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Article

A Credit-Risk Valuation under the Variance-Gamma Asset Return

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Abstract: This paper considers risks of the investment portfolio, which consist of distributed mortgages and sold European call options. It is assumed that the stream of the credit payments could fall by a jump. The time of the jump is modeled by the exponential distribution. We suggest that the returns on stock are variance-gamma distributed. The value at risk, the expected shortfall and the entropic risk measure for this portfolio are calculated in closed forms. The obtained formulas exploit the values of generalized hypergeometric functions.

Keywords: variance-gamma distribution; credit risk; call option; exponential distribution; shortfall risk; generalized hyperbolic function

1. Introduction

As there are a lot of various types of trading financial instruments on financial markets, a number of different investment strategies can happen. It is supposed throughout this paper that the investor accumulates money by selling call options and issues loans on the sum. We discuss the risk of such a portfolio. It is connected with the possible fall of mortgage payments and the growth of the asset price. The time of negative jump of the payment rate is assumed to be exponentially distributed.

The exponential distribution is exploited widely for the modeling of credit risk. The area of use covers both the stock (see [Bäurer 2015](#); [Carr and Wu 2010](#)) and interest rate ([Eberlein et al. 2013](#); [Grbac 2009](#); [Madan 2014](#)) frameworks. Let us notice that the hazard rate theory is an extension of the idea of default time modeling by the exponential distribution, see, for example, the book by [Bielecki and Rutkowski \(2002\)](#).

The variance-gamma distribution was first introduced in [Madan and Seneta \(1990\)](#) and [Madan and Milne \(1991\)](#), who discussed its symmetric version in the context of various problems of mathematical finance. [Madan et al. \(1998\)](#) defined the variance-gamma process as a time-changed Brownian motion with drift, or as a difference of two gamma processes, and supplied its properties. For a review of various investigations for the variance-gamma process and its properties, see also [Seneta \(2007\)](#).

There is much work that supports the use of the variance-gamma process in finance. Among others, let us mention the papers by [Daal and Madan \(2005\)](#), [Finlay and Seneta \(2006\)](#), [Linders and Stassen \(2016\)](#), [Luciano et al. \(2016\)](#), [Luciano and Schoutens \(2006\)](#), [Moosbrucker \(2006\)](#), [Mozumder et al. \(2015\)](#), [Rathgeber et al. \(2016\)](#), and [Wallmeier and Diethelm \(2012\)](#), where the variance-gamma distribution is confirmed as a very good model to make out the statistics. For approximations of processes by the variance-gamma one, see [Eichelsbacher and Thäle \(2015\)](#).

Various procedures can be exploited for the computing in the variance-gamma model. See, for example, [Avramidis et al. \(2003\)](#), [Fu \(2007\)](#) and the monograph by [Korn et al. \(2010\)](#) on Monte-Carlo methods, [Carr and Madan \(1999\)](#), and the review paper by [Eberlein \(2014\)](#) and [Wang \(2009\)](#) on Fourier transform methodologies. American-style options are discussed in [Hirsa and Madan \(2004\)](#) and

Almendral and Oosterlee (2007). Closed-form solutions are presented in Madan et al. (1998) and then developed in Ivanov (2018), Ivanov and Anov (2016) and Ivanov and Temnov (2016).

This paper continues the direction of Madan et al. (1998), giving analytical expressions that allow one to compute the value at risk, the expected shortfall risk and the entropic risk measure for the portfolio that is introduced above.

2. Model

It is assumed that at the start of investment, the banker has a portfolio with the value

$$X_0 = xc,$$

where $x > 0$ is the number of sold European call options and $c > 0$ is the price of one option. The strategy of the investor is to hand out credits on this sum of money, that is, the banker has the capital

$$X_1 = \int_0^1 R_t dt - x(S - K)^+,$$

at the end of investment, where $K \geq 0$. Normalizing with x , one can see that it is enough to estimate risks of the return,

$$X = \int_0^1 r_t dt - (S - K)^+, \quad (1)$$

where $r_t = R_t/x$.

It is supposed that the stock return is variance-gamma distributed, that is,

$$S = \mu + \theta\gamma + \sigma N_\gamma, \quad (2)$$

where γ has the gamma distribution with unit mean, that is, with the probability density function

$$f(\gamma, x) = \frac{a^a}{\Gamma(a)} x^{a-1} e^{-ax}, \quad (3)$$

where $\Gamma(u)$, $u > 0$, is the gamma function. Thus, there are the sold European call options on interest rate (for example, swaptions or caplets) in the Merton variance-gamma model, or even the options on stock if the stock tail is modeled directly by the variance-gamma distribution but not by its exponential.

The process r_t is presumed to stay constant $r_t = r_0 \geq 0$ if the credit return does not change over the investment period, and to become $r_t = r_1 \geq 0$, $r_1 \leq r_0$ if the return falls. It is assumed that the time τ of the return jump is exponentially distributed with the intensity λ . It is easy to see, then, that

$$X = r_0 I_{\{\tau > 1\}} + (r_0\tau + r_1(1 - \tau)) I_{\{\tau \leq 1\}} - (S - K)^+. \quad (4)$$

Next, we set

$$\text{sign}(u) = \begin{cases} 1 & \text{if } u > 0, \\ 0 & \text{if } u = 0, \\ -1 & \text{if } u < 0, \end{cases}$$

and use notations

$$\Psi(u), u \in \mathbb{R}, \quad \mathbf{B}(u_1, u_2), u_1 > 0, u_2 > 0, \quad \mathbf{K}_{u_1}(u_2), u_1 \in \mathbb{R}, u_2 > 0$$

for the normal distribution function, the beta function and the MacDonald function (the modified Bessel function of the second kind), respectively. The hypergeometric Gauss function is denoted as

$$F(u_1, u_2, u_3; u_4), \quad u_1, u_2, u_3 \in \mathbb{R}, u_4 < 1.$$

Also, the degenerate Appell functions (or the Humbert series), which is the double sum,

$$\Phi(u_1, u_2, u_3; u_4, u_5) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(u_1)_{m+n} (u_2)_m}{m! n! (u_3)_{m+n}} u_4^m u_5^n,$$

with $u_1, u_2, u_3, u_5 \in \mathbb{R}$ and $|u_4| < 1$, and where $(u)_l, l \in \mathbb{N} \cup \{0\}$, the Pochhammer’s symbol, is exploited. For more information on the special mathematical functions above, see the monographs by [Bateman and Erdélyi \(1953\)](#), [Whittaker and Watson \(1990\)](#) and also [Gradshteyn and Ryzhik \(1980\)](#).

Also, we need to represent the following constants and functions for the formulating of results. Let

$$w = \frac{1}{2} \left(\frac{\sigma\lambda}{r_1 - r_0} \right)^2 + \frac{\lambda\theta}{r_1 - r_0}, \quad z = \frac{\sigma\lambda}{r_1 - r_0} + \frac{\theta}{\sigma}, \tag{5}$$

$$\hat{w} = \frac{(\zeta\sigma)^2}{2} + \zeta\theta, \quad \hat{z} = \zeta\sigma + \frac{\theta}{\sigma}, \tag{6}$$

$$\begin{aligned} \mathfrak{U}_j &= \\ &= \frac{\Gamma(a+j+\frac{1}{2})}{a^{a+j}} \left(\frac{B(\frac{1}{2}, a+j)}{\sqrt{2}} + \frac{\theta}{\sigma\sqrt{a}} F\left(a+j+\frac{1}{2}, \frac{1}{2}, \frac{3}{2}; -\frac{\theta^2}{2a\sigma^2}\right) \right), \end{aligned} \tag{7}$$

$$\mathfrak{W} = \frac{\Gamma\left(a+\frac{1}{2}\right)}{(a-w)^a} \left(\frac{B\left(\frac{1}{2}, a\right)}{\sqrt{2}} + \frac{z}{\sqrt{a-w}} F\left(a+\frac{1}{2}, \frac{1}{2}, \frac{3}{2}; \frac{z^2}{2(w-a)}\right) \right), \tag{8}$$

$$\begin{aligned} Y_j &= \frac{|k_j|^{a-\frac{1}{2}} e^{\frac{1}{2}k_j(1+v)^a}}{(a-w)^a} \left\{ B(a, 1) \left(|k_j| K_{a+\frac{1}{2}}(|k_j|) + k_j K_{a-\frac{1}{2}}(|k_j|) \right) \times \right. \\ &\quad \times \Phi\left(a, 1-a, a+1; \frac{1+v}{2}, -k_j(1+v)\right) - (1+v)k_j B(a+1, 1) \times \\ &\quad \left. \times K_{a-\frac{1}{2}}(|k_j|) \Phi\left(a+1, 1-a, a+2; \frac{1+v}{2}, -k_j(1+v)\right) \right\} \end{aligned} \tag{9}$$

with $v = \frac{z}{\sqrt{z^2+2(a-w)}}$ and $k_j = (\mu + u - r^j - K)\sqrt{z^2+2(a-w)}$,

$$\begin{aligned} \Lambda_{jl} &= \frac{|s_j|^{a+l-\frac{1}{2}} e^{\frac{1}{2}s_j(1+q)^{a+l}}}{a^{a+l}} \left\{ B(a+l, 1) \left(|s_j| K_{a+l+\frac{1}{2}}(|s_j|) + \right. \right. \\ &\quad \left. \left. + s_j K_{a+l-\frac{1}{2}}(|s_j|) \right) \Phi\left(a+l, 1-a-l, a+l+1; \frac{1+q}{2}, -s_j(1+q)\right) - \right. \\ &\quad \left. - (1+q)s B(a+l+1, 1) K_{a+l-\frac{1}{2}}(|s_j|) \times \right. \\ &\quad \left. \times \Phi\left(a+l+1, 1-a-l, a+l+2; \frac{1+q}{2}, -s_j(1+q)\right) \right\}, \end{aligned} \tag{10}$$

where $q = \frac{\theta}{\sqrt{\theta^2 + 2a\sigma^2}}$, $s_j = \frac{(\mu + u - r^j - K)\sqrt{\theta^2 + 2a\sigma^2}}{\sigma}$,

$$\begin{aligned} \Xi_j &= \frac{e^{\frac{z(r^j + K - \mu - u)}{\sigma}} |r^j + K - u - \mu|^{a + \frac{1}{2}}}{a\sigma^{a + \frac{1}{2}} |z|^{a - \frac{1}{2}}} \left(K_{a + \frac{1}{2}} \left(\frac{|z(r^j + K - u - \mu)|}{\sigma} \right) + \right. \\ &\quad \left. + \text{sign}(\mu + u - r^j - K) K_{a - \frac{1}{2}} \left(\frac{|z(r^j + K - u - \mu)|}{\sigma} \right) \right), \end{aligned} \tag{11}$$

$$\mathfrak{Y} = \frac{2^{a - \frac{1}{2}} \Gamma\left(a + \frac{1}{2}\right)}{a(z^2 - 2(w - a))^a} F\left(\frac{1}{2}, a, a + 1; \frac{2(w - a)}{2(w - a) - z^2}\right), \tag{12}$$

$$\begin{aligned} \Theta_j &= \frac{(c - 1)^a |l_j|^{a - \frac{1}{2}} e^{l_j}}{(w - a)^a} \left[B(a, 1) \Phi\left(a, 1 - a, a + 1; \frac{1 - c}{2}, l_j(1 - c)\right) \times \right. \\ &\quad \times \left(|l_j| K_{a + \frac{1}{2}}(|l_j|) + l_j K_{a - \frac{1}{2}}(|l_j|) \right) + (c - 1) l_j K_{a - \frac{1}{2}}(|l_j|) B(a + 1, 1) \times \\ &\quad \left. \times \Phi\left(a + 1, 1 - a, a + 2; \frac{1 - c}{2}, l_j(1 - c)\right) \right], \end{aligned} \tag{13}$$

where $c = -\frac{z}{\sqrt{z^2 - 2(w - a)}}$ and $l_j = (\mu + u - r^j - K)\sqrt{z^2 - 2(w - a)}$,

$$\begin{aligned} \tilde{\Lambda}_l &= \frac{|s|^{a + l - \frac{1}{2}} e^{s(1 + q)^{a + l}}}{a^{a + l}} \left\{ B(a + l, 1) \left(|s| K_{a + l + \frac{1}{2}}(|s|) + \right. \right. \\ &\quad \left. \left. + s K_{a + l - \frac{1}{2}}(|s|) \right) \Phi\left(a + l, 1 - a - l, a + l + 1; \frac{1 + q}{2}, -s(1 + q)\right) - \right. \\ &\quad \left. - (1 + q) s B(a + l + 1, 1) K_{a + l - \frac{1}{2}}(|s|) \times \right. \\ &\quad \left. \times \Phi\left(a + l + 1, 1 - a - l, a + l + 2; \frac{1 + q}{2}, -s(1 + q)\right) \right\} \end{aligned} \tag{14}$$

with $s = \frac{(\mu - K)\sqrt{\theta^2 + 2a\sigma^2}}{\sigma}$,

$$\begin{aligned} \tilde{\Upsilon} &= \frac{|k|^{a - \frac{1}{2}} e^{k(1 + v)^a}}{(a - w)^a} \left\{ B(a, 1) \left(|k| K_{a + \frac{1}{2}}(|k|) + k K_{a - \frac{1}{2}}(|k|) \right) \times \right. \\ &\quad \times \Phi\left(a, 1 - a, a + 1; \frac{1 + v}{2}, -k(1 + v)\right) - (1 + v) k B(a + 1, 1) \times \\ &\quad \left. \times K_{a - \frac{1}{2}}(|k|) \Phi\left(a + 1, 1 - a, a + 2; \frac{1 + v}{2}, -k(1 + v)\right) \right\}, \end{aligned} \tag{15}$$

where $k = (\mu - K)\sqrt{z^2 + 2(a - w)}$,

$$\begin{aligned} \tilde{\Xi} &= \frac{e^{\frac{z(K - \mu)}{\sigma}} |K - \mu|^{a + \frac{1}{2}}}{a\sigma^{a + \frac{1}{2}} |z|^{a - \frac{1}{2}}} \left(K_{a + \frac{1}{2}} \left(\frac{|z(K - \mu)|}{\sigma} \right) + \right. \\ &\quad \left. + \text{sign}(\mu - K) K_{a - \frac{1}{2}} \left(\frac{|z(K - \mu)|}{\sigma} \right) \right), \end{aligned} \tag{16}$$

$$\begin{aligned} \tilde{\Theta} &= \frac{(c-1)^a |l|^{a-\frac{1}{2}} e^l}{(w-a)^a} \left[\text{B}(a, 1) \Phi \left(a, 1-a, a+1; \frac{1-c}{2}, l(1-c) \right) \times \right. \\ &\times \left(|l| \text{K}_{a+\frac{1}{2}}(|l|) + l \text{K}_{a-\frac{1}{2}}(|l|) \right) + (c-1) l \text{K}_{a-\frac{1}{2}}(|l|) \text{B}(a+1, 1) \times \\ &\left. \times \Phi \left(a+1, 1-a, a+2; \frac{1-c}{2}, l(1-c) \right) \right] \end{aligned} \tag{17}$$

with $l = (\mu - K) \sqrt{z^2 - 2(w - a)}$,

$$\begin{aligned} \mathfrak{A}_j &= 2e^{\frac{\theta(r^j+K-u-\mu)}{\sigma^2}} \left(\frac{|r^j+K-u-\mu|}{\sqrt{\theta^2+2\sigma^2a}} \right)^{a+\frac{1}{2}} \times \\ &\times \text{K}_{a+\frac{1}{2}} \left(\frac{|r^j+K-u-\mu|\sqrt{\theta^2+2\sigma^2a}}{\sigma^2} \right), \end{aligned} \tag{18}$$

$$\tilde{\mathfrak{A}} = 2e^{\frac{\theta(K-\mu)}{\sigma^2}} \left(\frac{|K-\mu|}{\sqrt{\theta^2+2\sigma^2a}} \right)^{a+\frac{1}{2}} \text{K}_{a+\frac{1}{2}} \left(\frac{|K-\mu|\sqrt{\theta^2+2\sigma^2a}}{\sigma^2} \right), \tag{19}$$

$$\widehat{\mathfrak{W}} = \frac{\Gamma \left(a + \frac{1}{2} \right)}{(a - \widehat{w})^a} \left(\frac{\text{B} \left(\frac{1}{2}, a \right)}{\sqrt{2}} + \frac{\widehat{z}}{\sqrt{a - \widehat{w}}} \text{F} \left(a + \frac{1}{2}, \frac{1}{2}, \frac{3}{2}; \frac{\widehat{z}^2}{2(a - \widehat{w})} \right) \right) \tag{20}$$

and

$$\begin{aligned} \widehat{Y}_j &= \frac{|\widehat{k}_j|^{a-\frac{1}{2}} e^{\widehat{k}_j(1+\widehat{v})}}{(a-\widehat{w})^a} \left\{ \text{B}(a, 1) \left(|\widehat{k}_j| \text{K}_{a+\frac{1}{2}}(|\widehat{k}_j|) + \widehat{k}_j \text{K}_{a-\frac{1}{2}}(|\widehat{k}_j|) \right) \times \right. \\ &\times \Phi \left(a, 1-a, a+1; \frac{1+\widehat{v}}{2}, -\widehat{k}_j(1+\widehat{v}) \right) - (1+\widehat{v}) \widehat{k}_j \text{B}(a+1, 1) \times \\ &\left. \times \text{K}_{a-\frac{1}{2}}(|\widehat{k}_j|) \Phi \left(a+1, 1-a, a+2; \frac{1+\widehat{v}}{2}, -\widehat{k}_j(1+\widehat{v}) \right) \right\} \end{aligned} \tag{21}$$

with $\widehat{v} = \frac{\widehat{z}}{\sqrt{\widehat{z}^2+2(a-\widehat{w})}}$ and $\widehat{k}_j = (\mu + u - r^j - K) \sqrt{\widehat{z}^2 + 2(a - \widehat{w})}$.

3. Results

First, let us discuss the basic monetary risk measure for an investment portfolio, that is, the value at risk (VaR). Properties and methods of computing of the VaR in various models are discussed in particular in the papers by Berkowitz et al. (2002), Chen and Tang (2005), Pritsker (1997) and Tsyurmasto et al. (2014). If the financial losses are modeled by a random variable X and $0 < \alpha < 1$, the α -VaR of X , $\text{VaR}_\alpha(X)$, is defined as the low α -quantile of the distribution of X with minus sign, that is,

$$\text{VaR}_\alpha(X) = -u_\alpha, \quad \text{where } u_\alpha = \inf\{u \in \mathbb{R} : P(X \leq u) \geq \alpha\}.$$

Thereby, one needs to be able to compute the probability $P(X \leq u)$ if they want to derive $\text{VaR}_\alpha(X)$.

Theorem 1. Let the random variable X be defined by (1)–(4). Then the probability $P(X \leq u)$ can be calculated with respect to the next ratios, where the constant and functions $w, z, \mathfrak{A}_j, \mathfrak{W}, Y_j, \Lambda_{jl}, \Xi_j, \mathfrak{Y}, \Theta_j, \tilde{\Lambda}_l, \tilde{Y}, \tilde{\Xi}, \tilde{\Theta}$ are defined in (5) and (7)–(17).

Ratio 1: $r_0 > r_1$.

Ratio 1.1: $u < r_1$.

Ratio 1.1.1: $a > v$.

If $K + r_1 = \mu + u$,

$$P(X \leq u) = \frac{a^a}{\Gamma(a)\sqrt{2\pi}} (\mathfrak{U}_0 + Y_0 - \mathfrak{W}). \tag{22}$$

When $K + r_0 = \mu + u$,

$$P(X \leq u) = \frac{a^a}{\Gamma(a)\sqrt{2\pi}} \left(\Lambda_{10} + e^{\frac{\lambda(r_1-u+K-\mu)}{r_0-r_1}} (\mathfrak{W} - Y_1) \right). \tag{23}$$

If $K + r_1 \neq \mu + u$ and $K + r_0 \neq \mu + u$,

$$P(X \leq u) = \frac{a^a}{\Gamma(a)\sqrt{2\pi}} \left(\Lambda_{10} + e^{\frac{\lambda(r_1-u+K-\mu)}{r_0-r_1}} (Y_0 - Y_1) \right). \tag{24}$$

Ratio 1.1.2: $a = w$.

When $K + r_1 = \mu + u$,

$$P(X \leq u) = \frac{a^a}{\Gamma(a)\sqrt{2\pi}} \left(\mathfrak{U}_0 + \Xi_0 - \frac{2^{a-\frac{1}{2}}\Gamma\left(a+\frac{1}{2}\right)}{a|z|^{2a}} \right). \tag{25}$$

If $K + r_0 = \mu + u$,

$$\begin{aligned} P(X \leq u) &= \\ &= \frac{a^a}{\Gamma(a)\sqrt{2\pi}} \left(\Lambda_{10} + e^{\frac{\lambda(r_1-u+K-\mu)}{r_0-r_1}} \left(\frac{2^{a-\frac{1}{2}}\Gamma\left(a+\frac{1}{2}\right)}{a|z|^{2a}} - \Xi_1 \right) \right). \end{aligned} \tag{26}$$

When $K + r_1 \neq \mu + u$ and $K + r_0 \neq \mu + u$,

$$P(X \leq u) = \frac{a^a}{\Gamma(a)\sqrt{2\pi}} \left(\Lambda_{10} + e^{\frac{\lambda(r_1-u+K-\mu)}{r_0-r_1}} (\Xi_0 - \Xi_1) \right). \tag{27}$$

Ratio 1.1.3: $a < w, \theta \neq 0$.

If $K + r_1 = \mu + u$,

$$P(X \leq u) = \frac{a^a}{\Gamma(a)\sqrt{2\pi}} (\mathfrak{U}_0 + \Theta_0 - \mathfrak{V}). \tag{28}$$

When $K + r_0 = \mu + u$,

$$P(X \leq u) = \frac{a^a}{\Gamma(a)\sqrt{2\pi}} \left(\Lambda_{10} + e^{\frac{\lambda(r_1-u+K-\mu)}{r_0-r_1}} (\mathfrak{V} - \Theta_1) \right). \tag{29}$$

If $K + r_1 \neq \mu + u$ and $K + r_0 \neq \mu + u$,

$$P(X \leq u) = \frac{a^a}{\Gamma(a)\sqrt{2\pi}} \left(\Lambda_{10} + e^{\frac{\lambda(r_1-u+K-\mu)}{r_0-r_1}} (\Theta_0 - \Theta_1) \right). \tag{30}$$

Ratio 1.2: $r_1 \leq u < r_0$.

Ratio 1.2.1: $a > w$.

When $K = \mu$,

$$P(X \leq u) = 1 - e^{-\frac{\lambda(r_1-u)}{r_0-r_1}} + \frac{a^a e^{\frac{\lambda(r_1-u)}{r_0-r_1}}}{\Gamma(a)\sqrt{2\pi}} (\mathfrak{U}_0 + Y_0 - \mathfrak{W}). \tag{31}$$

If $K + r_0 = \mu + u$,

$$P(X \leq u) = 1 - e^{-\frac{\lambda(r_1-u)}{r_0-r_1}} + \frac{a^a e^{\frac{\lambda(r_1-u)}{r_0-r_1}}}{\Gamma(a)\sqrt{2\pi}} \left(\tilde{\Lambda}_0 + e^{\frac{\lambda(K-\mu)}{r_0-r_1}} (\mathfrak{W} - \tilde{Y}) \right). \tag{32}$$

When $K \neq \mu$ and $K + r_0 \neq \mu + u$,

$$P(X \leq u) = 1 - e^{-\frac{\lambda(r_1-u)}{r_0-r_1}} + \frac{a^a e^{\frac{\lambda(r_1-u)}{r_0-r_1}}}{\Gamma(a)\sqrt{2\pi}} \left(\tilde{\Lambda}_0 + e^{\frac{\lambda(K-\mu)}{r_0-r_1}} (Y_0 - \tilde{Y}) \right). \tag{33}$$

Ratio 1.2.2: $a = w$.

When $K = \mu$,

$$P(X \leq u) = 1 - e^{-\frac{\lambda(r_1-u)}{r_0-r_1}} + \frac{a^a e^{\frac{\lambda(r_1-u)}{r_0-r_1}}}{\Gamma(a)\sqrt{2\pi}} \left(\mathfrak{U}_0 + \Xi_0 - \frac{2^{a-\frac{1}{2}} \Gamma\left(a + \frac{1}{2}\right)}{a|z|^{2a}} \right). \tag{34}$$

If $K + r_0 = \mu + u$,

$$P(X \leq u) = 1 - e^{-\frac{\lambda(r_1-u)}{r_0-r_1}} + \frac{a^a e^{\frac{\lambda(r_1-u)}{r_0-r_1}}}{\Gamma(a)\sqrt{2\pi}} \left(\tilde{\Lambda}_0 + e^{\frac{\lambda(K-\mu)}{r_0-r_1}} \left(\frac{2^{a-\frac{1}{2}} \Gamma\left(a + \frac{1}{2}\right)}{a|z|^{2a}} - \tilde{\Xi} \right) \right). \tag{35}$$

When $K \neq \mu$ and $K + r_0 \neq \mu + u$,

$$P(X \leq u) = 1 - e^{-\frac{\lambda(r_1-u)}{r_0-r_1}} + \frac{a^a e^{\frac{\lambda(r_1-u)}{r_0-r_1}}}{\Gamma(a)\sqrt{2\pi}} \left(\tilde{\Lambda}_0 + e^{\frac{\lambda(K-\mu)}{r_0-r_1}} (\Xi_0 - \tilde{\Xi}) \right). \tag{36}$$

Ratio 1.2.3: $a < w, \theta \neq 0$.

If $K = \mu$,

$$P(X \leq u) = 1 - e^{-\frac{\lambda(r_1-u)}{r_0-r_1}} + \frac{a^a e^{\frac{\lambda(r_1-u)}{r_0-r_1}}}{\Gamma(a)\sqrt{2\pi}} (\mathfrak{U}_0 + \Theta_0 - \mathfrak{V}). \tag{37}$$

When $K + r_0 = \mu + u$,

$$P(X \leq u) = 1 - e^{-\frac{\lambda(r_1-u)}{r_0-r_1}} + \frac{a^a e^{\frac{\lambda(r_1-u)}{r_0-r_1}}}{\Gamma(a)\sqrt{2\pi}} \left(\tilde{\Lambda}_0 + e^{\frac{\lambda(K-\mu)}{r_0-r_1}} (\mathfrak{V} - \tilde{\Theta}) \right). \tag{38}$$

If $K \neq \mu$ and $K + r_0 \neq \mu + u$,

$$P(X \leq u) = 1 - e^{-\frac{\lambda(r_1-u)}{r_0-r_1}} + \frac{a^a e^{\frac{\lambda(r_1-u)}{r_0-r_1}}}{\Gamma(a)\sqrt{2\pi}} \left(\tilde{\Lambda}_0 + e^{\frac{\lambda(K-\mu)}{r_0-r_1}} (\Theta_0 - \tilde{\Theta}) \right). \tag{39}$$

Ratio 1.3: $u \geq r_0$.

Then

$$P(X \leq u) = 1.$$

Ratio 2: $r_0 = r_1$.

Ratio 2.1: $u < r_0$.

When $K + r_0 = \mu + u$,

$$P(X \leq u) = \frac{a^a}{\Gamma(a)\sqrt{2\pi}} \mathfrak{L}_0.$$

If $K + r_0 \neq \mu + u$,

$$P(X \leq u) = \frac{a^a}{\Gamma(a)\sqrt{2\pi}} \Lambda_{00}.$$

Ratio 2.2: $u \geq r_0$.

Then

$$P(X \leq u) = 1.$$

Because $\text{VaR}_\alpha(X)$ does not take into account the size of losses, the monetary measure $\text{ES}_\alpha(X)$, the expected shortfall, which is defined as

$$\text{ES}_\alpha(X) = -\frac{1}{\alpha} \left[\text{EXI}_{\{X \leq u_\alpha\}} + u_\alpha (\alpha - P(X \leq u_\alpha)) \right],$$

was recommended to financial institutions as a preferable alternative to the VaR in Basel III (2011). For continuous distributions, $\text{ES}_\alpha(\cdot)$ coincides with the measure $\text{CVaR}_\alpha(\cdot)$, the conditional value at risk, which is the mathematical expectation of “the tail” of the distribution:

$$\text{CVaR}_\alpha(X) = -\frac{1}{\alpha} \text{EXI}_{\{X \leq u_\alpha\}}.$$

The CVaR monetary measure was first introduced and studied for portfolio losses that are determined by the normal distribution in [Rockafellar and Uryasev \(2000\)](#). General properties of the CVaR were investigated in [Rockafellar and Uryasev \(2002\)](#). For methods of numerical estimation of the CVaR, see the works by [Chernozhukov and Umantsev \(2001\)](#) and [Chun et al. \(2012\)](#). For more recent studies, applications and investigations of the ES in various models, see in particular the works by [Drapeau et al. \(2014\)](#), [Ivanov \(2018\)](#), [Kalinchenko et al. \(2012\)](#) and [Mafusalov and Uryasev \(2016\)](#).

Obviously, it is required to obtain $E\left(XI_{\{X \leq u\}}\right)$ for calculating both the expected shortfall and the conditional value at risk. Set

$$\mathfrak{D} = \left(\frac{r_1 - r_0}{\lambda} - u \right) e^{\frac{\lambda(r_1 - u + K - \mu)}{r_0 - r_1}}, \tag{40}$$

$$\begin{aligned} \mathbb{W}_1 = & \left(\frac{r_0 - r_1}{\lambda} + r_1 + K - \mu \right) \mathfrak{L}_0 - \frac{r_0 - r_1}{\lambda e^\lambda} \Lambda_{00} - \\ & - \theta \mathfrak{L}_1 - \sigma \left(\frac{2\sigma^2}{\theta^2 + 2\sigma^2 a} \right) \Gamma \left(a + \frac{1}{2} \right), \end{aligned} \tag{41}$$

$$\mathbb{W}_2 = \left(\frac{r_0 - r_1}{\lambda} + r_1 + K - \mu \right) \Lambda_{10} - \frac{r_0 - r_1}{\lambda e^\lambda} \mathfrak{L}_0 - \theta \Lambda_{11} - \sigma \mathfrak{L}_1, \tag{42}$$

$$\mathbb{W}_3 = \left(\frac{r_0 - r_1}{\lambda} + r_1 + K - \mu \right) \Lambda_{10} - \frac{r_0 - r_1}{\lambda e^\lambda} \Lambda_{00} - \theta \Lambda_{11} - \sigma \mathfrak{L}_1. \tag{43}$$

The next theorem is valid.

Theorem 2. *If X is defined by (1)–(4) and $u < r_1$, the expectation $E(XI_{\{X \leq u\}})$ is computed with respect to the following ratios, where the constants and functions $w, z, \mathfrak{A}_j, \mathfrak{B}, Y_j, \Lambda_{jl}, \Xi_j, \mathfrak{Y}, \Theta_j, \mathfrak{A}_j, \mathfrak{D}, \mathbb{W}_1, \mathbb{W}_2, \mathbb{W}_3$ are defined in (5), (7)–(13), (18) and (40)–(43).*

Ratio 1: $r_0 > r_1$.

Ratio 1.1: $a > w$.

If $r_1 + K = \mu + u$,

$$E(XI_{\{X \leq u\}}) = \frac{a^a}{\Gamma(a)\sqrt{2\pi}} (\mathbb{W}_1 - \mathfrak{B} + Y_0). \tag{44}$$

When $r_0 + K = \mu + u$,

$$E(XI_{\{X \leq u\}}) = \frac{a^a}{\Gamma(a)\sqrt{2\pi}} (\mathbb{W}_2 + \mathfrak{D} (\mathfrak{B} - Y_1)). \tag{45}$$

If $r_1 + K \neq \mu + u$ and $r_0 + K \neq \mu + u$,

$$E(XI_{\{X \leq u\}}) = \frac{a^a}{\Gamma(a)\sqrt{2\pi}} (\mathbb{W}_3 + \mathfrak{D} (Y_0 - Y_1)). \tag{46}$$

Ratio 1.2: $a = w$.

When $r_1 + K = \mu + u$,

$$E(XI_{\{X \leq u\}}) = \frac{a^a}{\Gamma(a)\sqrt{2\pi}} \left(\mathbb{W}_1 - \frac{2^{a-\frac{1}{2}} \Gamma(a + \frac{1}{2})}{a|z|^{2a}} + \Xi_0 \right). \tag{47}$$

If $r_0 + K = \mu + u$,

$$\begin{aligned} E(XI_{\{X \leq u\}}) &= \\ &= \frac{a^a}{\Gamma(a)\sqrt{2\pi}} \left(\mathbb{W}_2 + \mathfrak{D} \left(\frac{2^{a-\frac{1}{2}} \Gamma(a + \frac{1}{2})}{a|z|^{2a}} - \Xi_1 \right) \right). \end{aligned} \tag{48}$$

When $r_1 + K \neq \mu + u$ and $r_0 + K \neq \mu + u$,

$$E(XI_{\{X \leq u\}}) = \frac{a^a}{\Gamma(a)\sqrt{2\pi}} (\mathbb{W}_3 + \mathfrak{D} (\Xi_0 - \Xi_1)). \tag{49}$$

Ratio 1.3: $a < w, \theta \neq 0$.

If $r_1 + K = \mu + u$,

$$E(XI_{\{X \leq u\}}) = \frac{a^a}{\Gamma(a)\sqrt{2\pi}} (\mathbb{W}_1 - \mathfrak{Y} + \Theta_0). \tag{50}$$

When $r_0 + K = \mu + u$,

$$E(XI_{\{X \leq u\}}) = \frac{a^a}{\Gamma(a)\sqrt{2\pi}} (\mathbb{W}_2 + \mathfrak{D} (\mathfrak{Y} - \Theta_1)). \tag{51}$$

If $r_1 + K \neq \mu + u$ and $r_0 + K \neq \mu + u$,

$$E \left(XI_{\{X \leq u\}} \right) = \frac{a^a}{\Gamma(a)\sqrt{2\pi}} (\mathbb{W}_3 + \mathfrak{D} (\Theta_0 - \Theta_1)). \tag{52}$$

Ratio 2: $r_0 = r_1$.

When $r_0 + K = \mu + u$,

$$E \left(XI_{\{X \leq u\}} \right) = \frac{a^a}{\Gamma(a)\sqrt{2\pi}} \left((r_0 + K - \mu) \mathfrak{A}_0 - \theta \mathfrak{A}_1 - \sigma \left(\frac{2\sigma^2}{\theta^2 + 2\sigma^2 a} \right) \Gamma \left(a + \frac{1}{2} \right) \right).$$

If $r_0 + K \neq \mu + u$,

$$E \left(XI_{\{X \leq u\}} \right) = \frac{a^a}{\Gamma(a)\sqrt{2\pi}} ((r_0 + K - \mu) \Lambda_{00} - \theta \Lambda_{01} - \sigma \mathfrak{A}_0).$$

Remark 1. Theorem 2 gives us the value of $E \left(XI_{\{X \leq u\}} \right)$ for $u < r_1$. If $u \geq r_1$, similar formulas can be derived as well. They will be dependent then also on the meanings of the functions $\tilde{\Lambda}_1, \tilde{Y}, \tilde{\Xi}, \tilde{\Theta}, \tilde{\mathfrak{A}}$ which are defined in (14)–(17) and (19).

Next, the entropic risk measure (see for details and the importance of this measure, for example, the monograph by Föllmer and Schied (2004) and the works by Barrieu and El Karoui (2005), Föllmer and Schied (2010), Ivanov and Temnov (2017) and Ivanov (2018)) for the tail of distribution X is defined as

$$ERM_{\zeta}(X, u) = \frac{1}{\zeta} \log E \left(e^{-\zeta X} I_{\{X \leq u\}} \right), \quad \zeta > 0.$$

Let

$$\zeta \theta + \frac{\zeta^2 \sigma^2}{2} < a. \tag{53}$$

The validity of this inequality is the necessary and sufficient condition for the finiteness of expectation

$$E e^{\zeta S} < \infty.$$

Set

$$C_1 = \frac{\zeta (r_0 - r_1) e^{-\lambda - \zeta(r_0 + K - \mu)}}{\lambda + \zeta (r_0 - r_1)}, \tag{54}$$

$$C_2 = \frac{\lambda e^{-\zeta(r_1 + K - \mu)}}{\lambda + \zeta (r_0 - r_1)}, \tag{55}$$

$$C_3 = \frac{\lambda e^{\frac{\lambda(u + \mu - K - r_1)}{r_1 - r_0} - \zeta u}}{\lambda + \zeta (r_0 - r_1)}. \tag{56}$$

The next theorem allows to establish the value of the entropic risk measure for X .

Theorem 3. Let X be defined in (1)–(4). Assume that $u < r_1$ and (53) holds. Then the expectation $E\left(e^{-\zeta X} I_{\{X \leq u\}}\right)$ can be computed with respect to the following ratios, where the constants and functions $w, z, \widehat{w}, \widehat{z}, \mathfrak{W}, Y_j, \Xi_j, \mathfrak{Y}, \Theta_j, \widehat{\mathfrak{W}}, \widehat{Y}_j$ are defined in (5) and (6), (8) and (9), (11)–(13) and (20) and (21).

Ratio 1. $r_0 > r_1$.

Ratio 1.1. $a > w$.

When $r_1 + K = \mu + u$,

$$E\left(e^{-\zeta X} I_{\{X \leq u\}}\right) = \frac{a^a}{\Gamma(a)\sqrt{2\pi}} \left(C_1 \widehat{Y}_0 + C_2 \widehat{\mathfrak{W}} - C_3 (\mathfrak{W} - Y_0)\right). \tag{57}$$

If $r_0 + K = \mu + u$,

$$E\left(e^{-\zeta X} I_{\{X \leq u\}}\right) = \frac{a^a}{\Gamma(a)\sqrt{2\pi}} \left(C_1 \widehat{\mathfrak{W}} + C_2 \widehat{Y}_1 - C_3 (Y_1 - \mathfrak{W})\right). \tag{58}$$

When $r_1 + K \neq \mu + u$ and $r_0 + K \neq \mu + u$,

$$E\left(e^{-\zeta X} I_{\{X \leq u\}}\right) = \frac{a^a}{\Gamma(a)\sqrt{2\pi}} \left(C_1 \widehat{Y}_0 + C_2 \widehat{Y}_1 - C_3 (Y_1 - Y_0)\right). \tag{59}$$

Ratio 1.2. $a = w$.

If $r_1 + K = \mu + u$,

$$\begin{aligned} E\left(e^{-\zeta X} I_{\{X \leq u\}}\right) &= \\ &= \frac{a^a}{\Gamma(a)\sqrt{2\pi}} \left(C_1 \widehat{Y}_0 + C_2 \widehat{\mathfrak{W}} - C_3 \left(\frac{2^{a-\frac{1}{2}} \Gamma\left(a + \frac{1}{2}\right)}{a|z|^{2a}} - \Xi_0\right)\right). \end{aligned} \tag{60}$$

When $r_0 + K = \mu + u$,

$$\begin{aligned} E\left(e^{-\zeta X} I_{\{X \leq u\}}\right) &= \\ &= \frac{a^a}{\Gamma(a)\sqrt{2\pi}} \left(C_1 \widehat{\mathfrak{W}} + C_2 \widehat{Y}_1 - C_3 \left(\Xi_1 - \frac{2^{a-\frac{1}{2}} \Gamma\left(a + \frac{1}{2}\right)}{a|z|^{2a}}\right)\right). \end{aligned} \tag{61}$$

If $r_1 + K \neq \mu + u$ and $r_0 + K \neq \mu + u$,

$$E\left(e^{-\zeta X} I_{\{X \leq u\}}\right) = \frac{a^a}{\Gamma(a)\sqrt{2\pi}} \left(C_1 \widehat{Y}_0 + C_2 \widehat{Y}_1 - C_3 (\Xi_1 - \Xi_0)\right). \tag{62}$$

Ratio 1.3. $a < w, \theta \neq 0$.

When $r_1 + K = \mu + u$,

$$E\left(e^{-\zeta X} I_{\{X \leq u\}}\right) = \frac{a^a}{\Gamma(a)\sqrt{2\pi}} \left(C_1 \widehat{Y}_0 + C_2 \widehat{\mathfrak{W}} - C_3 (\mathfrak{Y} - \Theta_0)\right). \tag{63}$$

If $r_0 + K = \mu + u$,

$$E\left(e^{-\zeta X} I_{\{X \leq u\}}\right) = \frac{a^a}{\Gamma(a)\sqrt{2\pi}} \left(C_1 \widehat{\mathfrak{W}} + C_2 \widehat{Y}_1 - C_3 (\Theta_1 - \mathfrak{Y})\right). \tag{64}$$

When $r_1 + K \neq \mu + u$ and $r_0 + K \neq \mu + u$,

$$E \left(e^{-\zeta X} I_{\{X \leq u\}} \right) = \frac{a^a}{\Gamma(a)\sqrt{2\pi}} \left(C_1 \hat{Y}_0 + C_2 \hat{Y}_1 - C_3 (\Theta_1 - \Theta_0) \right). \tag{65}$$

Ratio 2. $r_0 = r_1$.

If $r_0 + K = \mu + u$,

$$E \left(e^{-\zeta X} I_{\{X \leq u\}} \right) = \frac{a^a e^{\zeta(\mu - r_0 - K)}}{\Gamma(a)\sqrt{2\pi}} \hat{\mathfrak{W}}. \tag{66}$$

When $r_0 + K \neq \mu + u$,

$$E \left(e^{-\zeta X} I_{\{X \leq u\}} \right) = \frac{a^a e^{\zeta(\mu - r_0 - K)}}{\Gamma(a)\sqrt{2\pi}} \hat{Y}_0. \tag{67}$$

The example below explains how the obtained results can be easily enough exploited by practitioners.

Example 1. *If we discuss the calculation of downside risks at practice as an application of our results, it is usual to assume that $u < 0$ in Theorems 1–3. Since $r_1 > 0$, we have that the condition $u < r_1$ of Theorems 2 and 3 holds.*

Also, it is natural to propose that the call option strike price $K > 0$. Hence a practitioner often has that $K + r_0 > \mu + u$ and $K + r_1 > \mu + u$. Therefore, if one wants to compute the expected shortfall of the considered portfolio, they need in fact only the formulas (46) and (52) instead of the result of Theorem 2. Similarly, the results of Theorems 1 and 3 are not too complicated either.

4. Conclusions

This paper introduces the formulas that allow one to calculate the VaR, the expected shortfall risk and the entropic risk measure for an investor who issues mortgages on the income earned from the sale of a package of European call options. It is assumed that the underlying stock return is variance-gamma distributed but the credit payment flow could drop down. The time of possible downside jump of credit payments is proposed to be distributed exponentially. The size of this jump is assumed to be constant and it is expected that the random size of the jump will be discussed in future investigations. Also, a number of jumps of the payment flow can be considered as well. Another idea is to model the stock return by the variance-gamma distribution with extra randomness in mean and volatility.

5. Proofs

Proof of Theorem 1. Since

$$P(X \leq u) = E \left(E \left(I_{\{X \leq u\}} | S \right) \right),$$

one could start from the computing of the function

$$\begin{aligned} g(S) &= E \left(I_{\{X \leq u\}} | S \right) = \\ &= P \left(r_0 I_{\{\tau > 1\}} + (r_0 \tau + r_1 (1 - \tau)) I_{\{\tau \leq 1\}} \leq u + (S - K)^+ | S \right). \end{aligned} \tag{68}$$

Let $r_0 = r_1$. Then, obviously,

$$g(S) = I_{\{r_0 \leq u + (S - K)^+\}} \tag{69}$$

and

$$P(X \leq u) = Eg(S) = I_{\{r_0 \leq u\}} + P(S \geq r_0 - u + K) I_{\{r_0 > u\}}. \tag{70}$$

Assume that $r_0 > r_1$. Then, if

$$r_0 \leq u, \tag{71}$$

the function

$$g(S) = 1. \tag{72}$$

Let $r_0 > r_1$ and (71) be not satisfied. Then,

$$\begin{aligned} g(S) &= e^{-\lambda} I_{\{r_0 \leq u+(S-K)^+\}} + P\left(\tau \leq \min\left\{\frac{u+(S-K)^+-r_1}{r_0-r_1}, 1\right\} \mid S\right) = \\ &= \left(1 - e^{-\frac{\lambda(r_1-(S-K)^+-u)}{r_0-r_1}}\right) I_{\left\{0 \leq \frac{u+(S-K)^+-r_1}{r_0-r_1} < 1\right\}} + \\ &+ (1 - e^{-\lambda}) I_{\left\{\frac{u+(S-K)^+-r_1}{r_0-r_1} \geq 1\right\}} + e^{-\lambda} I_{\{r_0 \leq u+(S-K)^+\}} \\ &= I_{\{r_0 \leq u+(S-K)^+\}} + \left(1 - e^{-\frac{\lambda(r_1-(S-K)^+-u)}{r_0-r_1}}\right) I_{\left\{0 \leq \frac{u+(S-K)^+-r_1}{r_0-r_1} < 1\right\}}. \end{aligned} \tag{73}$$

It follows from (73) that

$$g(S) = l_1(S) + l_2(S), \tag{74}$$

where

$$l_1(S) = I_{\{r_0 \leq u+(S-K)^+\}}$$

and

$$l_2(S) = \left(1 - e^{-\frac{\lambda(r_1-(S-K)^+-u)}{r_0-r_1}}\right) I_{\{r_1 \leq u+(S-K)^+ < r_0\}}.$$

Next, we have that

$$El_1(S) = EI_{\{S \geq K, r_0 \leq u+S-K\}} = P(S \geq \max\{K, K+r_0-u\}) \tag{75}$$

and

$$\begin{aligned} El_2(S) &= \left(1 - e^{-\frac{\lambda(r_1-u)}{r_0-r_1}}\right) EI_{\{S < K, r_1 \leq u < r_0\}} + \\ &+ E\left(1 - e^{-\frac{\lambda(r_1-S+K-u)}{r_0-r_1}}\right) I_{\{S \leq K, r_1 \leq u+S-K < r_0\}} = \\ &= \left(1 - e^{-\frac{\lambda(r_1-u)}{r_0-r_1}}\right) P(S < K) I_{\{r_1 \leq u < r_0\}} + \\ &+ E\left(\left(1 - e^{-\frac{\lambda(r_1-S+K-u)}{r_0-r_1}}\right) I_{\{\max\{K, r_1-u+K\} \leq S < r_0-u+K\}}\right). \end{aligned} \tag{76}$$

Combining together (69)–(76), one can observe that

$$\begin{aligned}
 P(X \leq u) &= I_{\{r_0 \leq u\}} + P(S \geq r_0 - u + K) I_{\{r_0 > u\}} + \\
 &+ \left(1 - e^{-\frac{\lambda(r_1 - u)}{r_0 - r_1}}\right) P(S < K) I_{\{r_0 > r_1, r_1 \leq u < r_0\}} + \\
 &+ P(\max\{K, r_1 - u + K\} \leq S < r_0 - u + K) I_{\{r_0 > r_1, r_0 > u\}} - \\
 &- e^{-\frac{\lambda(r_1 - u + K)}{r_0 - r_1}} E\left(e^{-\frac{\lambda S}{r_1 - r_0}} I_{\{\max\{K, r_1 - u + K\} \leq S < r_0 - u + K\}}\right) I_{\{r_0 > r_1, r_0 > u\}},
 \end{aligned}
 \tag{77}$$

and therefore it is necessary to compute

$$Ee^{AS} I_{\{S \geq C\}},
 \tag{78}$$

where $A \in \{0, \frac{\lambda}{r_1 - r_0}\} \leq 0$ and $C \in \mathbb{R}$.

It is easy to observe that

$$\begin{aligned}
 E\left(e^{AS} I_{\{S \geq C\}} | \gamma\right) &= e^{A(\mu + \theta\gamma)} E\left(e^{A\sigma N_\gamma} I_{\{\sigma N_\gamma \geq C - \mu - \theta\gamma\}} | \gamma\right) = \\
 &= e^{A(\mu + \theta\gamma)} \int_{C - \mu - \theta\gamma}^{\infty} e^{Ax} \frac{1}{\sigma\sqrt{2\pi\gamma}} e^{-\frac{x^2}{2\sigma^2\gamma}} dx = \\
 &= e^{A\mu + \frac{A(\sigma^2 A + 2\theta)}{2}\gamma} \int_{C - \mu - (\theta + \sigma^2 A)\gamma}^{\infty} \frac{1}{\sigma\sqrt{2\pi\gamma}} e^{-\frac{x^2}{2\sigma^2\gamma}} dx = \\
 &= e^{A\mu + \frac{A(\sigma^2 A + 2\theta)}{2}\gamma} \Psi\left(\frac{\mu - C + \gamma(\sigma^2 A + \theta)}{\sigma\sqrt{\gamma}}\right).
 \end{aligned}$$

Keeping in mind (3), one could notice that it is required to calculate for the computing of (78) the integral

$$I = \int_0^\infty x^\alpha e^{-(a-b)x} \Psi\left(h\sqrt{x} + \frac{p}{\sqrt{x}}\right) dx,
 \tag{79}$$

where $\alpha > -1, a > 0, p \in \mathbb{R}$ and

$$\begin{aligned}
 (b, h) &\in \left\{\left(0, \frac{\theta}{\sigma}\right), \left(\frac{\lambda}{2(r_1 - r_0)} \left(\frac{\lambda\sigma^2}{r_1 - r_0} + 2\theta\right), \frac{\lambda\sigma}{r_1 - r_0} + \frac{\theta}{\sigma}\right)\right\} = \\
 &= \left\{\left(0, \frac{\theta}{\sigma}\right), (w, z)\right\}.
 \end{aligned}$$

The integral (79) is the same as the one in the bottom of p. 207 of Ivanov and Ano (2016), which was partly computed for some values of the parameters therein. Further, we briefly summarize the results of Ivanov and Ano (2016) and calculate (79) for extra ratios between the parameters of the model.

Case 1. $a = b$.

Case 1.1. $h \geq 0$. Then, obviously, $I = \infty$.

Case 1.2. $h < 0, p = 0$. We have here that

$$\begin{aligned}
 I &= \int_0^\infty x^\alpha \Psi(h\sqrt{x}) dx = -\frac{h}{2(\alpha + 1)\sqrt{2\pi}} \int_0^\infty x^{\alpha + \frac{1}{2}} e^{-\frac{h^2 x}{2}} dx = \\
 &= \frac{2^\alpha \Gamma\left(\alpha + \frac{3}{2}\right)}{(\alpha + 1)|h|^{2(\alpha + 1)}\sqrt{\pi}}.
 \end{aligned}$$

Case 1.3. $h < 0, p \neq 0$. Integrating by parts, we have in this case that

$$\begin{aligned}
 I &= -\frac{1}{\alpha + 1} \int_0^\infty x^{\alpha+1} d\Psi \left(h\sqrt{x} + \frac{p}{\sqrt{x}} \right) = \\
 &= \frac{1}{2(\alpha + 1)\sqrt{2\pi}} \int_0^\infty \left(px^{\alpha-\frac{1}{2}} - hx^{\alpha+\frac{1}{2}} \right) e^{-\frac{(hx+p)^2}{2x}} dx.
 \end{aligned}$$

The formula 3.471.9 from Gradshteyn and Ryzhik (1980) includes the identity

$$\int_0^\infty x^{\vartheta_1-1} e^{-\frac{\vartheta_2}{x}-\vartheta_3x} dx = 2 \left(\frac{\vartheta_2}{\vartheta_3} \right)^{\frac{\vartheta_1}{2}} K_{\vartheta_1} \left(2\sqrt{\vartheta_2\vartheta_3} \right), \tag{80}$$

where $\vartheta_1 \in \mathbb{R}, \vartheta_2 > 0$ and $\vartheta_3 > 0$. Using (80), we get that

$$\begin{aligned}
 I &= \frac{e^{-hp}}{(\alpha + 1)\sqrt{2\pi}} \left(\frac{|p| |p|^{\alpha+\frac{1}{2}}}{|h|^{\alpha+\frac{1}{2}}} K_{\alpha+\frac{1}{2}}(|ph|) - \frac{h |p|^{\alpha+\frac{3}{2}}}{|h|^{\alpha+\frac{3}{2}}} K_{\alpha+\frac{3}{2}}(|ph|) \right) = \\
 &= \frac{e^{-hp} |p|^{\alpha+\frac{1}{2}}}{(\alpha + 1)\sqrt{2\pi} |h|^{\alpha+\frac{1}{2}}} \left(p K_{\alpha+\frac{1}{2}}(|ph|) + |p| K_{\alpha+\frac{3}{2}}(|ph|) \right).
 \end{aligned}$$

Case 2. $a \neq b, p = 0$. Then,

$$I = \frac{1}{|a - b|^{\alpha+1}} \tilde{I},$$

where $\tilde{h} = \frac{h}{\sqrt{|a-b|}}$ and $\tilde{I} = \int_0^\infty x^\alpha e^{-\text{sign}(a-b)x} \Psi(\tilde{h}\sqrt{x}) dx$.

Case 2.1. $b > a, \tilde{h} > -\sqrt{2}$. Then, $I = \infty$ with respect to Case 2.1 at p. 208 of Ivanov and Ano (2016).

Case 2.2. $b < a$. According to Case 2.2 at p. 208 of Ivanov and Ano (2016),

$$I = \frac{\Gamma\left(\alpha + \frac{3}{2}\right)}{\sqrt{2\pi}(a - b)^{\alpha+1}} \left(\frac{B\left(\frac{1}{2}, \alpha + 1\right)}{\sqrt{2}} + \frac{h}{\sqrt{a - b}} F\left(\alpha + \frac{3}{2}, \frac{1}{2}, \frac{3}{2}; \frac{h^2}{2(b - a)}\right) \right).$$

Case 2.3. $b > a, \tilde{h} < -\sqrt{2}$. We have that

$$\begin{aligned}
 \tilde{I} &= \int_0^\infty x^\alpha e^x \Psi(\tilde{h}\sqrt{x}) dx = \int_0^\infty x^\alpha e^x \left(\int_{-\infty}^{\tilde{h}} \frac{\sqrt{x}}{\sqrt{2\pi}} e^{-\frac{v^2x}{2}} dv \right) dx = \\
 &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\tilde{h}} \left(\int_0^\infty x^{\alpha+\frac{1}{2}} e^{-\left(\frac{v^2}{2}-1\right)x} dx \right) dv.
 \end{aligned} \tag{81}$$

Using 3.382.4 from Gradshteyn and Ryzhik (1980), we get from (81) that

$$\tilde{I} = \frac{\Gamma\left(\alpha + \frac{3}{2}\right)}{\sqrt{2\pi}} \int_{-\infty}^{\tilde{h}} \left(\frac{v^2}{2} - 1 \right)^{-\alpha-\frac{3}{2}} dv.$$

Set

$$u = \left(\frac{v^2}{2} - 1 \right)^{-1}.$$

Then,

$$v = -\sqrt{\frac{2}{u} + 2}, \quad dv = u^{-\frac{3}{2}}(2 + 2u)^{-\frac{1}{2}} du$$

and hence

$$\begin{aligned} \tilde{I} &= \frac{\Gamma(\alpha + \frac{3}{2})}{2\sqrt{\pi}} \int_0^{\tilde{h}^2-2} u^\alpha (1+u)^{-\frac{1}{2}} du = \\ &= \frac{2^\alpha \Gamma(\alpha + \frac{3}{2})}{(\alpha + 1)(\tilde{h}^2 - 2)^{\alpha+1} \sqrt{\pi}} F\left(\frac{1}{2}, \alpha + 1, \alpha + 2; \frac{2}{2 - \tilde{h}^2}\right), \end{aligned}$$

where the last identity follows from 3.194.1 in [Gradshteyn and Ryzhik \(1980\)](#). Therefore,

$$I = \frac{2^\alpha \Gamma(\alpha + \frac{3}{2})}{(\alpha + 1)(h^2 - 2(b-a))^{\alpha+1} \sqrt{\pi}} F\left(\frac{1}{2}, \alpha + 1, \alpha + 2; \frac{2(b-a)}{2(b-a) - h^2}\right).$$

Case 3. $a \neq b, p \neq 0$. We have that

$$I = \frac{1}{|a-b|^{\alpha+1}} \tilde{I},$$

where

$$\tilde{h} = \frac{h}{\sqrt{|a-b|}}, \quad \tilde{p} = p\sqrt{|a-b|}$$

and

$$\tilde{I} = \int_0^\infty x^\alpha e^{-\text{sign}(a-b)x} \Psi\left(\tilde{h}\sqrt{x} + \frac{\tilde{p}}{\sqrt{x}}\right) dx.$$

Case 3.1. $a < b, \tilde{h} > -\sqrt{2}$. We conclude that $\tilde{I} = \infty$ by Case 3.1 at p. 210 of [Ivanov and Ano \(2016\)](#).

Case 3.2. $a > b$. With respect to (21) of [Ivanov and Ano \(2016\)](#),

$$\begin{aligned} I &= \frac{|s|^{\alpha+\frac{1}{2}} e^{s(1+q)^{\alpha+1}}}{\sqrt{2\pi}(a-b)^{\alpha+1}} \left[\text{B}(\alpha + 1, 1) \left(|s| \text{K}_{\alpha+\frac{3}{2}}(|s|) + \right. \right. \\ &+ s \text{K}_{\alpha+\frac{1}{2}}(|s|) \left. \right) \Phi\left(\alpha + 1, -\alpha, \alpha + 2; \frac{1+q}{2}, -s(1+q)\right) - \\ &\left. \left. (1+q)s \text{B}(\alpha + 2, 1) \text{K}_{\alpha+\frac{1}{2}}(|s|) \Phi\left(\alpha + 2, -\alpha, \alpha + 3; \frac{1+q}{2}, -s(1+q)\right) \right], \end{aligned} \tag{82}$$

where

$$q = \frac{\tilde{h}}{\sqrt{\tilde{h}^2 + 2}} \quad \text{and} \quad s = \tilde{p}\sqrt{\tilde{h}^2 + 2}.$$

Case 3.3. $a < b, \tilde{h} < -\sqrt{2}$. Let $D(v)$ and $H(v), v \leq V \in \mathbb{R}$ be two differentiable functions with $D(V) = \tilde{p}, H(V) = \tilde{h}$ and $\frac{D(-\infty)}{\sqrt{x}} + H(-\infty)\sqrt{x} = -\infty$. Since

$$\begin{aligned} \Psi\left(\frac{D(v)}{\sqrt{x}} + H(v)\sqrt{x}\right) &= \int_{-\infty}^v \Psi'_u\left(\frac{D(u)}{\sqrt{x}} + H(u)\sqrt{x}\right) du = \\ &= \int_{-\infty}^v \Psi'_{\frac{D(u)}{\sqrt{x}} + H(u)\sqrt{x}}\left(\frac{D(u)}{\sqrt{x}} + H(u)\sqrt{x}\right) \left(\frac{D'_u(u)}{\sqrt{x}} + H'_u(u)\sqrt{x}\right) du, \end{aligned}$$

we get that

$$\Psi\left(\frac{\tilde{p}}{\sqrt{x}} + \tilde{h}\sqrt{x}\right) = \int_{-\infty}^V \frac{1}{\sqrt{2\pi}} e^{-D(v)H(v) - \frac{D^2(v)}{2x} - \frac{H^2(v)x}{2}} \left(\frac{D'_v(v)}{\sqrt{x}} + H'_v(v)\sqrt{x}\right) dv.$$

Set $V = \tilde{h}$ and

$$H(v) = v, \quad D(v) = \frac{\tilde{p}\sqrt{\tilde{h}^2 - 2}}{\sqrt{v^2 - 2}}, \quad v \leq \tilde{h}.$$

Then,

$$\begin{aligned} \tilde{I} &= \frac{1}{\sqrt{2\pi}} \int_0^\infty x^\alpha e^x \left[\int_{-\infty}^{\tilde{h}} e^{-\frac{\tilde{p}v\sqrt{\tilde{h}^2 - 2}}{\sqrt{v^2 - 2}} - \frac{\tilde{p}^2(\tilde{h}^2 - 2)}{2(v^2 - 2)x} - \frac{v^2x}{2}} \times \right. \\ &\quad \left. \times \left(\sqrt{x} - \frac{\tilde{p}v\sqrt{\tilde{h}^2 - 2}}{(v^2 - 2)^{\frac{3}{2}}\sqrt{x}} \right) dv \right] dx. \end{aligned}$$

Let us consider the double integrals

$$J_1 = \int_0^\infty \int_{-\infty}^{\tilde{h}} \left(x^{\alpha + \frac{1}{2}} e^{x - \frac{\tilde{p}v\sqrt{\tilde{h}^2 - 2}}{\sqrt{v^2 - 2}} - \frac{\tilde{p}^2(\tilde{h}^2 - 2)}{2(v^2 - 2)x} - \frac{v^2x}{2}} \right) dv dx$$

and

$$J_2 = \int_0^\infty \int_{-\infty}^{\tilde{h}} \left(\frac{\tilde{p}x^{\alpha - \frac{1}{2}}v\sqrt{\tilde{h}^2 - 2}}{(v^2 - 2)^{\frac{3}{2}}} e^{x - \frac{\tilde{p}v\sqrt{\tilde{h}^2 - 2}}{\sqrt{v^2 - 2}} - \frac{\tilde{p}^2(\tilde{h}^2 - 2)}{2(v^2 - 2)x} - \frac{v^2x}{2}} \right) dv dx.$$

We can notice that both J_1 and J_2 are integrals of constant sign functions. Therefore, if the iterated integrals

$$\hat{J}_1 = \int_{-\infty}^{\tilde{h}} \left(\int_0^\infty x^{\alpha + \frac{1}{2}} e^{x - \frac{\tilde{p}v\sqrt{\tilde{h}^2 - 2}}{\sqrt{v^2 - 2}} - \frac{\tilde{p}^2(\tilde{h}^2 - 2)}{2(v^2 - 2)x} - \frac{v^2x}{2}} dx \right) dv$$

and

$$\hat{J}_2 = \int_{-\infty}^{\tilde{h}} \left(\int_0^\infty \frac{\tilde{p}x^{\alpha - \frac{1}{2}}v\sqrt{\tilde{h}^2 - 2}}{(v^2 - 2)^{\frac{3}{2}}} e^{x - \frac{\tilde{p}v\sqrt{\tilde{h}^2 - 2}}{\sqrt{v^2 - 2}} - \frac{\tilde{p}^2(\tilde{h}^2 - 2)}{2(v^2 - 2)x} - \frac{v^2x}{2}} dx \right) dv$$

exist, then $J_1 = \hat{J}_1, J_2 = \hat{J}_2$, and the Fubini's theorem can be exploited to J_1 and J_2 , that is,

$$\tilde{I} = \frac{1}{\sqrt{2\pi}} (\hat{J}_1 - \hat{J}_2). \tag{83}$$

Set $s = \tilde{p}\sqrt{\tilde{h}^2 - 2}$. Then we have, according to 3.471.9 from Gradshteyn and Ryzhik (1980), that

$$\hat{J}_1 = 2|s|^{\alpha + \frac{3}{2}} K_{\alpha + \frac{3}{2}}(|s|) \int_{-\infty}^{\tilde{h}} (v^2 - 2)^{-\alpha - \frac{3}{2}} e^{-\frac{sv}{\sqrt{v^2 - 2}}} dv$$

and

$$\widehat{J}_2 = 2s|s|^{\alpha+\frac{1}{2}}K_{\alpha+\frac{1}{2}}(|s|) \int_{-\infty}^{\tilde{h}} v (v^2 - 2)^{-\alpha-2} e^{-\frac{sv}{\sqrt{v^2-2}}} dv.$$

Next, we make a change of variables

$$v \rightarrow y, \quad y = -\frac{v}{\sqrt{v^2-2}}.$$

Then,

$$v = -\frac{y\sqrt{2}}{\sqrt{y^2-1}}, \quad v^2 - 2 = \frac{2}{y^2-1}, \quad dv = \frac{\sqrt{2}}{(y^2-1)^{\frac{3}{2}}} dy$$

and we get that

$$\widehat{J}_1 = 2^{-\alpha}|s|^{\alpha+\frac{3}{2}}K_{\alpha+\frac{3}{2}}(|s|) \int_1^{\frac{|\tilde{h}|}{\sqrt{\tilde{h}^2-2}}} (y^2 - 1)^\alpha e^{sy} dy$$

and

$$\widehat{J}_2 = -2^{-\alpha}s|s|^{\alpha+\frac{1}{2}}K_{\alpha+\frac{1}{2}}(|s|) \int_1^{\frac{|\tilde{h}|}{\sqrt{\tilde{h}^2-2}}} y (y^2 - 1)^\alpha e^{sy} dy.$$

Set

$$q = \frac{|\tilde{h}|}{\sqrt{\tilde{h}^2-2}} \quad \text{and} \quad u = \frac{y-1}{q-1}.$$

Then,

$$\begin{aligned} \widehat{J}_1 &= 2^{-\alpha}(q-1)|s|^{\alpha+\frac{3}{2}}e^sK_{\alpha+\frac{3}{2}}(|s|) \int_0^1 ((q-1)^2u^2 + 2(q-1)u)^\alpha e^{s(q-1)u} du \\ &= (q-1)^{\alpha+1}|s|^{\alpha+\frac{3}{2}}e^sK_{\alpha+\frac{3}{2}}(|s|) \int_0^1 u^\alpha \left(\frac{(q-1)u}{2} + 1\right)^\alpha e^{s(q-1)u} du \end{aligned} \tag{84}$$

and

$$\begin{aligned} \widehat{J}_2 &= \\ &= -2^{-\alpha}(q-1)s|s|^{\alpha+\frac{1}{2}}e^sK_{\alpha+\frac{1}{2}}(|s|) \left(\int_0^1 ((q-1)^2u^2 + 2(q-1)u)^\alpha e^{s(q-1)u} du \right. \\ &+ \left. (q-1) \int_0^1 u ((q-1)^2u^2 + 2(q-1)u)^\alpha e^{s(q-1)u} du \right) = \\ &= -(q-1)^{\alpha+1}s|s|^{\alpha+\frac{1}{2}}e^sK_{\alpha+\frac{1}{2}}(|s|) \left(\int_0^1 u^\alpha \left(\frac{(q-1)u}{2} + 1\right)^\alpha e^{s(q-1)u} du + \right. \\ &+ \left. (q-1) \int_0^1 u^{\alpha+1} \left(\frac{(q-1)u}{2} + 1\right)^\alpha e^{s(q-1)u} du \right). \end{aligned} \tag{85}$$

Applying 3.385 of Gradshteyn and Ryzhik (1980) to (84) and (85) and keeping in mind (83), we establish now that

$$\begin{aligned}
 I &= \frac{(q-1)^{\alpha+1} |s|^{\alpha+\frac{1}{2}} e^s}{(b-a)^{\alpha+1} \sqrt{2\pi}} \left[B(\alpha+1, 1) \Phi\left(\alpha+1, -\alpha, \alpha+2; \frac{1-q}{2}, s(1-q)\right) \times \right. \\
 &\times \left(|s| K_{\alpha+\frac{3}{2}}(|s|) + s K_{\alpha+\frac{1}{2}}(|s|) \right) + (q-1) s K_{\alpha+\frac{1}{2}}(|s|) B(\alpha+2, 1) \times \\
 &\left. \times \Phi\left(\alpha+2, -\alpha, \alpha+3; \frac{1-q}{2}, s(1-q)\right) \right].
 \end{aligned}$$

Now, one might observe that Ratio 1.1 follows from (77), since then

$$\begin{aligned}
 P(X \leq u) &= P(S \geq K + r_0 - u) + \\
 &+ P(r_1 - u + K \leq S \leq r_0 - u + K) - \\
 &- e^{\frac{\lambda(r_1-u+K)}{r_0-r_1}} E\left(e^{\frac{\lambda S}{r_1-r_0}} I_{\{r_1-u+K \leq S \leq r_0-u+K\}}\right) = \\
 &= P(S \geq K + r_1 - u) - \\
 &- e^{\frac{\lambda(r_1-u+K)}{r_0-r_1}} E\left(e^{\frac{\lambda S}{r_1-r_0}} I_{\{r_1-u+K \leq S \leq r_0-u+K\}}\right)
 \end{aligned}$$

and hence (22) results from Cases 2.2, 3.2, (23) comes after Case 3.2, 2.2, (24) issues from Case 3.2.

Further, one can notice that if $w \geq a$, then

$$\theta < \frac{\lambda \sigma^2}{2(r_0 - r_1)}$$

and hence $z < 0$. Also, when $w > a$, then

$$z^2 - 2\tau w = \frac{\theta^2}{\sigma^2} > 0,$$

and since $z < 0$, we have that $z < -\sqrt{2w}$. Therefore, one could derive (25) from Cases 2.2, 1.3, 1.2, (26) from Cases 3.2, 1.2, 1.3, (27) from Cases 3.2, 1.3, (28) from Cases 2.2, 3.3, 2.3, (29) from Cases 3.2, 2.3, 3.3, (30) from Cases 3.2, 3.3.

Next, Ratio 1.2 follows from (77), since then

$$\begin{aligned}
 P(X \leq u) &= P(S < K) \left(1 - e^{\frac{\lambda(r_1-u)}{r_0-r_1}}\right) + P(S \geq K + r_0 - u) + \\
 &+ P(K \leq S < r_0 - u + K) - e^{\frac{\lambda(r_1-u+K)}{r_0-r_1}} E\left(e^{\frac{\lambda S}{r_1-r_0}} I_{\{K \leq S < r_0-u+K\}}\right) = \\
 &= 1 - e^{\frac{\lambda(r_1-u)}{r_0-r_1}} \left(P(S < K) + e^{\frac{\lambda K}{r_0-r_1}} E\left(e^{\frac{\lambda S}{r_1-r_0}} I_{\{K \leq S < r_0-u+K\}}\right)\right) = \\
 &= 1 - e^{\frac{\lambda(r_1-u)}{r_0-r_1}} + e^{\frac{\lambda(r_1-u)}{r_0-r_1}} \left(P(S < K) - e^{\frac{\lambda K}{r_0-r_1}} E\left(e^{\frac{\lambda S}{r_1-r_0}} I_{\{K \leq S < r_0-u+K\}}\right)\right),
 \end{aligned}$$

and we get (31) from Cases 2.2, 3.2, (32) from Cases 3.2, 2.2, (33) from Case 3.2, (34) from Cases 2.2, 1.3, 1.2, (35) from Cases 3.2, 1.2, 1.3, (36) from Cases 3.2, 1.3, (33) from Case 3.2, (37) from Cases 2.2, 3.3, 2.3, (38) from Cases 3.2, 2.3, 3.3, (39) from Cases 3.2, 3.3 and Ratio 1.3 immediately from (77).

Finally, Ratio 2.1 comes after (77) and Cases 2.2, 3.2. Ratio 2.2 follows straightly from (77).

□

Proof of Theorem 2. One can see that

$$E\left(XI_{\{X \leq u\}}\right) = E\widehat{g}(S),$$

where

$$\begin{aligned} \widehat{g}(S) &= E\left(XI_{\{X \leq u\}}|S\right) = (r_0 - (S - K)^+) E\left(I_{\{\tau > 1, r_0 \leq u + (S - K)^+\}}|S\right) + \\ &+ (r_1 - (S - K)^+) E\left(I_{\{\tau \leq 1, (r_0 - r_1)\tau \leq u + (S - K)^+ - r_1\}}|S\right) + \\ &+ (r_0 - r_1)E\left(\tau I_{\{\tau \leq 1, (r_0 - r_1)\tau \leq u + (S - K)^+ - r_1\}}|S\right) = \\ &= r_0e^{-\lambda}I_{\{r_0 \leq u + (S - K)^+\}} + r_1E\left(I_{\{\tau \leq 1, (r_0 - r_1)\tau \leq u + (S - K)^+ - r_1\}}|S\right) - \\ &-(S - K)^+g(S) + (r_0 - r_1)E\left(\tau I_{\{\tau \leq 1, (r_0 - r_1)\tau \leq u + (S - K)^+ - r_1\}}|S\right), \end{aligned} \tag{86}$$

where $g(S)$ is defined in (68).

If $r_0 = r_1$, then it comes after (69) and (86) that

$$\begin{aligned} \widehat{g}(S) &= (r_0 - (S - K)^+) I_{\{r_0 \leq u + (S - K)^+\}} = \\ &= (r_0 - S + K) I_{\{S > K, r_0 \leq u + S - K\}}. \end{aligned}$$

Hence one can observe, since $u < r_1$, that

$$E\left(XI_{\{X \leq u\}}\right) = (r_0 + K) P(S \geq r_0 + K - u) - E\left(SI_{\{S \geq r_0 + K - u\}}\right). \tag{87}$$

When $r_0 > r_1$, then it follows from (86) that

$$\begin{aligned} \widehat{g}(S) &= r_0e^{-\lambda}I_{\{r_0 \leq u + (S - K)^+\}} - (S - K)^+g(S) + \\ &+ r_1I_{\{u + (S - K)^+ - r_1 \geq 0\}}E I_{\left\{\tau \leq \min\left\{1, \frac{u + (S - K)^+ - r_1}{r_0 - r_1}\right\}\right\}} + \\ &+ (r_0 - r_1)I_{\{u + (S - K)^+ - r_1 \geq 0\}}E\tau I_{\left\{\tau \leq \min\left\{1, \frac{u + (S - K)^+ - r_1}{r_0 - r_1}\right\}\right\}}. \end{aligned} \tag{88}$$

Since

$$\begin{aligned} \int_0^A \lambda x e^{-\lambda x} dx &= - \int_0^A x de^{-\lambda x} = -Ae^{-\lambda A} + \int_0^A e^{-\lambda x} dx = \\ &= \frac{1}{\lambda} - \left(\frac{1}{\lambda} + A\right) e^{-\lambda A}, \end{aligned}$$

we have from (88) that

$$\begin{aligned} \widehat{g}(S) &= r_0e^{-\lambda}I_{\{r_0 \leq u + (S - K)^+\}} - (S - K)^+g(S) + \\ &+ r_1I_{\{u + (S - K)^+ \geq r_0\}}(1 - e^{-\lambda}) + \\ &+ r_1I_{\{r_1 \leq u + (S - K)^+ < r_0\}}\left(1 - e^{\frac{\lambda(r_1 - u - (S - K)^+)}{r_0 - r_1}}\right) + \\ &+ (r_0 - r_1)I_{\{u + (S - K)^+ \geq r_0\}}\left(\frac{1}{\lambda} - \left(\frac{1}{\lambda} + 1\right)e^{-\lambda}\right) + \\ &+ (r_0 - r_1)I_{\{r_1 \leq u + (S - K)^+ < r_0\}} \times \\ &\times \left(\frac{1}{\lambda} - \left(\frac{1}{\lambda} + \frac{u + (S - K)^+ - r_1}{r_0 - r_1}\right) e^{\frac{\lambda(r_1 - u - (S - K)^+)}{r_0 - r_1}}\right). \end{aligned} \tag{89}$$

Substituting $g(S)$ from (74) into (89), one can observe that

$$\begin{aligned} \widehat{g}(S) &= \left(r_0 e^{-\lambda} + r_1 (1 - e^{-\lambda}) + (r_0 - r_1) \left(\frac{1}{\lambda} - \left(\frac{1}{\lambda} + 1 \right) e^{-\lambda} \right) - \right. \\ &\quad \left. - (S - K)^+ \right) I_{\{u+(S-K)^+ \geq r_0\}} + \left(r_1 \left(1 - e^{\frac{\lambda(r_1-u-(S-K)^+)}{r_0-r_1}} \right) + \right. \\ &\quad \left. + (r_0 - r_1) \left(\frac{1}{\lambda} - \left(\frac{1}{\lambda} + \frac{u+(S-K)^+ - r_1}{r_0 - r_1} \right) e^{\frac{\lambda(r_1-u-(S-K)^+)}{r_0-r_1}} \right) - \right. \\ &\quad \left. - (S - K)^+ \left(1 - e^{\frac{\lambda(r_1-u-(S-K)^+)}{r_0-r_1}} \right) \right) I_{\{r_1 \leq u+(S-K)^+ < r_0\}} \end{aligned}$$

and therefore

$$\begin{aligned} \widehat{g}(S) &= \left(\frac{r_0 - r_1}{\lambda} (1 - e^{-\lambda}) + r_1 - (S - K)^+ \right) I_{\{u+(S-K)^+ \geq r_0\}} + \\ &\quad + \left(r_1 + \frac{r_0 - r_1}{\lambda} + \left(\frac{r_1 - r_0}{\lambda} - u \right) e^{\frac{\lambda(r_1-u-(S-K)^+)}{r_0-r_1}} - (S - K)^+ \right) \times \\ &\quad \times I_{\{r_1 \leq u+(S-K)^+ < r_0\}}. \end{aligned}$$

Hence,

$$\begin{aligned} \widehat{g}(S) &= \left(\frac{r_0 - r_1}{\lambda} (1 - e^{-\lambda}) + r_1 - S + K \right) I_{\{S \geq r_0 + K - u\}} + \\ &\quad + \left(r_1 + \frac{r_0 - r_1}{\lambda} + \left(\frac{r_1 - r_0}{\lambda} - u \right) e^{\frac{\lambda(r_1-u-S+K)}{r_0-r_1}} - S + K \right) \times \\ &\quad \times I_{\{r_1 - u + K \leq S < r_0 - u + K\}}. \end{aligned}$$

Because of $E(XI_{\{X \leq u\}}) = E\widehat{g}(S)$, one can see that

$$\begin{aligned} E(XI_{\{X \leq u\}}) &= \left(\frac{r_0 - r_1}{\lambda} (1 - e^{-\lambda}) + r_1 + K \right) P(S \geq r_0 + K - u) - \\ &\quad - E(SI_{\{S \geq r_0 + K - u\}}) - E(SI_{\{r_1 - u + K \leq S < r_0 - u + K\}}) + \left(\frac{r_1 - r_0}{\lambda} - u \right) \times \\ &\quad \times e^{\frac{\lambda(r_1-u+K)}{r_0-r_1}} E\left(e^{\frac{-\lambda S}{r_1-r_0}} I_{\{r_1-u+K \leq S < r_0-u+K\}} \right) + \\ &\quad + \left(r_1 + \frac{r_0 - r_1}{\lambda} + K \right) P(r_1 - u + K \leq S < r_0 - u + K) \end{aligned}$$

and therefore

$$\begin{aligned} E(XI_{\{X \leq u\}}) &= \left(r_1 + \frac{r_0 - r_1}{\lambda} + K \right) P(S \geq r_1 - u + K) + \\ &\quad + \frac{r_1 - r_0}{\lambda e^\lambda} P(S \geq r_0 - u + K) - E(SI_{\{S \geq r_1 + K - u\}}) + \left(\frac{r_1 - r_0}{\lambda} - u \right) \times \\ &\quad \times e^{\frac{\lambda(r_1-u+K)}{r_0-r_1}} E\left(e^{\frac{-\lambda S}{r_1-r_0}} I_{\{r_1-u+K \leq S < r_0-u+K\}} \right). \end{aligned}$$

Next, one can see that for $C \in \mathbb{R}$,

$$\begin{aligned} E\left(SI_{\{S \geq C\}}|\gamma\right) &= (\mu + \theta\gamma) P(S \geq C|\gamma) + \int_{C-\mu-\theta\gamma}^{\infty} \frac{x}{\sigma\sqrt{2\pi\gamma}} e^{-\frac{x^2}{2\sigma^2\gamma}} dx = \\ &= (\mu + \theta\gamma) P(S \geq C|\gamma) + \frac{\sigma\sqrt{\gamma}}{\sqrt{2\pi}} e^{-\frac{(C-\mu-\theta\gamma)^2}{2\sigma^2\gamma}}. \end{aligned}$$

Let $C \neq \mu$. Then with respect to (80),

$$\begin{aligned} E\left(SI_{\{S \geq C\}}\right) &= \mu P(S \geq C) + \frac{\theta a^a}{\Gamma(a)} \int_0^{\infty} x^a e^{-ax} \Psi\left(\frac{\mu - C + \theta x}{\sigma\sqrt{x}}\right) dx \\ &+ \frac{\sigma a^a}{\Gamma(a)\sqrt{2\pi}} \int_0^{\infty} x^{a-\frac{1}{2}} e^{-\frac{(C-\mu-\theta x)^2}{2\sigma^2 x} - ax} dx = \mu P(S > C) + \\ &+ \frac{\theta a^a}{\Gamma(a)} \int_0^{\infty} x^a e^{-ax} \Psi\left(\frac{\mu - C + \theta x}{\sigma\sqrt{x}}\right) dx + \\ &+ \frac{\sigma a^a e^{\frac{\theta(C-\mu)}{\sigma^2}} \sqrt{2}}{\Gamma(a)\sqrt{\pi}} \left|\frac{C - \mu}{\sqrt{\theta^2 + 2\sigma^2 a}}\right|^{a+\frac{1}{2}} K_{a+\frac{1}{2}}\left(\frac{|C - \mu|\sqrt{\theta^2 + 2\sigma^2 a}}{\sigma^2}\right). \end{aligned}$$

If $C = \mu$, then obviously

$$\begin{aligned} E\left(SI_{\{S \geq C\}}\right) &= \mu P(S > C) + \frac{\theta a^a}{\Gamma(a)} \int_0^{\infty} x^a e^{-ax} \Psi\left(\frac{\theta}{\sigma}\sqrt{x}\right) dx + \\ &+ \frac{\sigma a^a}{\Gamma(a)\sqrt{2\pi}} \left(\frac{2\sigma^2}{\theta^2 + 2\sigma^2 a}\right)^{a+\frac{1}{2}} \Gamma\left(a + \frac{1}{2}\right). \end{aligned}$$

Thus, we get (44) from Cases 2.2, 3.2, (45) from Cases 3.2, 2.2, (46) from Case 3.2, (47) from Cases 2.2, 3.2, 1.2, 1.3, (45) from Cases 3.2, 2.2, 1.3, 1.2, (46) from Cases 3.2, 1.3, (50) from Cases 2.2, 3.2, 2.3, 3.3, (51) from Cases 3.2, 2.2, 3.3, 2.3, (52) from Cases 3.2, 3.3.

Finally, Ratio 2 follows from (87) and Cases 2.2, 3.2. \square

Proof of Theorem 3. If $r_0 = r_1$, then

$$\begin{aligned} E\left(e^{-\zeta X} I_{\{X \leq u\}}\right) &= e^{-\zeta r_0} E\left(e^{\zeta(S-K)^+} I_{\{(S-K)^+ \geq r_0 - u\}}\right) = \\ &= e^{-\zeta(r_0+K)} E\left(e^{\zeta S} I_{\{S \geq r_0+K-u\}}\right). \end{aligned} \tag{90}$$

Let $r_0 > r_1$. Then,

$$\begin{aligned} E\left(e^{-\zeta X} I_{\{X \leq u\}}|S\right) &= e^{\zeta((S-K)^+ - r_0)} I_{\{(S-K)^+ \geq r_0 - u\}} P(\tau > 1) + \\ &+ e^{\zeta((S-K)^+ - r_1)} E\left(e^{\zeta(r_1 - r_0)\tau} I_{\{\tau \leq \min\left\{1, \frac{u+(S-K)^+ - r_1}{r_0 - r_1}\right\}\}}|S\right) = \\ &= e^{-\lambda - \zeta(r_0+K) + \zeta S} I_{\{S \geq r_0+K-u\}} + e^{\zeta(S-K-r_1)} \times \\ &\times I_{\{S \geq r_1+K-u\}} E\left(e^{\zeta(r_1 - r_0)\tau} I_{\{\tau \leq \min\left\{1, \frac{u+S-K-r_1}{r_0 - r_1}\right\}\}}|S\right). \end{aligned}$$

Therefore,

$$\begin{aligned}
 E\left(e^{-\zeta X} I_{\{X \leq u\}} | S\right) &= e^{-\lambda - \zeta(r_0 + K) + \zeta S} I_{\{S \geq r_0 + K - u\}} + e^{\zeta(S - K - r_1)} \times \\
 &\times I_{\{S \geq r_0 + K - u\}} E\left(e^{\zeta(r_1 - r_0)\tau} I_{\{\tau \leq 1\}}\right) + e^{\zeta(S - K - r_1)} \times \\
 &\times I_{\{r_1 + K - u \leq S < r_0 + K - u\}} E\left(e^{\zeta(r_1 - r_0)\tau} I_{\left\{\tau \leq \frac{u + S - K - r_1}{r_0 - r_1}\right\}} | S\right)
 \end{aligned}
 \tag{91}$$

in this case.

Since

$$E\left(e^{A\tau} I_{\{\tau \leq C\}}\right) = \int_0^C \lambda e^{(A-\lambda)x} dx = \frac{\lambda}{A - \lambda} \left(e^{(A-\lambda)C} - 1\right),$$

we have from (91) that if $r_0 > r_1$, then

$$\begin{aligned}
 E\left(e^{-\zeta X} I_{\{X \leq u\}} | S\right) &= e^{-\lambda - \zeta(r_0 + K) + \zeta S} I_{\{S \geq r_0 + K - u\}} + \\
 &+ \frac{\lambda e^{-\zeta(r_1 + K - S)}}{\lambda + \zeta(r_0 - r_1)} \left(1 - e^{-\lambda - \zeta(r_0 - r_1)}\right) I_{\{S \geq r_0 + K - u\}} + \\
 &+ \frac{\lambda e^{-\zeta(r_1 + K - S)}}{\lambda + \zeta(r_0 - r_1)} \left(1 - e^{\frac{(\lambda + \zeta(r_0 - r_1))(u + S - K - r_1)}{r_1 - r_0}}\right) I_{\{r_1 + K - u \leq S < r_0 + K - u\}}
 \end{aligned}$$

and

$$\begin{aligned}
 E\left(e^{-\zeta X} I_{\{X \leq u\}} | S\right) &= \frac{\zeta(r_0 - r_1) e^{-\lambda - \zeta(r_0 + K - S)}}{\lambda + \zeta(r_0 - r_1)} I_{\{S \geq r_0 + K - u\}} + \\
 &+ \frac{\lambda e^{-\zeta(r_1 + K - S)}}{\lambda + \zeta(r_0 - r_1)} I_{\{S \geq r_1 + K - u\}} - \frac{\lambda e^{-\zeta(r_1 + K - S)}}{\lambda + \zeta(r_0 - r_1)} \times \\
 &\times e^{\frac{(\lambda + \zeta(r_0 - r_1))(u + S - K - r_1)}{r_1 - r_0}} I_{\{r_1 + K - u \leq S < r_0 + K - u\}}.
 \end{aligned}$$

Hence

$$\begin{aligned}
 E\left(e^{-\zeta X} I_{\{X \leq u\}}\right) &= \frac{\zeta(r_0 - r_1) e^{-\lambda - \zeta(r_0 + K)}}{\lambda + \zeta(r_0 - r_1)} E\left(e^{\zeta S} I_{\{S \geq r_0 + K - u\}}\right) + \\
 &+ \frac{\lambda e^{-\zeta(r_1 + K)}}{\lambda + \zeta(r_0 - r_1)} E\left(e^{\zeta S} I_{\{S \geq r_1 + K - u\}}\right) - \frac{\lambda e^{\frac{\lambda(u - K - r_1)}{r_1 - r_0} - \zeta u}}{\lambda + \zeta(r_0 - r_1)} \times \\
 &\times E\left(e^{\frac{\lambda S}{r_1 - r_0}} I_{\{r_1 + K - u \leq S < r_0 + K - u\}}\right)
 \end{aligned}
 \tag{92}$$

if $r_0 > r_1$.

Thereby, we get (57) from Cases 3.2, 2.2, (58) from Cases 2.2, 3.2, (59) from Case 3.2, (60) from Cases 3.2, 2.2, 1.3, (61) from Cases 2.2, 3.2, 1.2, (62) from Cases 3.2, 1.3, (63) from Cases 3.2, 2.2, 3.3, (64) from Cases 2.2, 3.2, 2.3, (65) from Cases 3.2, 3.3, (66) from Case 2.2 and (67) from Case 3.2. □

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