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PAUL SAMUELSON'S WAYS TO MACROECONOMIC DYNAMICS

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Paul Samuelson's ways to macroeconomic dynamics

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Abstract. Samuelson kept optimization-based problems separated from macroeconomic dynamics in his *Foundations*, where dynamics were defined in terms of difference and differential equations. Despite some criticism of his “correspondence principle” of stability analysis by D.F. Gordon, D. Patinkin and others, it was only in the 1970s that Samuelson’s separation was effectively challenged, particularly by R. Lucas. After the *Foundations*, Samuelson developed dynamic optimization models, sometimes featuring representative agents, but he did not extend that to the study of macroeconomic fluctuations. Neither did he accept market clearing inter-temporal maximization as a solution to the microfoundations problem that beset his models of macroeconomic dynamics. His last contribution to macro dynamics was his 1988 nonlinear non-optimizing business cycle model. Eventually, he disentangled his 1965 “efficient market hypothesis” from rational expectations and claimed that the former should form one of the pillars of macroeconomic dynamics, together with imperfectly competitive markets for goods and labour.

Keywords. Samuelson, macroeconomic dynamics, stability, multiplier-accelerator

JEL classification. B22, B31, B41

It is the essence of dynamics that *economic variables at different points of time are functionally related*; or what is the same thing, that *there are functional relationships between economic variables and their rates of change, their “velocities”, “accelerations”, or higher “derivatives of derivatives”*. (Samuelson 1948a; italics in the original)

1. Are dynamic and optimization problems distinct in economics?

The answer to that question, from Paul A. Samuelson's perspective in the 1930s and 1940s, when he wrote his path-breaking *Foundations of Economic Analysis*, was a definite “yes”. Indeed, *Foundations* was separated into Part I, about maximization and optimization-based problems, and Part II dealing with dynamic issues. Samuelson's definition of economic dynamics, as quoted above (see also Samuelson 1942, p. 59, and 1947, p. 314), pointed the way to differential and difference equations as key tools in the study of economic stability and changes of economic variables over time, as developed in the second part of that book. The new tools were particularly useful for the investigation of business cycles, as forcefully illustrated by Samuelson's (1939a, b) multiplier-accelerator model. That was distinct from the mathematical and economic frameworks deployed in the static microeconomic theory of constrained maximizing choices by individual agents, discussed in the first part of *Foundations*. Samuelson (1947, p. 5) argued that meaningful operational propositions in economics were based on two different types of hypotheses. The first was that the conditions of equilibrium are equivalent to the maximization of some amount.

However, “when we leave single economic units, the determination of unknowns is found to be unrelated to an extreme position”. Instead, the “dynamical properties of the system are specified, and the hypothesis is made that the system is in ‘stable’ equilibrium or motion”.

The concept of equilibrium was involved in both types of hypotheses, but in different ways. In the dynamic realm, equilibrium was related to stability instead of optimum (second-order) conditions. That was the role of Samuelson’s “Correspondence Principle” between comparative statics and dynamics, which restricts the values of the parameters of a system by assuming dynamic stability (1947, p. 5). Samuelson (1947, p. 284; 1942, p. 1) regarded that principle as the continuation and further refinement of the “revolution” from static to dynamic modes that Ragnar Frisch (1933) had started.

Samuelson’s (1942, 1947, 1948a) definition of economic dynamics, as he acknowledged, built on Frisch’s (1933, p. 171) remarks that dynamic theory considers “the magnitudes of certain variables in different points of time” by means of equations which “embrace at the same time several of these magnitudes belonging to different instants”. Although equilibrium and stability were relevant to Frisch, they played a different role in Samuelson’s dynamic framework. Dynamic models – of the kind put forward by Samuelson, Hicks, Lange, Goodwin, Domar, Metzler and others at the time – studied the “stability and fluctuating deviations around any defined equilibrium”, encompassing the fields of price theory, business cycles and income determination (Samuelson 1948a, p. 353).

Frisch’s “macro-dynamics”, together with the “Keynesian system” of income determination and Walrasian general equilibrium equations (particularly in their dynamic tâtonnement version) dealt with the “interaction between individuals”, not

with optimizing action “within an economic unit” (Samuelson 1941, p. 98; 1947, pp. 138, 258, 351). Equilibrium systems such as Keynesian macroeconomics entailed non-optimizing foundations, provided by the correspondence principle. Aggregate behaviour could be neither understood as the result of maximization or “extremum” problems nor “converted into this form” (1947, p. 138). The problem of aggregation over heterogeneous agents was only implicit in the argument; it became explicit after Samuelson (1956) took into account Gorman’s (1953) criterion that perfect aggregation required identical (quasi) homothetic utility functions.

Samuelson’s (1941, 1942, 1947) rigorous separation of static maximization and dynamic stability became a hallmark of both general equilibrium and macroeconomic analyses from the 1940s to the 1970s, even if the correspondence principle faced criticism from the beginning. As argued by Roy Weintraub (1991), the meanings of the terms “dynamic” – as the specification of the system in terms of differential or difference equations – and “equilibrium” – as the limit of the dynamic behaviour as the system reaches a position of rest and the equations are solved – was stabilized between the 1940s and 1960s, building and expanding on Samuelson’s *Foundations*. Oskar Lange (1944) followed Samuelson’s dynamic stability analysis closely. William Baumol’s (1951) *Economic Dynamics*, the first textbook on the subject, was largely organized around those concepts, with a part III about “process analysis”, the same title of Samuelson’s (1948a) survey.¹ Part 5 of Alpha Chiang’s ([1967] 1984) well-known textbook of mathematical economics addressed “Dynamic Analysis”. It was entirely about differential and difference equations, whereas Part 4

¹ Baumol’s book opened with a lengthy discussion of what he famously called the “magnificent dynamics” of classical economists, Marx, Schumpeter and Harrod, which did not easily fit into Samuelsonian dynamics.

on “Optimization Problems” was strictly static, following along with Samuelson’s agenda.

Samuelson’s (1947) approach to macroeconomic dynamics as separate from optimization remained essentially unchallenged until the mid 1970s, despite some important criticism of the correspondence principle, the dynamic stability hypothesis and tâtonnement price dynamics by Don Patinkin (1952, 1965), Donald F. Gordon (1955), Takashi Negishi (1962), Kenneth Arrow (1959, 1967), and D.F. Gordon and J. Allan Hynes (1970). Careful reviewers of the *Foundations*, such as Lloyd Metzler (1948) and Kenneth Boulding (1948), called attention to and approved of the distinction between micro maximization and macro dynamics, without proposing a bridge between them. Metzler (1948, p. 905), Samuelson’s colleague at Harvard in the 1930s and 1940s, supported Samuelson’s contentions that “most of the important economic problems, including practically all of those which deal with the economic system as a whole, cannot be reduced to simple problems of maximization”, and that the equilibrium of prices and quantities for the economy as whole cannot be proved to “represent a maximum position for some variable” in the system (see also Backhouse 2017, p. 477). Boulding (1948, p. 194) stressed the role of difference equations in dynamic (as opposed to marginal) analysis and argued that maximization had a very limited role, if any, in macroeconomics.

It was only in the mid 1970s, after Robert Lucas² launched new classical macroeconomics, that the answer to the question posed in the title of this section turned into a resounding “no”. Instead of Samuelson’s separation and of tâtonnement analysis, Lucas argued for dynamic optimization in an inter-temporal setting in permanent equilibrium. Lucas ([1980] 1981) – who had learned his economics from

² See articles collected in Lucas 1981.

the *Foundations*, upon graduating in history at the University of Chicago in the late 1950s – recollected how his new concept of stochastic contingent-claim equilibrium was constructed as a critical reaction to what he perceived as Samuelson’s static view of competitive equilibrium and loose dynamic analysis. Lucas believed he was closer to John Hicks’ ([1939] 1946) notion of equilibrium, with its emphasis on expectations. Hicks ([1939] 1946) had reacted critically to Samuelson’s (1941, 1942, 1943) analysis of dynamic stability, which he found too “mechanical”, the same term Lucas used to describe the *Foundations*.

There had been early unsuccessful attempts to discuss economic fluctuations in terms of the maximizing behaviour of (what we now call) a representative agent, most notably by Vilfredo Pareto ([1896-97] 2005). Pareto’s business cycle model was based on a disaggregated general equilibrium system with dynamic behaviour of the representative consumer over time determined by frictions (“inertia”), in analogy with D’Alambert Principle of mechanics. However, as Wicksell pointed out, he got the mathematics wrong (see Boianovsky 2013). Samuelson (1947, p. 311) did not follow that path. He warned against attempts to search for dynamic economic notions analogous to energy, inertia, force and other concepts from theoretical physics, although he referred to Frank Knight, not to Pareto, in that connection.

Lucas’s ([1980] 1981) contrast between his and Samuelson’s approaches to macroeconomic dynamics made clear the general features of the framework introduced by the MIT economist, which had often been taken for granted or just passed over. Hence, Axel Leijonhufvud’s (1998) distinction between the “Classical” and “Modern” traditions in economics was not yet clear to him by the time he wrote his influential 1968 book on the economics of Keynes. Leijonhufvud (1998) identified the “Classical” approach with the study of the “laws of motion” of society,

stemming from classical British economics' "magnificent dynamics" and continuing through Marshall and Keynes, with emphasis on adaptive behaviour. The hallmarks of the "Modern" tradition were optimization, equilibrium and choice, as illustrated by Arrow, Debreu and Lucas.

According to Leijonhufvud (1998, pp. 172-73), parts I and II of Samuelson's *Foundations* belonged in the "Modern" and "Classical" traditions respectively. From that perspective, the "correspondence principle" indicated that the comparative static equilibria of the "Modern" tradition were operationally meaningful only if they worked as the "attractors" of a "Classical" dynamic mode. Peaceful coexistence between those two ways of thinking, as displayed by Samuelson's *Foundations*, lasted until the 1970s, when "Modern" theory came to displace "Classical" ways of theorizing almost completely, suggested Leijonhufvud (p. 181) with the benefit of hindsight. Whereas "Keynesian economics", as developed by Samuelson and other supporters of the IS-LM model in the post-war period, was the main target of Leijonhufvud (1968), he later shifted the focus of his criticism to Lucas's "Modern" macroeconomic dynamics.

A couple of years after Lucas's 1980 essay, as part of a Festschrift for Samuelson, Frank Hahn (1983) and James Tobin (1983) provided thorough assessments of Samuelson's contributions to equilibrium analysis, stability and macroeconomics, with references to Lucas's new classical macroeconomics. That was followed by Stanley Fischer's (1987) dictionary entry, which shared Hahn's and Tobin's concern over Samuelson's implicit assumption of given prices and wages in much of his macroeconomics. By the late 1980s and early 1990s, Bruna Ingraio and Giorgio Israel (1990) and Roy Weintraub (1991) located Samuelson's *Foundations* as an essential part of the history of mathematical general equilibrium and dynamic

stability analyses. More recently, expanding on some points advanced by Hahn (1983) and Leijonhufvud (1998), Wade Hands (2010, 2012, 2016) and Roger Backhouse (2015a, b; 2017, chapters 14 and 22)³ have examined in detail Samuelson's separation between individual optimization and aggregative dynamics, with comparisons drawn to Lucas (see also Backhouse and Boianovsky 2013, pp. 96-99).

Samuelson (1972) came back to the "separation" issue in his 1970 Nobel Lecture. The topic of that lecture was "maximum principles", which covered matters related to the first part of *Foundations*. However, towards the end of the lecture, he brought in two sections on "nonmaximum problems" and "dynamics and maximizing". In those sections, Samuelson (1972) reaffirmed his point that there are many areas in economics in which maximum principles do not apply, as forcefully illustrated by the multiplier-accelerator model. Moreover, moving beyond the original framework of the *Foundations*, he pointed out that there were important dynamic problems that could be related to maximizing, as indicated by the "turnpike theorems" of inter-temporal efficiency and optimal growth he developed in the 1950s and 1960s. However, Samuelson kept that apart from macroeconomic dynamics properly.

As expected, Samuelson (1983) reacted negatively to Lucas's ([1980] 1981) attack and to rational expectations market clearing macroeconomics as a whole. That was despite Samuelson's (1965) formulation of the "efficient market hypothesis", which has been perceived by many (but not by Samuelson) as very close to rational expectations. Samuelson would acknowledge that traditional Keynesian business cycle and income determination models lacked proper microfoundations of price and

³ See also Michael Brady (2018).

wage dynamics, but not of the kind proposed by Lucas. However, he never settled the matter to his satisfaction. In his last contribution to macroeconomic dynamics, Samuelson (1988) came back full circle to the theme of his seminal multiplier-accelerator model (Samuelson 1939a, b). In both cases, he attempted to make sense of Alvin Hansen's macroeconomics by means of non-optimizing models based on difference and differential equations. However, in 1988 he would set his focus on a non-linear locally unstable model with anti-damped roots, instead of the 1939 linear multiplier-accelerator formulation. In that sense, Samuelson (1988) represented a further move away from the limits of the "dogma" of stability (cf. Samuelson 1955), conspicuous in the structure of *Foundations*.

2. Output and prices: two types of economic dynamics

In his influential survey of stability analysis, Negishi (1962, p. 637) observed that there were two different sorts of dynamic models deployed at the time. One set of models contained the "magnificent" dynamics of trade cycles and economic growth, whereas the other type of models tackled the dynamics of the market clearing process, such as the Walrasian tâtonnement. The focus of Negishi's survey was the behaviour of the short-run market clearing adjustment process towards equilibrium. On the other hand,

The models of trade cycles and economic growth generate time paths of outputs, capital stocks and prices, which are of a dynamic equilibrium type, in which the supply of and demand for each commodity are assumed to be continuously equal in every market. This abstraction from the market clearing

process, which may be considered as a short run phenomenon than the one under consideration, may be justified if the former is rapidly damped and can be supposed to have worked out its effects. (Negishi 1962, p. 637)

That was a crucial distinction, made by Negishi with reference to Samuelson's (1947, pp. 331-32) discussion of equilibrium processes with different speed. Samuelson argued that economists may abstract from the behaviour of processes much faster than those under examination, if it is either assumed that they are "rapidly damped" and stable or by including them in the dynamical differential equations of the behaviour of the system out of equilibrium. He often adopted the former procedure. Samuelson (1947, p. 263; 1941, p. 102) wrote price adjustment as the differential equation

$$\dot{p} = \frac{dp}{dt} = H(q_D - q_S) = H[D(p, \alpha) - S(p)]$$

where α is a parameter, $H(0) = 0$ and $H' > 0$.

Hicks had provided, in the first edition of *Value and Capital*, the most comprehensive formal treatment of dynamic stability previous to the *Foundations*. However, from Samuelson's (1941; 1947, chapter IX) perspective, Hicks did not specify the dynamics of Walrasian general equilibrium and therefore was unable to present a correct treatment of stability conditions. Samuelson's formal stability analysis has been seen as the first full mathematical account of the tâtonnement, even if restricted to local stability, or "in the small", although Samuelson did not explicitly refer to the Walrasian tâtonnement in that connection (Patinkin 1965, pp. 539-40;

Hahn 1983, pp. 48-50; Ingrao and Israel 1990, chapter IX.5; Weintraub 1991, chapters 2 and 3).

Samuelson (1941, pp. 113-20; 1947, pp. 276-83) applied his formal stability apparatus and correspondence principle to provide a first dynamic version of the comparative-statics properties of the IS-LM Keynesian system in terms of differential and difference equations. The first model, with differential equations, yielded a dynamic system of the form:

$$\dot{Y} = I - [Y - C(i, Y) - \alpha]$$

$$0 = L(Y, i, M)$$

where Y is income, i , the interest rate, M , the stock of money, I , investment, C , consumption, L , the excess demand for money, and α is a shift parameter. It is remarkable that prices are conspicuously absent from the model. Indeed, Samuelson (1946a, p. 199; 1949a, p. 135) compared the equation of income change (illustrated by Samuelson's [1939b, p. 790] well-known "Keynesian cross" diagram) to the equation of price dynamics (as represented graphically by the "Marshallian cross" of supply and demand). They were considered "logically" equivalent, but operating on different levels. In view of Samuelson's (1947, pp. 331-32) and Negishi's (1962, p. 637) remarks quoted above, the absence of price changes from the income determination IS-LM model may be understood as an abstraction of the supposedly faster short-run processes of price determination, with incessant equilibrium in the commodities markets, instead of interpreted as a fixed-price assumption. The same applies to Samuelson's (1939 a, b) multiplier-accelerator model. In any event, the results of Samuelson's dynamic IS-LM model, as an exercise in the application of the

correspondence principle, have been regarded as ambiguous, not least because of the implicit constancy of money prices (Hahn 1983, pp. 51-52; Tobin 1983, p. 194).

The basic assumption behind the correspondence principle was that the economy is stable. Like Hicks before him, Samuelson restricted himself to postulating the existence of equilibrium, without attempting its demonstration. That would be the task performed by Arrow, Hurwicz, Debreu and others in the 1950s, a research program that did not engage Samuelson (Negishi 1962; Ingrao and Israel 1990; Weintraub 1991). The “stability hypothesis” had no “teleological or normative significance”, as the “stable equilibrium might be at a fifty per cent unemployment” (Samuelson 1947, p. 5). The stability conditions were deemed essential to allow for the investigation of the dynamic properties of non-optimizing systems formed by the interaction between economic units, such as Keynesian macroeconomics. Moreover, stability was considered an empirically plausible hypothesis, since positions of unstable equilibrium are “non-persistent states” and less likely to be observed. “How many times has the reader seen an egg standing upon its end?”, asked Samuelson rhetorically (1947, p. 5).

Samuelson’s argument that the rate of change of income is proportional to the difference between investment and saving, expressed in the equation above, is directly related to the assumption that the marginal propensity to consume is a positive fraction. This follows from the assumption of stability of income determination and a positive multiplier in comparative statics (Samuelson 1948, p. 375). Keynes (1936, p. 250) had stated that a key “stability condition” of his system, that prevents income from fluctuating widely, is the fractional value of the marginal propensity to consume. According to Metzler (1946), that was the main contribution of the “modern theory of employment” to business cycle theory, which had been

dominated by unstable Wicksellian cumulative self-reinforcing processes of expansion and contraction, with turning points explained by external factors. In the Keynesian system, unlike the old business cycle literature, a decline in output caused by excess aggregate supply will reduce supply more than demand (because of the consumption function) and bring the economy to equilibrium at less than full employment.⁴

As Lawrence Klein (2006) recollected, Samuelson, as adviser to his 1944 MIT PhD thesis (published 1947), suggested him that the core of Keynesian macroeconomics was the proposition that there is no positive value of the rate of interest able to equilibrate saving and investment at full-employment income. The economy converges to unemployment dynamic equilibrium through downward changes in income and ensuing shifts of the saving and investment curves, as developed by Klein (1947). Despite the practice described by Negishi (1962) under Samuelson's (1947) influence, price and output dynamics in market clearing and business cycle analyses were intertwined, as became clear in Samuelson's 1946 correspondence with Patinkin, who sent him a paper on price adjustment equations, later published in condensed form (Patinkin 1946, 1947). Patinkin criticized Samuelson's (1941) argument that a dynamic version of the Walrasian system could be found by replacing the condition of equality between supply and demand by an equation determining the rate of change of prices as a function of excess demand in each market.

The main problem was that "the concept of excess demand is inconsistent with our definitions of the demand and supply curves", which are based on the notion

⁴ As put by Patinkin (1982, p. 10), the equilibrating effect of the contraction in aggregate income is the nucleus of Keynes's theory of effective demand, expressed by the stability of the equation $dY/dt = f[F(Y) - Y]$, where $F(Y)$ is aggregate demand and $f' > 0$.

that agents are always on their curves (Patinkin 1946, p. 4). Samuelson's market-adjusting equation "is not localized in any specified behaviour unit except in a few markets where there is an official auctioneer" (Patinkin 1947, p. 172). That was the starting-point of Patinkin's off-curve analysis, which would eventually help to launch disequilibrium macroeconomics (Boianovsky 2006, pp. 206-08). Arrow (1959) would take up some of the points Patinkin raised, in an implicit criticism of Samuelson (1947) that pointed to imperfect competition as the only way to model non-market clearing behaviour (see Backhouse and Boianovsky 2013, pp. 106-10).

Samuelson (1946b) welcomed Patinkin's (1946) paper, but reaffirmed his distinction between static maximization and dynamics. He agreed that "total market demand curves often have in their background maximizing behaviour of producing or consuming units". Nevertheless, he argued, "We may study their properties independently of that fact. Theoretical economics is more than the study of maximizing units" (1946b, p. 1). That should not prevent the study of "how *markets* behave when not in long-run equilibrium". Samuelson acknowledged that his dynamical hypothesis of price change, although recording "what appears in every elementary textbook", when one critically puts a "microscope upon" the process it is found to be a "difficult one" (ibid, p. 2).

In a perfectly competitive market, no one is supposed to have any reason to "bid up" or "bid down" a price ... Certainly, the assumption that each (P,Q) observation represents the intersection of shifting supply and demand curves is empty, unless we have some hypothesis of shift. The problem is extraordinarily complex from a theoretical point of view ... (Samuelson 1946b, p. 3)

Samuelson and Patinkin seemed to agree that “if we assume that both demand and supply equilibrium curves always hold instantaneously, than no dynamics is possible”. That looked like a criticism of Hicks’s ([1939] 1946) assumption – repeated by Lucas ([1980] 1981) in a different context – that the process of adjustment to (temporary) equilibrium is completed within a short period (“week”), with no analysis of the movements of prices within the week. In the same year as Patinkin (1946), Hicks reacted to Samuelson’s (1941, 1942, 1944) criticism of the dynamic stability analysis of the first edition of *Value and Capital*. Hicks showed no concern for price adjustment and tâtonnement issues that exercised Patinkin and Samuelson (see Ingrao and Israel 1990, chapter VIII.5; Weintraub 1991, pp. 29-37).

In the additional note C, about “Professor Samuelson’s Dynamic Theory”, Hicks ([1939] 1946, p. 336) noticed Samuelson’s assumption that rates of price change are functions of differences between demands and supplies – instead of Hicks’s own assumption of quick passage to temporary equilibrium. This allowed the discussion of dynamic stability in terms of differential and difference equations, unlike Hicks’s own analysis. But Hicks was not convinced.

By my hypothesis of essentially instantaneous adjustment, I reduced the purely mechanical part of my dynamic theory to the simplest terms ... But in so doing I did leave myself free to make some progress with the less mechanical part – expectations and so on. (Hicks [1939] 1946, p. 337)

From Hicks’s perspective, Samuelson’s dynamics was too mechanical, lacking a treatment of the “behaviour of people” concerning expectations and other motives of conduct (ibid). It was clear to Hicks ([1939] 1946, p. 337) the parallels

between Samuelson's work in dynamic stability and business cycle "econometric" formal models developed by Frisch, Tinbergen and Kalecki in the 1930s.⁵ A central question to be settled was the choice between that approach to business cycles, in terms of "mechanical periodicities" expressed by difference equations, or Hicks's method of temporary equilibrium theory. That would also settle, Hicks expected, the theoretical issue between himself and Samuelson on dynamics. However, it is ironical that a few years later Hicks (1950) would join the "econometricians" and develop a difference equations (nonlinear) business cycle model significantly influenced by Samuelson's (1939a, b) multiplier-accelerator formulation.

A crucial feature of Samuelson's (1948a, p. 354) economic dynamics is the "self-generating" development of each dynamic system produced over time, as an autonomous response to the initial conditions or as a reaction to changing external conditions. One of the results of linear business cycle mathematical models – put forward by Frisch, Tinbergen, Metzler and Samuelson – was to show that no separate theory of the turning points of the phases of the cycle was necessary, against the verbal nonlinear accounts that prevailed until the 1940s (cf. Haberler's [1946, pp. 473-80] sceptical reaction). Samuelson's (1939a, b; 1947, pp. 340-42) multiplier-accelerator model, produced under the stimulus of his Harvard mentor Alvin Hansen, was the first fully endogenous model of the movement from boom to depression and back (see Tobin 1983, p. 195; Heertje and Heemeijer 2002; Morgan 2012, pp. 228-32; Backhouse 2017, pp. 262-65; see also Chiang [1967] 1984, pp. 585-91, for the mathematical solution, which is only implicit in Samuelson). Like Samuelson's dynamized IS-LM model, it did not feature price adjustment equations. The multiplier-accelerator model was formed by a system of equations:

⁵ Albert Hart (1951, p. viii) would comment on the absence of anticipations in Samuelson's (aaa) survey of economic dynamics (see Kregel 1980, pp. 27-29).

$$Y_t = C_t + I_t + G_0$$

$$C_t = \alpha (Y_{t-1})$$

$$I_t = \beta (C_t - C_{t-1})$$

where G is government expenditure.

Samuelson (1939a) first used arithmetical simulations with different values of the parameters α and β and then solved the model analytically by means of a second-order linear difference equation that permitted the existence of conjugate complex roots for a periodic solution. That was illustrated by a two-dimensional space and diagram depicting pattern reactions to changes in government expenditure (hence, it was a “mixed exogenous-endogenous” model). As Samuelson (2002, p. 220 n.1) recollected, what “did the trick” was his borrowing from Hansen (1938) the “idiosyncratic” formulation that related consumption (not income) growth to induced investment. Otherwise the model would generate a much more complex third-order difference equation. Mathematical analysis of stability conditions was able to provide a solution in terms of the several combinations of values of the parameters. The values Hansen had suggested for α and β (one-half and 2) lied on the boundary between damped and anti-damped cycles in the stability diagram. Instead of Hansen’s conclusion – that the 1937 American consumption-based recovery and upturn would be short-lived and followed by a permanent downturn – Samuelson showed that perpetual oscillation was the analytical result obtained for those parameter values.

Samuelson (1972, pp. 258-59) came back to the “fundamental” multiplier-accelerator model in his Nobel Lecture, while pointing out that “it provides a typical example of a dynamic system that can in no useful sense be related to a maximum problem”, along the lines of the *Foundations*. However, he now mentioned some of

the drawbacks and limitations caused by that. The fact that “the accelerator-multiplier *cannot be* related to maximizing takes its toll in terms of the intractability of the analysis”. Samuelson referred to Richard Eckaus’s (1954) PhD dissertation, written under his supervision, which, in his view, extracted from the multiplier-accelerator model all that could be obtained. Yet, “few grand simplicities emerged”, as there was an extensive range of possibilities of what could happen (see also Eckaus 1957). Hence, if Europe in 1970 and 1950 were stable multiplier-accelerator systems (with damped characteristic roots), and if the coefficients of the model for 1960 were the arithmetic mean of the 1950 and 1970 coefficients, one would expect the 1960 system to be stable as well. However, the model did not warrant that prediction. The paradox is solved when it is realized that

The determinantal conditions for stability of a system [Samuelson 1947, p. 436] do not define a stability region in terms of the coefficient of the system that is a convex region. Hence a point half-way between two points in the region may itself fall outside that region. This sort of thing does not arise in the case of well-behaved maximum systems. (Samuelson 1972, p. 259)

Without optimization, stability conditions of the Jacobian determinant do not produce a tractable convex set. That was indirectly related to the problem of “free parameters” in macroeconomic dynamics, discussed in the next section.

3. Samuelsonian macroeconomic dynamics criticized

As discussed above, critical reactions to Samuelson's research program of dynamic stability analysis started to appear even before the publication of the *Foundations*, especially by Patinkin and Hicks. However, it was only in the 1970s that criticism – led by Lucas under Hicks's partial inspiration – reached the core of that program as based on the separation between optimization and macroeconomic dynamics. Patinkin, who had been the first to criticize Samuelson's price adjustment analysis, articulated a few years later, en route to his seminal *Money, Interest, and Prices*, a full critical assessment of the correspondence principle. Patinkin (1965) made extensive use of Samuelson's correspondence principle in deriving comparative statics propositions, but at the same time called attention to its limitations, particularly in the presence of spillover effects across markets (Patinkin 1952; 1965, mathematical appendix to chapter XII).

As Patinkin (1952, p. 37) pointed out, the meaning of Samuelson's correspondence principle is that it is sometimes impossible to compare equilibrium positions unless restrictions are imposed the functions of the system by the condition that it converges. He set out to show that there are important cases in which dynamics does not succeed in casting the necessary light on comparative statics. Samuelson's stability analysis depended on the tacit assumption that excess demand in one market affects only the price of that market. That assumption is warranted only if the individuals who form the market are able to buy or sell as much as they desire at the prevailing prices, which does not obtain in dynamic disequilibrium conditions, when there are unsatisfied buyers and sellers.

Patinkin (1965, p. 235) concept of “spillover effects” – the notion that an agent's actual demand and supply in one market will depend upon the transactions actually performed in other markets, in the sense that a constraint in one market will

affect behaviour in other markets – would turn into one of the stepping stones of disequilibrium macroeconomics (see Backhouse and Boianovsky 2013, chapter 3). Although Patinkin did not express it in those terms, such limitation of the correspondence principle results if the price adjustment process, because of inter-market pressures, is not a Walrasian tâtonnement. Under these circumstances, there is no correspondence between the Jacobian matrix of the static system and terms in the determinant relevant for the dynamic (non-tâtonnement) adjustment process, as Patinkin (1952) showed.⁶

Shortly after Patinkin's (1952), Gordon (1955) critically tackled again the correspondence principle, this time from the broader perspective of Samuelson's methodology of operationalism. Gordon's assessment has been influential, among other reasons because it elicited a reply from Samuelson (1955) acknowledging its pertinence (see e.g. Blaug 1980, pp. 101-03). Among other points of criticism, Gordon challenged the empirical and theoretical validity of Samuelson's key assumption that the real world is dynamically stable. Gordon (1955, p. 308) pointed out that "recent theories of the business cycle ... suggest that actual economic variables may possess *no* stable equilibrium values over the observable range, yet the values observed may all be points on stable functions." As Samuelson (1955, p. 313) remarked, Gordon was referring to auto-relaxation business cycle models of the kind proposed by Nicholas Kaldor, Hicks and especially Richard Goodwin, based on local instability at their stationary levels, and featuring limited oscillations because of nonlinearities. That was distinct from Samuelson's linear multiplier-accelerator model. Gordon's point was that – instead of Samuelson's (1947, p. 5) claim that

⁶See also Brown and Rogers 1978 for a survey of critical assessments of the correspondence principle from the 1950s to early 1970s. Negishi (1982) compares Samuelson's stability analysis with non-Walrasian disequilibrium macroeconomics, without mentioning Patinkin though.

actual observations are either points of dynamically stable or unstable equilibrium, which makes the latter very unlikely to be observed – what we may actually observe, as implied by the mentioned business cycle models, are neither.

As recalled by Tobin (1983, p. 193) – who attended, together with Samuelson and other fellow students, Schumpeter’s course on general equilibrium at Harvard (Klamer 1983, p. 100) – Schumpeter’s reproach to Samuelson’s empirical stability argument was: “Who could claim that capitalism is stable?” In the same year as Gordon’s critical assessment, Tobin and Hall ([1955] 1987) proposed an interpretation of the correspondence principle that diverged from Samuelson’s own. They considered a “perversion” of that principle the dominant view that stability conditions placed limitations on empirical possibilities. Instead, they placed limitations only on the “relevance of static theories”. If the values of the parameters of a static system lead to an unstable equilibrium, this does not mean that such values are not empirically valid, but that the economy “cannot be represented by a static model” (Tobin and Hall, p. 75). In particular, they rejected arguments (made by Keynes and Samuelson alike) of the kind that the marginal propensity to consume must be a positive fraction to avoid real world instability. Stability should not be taken for granted, as Tobin’s (1975) model of depressions illustrated (see also Scarth 1991, p. 28).

Gordon⁷ came back to Samuelson’s dynamic analysis in his 1970 joint paper with Allan Hynes, published in the well-known Phelps volume but circulated since the mid 1960s. Gordon and Hynes (1970) may be seen as the main piece of criticism of Samuelson’s price adjustment analysis after Patinkin (1946, 1947). According to Gordon and Hynes (p. 370), Samuelson’s (1941) “seemingly innocuous clarification”

⁷ Donald F. Gordon (1923-2001) did his PhD at Cornell University and spent a significant part of his career at the University of Washington.

of the law of supply and demand by means of his price adjustment equation provided the basis for research in the theory of price dynamics and for empirical work on labour market disequilibrium and the Phillips curve. They pointed a number of “conceptual weaknesses” in Samuelson’s framework. Firstly, whereas the properties of static demand and supply functions are derived from maximization, the dynamic properties are not deduced as the “maximizing response of economic units to changing data”. Moreover, Samuelson left unexplained who is the economic unit whose behaviour is described by the equation of excess demand, unless he assumed the *deus ex machina* auctioneer with its ad hoc dynamics (ibid, pp. 371-72).

Such pieces of criticism are reminiscent of Patinkin (1946, 1947), and Arrow (1959), whom Gordon and Hynes cited. Their third and last critical remark was quite distinct from Patinkin or Arrow. It referred to the role of information and anticipations in price dynamics. The hypothesis of stability of a function like Samuelson’s price adjustment equation “makes little sense in a private market inhabited by maximizing traders”, since it implies that given the initial price and excess demand level “the course of future prices is predictable” (Gordon and Hynes 1970, p. 372). But, in such situation, traders would exploit profit opportunities and “destroy the stability of the hypothetical differential equation”, just like in the argument that stock market prices may be understood as a random walk (ibid).

Surprisingly, Gordon and Hynes (1970) did not refer to Samuelson’s (1965a) argument (further discussed in the next section below) that stock market prices follow a random walk. Lucas ([1980] 1981, pp. 292-93, n. 4) would point out that “Gordon and Hynes’s criticism of the use of Samuelsonian disequilibrium price dynamics as a description of observed price paths received central support from” Samuelson (1965a). Like Gordon and Hynes (1970), Lucas rejected the dynamic analysis

advanced in the *Foundations*, which, in his view, was behind the “neoclassical synthesis” that dominated macroeconomics until the 1970s.⁸

The synthesis consisted of the addition of “free parameters” – in the sense of parameters describing economic behavior that are not derived from optimization – to a static general equilibrium neoclassical system, which allowed for a diversity of Keynesian business cycle models understood as movements out of stationary equilibrium. From the perspective of both Lucas and real business cycle theory, the multiplier-accelerator model suffered from the problem (shared by other business cycle models of the 1930s put forward by Frisch and Tinbergen) that its quantitative behavior depended upon the assumed values of the coefficients of the variables in the equation. In that sense, “pure theory was not providing sufficient discipline” (Kydland and Prescott 1991, p. 165). Samuelson (1972) was aware of some aspects of the “free parameters” problem, as indicated by his comments about the non-maximizing multiplier-accelerator discussed in the previous section.

According to Lucas ([1908] 1981), the neoclassical synthesis should be traced back to the *Foundations*, with its combination of the study of static maximum problems (with equilibrium defined as “rest” as in mechanics) in the first part and non-maximizing dynamic theory in the second (that is different from Samuelson’s original meaning of “neoclassical synthesis” introduced in 1955). Samuelson attempted to solve the “disparity” through his model of price (and implicitly quantity) dynamics, which “introduced sufficient additional (to those needed to describe tastes

⁸ Under Gordon’s influence (his colleague at the University of Washington), Silberberg (1978, pp. 527-28) criticized Samuelson’s stability hypothesis for replacing the explicit assertions of maximizing behavior for a weaker account of how the markets operate. This appeal to some “mystical stability properties” represented a “departure from the explicitly choice-theoretic microeconomic paradigm”. Moreover, according to Silberberg, Samuelson’s dynamics conflicted with the notion that there can be no disequilibrium in a world of utility or wealth maximizers.

and technology) parameters to the equilibrium system” (Lucas [1908] 1981, p. 278). The flexibility of free parameters was also its disadvantage, since such parameters reflected past behavior and would not remain stable when the system was exposed to shocks of several kinds, including economic policy changes along the lines of the well-known “Lucas Critique”. The application of Samuelson’s correspondence principle was handicapped by the fact that more than one dynamic model could be specified. Such “arbitrariness” would be avoided if a “clear microeconomic rationale for the model” was provided (Scarth 1991, p. 28).⁹

Lucas (pp. 284-85) argued for a return to the dynamic view of competitive equilibrium, outlined by Hicks ([1930] 1946) and further developed by Arrow and Gerard Debreu in the 1950s, as contingent-claim, with planned choices over sequences of dated goods and their expected prices. The contingent-claim view of equilibrium should replace Samuelsonian modeling of price dynamics as responses to static excess demands. Lucas’s market clearing models of business cycles assumed instead that prices and quantities are always in equilibrium (see also Backhouse and Boianovsky 2013, pp. 96-99).

Interestingly enough, Hahn (1983, p. 33) suggested that Lucas shared with Samuelson the view that only equilibrium states are observable, and, therefore, that economics should be concerned with “equilibrium states” only. However, unlike Lucas, Samuelson (1947) was at pains to stress that his notion of equilibrium encompassed unemployment and other non-maximum states. In fact, the comparative statics properties of the system were more important to Samuelson than reference to a

⁹ Attempts to rehabilitate the correspondence principle under dynamic stochastic rational expectations models have been made by Brock (1987) and Evans and Honkapohja (2007). Apart from the dynamic issue, the content of the principle is dubious because any continuous function can be an excess demand function (the Sonnenschein-Mantel-Debreu Theorem; see Ingrao and Israel 1990, chapter XI).

particular point called equilibrium. As pointed out by Weintraub (1991, p. 104), “like physicists mistrustful of mathematicians, Samuelson believed that his equations characterized ‘reality’ or the ‘real economic situation’”, which prominently included unemployment configurations.

Lucas ([1980] 1981, p. 292, n. 4) observed how Samuelson (1947, p. 5) deployed the rolling egg metaphor in order to illustrate the use of the correspondence principle and stability analysis as a criterion to decide which equilibrium points are actually observed. “Here the idea is clearly to decide which static egg-equilibria are empirically interesting, not to offer an empirically useful dynamic model of rolling or wobbling eggs”, Lucas (*ibid*) distinguished. The analogy to the latter was Lucas’s own inter-temporal optimization approach. Lucas (2004) recalled how he, as many others, had learned his economics in the late 1950s from Samuelson (1947) and the first 1956 edition of Patinkin’s *Money, Interest, and Prices*. However, he found Patinkin’s model too complicated to work out its predictions. “All the dynamics are the mechanical auctioneer dynamics that Samuelson introduced, where *anything* can happen” (Lucas 2004, p. 15, italics in the original). Patinkin’s verbal discussion in the book indicated to Lucas that there were some significant economic arguments into those dynamics

What are people thinking? What are they expecting? He’s too good an economist to take the Samuelsonian dynamics literally. He’s really thinking about intertemporal substitution. He doesn’t know *how* to think about it well, but he’s trying to. So in some sense Patinkin’s book is less mechanical than it looks. (Lucas 2004, p. 16, italics in the original)

From Lucas's perspective, his task in the 1970s was to fulfill the unfinished job Patinkin had started, but devoid of its Keynesian macroeconomic disequilibrium features. Lucas ([1980] 1981, pp. 288-99 and p. 293, n. 10) considered the possibility of a "synthesis" between the then new equilibrium models and a Samuelson-like model of disequilibrium price adjustment, as in Malinvaud's disequilibrium macroeconomics approach. But he was wary of the addition of free parameters to the analysis. Indeed, by the mid 1980s explicit use of Samuelson's excess demand equation of price change was largely gone, replaced either by explicit models of price setting by firms and workers (as in New-Keynesian macroeconomics) or continuous market-clearing (as in New-Classical macroeconomics). As put by Fischer (1987, p. 237), "the older approach is used in disequilibrium macroeconomics, but is typically regarded as suspect". Equally "suspect" was what was seen as the lack of microfoundations of Samuelson's multiplier-accelerator and dynamized IS-LM models, as well as of his famous Keynesian cross diagram, which implicitly assumed fixed prices and wages.¹⁰

The want of a clear link between Samuelson's general equilibrium (value) theory and his macroeconomic models was noticed and criticized first by Arrow (1967) and then by Hahn (1983, p. 51) and Tobin (1983, pp. 195-96), who pointed out critically that Samuelson "never found the existence of excess-supply disequilibrium in the labor market a surprising departure from Walrasian equilibrium worthy of defense or of theoretical investigation". From that perspective, the neoclassical synthesis wasn't (see also Backhouse and Boianovsky 2013, pp. 41-44).

¹⁰ Endogenous business cycle models, of the kind inaugurated by Samuelson's (1939a, b), were developed along different lines in the 1980s (see Grandmont 1985), since the 1939 model was perceived as lacking "rigorous microfoundations" (Benassy 2011, pp. 206-07). Ironically enough, Grandmont based his model on the overlapping generations framework devised by Samuelson (1958).

Tobin (1983, p. 194), one of the foremost critics of Lucas's equilibrium macroeconomics, agreed, with reference to Lucas ([1980] 1981), that without the constraints of maximization assumptions, Samuelson's dynamic stability analysis contained an "embarrassing abundance of free parameters on whose values the model-builder has few clues". Hence, the "Frischian revolution" of explicit dynamic macroeconomic models hailed and further elaborated by Samuelson remained uncertain.

4. Samuelson's dynamics after the *Foundations*

Upon discussing the non-maximization features of his multiplier-accelerator model, Samuelson (1972, p. 259) observed that "this does not deny that there is rich dynamics which can be related to maximizing". That remark was followed by an account of the circumstances surrounding his "turnpike theorems" of inter-temporal efficiency, Samuelson's main contribution to optimizing dynamics, written under the influence of von Neumann ([1937] 1946). That was not part of the 1947 *Foundations* – it was surveyed, together with other post-1947 contributions by Samuelson and others to mathematical economics, in the "mathematical appendix C" to the 1983 enlarged edition (Samuelson [1947] 1983).¹¹ Growth economics was conspicuously absent from the *Foundations*, if only because Evsey Domar's (1946) path-breaking

¹¹ As Samuelson (1971, p. 691) observed, were the *Foundations* "written today with knowledge of the revival of interest in Ramsey growth models, I would certainly have added a chapter on optimal-control theory and similar dynamic maximization matters. And then instead of being preoccupied with the problem of damped stability of dynamic motions, I would have been interested as well in stationary points which are saddle-points surrounded by dynamic motions of the catenary type that we associate with modern turnpike theory".

article had just appeared. It is worth noting, though, that the very last phrase of the book reflected Samuelson's (1947, p. 355) hope that comparative dynamics should be able to illuminate the "majestic problems of economic development" (see also Boianovsky and Hoover 2014; Boianovsky 2019).

Domar introduced into economics the "method of growth theory", based on the notion of the equilibrium of an expanding economy whose component parts grow at the same steady rate and retain some proper relationship to each other, i.e., according to an exponential function. His 1946 article became a favorite illustration of the use of differential equations in non-optimizing economic dynamics, as discussed in Samuelson (1948a, pp. 361-63). Harrod's (1939) approach to cyclical growth became well known in the US shortly after, leading to the "Harrod-Domar growth model" literature (Boianovsky 2017).

The Harrod-Domar model – particularly in Harrod's version – shared with Samuelson (1939a, b) the notion that the dynamic path is determined by the interaction between the multiplier and the accelerator. However, Samuelson did not develop or anticipate that growth model, as he was concerned, under Frisch's influence, with damped-root stability. Samuelson (1974, p. 10; see also 1955, pp. 312-23; 1972, p. 259; 1988, p. 17) recollected how he and his Harvard colleague Metzler "fell into the dogma ... that all economic business-cycle models" should be stable, in the sense of having "*damped* roots".

Samuelson and Metzler accepted Frisch's criticism of Kalecki's practice of imposing constraints on his parameter-estimating equations so that roots would be neither damped nor undamped (cf. Samuelson 1947, p. 337). Moreover, the "dogma" was empirically inspired by the behavior of the American economy in 1933-40, when it seemed incapable of "self-fulfilling bootstrap returns to prosperity" (Samuelson

1974, p. 10). The stability dogma led Samuelson to delay recognition of the full relevance of non-linear auto-relaxation models, which are briefly mentioned in the *Foundations* (pp. 338-39) but only tackled upfront in his 1988 Hansen anniversary article.

The “dogma” led as well to Samuelson’s “suppressing development of the Harrod-Domar exponential growth aspects that kept thrusting themselves on anyone who worked with accelerator-multiplier systems” in the 1930s (ibid; see also Samuelson [1947] 1983, p. 480, on the Harrod-Domar model). The same applied to his critical reception of Hansen’s secular stagnation hypothesis in the 1930s and 1940s (Samuelson 1988, p. 17, n. 2). Already in his reaction to Gordon, Samuelson (1955, p. 312) acknowledged that he was “no longer so sure” that the hypothesis of dynamic stability was “realistic”.

Well, maybe the [economic] system *is* unstable ... These are important empirical questions that cannot be answered by dividing dichotomously the world’s possibilities into categories of unstable and stable and inferring that our observed world by its not having exploded away is necessarily in the stable category (Samuelson 1955, p. 313)

In 1945 Samuelson attended at Harvard a seminar by von Neumann about his general equilibrium growth model about to be published in translation. He then challenged von Neumann’s remark that the model involved new mathematical techniques unrelated to the traditional mathematics of physics and maximization (Samuelson 1972, p. 260). A few years later, Samuelson (1949) conjectured his first “turnpike theorem”, further elaborated in chapter 12 of Dorfman, Samuelson and

Solow (1958), and formally proved by Roy Radner in 1961.¹² Unlike the “positivistic” multiplier-accelerator and Harrod-Domar growth models, this was a case of a maximizing model, in the sense of inter-temporal efficiency. It featured a catenary motion around a saddle-point, which removed the possibility that the dynamic characteristic roots could be all damped (Samuelson 1972, p. 259).

Samuelson (1965b, pp. 494-95) “followed the crowd” in assuming the maximization of a Ramsey inter-temporal utility function of the “representative man”. He would use the same tool in an article about neoclassical monetary theory featuring real money balances in an inter-temporal utility function of the “representative man” (Samuelson 1968). As noticed by Samuelson (1968, p. 7, n. 4), his colleague Miguel Sidrauski (1967) had independently arrived at a similar dynamic formulation of money demand and growth in a monetary economy. Sidrauski’s article is often regarded as one of the first to have used the notion of a representative agent in macroeconomics. Samuelson (1968, p. 11), however, was careful to point out that he had assumed so far in his article that “every man are exactly alike”, as in a Robinson Crusoe economy.

If the extreme symmetry assumption is relaxed, the notion of collective indifference curves, necessary for his argument, is only valid if all income elasticities, including that for money, are (near) unity (*ibid*), which allows to abstract from income effects. The main argument was constructed from the point of view of *microeconomic* maximization, as Samuelson (1947, pp. 117-122) had done in his

¹² Samuelson’s production turnpike theorem was a dynamic generalization of von Neumann’s closed system, in the sense that whatever composition of consumption and capital goods the planner would like to achieve, one obtains the most of all goods if the (efficient) growth path is close to the von Neumann path for most of the time. Samuelson (1965b) later established as well consumption turnpikes with the help of Ramsey’s (1928) model of optimal saving (for a comprehensive survey of the literature see Turnovsky 1970).

treatment of money demand – with the difference that he now dealt with an inter-temporal setting. The same may be said of Samuelson's (1969) discussion of portfolio and consumption decisions under uncertainty, which also takes the Ramsey model as its starting-point (see Blanchard and Fischer 1989, pp. 279-83).

Unitary income elasticity is equivalent to assuming that individual utilities are the same and homothetic, a key condition for aggregation, as Samuelson (1956) had pointed out (see also Hahn 1983, p. 34).¹³ In fact, Samuelson (1948b, pp. 8-9; kept in all further editions) called attention to the logical “fallacy of composition” (in which what is true of a part is alleged to be also true of the whole) involved in economic aggregation problems. “Very definitely”, he wrote, “in the field of economics, it turns out that what seems to be true for individuals is not always true for society as a whole; and conversely, what seems to be true for all may be quite false for any one individual”.

As pointed out by Hartley (1997, pp. 173-74), that was an early criticism of the notion of the representative agent. Samuelson was ready to use that notion in certain contexts and under certain conditions (see Hands 2016, pp. 433-34), but those did not include Walrasian general equilibrium and Keynesian macroeconomics. Hence, consistently with the original framework of the *Foundations*, he did not extend dynamic optimization to deal with “positivist” macroeconomic determination of prices and output, as opposed to optimal growth paths and inter-temporal consumption decisions by individuals.

¹³ Samuelson (1956, p. 5, n. 2) referred to Wicksell's ([1893] 1954, pp. 72-74) early work on homothetic utility and preference aggregation. Wicksell used the notion of a representative agent in his discussion of optimal capital accumulation, but not in his macroeconomics (see Boianovsky 2016). That suggests an interesting parallel with Samuelson.

The “loose heuristic” correspondence principle (Samuelson [1947] 1983, p. 479) remained one of the pillars of Samuelson’s macroeconomic dynamics. While recollecting how *Foundations* came to be, Samuelson (1998, p. 1384) reaffirmed that “no one associates a Keynesian system with a maximizing single mind or even to an as-if-pretend maximizing system”. But that did not prevent drawing comparative statics predictions based on the *General Theory*’s “stable dynamics” and “heuristic ‘correspondences’ between dampening in dynamics and qualitative direction of ... equilibrium responses to exogenous perturbations”.

However, not all was well in the kingdom of Keynesian macroeconomic dynamics, even from Samuelson’s perspective. It lacked proper microfoundations, as he acknowledged in the section “Microfoundations of unemployment and inflation” (Samuelson 1976, pp. 828-29), under the impact of the then new literature – partly motivated by the attempt to understand the “complexities of the so-called Phillips curve problem” – on search, information and imperfect price adjustment by Phelps, Alchian and Leijonhufvud, among others. That was not the same as the approach to microfoundations based on modeling households and firms as optimizing agents operating in perfectly competitive markets (see also Backhouse and Boianovsky 2013, chapter 1).

Samuelson pointed out that Keynesian unemployment was incompatible with the “perfect market clearing mechanism”, in which prices and wages are always determined by the intersection of supply and demand schedules, as in abstract general equilibrium theory.

Keynes’ great breakthrough was to let the facts oust a beautiful but somewhat irrelevant theory. He could not present an elegant theory of price and wage rigidity that would explain exactly how unemployment and job

vacancies are possible. So he simply assumed in an unexplained way that pricing would be such as to permit of a mismatch between job seekers and jobs, between the goods that firms want to sell and what they succeed in finding customers for. (Samuelson 1976, p. 828)

Samuelson had been bothered by those issues from the beginning – as indicated by his 1946 correspondence with Patinkin discussed above – although he decided to move away from them. In fact, the contrast between the actual relevance of the *General Theory* and its analytical shortcomings applied to important aspects of Samuelson's own work in macroeconomic dynamics. Reacting to criticism of Keynesian (and Samuelsonian) economics by Lucas, Sargent, Barro and others, Samuelson (1991, pp. 402-03) argued for the superiority of Keynes's principle of effective demand as a reduced form description of income determination, over the New Classical market clearing rationalization. Lucas and other "modern critics of Keynes have rediscovered what we early converts to Keynes knew but chose to conveniently forget: the system [of equations of income determination through the multiplier] does lack firm 'foundations', micro or otherwise. Better no foundations than bad ones!"¹⁴ Whereas there was an "isomorphism" between Friedman's monetarist dynamics on one side and Keynesian models put forward by Tobin, Solow, Klein, Modigliani and Samuelson himself on the other, equilibrium business cycle models – as influenced by the Walras-Debreu system or even the Knight-Viner

¹⁴ This confirms Hahn's (1983, p. 51) feeling that Samuelson in the 1930s and 1940s had refrained from discussing the problem – that there is no meaning to 'lack of effective demand' in a world where agents can buy and sell as much as they wish at current prices – because he, "like everyone today, did not know how to proceed to a resolution". In any event, pointed out Hahn, business cycle theory without prices (as in the multiplier-accelerator model) "now seems inconclusive". Samuelson (1988, p. 12, n. 1) acknowledged as much.

approach of Samuelson's student days at Chicago – were distinctly different at the methodological level, as put by Samuelson (1983, p. 216).

In an interview with Samuelson (1996, pp. 161-62), David Colander and Harry Landreth asked him why he had not formalized price and wage stickiness and its relation (if any) with Keynesian macroeconomic dynamics. To their amazement, he replied that “there was no need to”, since he took the “positivistic” attitude that “we know” that there are cyclical oscillations in capacity utilization and that the Keynesian model provided an apparatus to explain it. “Just because I don't understand the process of digestion, should I refuse my beefsteak?”, he asked. This is reminiscent of Samuelson's view of equilibrium as a feature of “reality” mentioned above.

He had decided very early that “life was more fruitful not worrying about” microfoundations, which was confirmed by his perception that the search for microfoundations in modern macroeconomics had not been successful (ibid). The same view was reported by his MIT colleague Stanley Fischer (1987, p. 239), who reported conversation with Samuelson in which he stated that he understood the behavior of the economy and gave policy advice on the basis the assumed fact of price stickiness, without seeing a payoff in researching the issue. Colander and Landreth found that attitude, coming from the author of *Foundations*, “schizophrenic”, to which Samuelson (p. 163) replied by invoking the correspondence principle and by referring to the notion by E.B. Wilson – who taught him mathematical economics and statistics at Harvard in the 1930s and influenced his mathematical dynamics deeply (see Weintraub 1991, pp. 57-62) – that “equilibrium” meant “very slowly disequilibrium”, so that the time period was involved. Samuelson

(1996, p. 161) clarified that his Keynesian system featured disequilibrium, in the sense that workers are off their labor-supply curves.¹⁵

Throughout several editions of his *Economics*, starting with the 5th 1964 edition, the multiplier-accelerator model featured prominently in the business cycles chapter, with attention drawn to its connections with the Harrod-Domar growth model. Unlike his 1939 original discussion, Samuelson (1976, pp. 259-63) took into account Hicks' (1950) reformulation, with nonlinear effects caused by the full-employment ceiling and a floor to disinvestment in the boom and depression respectively. By the last (posthumous) 2010 edition, written jointly with William Nordhaus, the multiplier-accelerator model still featured as a main endogenous business cycle mechanism, this time combined with Frisch's (1933) impulse-propagation and with a word of caution about the omission of prices and the supply side.

On the occasion of Alvin Hansen's centennial, Samuelson (1988) produced his last contribution to macroeconomic dynamics, an extensive reformulation of his first 1939 multiplier-accelerator articles. Whether Samuelson (1939a, b) became immediately influential, his "Keynes-Hansen-Samuelson (KHS) multiplier model of secular stagnation" did not attract much attention, not just because the modeling strategy of endogenous cycles had changed in the 1980s (see e.g. Grandmont 1985). Samuelson (1988) was written in the Richard Goodwin (1982) tradition of nonlinear auto-relaxation limit cycles – which Samuelson (1955) had mentioned in his response to Gordon (1955) – but it failed to engage even scholars belonging to that research

¹⁵ That was close to Patinkin (1965), although Samuelson did not elaborate. Significantly enough, in his *Economics* the chapters on income determination and economic fluctuations come *before* the microeconomic chapters on the firm and the consumer.

agenda, partly because it came out in a relatively obscure outlet.¹⁶ In 1997 Samuelson sent me an offprint of his 1988 article, as part of correspondence about Haberler's *Prosperity and Depression*:

For your interest, I enclose a 1988 reprint few have noticed. This Keynes-Hansen-Samuelson non-linear limit cycle captures the empirical content of the 1936 Harrod, 1930s Kalecki, 1940 Kaldor, 1940s Goodwin, 1950 Hicks cyclical model, avoiding certain infelicities and omissions; and it enabled me to discern (50 years later!) that decelerating population growth, at the same time that it lowered the acceleration-principle *investment* propensity, also lowered (by virtue of Modigliani's lifecycle theory of saving) the propensity to save. In principle, *Prosperity and Depression* would agree with its spirit. (Samuelson 1997)

One of the main goals of the 1988 model was to express the combined effects of a slowdown of population growth on investment demand (capital widening) à la Hansen and on savings à la Modigliani, in order to formalize Hansen's secular stagnation argument (see also Backhouse and Boianovsky 2016). Samuelson's (1939a, b) original linear multiplier-accelerator model, because of its concern with stability, could not make sense of secular stagnation (see Samuelson 1988, p. 17, n. 2). The 1988 model provided a mathematical non-optimizing treatment of dynamics and local instability of the nonlinear "Hansen limit-cycle". Samuelson and Goodwin were fellow students at Harvard, when Philippe LeCorbeiller (Goodwin's mentor) perfected the Van der Pol-Rayleigh limit-cycle theory of dynamics (ibid, p. 12; cf.

¹⁶ Samuelson (1988) was published in the inaugural issue of *Japan and the World Economy*, edited by Ryuzo Sato, Samuelson's co-author in papers about demand for money in the 1980s.

Samuelson 1947, pp. 339-40). In a footnote, Samuelson (1988, p. 12, n. 1) acknowledged that his KHS model lacked not just rigorous *microfoundations* but especially *macrofoundations*, in the sense that the price level could not be treated by convention as a quasi-constant when the economy approached the full-employment ceiling and inflation accelerated.¹⁷

Samuelson's (1965a) instrumental role in developing the "efficient market hypothesis" (EMH) of finance theory – that market prices fully reflect all available information – may be seen as contradictory with his critical stance against the "rational expectations hypothesis" (REH) under market clearing, with its implications that money is neutral in the short-run and that there is little deadweight loss involved in the business cycle. Indeed, EMH and REH are often seen as closely related or equivalent to one another (see Delcey and Sergi 2019, and references cited therein). His 1965 "Proof that properly anticipated prices fluctuate randomly" represented a significant change in his original 1947 framework about price dynamics, as expectations figure prominently in the new 1965 version (cf. Lucas [1980] 1981, p. 293, n. 4). Samuelson (1965a) showed that, in an informationally efficient market, price changes must be unpredictable if they are fully anticipated, in the sense that markets incorporate the information and expectations of all participants. In other words, prices follow a random walk.

Samuelson (1965a, p. 42) established the martingale property of a stochastic model of price change by proving the theorem that next-period's price differences are

¹⁷ Unlike Grandmont (1985), Samuelson did not use the overlapping generations (OLG) model in his KHS system, or in macroeconomic dynamics in general for that matter. OLG became an important alternative to Ramsey infinitely lived agents in monetary macroeconomic modeling, especially after Diamond (1965) extended Samuelson's (1958) original framework to include growth (see Blanchard and Fischer 1989, chapter 3). Samuelson, however, did not produce OLG macroeconomic models, with the partial exception of his 1975 article on social security and capital accumulation (Samuelson 1975; see Blanchard and Fischer, section 3.2).

uncorrelated with previous period's price differences. The intuitive wisdom behind the model is that in a competitive market, if one could be sure that a price will rise, it would have already risen. Upon proving his theorem, Samuelson (1965a, pp. 48-49) raised some skeptical points about its scope, such as where the basic probability distributions come from and in what sense they might be optimized. "Are they supposed to belong to the market as a whole? And what does that mean? Are they supposed to belong to the 'representative individual' and who is he?"

After the rational expectations revolution, Samuelson dissociated REH from EMH he had developed together with (but independently of) Eugene Fama. REH goes "beyond mere" EMH by assuming that "people form and act upon expectations that 'average out' to be 'correct' expectations" (Samuelson [1947] 1983, p. 483). It was an empirical issue. The evidence indicated, Samuelson (1984, p. 10) claimed, that the stock markets have historically displayed "micro" but not "macro" efficiency, in the sense that a minority who spot deviations from micro efficiency in individual stocks can make money and then wipe out the inefficiencies, but long persistent waves of aggregate indexes of security prices below and above their fundamental values are common (see Jung and Shiller 2005).¹⁸ Even if a particular speculator knew in advance that the whole "consensus crowd" was wrong, there is no way he or she could profit by betting against the crowd (Samuelson, 1984, p. 10). As he claimed,

¹⁸ As Samuelson (2009, p. 26; italics in the original) put it, "Markets do tend to be *micro* efficient. *Only when you know new correct news that others don't yet know you can capture easy returns in micro-efficient markets.* Does that mean that every rise or fall in the indexes of most stock prices are rational reactions to knowable correct news? Not at all. The big cumulative swings in mean prices that historians document – as in 1929-34 or 2007-08 – are well known features of historic business cycles ... Again, what makes *macro* efficiency impossible is the hard fact that economic history is at best *quasi-stationary* time series. That quasi kills all uncertainties."

The school of rational expectations has been accorded spurious honor because of the genuine honor earned by efficient-market theory. Saying this is not to say that rational expectations is without honor; it is to say that we must fairly identify what are its earned honors. (Samuelson 1984, p. 10)

The EMH was incorporated into the *Economics* textbook in the 1985 12th edition and kept ever since (see Samuelson and Nordhaus 2010, section 23.D) to explain price fluctuations in financial auction markets, whereas markets for goods and labor were treated as imperfectly competitive. That led to the suggestion that macroeconomic models should be based on a “new synthesis” formed by the assumptions that (i) labor and good markets display sticky wages and prices, (ii) prices in financial auction markets adjust quickly to economic shocks and expectations, and (iii) expectations in auction markets are forward-looking (Samuelson and Nordhaus 2010, p. 642). Samuelson’s ways to macroeconomic dynamics, opened in the 1930s, eventually converged to a view that reflected his long time search for hypotheses able to account for economic fluctuations in goods, labor and asset markets.

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