economy model. Although this formula is not valid in general, the authors show by numerical simulations that it provides rather robust carbon taxes when compared to the numerically calculated numbers under alternative preference and technology assumptions.

This paper attempts to extend the theory with a practically more useful formula for more general economic settings. We show that what matters most for the formula with regard to the economic module pins down to a single parameter i.e. the harmonic mean of the growth-corrected discount rates. First, we solve the adjoint equation for capital in (10) to obtain

$$\lambda_k(t) = \lambda_k(0) \exp\left(\int_0^t (\rho - r(v)) dv\right) \quad (17)$$

where $r(v) = D(T(v)) F_K(K(v), N(v) E(v)) - \delta$ is the real interest rate at time v . Then, for any time point $s \geq t$, we have

$$\frac{\lambda_k(s)}{\lambda_k(t)} e^{-\rho(s-t)} = e^{-\int_t^s r(v) dv} \quad (18)$$

which is a variant of the Keynes-Ramsey rule that links the utility rate of discount to the consumption rate of discount via the relative marginal utilities (cf. Blanchard and Fisher, 1989; Weitzman, 2009a; and Lofgren and Li, 2011).

Second, we express the future GDP at any time $s \geq t$ as

$$Y(s) = Y(t) e^{\int_t^s g(v) dv} \quad (19)$$

where $g(v)$ is the instant GDP growth rate at time $\nu \in (t, s)$, with which we can rewrite the optimal tax formula in (15) as

$$\tau(t) = \sigma\gamma Y(t) \int_t^\infty e^{-\int_t^s \hat{\theta}(v) dv} ds \quad (20)$$

where $\hat{\theta}(v) = r(v) - g(v)$ is a growth-adjusted consumption rate of discount [Note that the growth rate in Nordhaus (2017) is concerned with growth in consumption rather than GDP as in this

paper] at time ν . To explore the implications of such a growth-adjusted discount rate, Nordhaus (2017) linearizes his DICE dynamic system for all the state variables to derive the equilibrium rate $\theta = r - g$ and then he shows that the optimal carbon tax at the equilibrium is proportional to the factor $(r - g)^{-1}$. For Golosov et al. (2014), the adjusted discount rate is always equal to the utility rate of discount due to the Ramsey rule $\rho = r - g$ such that the formula (16) holds even along the transitional path but only under the log-utility, Cobb-Douglas production and full capital depreciation assumptions.

Now, in the third step, we take advantage of the harmonic mean of the growth-adjusted discount rates to derive our explicit optimal tax formula for more general economic settings for all regular utility and production functions and capital depreciation rate and along the whole time path not limited only to the equilibrium. Let $x(t) = f(t) = \int_0^t \hat{\theta}(v) dv$ be the integrated discount rate, then $dx = \hat{\theta}(t) dt$. Obviously, $x(t)$ is a monotonically increasing function of time t for all positive discount rate. This implies that the inverse function $t(x) = f^{-1}(x)$ exists and we can always express $\hat{\theta}(t)$ as $\hat{\theta}(t) = \hat{\theta}(t(x)) = \theta(x)$. By the change of variables, we can rewrite the integral in the tax formula (20) as

$$\frac{1}{\bar{\theta}_t} = \int_0^\infty \frac{1}{\theta(\tilde{x})} e^{-x} dx, \text{ with } \tilde{x} = x(t) + x \quad (21)$$

where $\bar{\theta}_t$ corresponds to the harmonic mean of the growth adjusted discount rate weighted by the exponential weight function. By definition, this is the reciprocal of the weighted mean of the reciprocals of the growth-adjusted discount rate corresponding to

$$\bar{\theta}_t = \frac{\int_0^\infty e^{-x} dx}{\int_0^\infty \frac{1}{\theta(\tilde{x})} e^{-x} dx} \quad (22)$$

in which the numerator is always equal to one. In general, it is this type of the harmonic mean that correctly measures the average rate of change in physics and other fields. Consider, for example, a journey by car with the first 60 km at a speed of 20 km per hour and the second 30 km at a speed of 30 km per hour. What is the average speed? The arithmetic mean is simply $(20 + 30) / 2 = 25$ km per hour but this does not makes much sense. The true average speed should be equal to the total distance traveled divided by the total time spent i.e.

$$\frac{60 + 30}{\frac{60}{20} + \frac{30}{30}} = \frac{\frac{2}{3} + \frac{1}{3}}{\frac{1}{3} \frac{1}{20} + \frac{1}{3} \frac{1}{30}} = 22.5$$

which is the weighted harmonic mean speed. The first part of journey takes $60/20 = 3$ hours while the second part takes $30/30 = 1$ hour, so the true average speed is $90/4 = 22.5$ km per hour. With the arithmetic mean of 25 km per hour, the distance travelled over the 4 hours would be 100 km rather than 90!

Using equations (20) and (21), we arrive at the general carbon tax formula given by

$$\tau(t) = \frac{\sigma\gamma}{\theta_t} Y(t) \quad (23)$$

with the Golosov et al. (2014) and Nordhaus (2014) results as special cases. The whole economic module now pins down to the weighted harmonic mean discount rate irrespective of the form of the utility and production functions, any capital depreciation rate, or whether or not the economy is at equilibrium. The crucial assumption for this general formula is the exponential damage function with a constant damage exponent.

Based on the weighted harmonic mean discount rate formula (21), we may make certain theoretical predictions. First, the reciprocal of this mean measure is due to the Jensen inequality not symmetric with respect to interest rate and/or growth rate variations. A boost in GDP growth rate would lead to a larger optimal carbon tax increase than the tax decrease caused by a decline in the growth rate by the same amount. Second, the near term discount rate due to the exponential weighting plays a much large role on the optimal tax than the long-run ones.

An alternative way to conceptualize the formula (20) is to invoke the Weitzman (2001) ideal price index $P(s) = \frac{p(s)C_0}{p(0)C_0} = \frac{\lambda_k(0)}{\lambda_k(s)}$ where $(p(0), p(s))$ denote the prices (the dollar value of consumption) at time 0 and s , respectively. The reason for the second equality to hold is that with a time-invariant utility function we have $U(C_s) = U(C_0)$ and $U'(C_s) = U'(C_0)$ for $C_s = C_0$, and $p(s)\lambda_k(s) = p(0)\lambda_k(0)$. Suppose that the marginal utility as time s is a half of that at time 0, then the amount of money needs be doubled at time s to achieve the same utility level as at time 0 with an ideal price index $P(s) = \frac{p(s)C_0}{p(0)C_0} = \frac{\lambda_k(0)}{\lambda_k(s)} = 2$. With such an ideal price index, the optimal tax formula can now be written as

$$\tau(t) = \sigma\gamma \int_t^\infty \frac{Y(s)}{P(s|t)} e^{-\rho(s-t)} ds \quad (24)$$

where $P(s|t) = P(s)/P(t)$ denotes the ideal price index with time t as the base year. Let $g(s) = Y'(s)/Y(s)$ be the income growth rate and $\pi(s|t) = P'(s|t)/P(s|t)$ as the inflation rate, we can express the optimal tax as

$$\tau(t) = \sigma\gamma Y_t \int_t^\infty e^{-\int_t^s (\rho + \pi - g) du} ds = \frac{\sigma\gamma}{\theta_t} Y_t \quad (25)$$

where the denominator in the last expression corresponds to the weighted harmonic mean of $\rho + \pi - g$ with $\rho + \pi = r$ as the real interest rate! As a special case, Golosov et al. (2014) imposes a certain model structure such that $\pi = g$ which imposes $\bar{\theta}_t = \rho$ being a constant!

In the coming sections, we will demonstrate the use of this optimal tax formula in three different cases, a stylized climate-economy model of our own, the DICE-2016R model and real world economy outside the integrated modeling framework.

3 A numerical illustration with a stylized growth model

To demonstrate how the formula works, we specify explicit functional forms for the couple climate-economy model above. The utility function takes the CRRA form

$$U(C) = \frac{C^{1-\alpha}}{1-\alpha} \quad (26)$$

with $\alpha > 0$ measures the degree of relative risk aversion, and the gross production function takes the Cobb-Douglas form

$$F(K, N, E) = AK^a N^{1-a-b} (E + \bar{E})^b \quad (27)$$

with $a \in (0, 1)$, $b \in (0, 1)$ and $a+b \in (0, 1)$ as parameters. We assume that population N is constant and normalize it to unity and \bar{E} an exogenous supply of green energy with no carbon emissions. To be more realistic, we would be able to introduce some cost for gaining the green energy in terms of either capital or labor (or both) diverted from the final goods production. It is also possible to specify imperfect substitution between the green energy \bar{E} and the fossil energy E as a nested CES function (cf. Golosov et al., 2014). However, we abstract from these complications as they do not add insights into our optimal tax formula.

Under the constraints (3), (4) and (6), we maximize the intertemporal welfare function (1) with the utility and production functional forms in (26) and (27). We imagine the starting year for the optimization problem to be 2015 and thus use the same initial values as DICE-2016R (Nordhaus, 2017). The initial world capital is $K_0 = 223$ and GDP $Y_0 = 105.5$, both in trillions 2010 international dollars, and the temperature rise from the industrial revolution is $T_0 = 0.85^\circ C$. As there is no energy input in the DICE model, we refer to the numbers in Rogner (1997) and used in Golosov et al (2014). The total oil reserve is about 300 Gt corresponding to some 254 GtC carbon content. Adding the effective (economically profitable) reserve of coal, we take the total

fossil energy reserve as $R_0 = 500$ GtC in this study. The exogenous green energy input is taken to be 5 GtC per year.

For the utility and production functions, we assume that $\alpha = 1.45$ for the degree of relative risk aversion, $a = 0.3$ for capital share and $b = 0.03$ for energy share. For the climate damage function in (7), we calibrate the coefficient by using the lower-cost scenario (Nordhaus and Boyer, 2000; Engstrom and Gars, 2016) where a temperature rise by 2.5 degrees Celsius implies a 1.67% loss in GDP such that

$$\gamma = -\frac{\ln(0.9833)}{2.5} \approx 0.006736 \quad (28)$$

For the total factor productivity A in (27), we calibrate it by $Y_0 = \exp(-\gamma T_0) AK_0^a (E_0 + \bar{E})^b$ such that $A = 19.13$. Concerning the other economic parameters, we assume the capital depreciation rate as $\delta = 10\%$ per year, and the pure rate of time preference is $\rho = 1.5\%$ as in DICE-2016R.

With regard to the CCR function, Matthews et al. (2009) find that the CCR coefficient is on average about $\sigma = 0.0050$. For carbon flows, Golosov et al (2014) consider that some 20% of the carbon emitted would forever remain in the atmosphere, about 50% of the other 80% would "disappear" in the outer space almost instantly, and the rest would depreciate at some exponential rate. In this study, in the spirit of the CCR model, we consider the total accumulated carbon in the climate system without tracing its exchanges among the different mediums (atmosphere, surface and deeper oceans). With a share of about $(1 - 0.2) * 0.5 = 40\%$ disappeared in the outer space, we have 60% of the emitted carbon in the climate system so the net CCR coefficient becomes $\sigma = 0.005 * (1 - 0.4) = 0.003$. For convenience, we summarize the initial state and the relevant parameter values in Table 1.

We solve the dynamic optimization problem by using an IPOPT solver in the CasADi framework with Matlab R2014b. From the optimal solution depicted in Figure 1, we can see that capital, production and consumption all follow some inverted U-shaped form, which is consistent with Dasgupta and Heal (1976) on growth with depletable resources. A minor difference is that our solution converges to some positive rather than zero long-run steady state as we assume some perpetual exogenous inflow of green energy in the model. The optimal fossil stock and extraction are shown in Figure 2 which indicates that all the fossil energy would be effectively depleted in about hundred years, and after which the atmospheric temperature would stop rising (Figure 3) and converge to a state level due to the depletion of the fossil resource.

Table 1: The initial states and the model parameter values

Initial capital	K_0	223	Trillions 2010 dollars
Initial output	Y_0	105.5	Trillions 2010 dollars
Initial fossil reserve	R_0	500	Gigatons carbon (GtC)
Initial temperature rise	T_0	0.85	Celsius degrees
Total factor productivity	A	19.13	-
Degree of risk aversion	α	1.45	-
Pure rate of time preference	ρ	1.5%	-
Capital depreciation rate	δ	10%	-
Capital share in production	a	0.3	-
Energy share in production	b	0.03	-
The CCR coefficient	σ	0.003	-
Exponential damage coefficient	γ	0.006736	-

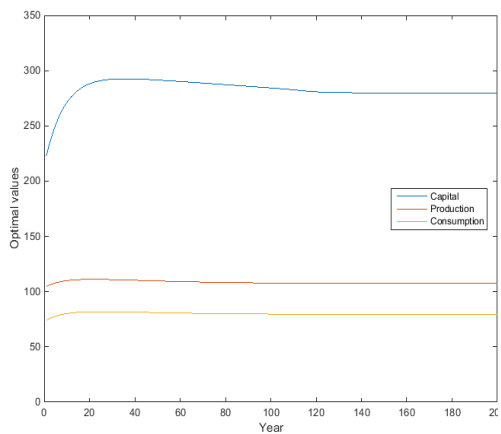


Figure 1: Optimal capital, production and consumption

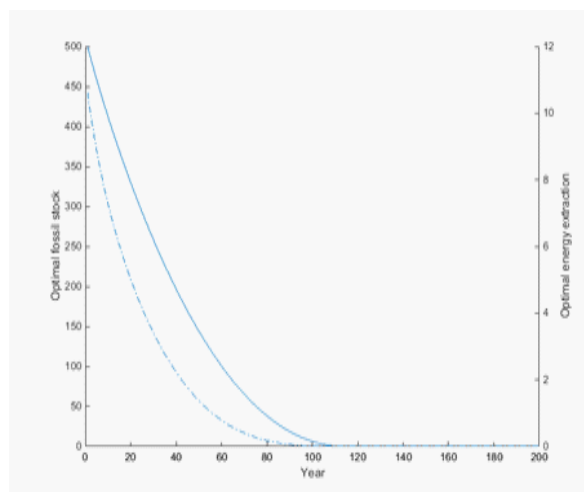


Figure 2. Optimal fossil stock and extraction (GtC)

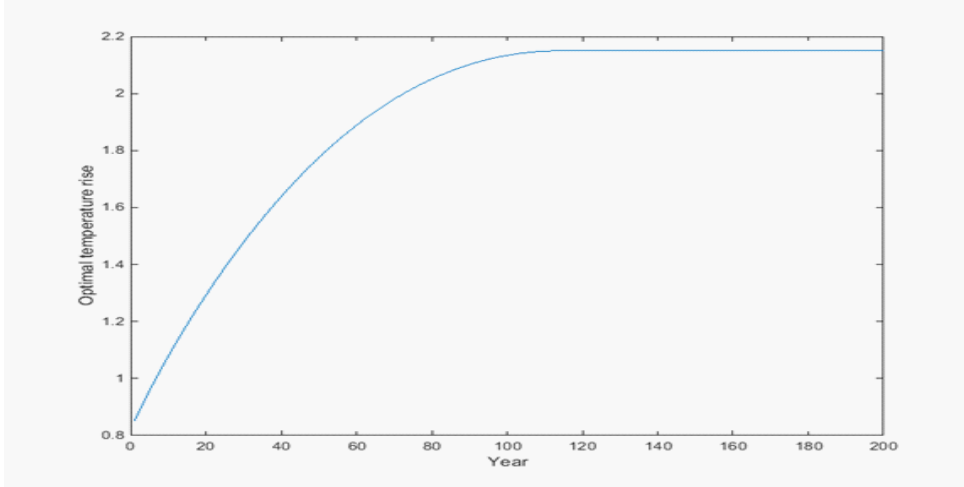


Figure 3. Optimal temperature rise

Figure 4 shows the trend of real interest rate, GDP growth rate and the growth-adjusted consumption rate of discount. Due to the higher initial capital productivity, the interest rate starts very high but then decreases rapidly and so is the GDP growth rate which is much lower due the nature of the depletable resource and we do not assume any technical progress in the model. The difference between these two rates corresponds to the growth-adjusted discount rate (cf. Nordhaus, 2017) that almost completely overlaps with the real interest rate due the lower GDP growth rate.

Using the formula (21), we calculate the weighted harmonic mean of the growth-adjusted discount rate using the Matlab *trapezoidal numerical integration procedure*. As of time 0, for example, we have

$$\bar{\theta}_0 = \left(\int_0^{\infty} \frac{1}{\theta(x)} e^{-x} dx \right)^{-1} = 0.0163 \quad (29)$$

As the harmonic mean is a forward-looking measure, it has a much flatter form as compared to the raw growth-adjusted rate of discount (Figure 5). It has a somewhat weak U-shaped form first and after about 100 years seems to converge to a steady state overlapping with the constant raw growth-adjusted rate. With such harmonic mean discount rates, we calculate the tax-GDP ratio $\sigma\gamma/\bar{\theta}_t$ in (23). For the starting year, with $\bar{\theta}_0 = 0.0163$ for example, the tax-GDP ratio becomes $0.003 \cdot 0.0067364 / 0.0163 = 1.2398 \times 10^{-3}$. Multiplying it with the GDP value 105.5 trillion dollars, we obtain the optimal carbon tax at this initial year as $1.2398 \times 10^{-3} \cdot 105 \times 10^3 \approx \130 per ton carbon i.e. about $130 \div 3.667 = \$35.5$ per ton carbon dioxide. The optimal taxes over time are

depicted in Figure 6, which displays an inverted U-shape in the first phase, similar to the optimal path of capital, GDP and consumption, and then converges to a steady state level about \$34 (2010 dollars).

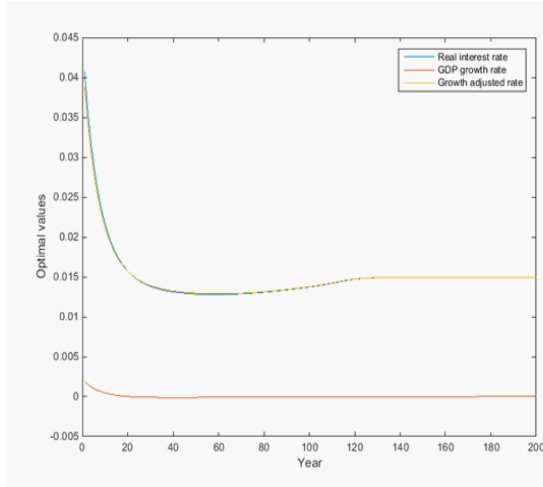


Figure 4. Real interest, GDP growth and growth-adjusted discount rate

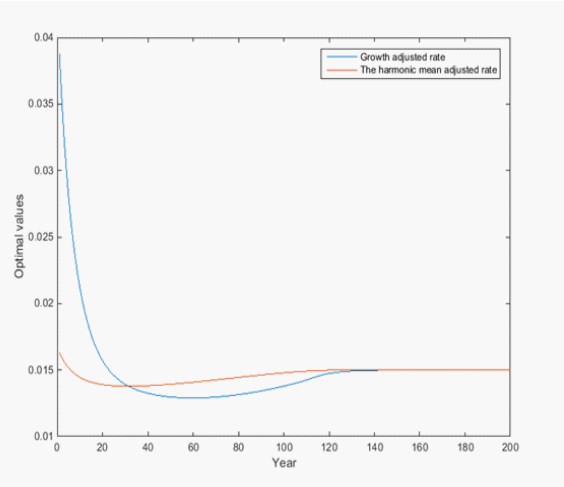


Figure 5. Raw and weighted harmonic mean growth-adjusted discount rate

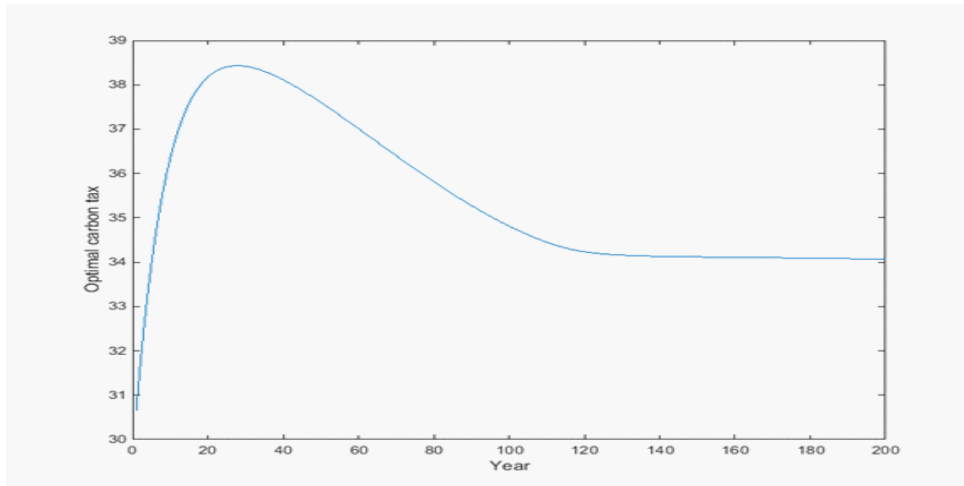


Figure 6. Optimal tax per ton carbon dioxide

4 Application in the DICE model and the real world

The new version of DICE 2016 has modified the climate module based on the updated scientific knowledge from the most recent IPCC report (IPCC Fifth Assessment Report, Science, 2013) and incorporated the possibility of negative emissions (Nordhaus, 2016). As touched upon above, the starting year of the model is 2015 with capital and GDP etc. all measured in 2010 international dollars. The time horizon is 100 five-year periods i.e. 500 years beyond which the present discounted value of future consumption flows is virtually zero. In this section, we first test our theory with the DICE-2016R model and then attempt to apply it to the real world beyond the integrated assessment modeling framework.

By running the DICE-2016R model without allowing for negative emissions, we use the Matlab version of DICE (Kellett et al., 2016) to obtain the optimal solution for all the state and control variables such as capital, carbon stocks in the atmosphere, surface ocean and deeper ocean, the surface and water temperature, and consumption and the emission control rates. The CCR coefficient is calibrated by a ratio estimator between the optimized temperature rises and the accumulated carbon emissions from industry and land which gives a net CCR coefficient $\sigma = 0.0031$. We can then approximate the temperature dynamics using just one difference equation $T(t) = T_0 + \sigma \sum_{i=0}^t E_i$ for each future year t instead of the multiple difference equations for carbon and temperature. From Figure 7, it can be seen that our simple difference equation approximates the DICE optimal temperature changes fairly well. A subtle difference is that our calibrated one-equation CCR model involves an instant temperature rise to any addition of carbon emission while the multiple equations model in DICE implies certain hysteresis effect. To start with, temperature rises more rapidly according to the CCR model and after about 120 years where the net emission turns to zero the atmospheric temperature rise would stay constant at about 4.15 degrees Celsius. In contrast, the DICE temperature path reacts a bit slowly on carbon emissions in the first phase but keep rising even after carbon emissions have completely ceased. For climate damages, we fit the DICE quadratic damage function D_t with a single-parameter exponential function (cf. Golosov et al., 2014) with $\gamma = -\ln(1 - D_t)/T_t$, as shown in Figure 8.

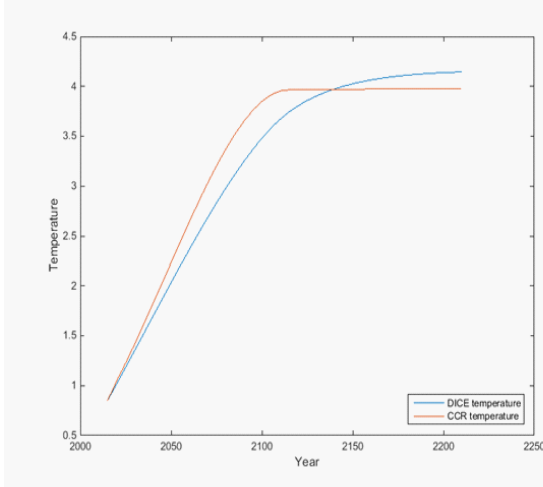


Figure 7. Temperature dynamics predicted by DICE and the CCR function

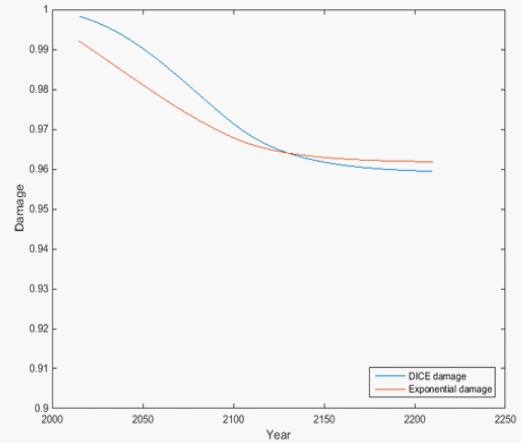


Figure 8. Exponential vs the DICE damage functions

Note that the DICE model assumes certain exogenous population growth, productivity growth and emission efficiency improvement. In our theoretical model, we have abstracted from these changes as they provide no insights for our formula but when needed they can be readily incorporated by introducing extra state variables. What is essential is the resulting stream of growth-adjusted discount rate. Also, the DICE model assumes a total utilitarian welfare function in which the social utility is the sum of all individuals' utilities. To calculate the real interest, we take advantage of Keynes-Ramsey rule

$$r_t = \rho + \alpha \frac{\dot{c}_t}{c_t} = \rho + \alpha \left(\frac{\dot{C}_t}{C_t} - \frac{\dot{N}_t}{N_t} \right) \quad (30)$$

where c is the per-capita consumption, N the population (labor) size, and the dot denotes their annual change. With $g_t = \dot{Y}_t/Y_t$ as the annual growth rate of GDP (from year t to $t + 1$), we calculate the growth-adjusted growth rate $\hat{\theta}_t = r_t - g_t$ and the harmonic mean values as shown in Figure 9.

At the starting year 2015, our approximation of the optimal tax based on the simple formula (23) is about \$44.02 per ton CO₂, which is comparable to the DICE value being about \$30.75. As time goes, our simple formula seems to approximate the DICE result rather well over the whole future. The somewhat over-estimation at the beginning may depend on our calibrated CCR coefficient

that generates a higher temperature rise in the "near" future and the "near" term, according to our weighted harmonic mean formula, has a larger effect on the optimal carbon tax than the long-run temperature change.

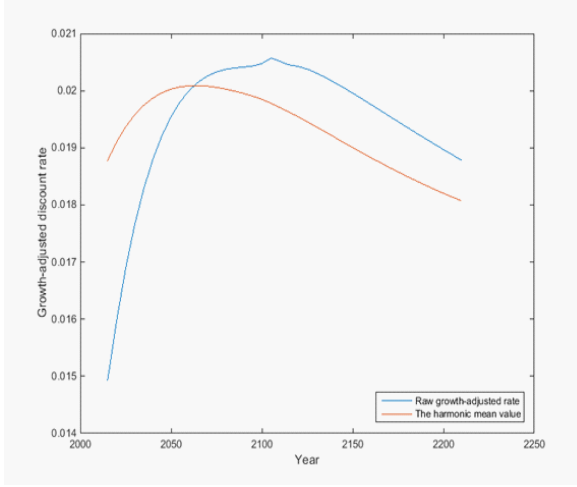


Figure 9. Raw and weighted harmonic mean growth-adjusted discount rate

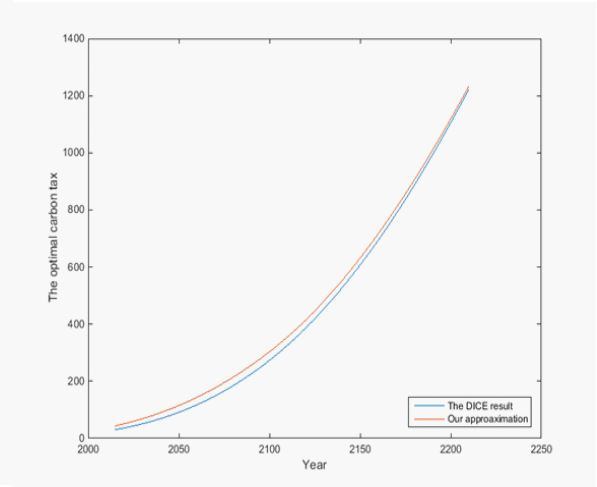


Figure 10. Optimal taxes from DICE 2016 and our tax formula

Using the DICE-2016R version in Matlab (Kellett, 2017), we also test our formula under the possibility of negative emissions (with backstop technology) after year 2150. The calibrated CCR and damage functions are depicted in Figures 11 and 12. Concerning temperature change, it is seen that after the year 2150 when negative emissions are allowed for, our CCR predicts an immediate temperature drop while the DICE model predicts some delayed effects in temperature fall. For this version of the DICE model, our formula provides an optimal carbon tax at the starting year 2015 as \$34.50 per ton CO₂ which is rather close to the DICE result \$30.75. The time path of optimal taxes based on our formula follows closely to the DICE value until year 2075 or so but then starts to under-estimate the taxes. This may be due to the interaction of the temperature hysteresis effect and the acceleration effect in the DICE damage function based on the square term of temperature rise which were not taken into account in our model. While the effect is small with smaller temperature rise, the effect will loom larger with larger temperature rises with increased risk of catastrophic effects.

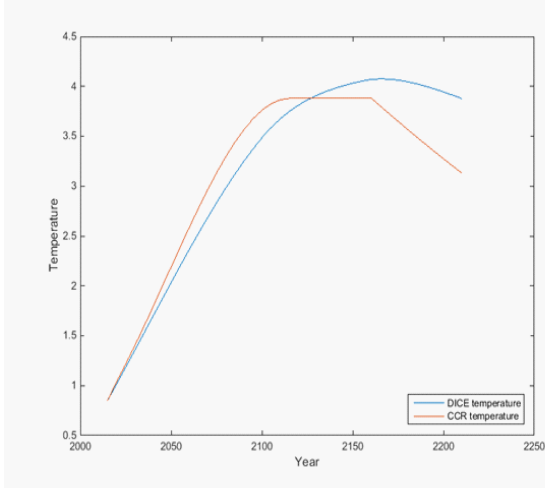


Figure 11. Temperature dynamics with negative emissions

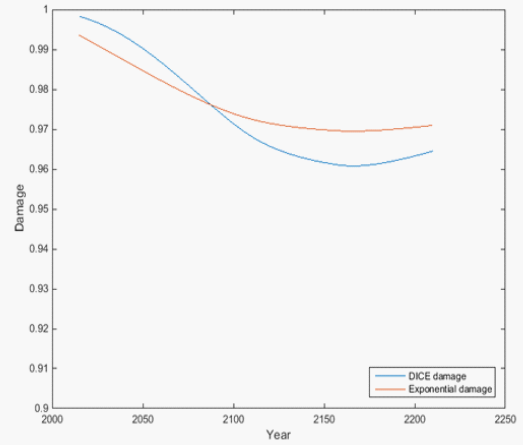


Figure 12. Damage functions with negative emissions

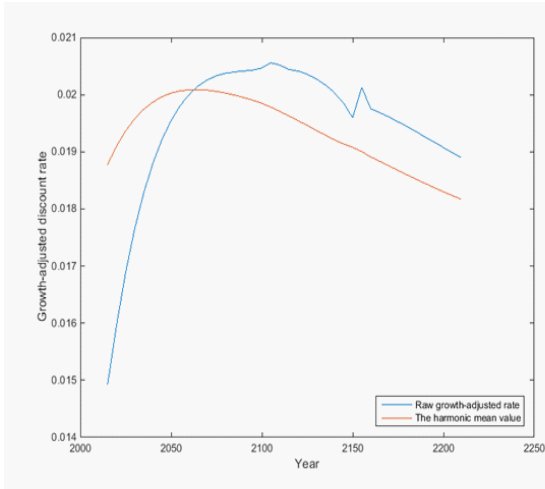


Figure 13. Growth-adjusted discount rate with negative emissions

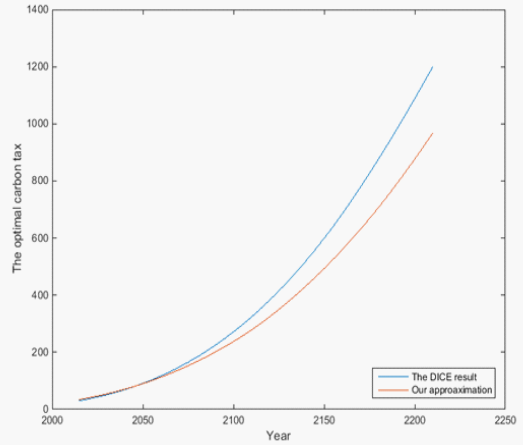


Figure 14. Optimal taxes with negative emissions

In the rest of this section, we attempt to apply our formula to the real world economy beyond any integrated modeling framework. First, we take the time-varying consumption rate of discount as suggested by the UK government (HM Treasury, 2003) for social cost-benefit analysis of public projects (corresponding to the r_t values in our model) as shown in Figure 15. For the world

economy, the World Bank Group (2018) shows that the annual growth rate of global GDP (in 2011 international dollars) to be about 4.3% and the inflation rate about 2% for 2015. We assume that this trend would continue with a growth-adjusted rate of discount about 2.3% for some 125 years at the first stage, and then drop to some 1.25% up to year 300, and then to 0.5%. With these numbers, we calculate the weighted harmonic mean discount rate by the Matlab *trapezoidal numerical integration procedure* to be about 0.02897 for the starting year 2015. With a lower-bound damage coefficient $\gamma = 0.006736$ (section 3) and the net CCR coefficient $\sigma = 0.003$, we calculate the optimal tax using our formula (23) as

$$\frac{0.003 \times 0.006736}{0.02897} \times 105 \times 10^3 = \$73.24/tC = \$19.98/tCO_2 \quad (31)$$

Using the damage coefficient corresponding to DICE with $\gamma = 0.009383$, the number becomes

$$\frac{0.003 \times 0.009383}{0.02897} \times 105 \times 10^3 = \$102.02/tC = \$27.822/tCO_2 \quad (32)$$

Another country that endorsed the hyperbolic discounting scheme for social cost-benefit analysis is France (Lebègue, 2005) as shown in Figure 16. Over the short term with 25 years, the real consumption rate of discount is suggested to be 4%, which is higher than the UK rate. However, after this period, the country adopts a much lower discount rate with 2%. By calculating world GDP growth rate as 2.3% over the first period of 25 years and 1.45% from year 26 up to year 350, we calculate the harmonic mean of the growth-adjusted rate of discount as 0.01274, which is less than a half of the UK number. The resulting carbon taxes are thus

$$\frac{0.003 \times 0.006736}{0.01274} \times 105 \times 10^3 = \$166.55/tC = \$45.42/tCO_2 \quad (33)$$

$$\frac{0.003 \times 0.009383}{0.01274} \times 105 \times 10^3 = \$232.00/tC = \$63.26/tCO_2 \quad (34)$$

for the lower-bound damage coefficient with $\gamma = 0.006736$ and the DICE-2016R equivalent one with $\gamma = 0.009383$, respectively.

The applications here indicate that the exact tax value is sensitive to the key economic parameter i.e. the weighted harmonic mean of the growth-adjusted discount rate. The larger the real interest rate or the lower the GDP growth rate implies a lower carbon tax, and the combination with a lower real interest rate and a higher growth rate implies a large carbon tax. In addition to the economic parameter, the temperature-economy effect namely the damage coefficient as well as the CCR coefficient also play a vital role for the carbon tax. To determine their exact tax values,

with both large scientific and economic uncertainties in the future, is a very challenging task (cf. Nordhaus, 2013).

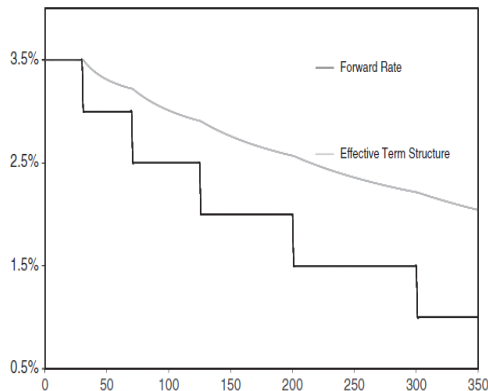


Figure 15. Social rate of discount in the UK

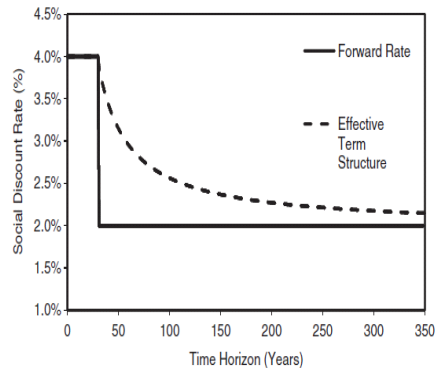


Figure 16. Social rate of discount in France

5 Concluding remarks

In this paper, we have attempted to contribute to the literature of analytical integrated assessment by developing an implicit formula for optimal carbon taxes under general economic settings. The formula involves, in addition to GDP, three key parameters i.e. the CCR coefficient, the exponential damage coefficient, and the weighted harmonic mean of the growth-adjusted consumption rate of discount. As compared to Golosov et al. (2014), this formula imposes no particular utility and production functional forms and does not assume any full capital depreciation. Related to Nordhaus (2017), we consider the whole time path of the growth-adjusted consumption rate of discount rather than only the steady state. Except our simple treatment of the climate module, therefore, our formula generalizes the earlier literature from an economic standpoint.

We show that all that matters for the tax-output ratio from the economic module is the weighted harmonic mean of the growth-adjusted consumption rate of discount. Our formula provides a useful experimental tool for examining the role of specific model parameters such as the pure rate of time preference, risk aversion parameter, and the rate of capital depreciation. If different parameter combinations would generate the same harmonic mean discount rate, then the optimal carbon tax would be the same, conditional on the same climate parameters and GDP. The formula is

transparent in other aspects as well. For example, it is the product of the CCR and the exponential damage coefficients that counts for the optimal tax value. If one parameter increases in value by the same percent as the decrease of the other, the optimal tax rate would not change.

Based on a stylized growth model involving a (composite) depletable fossil resource and climate changes, we demonstrate how the formula works with special focus on the role of the harmonic mean discount rate. Typical for such growth models, the optimal path of capital, production and consumption has shown some inverted U-shaped form over time. As the resulting harmonic mean and thereby the tax-GDP ratio do not fluctuate very much, the optimal carbon tax also exhibits an inverted U-form initially and then converge to its steady state value.

After the demonstration with the stylized growth model, we have moved to the full-fledged numerical integrated assessment model DICE 2016 (Nordhaus, 2016; 2017; Kellet, 2017) to test our formula. For the climate module, we find that the CCR function roughly traces the optimal temperature dynamics though without the hysteresis effect. The calibrated CCR coefficient predicts a slightly larger temperature increase for the first 125 years and a somewhat smaller increase afterwards as compared to DICE model. Concerning the damage effect, our one-parameter exponential damage function generates a higher damage first and a lower one later on. The harmonic mean, as expected, is smoother than the raw growth-adjusted discount rate. The overall optimal tax values over time, however, show a consistent trends with the DICE 2016. With negative emissions incorporated, we find a similar pattern, though the resulting carbon tax values become a bit closer to the DICE results initially but divert more after some 75 years from today.

We have also attempted to go beyond any integrated modeling framework by looking at the real world economy with data from the World Bank and the discount schemes from the UK and French governments for social-cost benefit analysis of public projects. We calculate the harmonic mean values of the growth-adjusted discount rates, and then calculate the optimal carbon taxes under different discounting schemes and using different damage parameters. The results show that the carbon tax values are sensitive to the parameter values ranging from about \$20 to \$64 per ton carbon dioxide for 2015, or equivalently \$73 to \$232 per ton carbon.

Note that the formula developed in this paper deals with gradual climate changes without explicitly taking into account any tipping points and catastrophic effects in the model. To develop analytical integrated assessment models with these elements and under general economics settings should be interesting for future research.

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