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Extreme Value Theory: An Application to the Peruvian Stock Market Returns

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ABSTRACT

Using daily observations of the index and stock market returns for the Peruvian case from January 3, 1990 to May 31, 2013, this paper models the distribution of daily loss probability, estimates maximum quantiles and tail probabilities of this distribution, and models the extremes through a maximum threshold. This is used to obtain the better measurements of the Value at Risk (VaR) and the Expected Short-Fall (ES) at 95% and 99%. One of the results on calculating the maximum annual block of the negative stock market returns is the observation that the largest negative stock market return (daily) is 12.44% in 2011. The shape parameter is equal to -0.020 and 0.268 for the annual and quarterly block, respectively. Then, in the first case we have that the non-degenerate distribution function is Gumbel-type. In the other case, we have a thick-tailed distribution (Fréchet). Estimated values of the VaR and the ES are higher using the Generalized Pareto Distribution (GPD) in comparison with the Normal distribution and the differences at 99.0% are notable. Finally, the non-parametric estimation of the Hill tail-index and the quantile for negative stock market returns shows quite instability.

Keywords: Extreme Value Theory; Value-at-Risk (VaR); Expected Short-Fall (ES); Generalized Pareto Distribution (GPD); Gumbel Distribution; Exponential Distribution; Fréchet Distribution; Extreme Loss; Peruvian Stock Market.

JEL classification: C22; C58; G32.

MSC2010: 62M10; 62P20; 91G70; 91G80.

Teoría de valores extremos: una aplicación a los retornos bursátiles peruanos

RESUMEN

Usando observaciones diarias del índice y los retornos bursátiles para el caso Peruano desde el 3 de enero de 1990 hasta el 31 de mayo de 2013, este documento modela la distribución de las probabilidades de pérdidas diarias, estima los cuantiles máximos y las colas de la distribución y finalmente, modela los valores extremos usando un umbral máximo. Todo esto es usado para obtener una mejor medida del valor en riesgo (VaR) y de la pérdida esperada (ES) al 95% y 99% de confianza. Uno de los resultados de calcular el bloque máximo anual de los retornos bursátiles negativos es la observación que el retorno bursátil más negativo (diario) es 12.44% en el 2011. El parámetro de forma es igual a -0.020 y 0.268 para los bloques anuales y trimestrales, respectivamente. En consecuencia en el primer caso tenemos una distribución de tipo Gumbel. En el otro caso se tiene una distribución de cola pesada (Fréchet). Los valores estimados para el VaR y el ES son más elevados utilizando una distribución de tipo Pareto Generalizada (GPD) en comparación con la distribución normal y las diferencias al 99% son remarcables. Finalmente, la estimación no paramétrica del índice de cola de Hill y del cuantil para retornos negativos muestra ser bastante inestable.

Palabras claves: Teoría de valores extremos; valor en riesgo (VaR); pérdida esperada (ES); distribución de Pareto Generalizada (GPD); distribución de Gumbel; distribución exponencial; distribución de Fréchet; pérdida extrema; mercado bursátil peruano.

Clasificación JEL: C22; C58; G32.

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1 Introduction

As part of the Peruvian economy's good performance in recent years, the financial sector has played a significant role in terms of the objective of economic growth and capital accumulation. Nonetheless, the global financial crisis that began in the fourth quarter of 2007 affected the Peruvian capitals market and brought about a sharp fall in the General Index of the Lima Stock Exchange (IGBVL) of 59.78%, and in the Selective Index (ISBVL) of 59.73%. This event illustrates that big losses occur as a result of extreme movements in the markets, and hence that financial risk is related to the possible losses that investors can suffer in these markets; see Jorion (2001).

In general, the series of stock market returns have heavy-tailed distribution, due to which, unlike traditional distributions, the distribution of stock market returns possess greater probabilistic density on the tails. The above has, as a consequence, greater probability of extreme losses and it is necessary to analyze the tails of the distribution through the use of methodologies in the context of the Extreme Values Theory (EVT). I seek to capture in the best way possible the sudden movements of the performances of financial assets associated with the tails of the distribution, and thus allow better measurement of the behavior of financial asset performance¹. The recent financial crisis put in evidence the existence of multiple faults in the form of risk modeling, and this in turn prompted notable criticism of the different mathematical models and traditional statistics employed by companies in attempts to predict the risk. In 1993, the members of the Bank for International Settlements (BIS) gathered in Basel and amended the Basel Accords to require that banks and other financial institutions keep sufficient capital in reserve to cover ten days of potential losses based on the 10-day Value at Risk (VaR)².

The estimation of VaR by way of traditional models is not entirely adequate, because many of the techniques employed are based on the assumption that the financial returns follow a normal distribution. In this context, the measurement of risk through traditional measures occasions large losses to market participants because of the unexpected falls in financial market returns. Another measure of risk is that proposed by Artzner *et al.* (1999), called expected shortfall or expected loss (Expected Shortfall - ES) which is an expectation of loss conditioned to exceeding the indicated VaR level. One of the objectives of financial risk management is the exact calculation of the magnitudes and probabilities of big financial losses that are produced at times of financial crisis. It is thus of relevance to model the probability of loss distribution and estimate the maximum quantiles and tail probabilities associated with this distribution; see Zivot and Wang (2006).

The modern EVT started with von Bortkiewicz (1922). Thereafter, Fisher and Tippett (1928) laid the foundations of the asymptotic theory of the distributions of extreme values. Hill (1975) introduces a general approach for inference around the behavior of the tail of a distribution, while Danielsson and De Vries (1997) believe that a specific estimation of the form of the tail of foreign currency returns is of vital importance for adequate risk assessment.

¹Important texts include Embrechts *et al.* (1997) and Coles (2001). Other references applied to finance and financial risk management are Diebold *et al.* (1998), Danielsson and De Vries (1997), McNeil (1998a, 1998b) and Longin (2000).

²Danielsson *et al.* (2001) hold that the Committee on Banking Supervision was wrong to consider the risk to be endogenous and affirm that VaR can destabilize an economy and generate breaks which would not otherwise occur. In this way, the authors leave open the possibility that traditional financial models employed to measure and diagnose the risk have a certain degree of inconsistency, primarily because certain assumptions of these models are incapable of capturing the behavior of the indices that are used to measure risk. In particular, it is found that traditional models have a poor performance against sudden movements of these indices in a context of crisis.

On the other hand, Embrechts *et al.* (1997) present the probabilistic models and techniques with the aim of mathematically describing extreme events in the unidimensional case. McNeil (1998a) reduces data from the S&P500 index to 28 annual maximums corresponding to the period 1960-1987, and adjusts them to a Fréchet distribution. In this way, they calculate the estimations of various levels of returns, as well as the confidence interval at 95% for a 50-year level of return—which on average must be exceeded in just one year— every fifty years. The most probable calculated value is 7.4, but there is a great deal of uncertainty in the analysis as the confidence interval is approximately [4.9, 24].

Moreover, McNeil (1998b) considers the estimation of quantiles in the marginal distribution tail in the series of financial returns, utilizing statistical methods of extreme values based on the distribution limit of maximum blocks of stationary time series. The author proposes a simple methodology for the quantification of the worst possible scenarios, with losses of ten or twenty years.

Diebold *et al.* (1998) hold that the literature on the EVT is more accurate for the exact estimation of the extreme quantiles and tail probabilities of the financial assets³. McNeil (1999) shows a general vision of the EVT in the management of risks as a method for modeling and measuring extreme risks, concentrating on the peaks through a threshold. McNeil and Frey (2000) propose a method for estimating VaR and relate it to the risk measurements that describe the conditional distribution tail of a series of heteroskedastic financial yields.

Moreover, Longin (2000) present an application of the EVT to calculate VaR of a position in the market. For Embrechts *et al.* (2002), the modern risk management requires an understanding of stochastic dependence. The authors conduct a discussion on joint distributions and the use of copulas as descriptions of dependency among random variables.

Tsay (2002) applies the EVT to the logarithm of profitability of IBM shares for the period from July 3, 1962 to December 31, 1998 and finds that the range of fluctuation of the daily yields, excluding the crisis of 1987, fluctuates between 0.5% and 13%. He also estimates the Hill estimator and finds stable results for a minimum and a maximum value of the biggest n -th observation of this estimator. Tsay (2002) performs the estimation for different sample sizes (monthly, quarterly, weekly, and yearly) and concludes that the estimation of the scale and location parameters increase in modulus when the sample size increases. The shape parameter is stable for extreme negatives values when the sample size is greater than 62 and is approximately equal to a -0.33. The estimator of the shape parameter is small, significantly different to zero, and less stable for positive extremes. The result for the annual sample size has high variability when the number of subperiods is relatively small.

According to Delfiner and Gutiérrez Girault (2002), the returns in developing markets are characterized by being more leptokurtic compared to the returns of more developed economies; see also Humala and Rodríguez (2013) for stylized facts in the Peruvian stock market. The authors estimate an autoregressive AR-GARCH model of stochastic volatility, and then apply the EVT to the distribution tail of standardized residuals of the model by estimating a generalized Pareto distribution with a view to obtaining a better estimation of the probability when extreme losses are presented.

Finally, McNeil *et al.* (2005) provide two main types of models of extreme values. The most traditional models are maximum block, which are models for the biggest ordered observations of

³Diebold *et al.* (1998) demonstrate the existence of a trade off between the bias error and the variance when the largest n -th observation increases in Hill's tail index estimator (Hill, 1975).

big samples of identically distributed observations. The other group of models are for threshold exceedances and apply to all big observations that exceed a high level. They are generally considered very useful for practical applications, given their more efficient use (often limited) of the data on the extreme results.

Using daily observations of the index and stock market returns for the Peruvian case from January 3, 1990 to May 31, 2013, this paper models the distribution of daily loss probability, estimates maximum quantiles and tail probabilities of this distribution, and models the extremes through a maximum threshold. This is used to obtain the best measurements of the Value at Risk (VaR) and the Expected Short-Fall (ES) at 95% and 99%. One of the results on calculating the maximum annual block of the negative stock market returns is the observation that the largest negative stock market return (daily) is 12.44% in 2011. The shape parameter is equal to -0.020 and 0.268 for the annual and quarterly block, respectively. Then, in the first case, we have that the non-degenerate distribution function is Gumbel-type. In the other case, we have a thick-tailed distribution (Fréchet). Estimations of VaR and ES are higher using the Generalized Pareto Distribution (GPD) in comparison with the normal distribution and the differences at 99.0% are notable. Finally, the non-parametric estimation of the Hill tail-index and the quantile for negative stock market returns shows quite instability.

This paper is structured as follows: Section 2 describes the main definitions associated with EVT, as well as the method for estimating the main measurements of risk, VaR and ES. Section 3 presents the results, utilizing a sample of daily returns of the Peruvian stock market. Section 4 presents the main conclusions.

2 Methodology

In this Section, we closely follow and employ the notation in Zivot and Wang (2006). The EVT provides the statistical tools to model the unknown cumulative distribution function of the random variables that represent the risk or losses, especially in those situations where large losses are produced. Let $\{X_1, X_2, \dots, X_n\}$ independent and identically distributed (*i.i.d.*) random variables that symbolize the risk or expected losses, which have an unknown cumulative distribution function $F(x) = \Pr[X_i \leq x]$. $M_n = \max[X_1, X_2, \dots, X_n]$ is specified as the worst loss in a n -size sample of losses. In virtue of the assumption of *i.i.d.*, the cumulative distribution function of M_n is

$$\Pr[M_n \leq x] = \Pr[X_1 \leq x, X_2 \leq x, \dots, X_n \leq x] = \prod_{i=1}^n F(x) = F^n(x).$$

It is assumed that the function F^n is unknown, and, moreover, it is known that the function of empirical distribution is not a good approximation of $F^n(x)$. According to Fisher-Tippett Theorem (Fisher and Tippet, 1928)⁴, an asymptotic approximation is obtained for F^n based on the standardization of the maximum value; that is, $Z_n = \frac{M_n - \mu_n}{\sigma_n}$ where $\sigma_n > 0$ and μ_n are measurements of scale and position, respectively. In this way, for Fisher and Tippet (1928), the maximum standardized value converges to a distribution function of generalized extreme value (GEV), defined as:

$$H_\xi(z) = \begin{cases} \exp[-(1 + \xi z)^{-1/\xi}], & \text{for } \xi \neq 0 \text{ and } 1 + \xi z > 0 \\ \exp[-\exp(-z)], & \text{for } \xi = 0 \text{ and } -\infty \leq z \leq \infty \end{cases}$$

⁴This theorem is analogous to the Central Limit Theorem for extreme values.

where ξ is denominated the shape parameter and determines the behavior of the tail of $H_\xi(\cdot)$ ⁵. This distribution is not degenerated and is generalized in the sense that the parametric shape summarizes three types of known distributions. Moreover, if $\xi = 0$, H is a Gumbel distribution; if $\xi > 0$, H is a Fréchet distribution; and finally if $\xi < 0$, H is a Weibull distribution.

The parameter shape ξ is associated with the behavior of the tail of the distribution F and decays exponentially for a function of power $1 - F(x) = x^{-1/\xi}L(x)$ where $L(x)$ changes slowly. The GEV distribution is not changed for the transformations of location and scale: $H_\xi(z) = H_\xi(\frac{x-\mu}{\sigma}) = H_{\xi,\mu,\sigma}(x)$. For a large size n , Fisher-Tippett Theorem (Fisher and Tippet, 1928) can be interpreted as follows: $\Pr[Z_n < z] = \Pr[\frac{M_n - \mu_n}{\sigma_n} < z] \approx H_\xi(z)$. Assuming $x = \sigma_n z + \mu_n$, then: $\Pr[M_n < x] \approx H_{\xi,\mu,\sigma}(\frac{x-\mu_n}{\sigma_n}) = H_{\xi,\mu_n,\sigma_n}(x)$. This expression is useful for performing inference related to the maximum loss M_n . The expression depends on the parameter of ξ in form and the standardized constants σ_n and μ_n , which are estimated for maximum likelihood.

To perform the estimation of maximum likelihood, it is supposed to be a set of identically distributed losses from a sample of size T represented for $\{X_1, X_2, \dots, X_T\}$ that have an cumulative density function F . A sub-sample method is utilized to form the likelihood function for the parameters ξ, σ_n and μ_n from the GEV distribution for M_n . In this way, the sample is divided into m non-overlapping blocks of equal size $n = T/m$, with which we have $[X_1, \dots, X_n | X_{n+1}, \dots, X_{2n} | \dots | X_{(m-1)n+1}, \dots, X_{mn}]$ and where $M_n^{(j)}$ is defined as the maximum value of X_i in the block $j = 1, \dots, m$. The likelihood function for the parameters ξ, σ_n and μ_n of the GEV distribution is constructed from the maximum block sample of $\{M_n^{(1)}, \dots, M_n^{(m)}\}$. The likelihood log function assuming *i.i.d.* observations of the GEV distribution when $\xi \neq 0$ is

$$\log(\mu, \sigma, \xi) = -m \log(\sigma) - (1 + \frac{1}{\xi}) \sum_{i=1}^m \log[1 + \xi(\frac{M_n^{(i)} - \mu}{\sigma})] - \sum_{i=1}^m [1 + \xi(\frac{M_n^{(i)} - \mu}{\sigma})]^{-1/\xi}$$

with the restriction $1 + \xi(\frac{M_n^{(i)} - \mu}{\sigma}) > 0$. When $\xi = 0$, we obtain a Weibull distribution⁶.

It is important to discuss the limit distribution of extremes on high thresholds and the generalized Pareto distribution (GPD). When there is a succession of *i.i.d.* random variables $\{X_1, X_2, \dots, X_n\}$ associated with an unknown function of distribution $F(x) = \Pr[X_i \leq x]$, the extreme values are defined as the X_i values that exceed the high threshold κ . So, the variable $X - \kappa$ represents the excesses on this threshold. The distribution of excesses on the threshold κ is defined as a conditional probability: $F_\kappa(y) = \Pr[X - \kappa \leq y | X > \kappa] = \frac{F(y+\kappa) - F(\kappa)}{1 - F(\kappa)}$ for $y > 0$. This is interpreted as the probability that a loss exceeds the threshold κ for a value that is equal to or less than y , given that the threshold of κ has been exceeded. For $M_n = \max\{X_1, X_2, \dots, X_n\}$, defined as the worst loss in a n -sized sample of losses, the distribution function F satisfies Fisher-Tippett Theorem (Fisher and Tippet, 1928) and, for a sufficiently large κ , there is a positive function $\beta(\kappa)$. Thus, the surplus distribution is approximated through the GPD

$$G_{\xi,\beta(\kappa)}(y) = \begin{cases} 1 - [1 + \xi y / \beta(\kappa)]^{-1/\xi}, & \text{for } \xi \neq 0 \text{ and } \beta(\kappa) > 0 \\ 1 - \exp[-y / \beta(\kappa)], & \text{for } \xi = 0 \text{ and } \beta(\kappa) > 0 \end{cases}$$

⁵The expression $H_\xi(\cdot)$ is continuous in the shape parameter ξ .

⁶Distribution in the domain of attraction of the Gumbel-type distribution are thin-tailed distributions where practically all moments exist. If they are Fréchet-type, they include fat-tailed distributions such as Pareto, Cauchy or t-Student, among others. Some moments do not exist for these distributions.

defined for $y \geq 0$ when $\xi \geq 0$ and $0 \leq y \leq -\beta(\kappa)$ when $\xi < 0$. For a sufficiently high threshold κ , it is found that $F_\kappa(y) \approx G_{\xi, \beta(\kappa)}(y)$ for a wide range of loss functions $F(\cdot)$. To apply this result, the value of the threshold must be specified and the estimates of ξ and $\beta(\kappa)$ can be obtained.

There is a close connection between the GEV limit distribution for maximum blocks and the GPD for excesses with respect to the threshold. For a given value of κ , the parameters ξ , μ and σ of the GEV distribution determine the parameters ξ and $\beta(\kappa)$. It is clear that the shape parameter ξ of the GEV distribution is the same parameter ξ in the GPD and is independent of the threshold value κ . In consequence, if $\xi > 0$, the function F is Fréchet-type and the expression $G_{\xi, \beta(\kappa)}(y)$ is denominated classic Pareto distribution; when $\xi = 0$, the function F is Gumbel-type and $G_{\xi, \beta(\kappa)}(y)$ follows an exponential distribution. Finally, it is found that $0 \leq y \leq -\beta(\kappa)/\xi$ when $\xi < 0$, so the function F is Weibull-type and $G_{\xi, \beta(\kappa)}(y)$ is a type-II Pareto distribution. The parameter ξ is the shape or tail-index parameter and is associated with the rate of decay of the tail of the distribution, and the decreasing parameter β is the shape parameter and is associated with the position of the threshold κ .⁷

Now, assuming that the parameter of form is $\xi < 1$, the mean excess function above the threshold κ_0 will be $E[X - \kappa_0 | X > \kappa_0] = \frac{\beta(\kappa_0)}{1-\xi}$ for any $\kappa > \kappa_0$, and it is found that the excess function of the mean $e(\kappa) = E[X - \kappa | X > \kappa] = \frac{\beta(\kappa_0) + \xi(\kappa - \kappa_0)}{1-\xi}$. Analogously, for any value of $y > 0$, the following conditions hold: $e(\kappa_0 + y) = E[X - (\kappa_0 + y) | X > \kappa_0 + y] = \frac{\beta(\kappa_0) + \xi y}{1-\xi}$. Therefore, to graphically deduce the threshold value for the GPD, we get the excess function of the empirical mean: $e_n(\kappa) = \frac{1}{n_\kappa} \sum_{i=1}^{n_\kappa} [x_{(i)} - \kappa]$, where $x_{(i)}$ ($i = 1, 2, \dots, n_\kappa$) is the value of x_i such that $x_i > \kappa$. With the previous expression, a graph representation of $e_n(\kappa)$ is constructed with the mean excess on the vertical axis. This graph can be interpreted as follows: if the slope is rising, it indicates thick tail behavior; but if there is a downward trend, this shows thin tail behavior in the distribution; and finally, if the slope of the line is equal to zero, the behavior of the tail is exponential. If the line is straight and has a positive slope located above the threshold, then it is an example of Pareto-type tail behavior.

For the values of the maximum losses that exceed the threshold (that is, when $x_i > \kappa$), the threshold excess is defined as $y_i = x_{(i)} - \kappa$ for $i = 1, \dots, k$, in which the values of x_1, \dots, x_n have been denoted as $x_{(1)}, \dots, x_{(k)}$. When the threshold value is sufficiently large, then the sample $\{y_1, \dots, y_k\}$ can be expressed within a likelihood that is based on the unknown parameters ξ and $\beta(\kappa)$; that is, a random sample of a GPD.

When $\xi \neq 0$, the log likelihood function of $G_{\xi, \beta(\kappa)}(y)$ has the following form

$$\log[\beta(\kappa)] = -k \log[\beta(\kappa)] - \left[1 + \frac{1}{\xi}\right] \sum_{i=1}^k \log\left[1 + \frac{\xi y_i}{\beta(\kappa)}\right]$$

where $y_i \geq 0$ when $\xi > 0$ and $0 \leq y_i \leq -\beta(\kappa)/\xi$ when $\xi < 0$. If the parameter of form is $\xi = 0$, then the log likelihood function is

$$\log[\beta(\kappa)] = -k \log[\beta(\kappa)] - \beta(\kappa)^{-1} \sum_{i=1}^k y_i.$$

⁷For $\xi > 0$ (the most relevant case for risk administration purposes), it can be shown that $E[X^k] = \infty$ for $k \geq \alpha = 1/\xi$. If $\xi = 0.5$, $E[X^2] = \infty$ and the distribution of losses X does not have finite variance. Analogously, if $\xi = 1$, then $E[X] = \infty$.

To estimate the tails of the loss distribution for $F(x)$, and where $x > \kappa$, we use $F(x) = [1 - F(\kappa)]G_{\xi, \beta(\kappa)}(y) + F(\kappa)$. The previous expression is fulfilled for a sufficiently large threshold and in which $F_\kappa(y) \approx G_{\xi, \beta(\kappa)}(y)$.⁸

There are two common risk measurements: Value at Risk (VaR) and Expected Shortfall (ES). The VaR is the largest quantile of the distribution of loss; that is, $VaR_q = F^{-1}(q)$.⁹ For a given probability $q > F(\kappa)$, it is found that $\widehat{VaR}_q = \kappa + \frac{\widehat{\beta}(\kappa)}{\widehat{\xi}}[\frac{n}{k}(1-q)^{-\widehat{\xi}} - 1]$. The ES is the expected size loss, given that VaR_q is exceeded: $ES_q = E[X|X > VaR_q]$. This equation is related to VaR_q in accordance with $ES_q = VaR_q + E[X - VaR_q|X > VaR_q]$, where the second term is the mean of excess of the distribution $F_{VaR_q}(y)$ on a threshold VaR_q . The approximation of the GPD (due to the translation property) to $F_{VaR_q}(y)$ has the shape parameter ξ and the scale parameter $\beta(\kappa) + \xi[VaR_q - \kappa] : E[X - VaR_q|X > VaR_q] = \frac{\beta(\kappa) + \xi(VaR_q - \kappa)}{1 - \xi}$ provided that $\xi > 1$. Moreover, it is found that the GPD approximates $\widehat{ES}_q = \frac{\widehat{VaR}_q}{1 - \widehat{\xi}} + \frac{\widehat{\beta}(\kappa) - \widehat{\xi}\kappa}{1 - \widehat{\xi}}$.¹⁰ It is also possible to perform the non-parametric estimation of the shape parameter ξ or the tail-index parameter $a = 1/\xi$ of the distributions $H_\xi(z)$ and $G_{\xi, \beta(\kappa)}(y)$ utilizing Hill method (Hill, 1975), in which $\xi > 0$ ($\alpha > 0$), is generated by the same thick-tailed distributions in the domain of attractions of a Fréchet GEV. Considering a sample of losses $\{X_1, X_2, \dots, X_T\}$, the statistical order is defined as $X_{(1)} \geq X_{(2)} \geq \dots \geq X_{(T)}$ for a positive whole k , and the Hill estimator of ξ is defined as $\widehat{\xi}^{Hill}(k) = \frac{1}{k} \sum_{j=1}^k [\log X_{(j)} - \log X_{(k)}]$. The Hill estimator of α is $\widehat{\alpha}^{Hill}(k) = \frac{1}{\widehat{\xi}^{Hill}(k)}$.¹¹

3 Empirical Evidence

Figure 1 shows the series for the closing prices of the General Index of the Lima Stock Exchange (IGBVL)¹². The series is of daily frequency and covers the period from January 3, 1990 to May 30, 2013. The returns are defined as $r_t = \log[\frac{P_t}{P_{t-1}}]$, which are shown in Figure 2. Empirically, the returns display certain properties as marginal thick-tailed distributions, nonexistence of correlation, and dependency across these; though they are highly correlated if it concerns the squared results or their absolute value; see Humala and Rodríguez (2013) for a more detailed description about the stylized facts.

By way of motivation, Figure 3 shows the GEV cumulative distribution function for the distribution function $H_\xi(\cdot)$, which adopts Fréchet, Weibull or Gumbel form of distribution when the shape parameter is $\xi = 0.5$, $\xi = -0.5$ or $\xi = 0$, respectively, and for general values of z , the parameter of position μ and the parameter of scale σ . In this particular case, the Fréchet distribution is defined for $z > -2$ and the Weibull distribution is only defined for $z < 2$. Figure 4 shows the GEV probability density function $H_\xi(\cdot)$ for the non-degenerate Fréchet, Weibull and Gumbell dis-

⁸It is assumed that $x = \kappa + y$.

⁹If $X \sim N(\mu, \sigma^2)$, then $VaR_{0.99} = \mu + \sigma q_{0.99}$.

¹⁰If $X \sim N(\mu, \sigma^2)$, it is found that $ES_{0.99} = \mu + \sigma \frac{\phi(z)}{1 - \Phi(z)}$.

¹¹It can be seen that if F is located in the domain of attraction of a GEV distribution, then $\widehat{\xi}^{Hill}(k)$ converges in probability to ξ when $k \rightarrow \infty$ and $\frac{k}{n} \rightarrow 0$, and $\widehat{\xi}^{Hill}(k)$ is normally asymptotically distributed with asymptotic variance: $avar[\widehat{\xi}^{Hill}(k)] = \frac{\xi^2}{k}$. Via the delta method, $\widehat{\alpha}^{Hill}(k)$ is normally asymptotically distributed with asymptotic variance $avar[\widehat{\alpha}^{Hill}(k)] = \frac{a^2}{k}$.

¹²The closing prices of the General Index of the Lima Stock Exchange are taken into account from Monday until the closure price on Friday. Moreover, it should be recalled that non-working days are not considered, and more generally the days on which the market was closed.



Figure 1: Closing prices of the General Index of the Lima Stock Exchange.

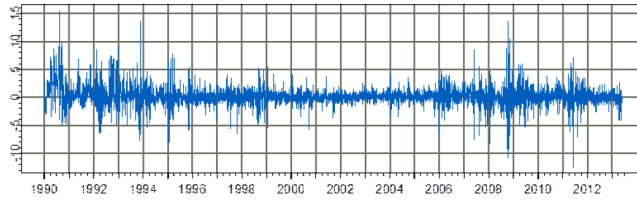


Figure 2: Daily Percentage Returns of the Stock Market of Peru.

tribution functions depending on the values $\xi = 0.5$, $\xi = -0.5$ and $\xi = 0$ for the shape parameter, respectively. The Fréchet and Weibull distributions are defined for $z > -2$ and $z < 2$, respectively.

Figure 5 shows the GEV density function for negative stock market returns. The horizontal axis represents the standardized value Z_n of the maximum value of the block M_n with respect to the measurements of scale and position. The vertical axis shows the probability associated with the GEV density function. It is observed that this distribution does not have the form of a known distribution and the maximum probability shows positive asymmetry.

The p -quantile of a distribution function G is defined by the value X_p such that $G(p) = X_p$; that is, the value of X_p that leaves the p -percentile of probability to its left. If a distribution function G is continuous and thus strictly growing, the quantile function is the inverse of the distribution function G and is usually denoted as G^{-1} . Figure 6 shows the q-q plot, taking the normal distribution as theoretical distribution to be contrasted with the distribution of stock market returns. Let us note that a straight diagonal line is not observed (approximately), and so it is concluded that the distribution of the variable is not the same as the comparison distribution, showing evidence that the distribution of negative stock market returns is unknown.

Subsequently, the annual maximum block of the negative stock market returns is calculated. Figure 7 shows four representations for this annual maximum block. In the upper-left graph, it is shown the largest negative return of the period analyzed, which reaches 12.44% in 2011. The upper-right graph shows the histogram where the horizontal axis represents the annual maximum blocks. In the lower-left representation, the q-q plot is shown, contrasting again the distribution of stock market returns for the period of analysis. In the vertical axis, the quantiles of the referential theoretical distribution are represented (Gumbel distribution, H_0), which satisfies $H_0^{-1}(p) = -\log[-\log(p)]$ and the horizontal axis represents the empirical quantiles for the annual

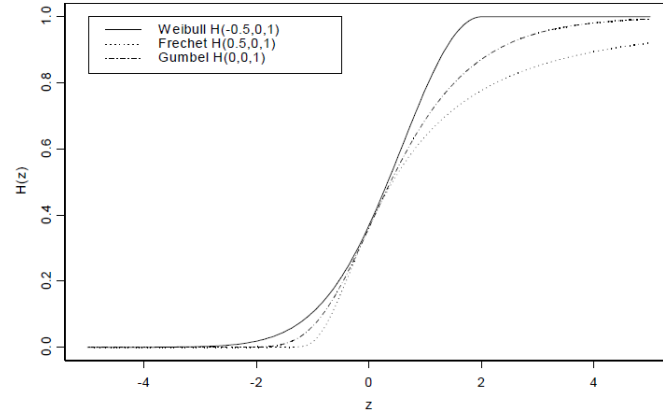


Figure 3: Generalized Extreme Value CDFs for Fréchet, Weibull and Gumbel.

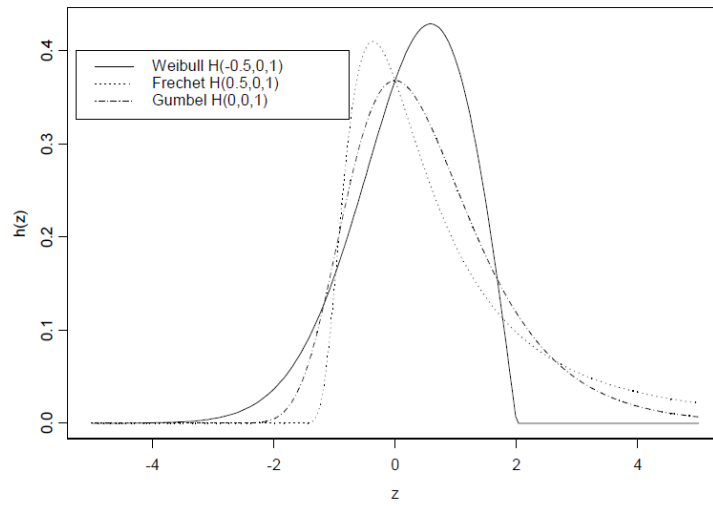


Figure 4: Generalized Extreme Value pdfs for Fréchet, Weibull and Gumbel.

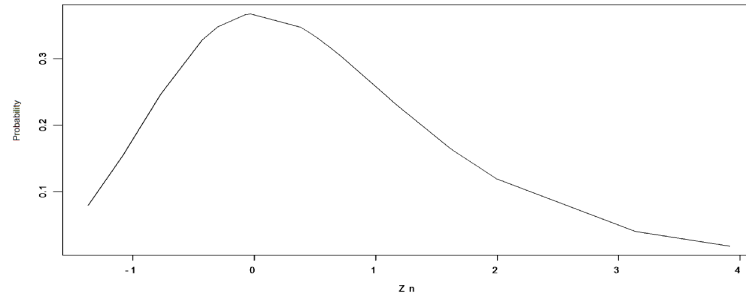


Figure 5: GEV pdf for Daily Returns in Peru.

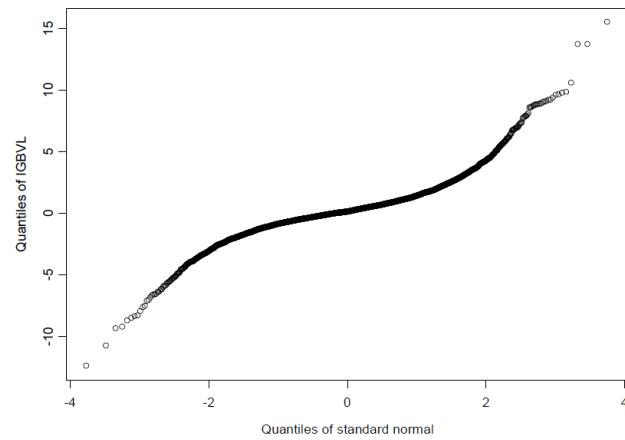


Figure 6: Normal q-q plot for the Daily Percentage Returns in Peru.

maximum blocks of the distribution of stock market returns. It is observed that the point cloud fits approximately the bisector for the axes, which suggests that the distribution of the variable of real data (empirical distribution) is the same as the distribution of comparison (Gumbel distribution). Finally, the lower-right graph shows the development of the records (new maximum) for the negative stock market returns, together with the expected number of returns for the *i.i.d.* data. In this graph, it is observed that the data was not within the confidence interval (delimited by dotted lines), due to which it can be concluded that the data is not consistent with the *i.i.d.* behavior.

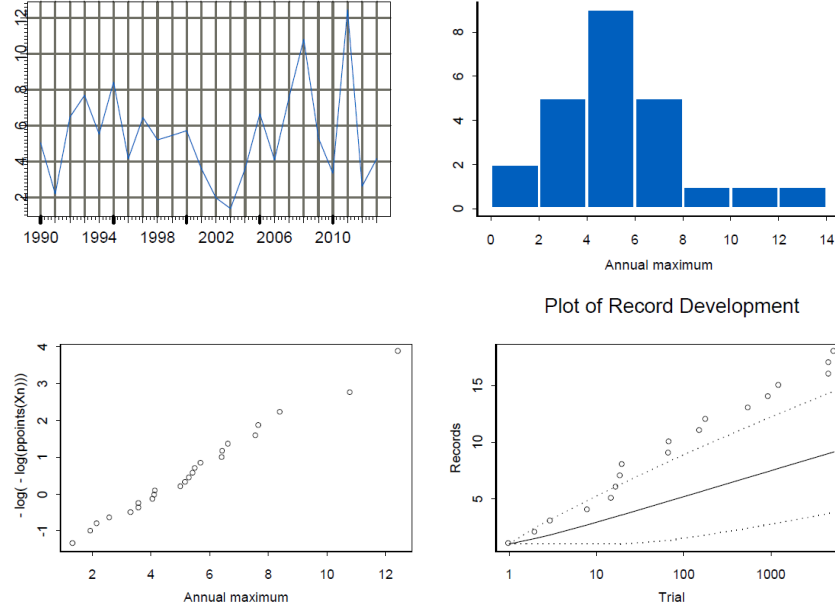


Figure 7: Annual block maxima, histogram, Gumbel q-q plot and records summary for the Daily Stock Returns in Peru.

Analogously to the lower-left graph in Figure 7, Figure 8 shows the *q-q plot*, using as referential distribution the Gumbel distribution H_0 . Unlike Figure 7, the horizontal axis represents the standardization of maximum value Z_n . As shown previously for the Gumbel distribution, the quantiles satisfy $H_0^{-1}(p) = -\log[-\log(p)]$ and the points of the quantiles correspond to the standardization of the maximum value Z_n and indicate a GEV distribution with $\xi = 0$.

Then, the entire annual value of the number of observations in each maximum block is determined by $M_n^{(i)}$ $i = 1, \dots, m$ for the stock market returns, with $m = 24$. The shape parameter ξ is statistically insignificant ($\hat{\xi} = -0.020$, $t_{\hat{\xi}} = -0.126$) and so the value of this parameter is equal to zero ($\xi = 0$). Moreover, the asymptotic interval at 95% of confidence for ξ is $[-0.337, 0.2968]$ and indicates the considerable uncertainty related to the value of ξ . This result determines the tail behavior of the GEV distribution function of stock market returns, and it is concluded that the non-degenerate distribution function is Gumbel-type. The position and scale parameters (standardized constants) are statistically significant: $\hat{\mu}_n = 4.232$, $t_{\hat{\mu}_n} = 8.713$ and $\hat{\sigma}_n = 2.098$, $t_{\hat{\sigma}_n} = 5.954$, respectively.

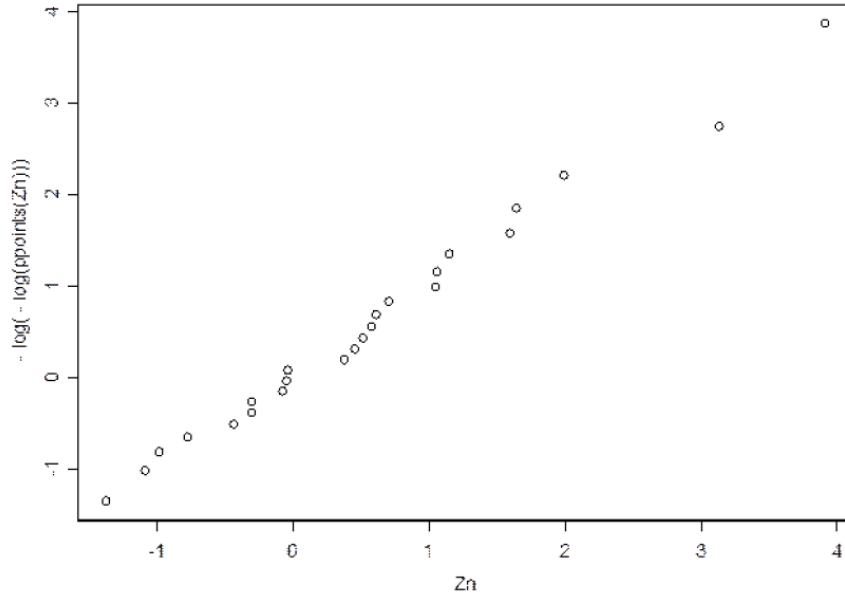


Figure 8: Gumbel q-q plot of Stock Daily Returns in Peru.

Utilizing the estimation by maximum likelihood of the adjusted GEV distribution for the maximum annual block of negative stock market returns, the following question can be answered: How probable is it that the maximum annual negative profitability for the following year exceeds the above negative returns? This probability is calculated utilizing the expression $H_{\xi, \mu_n, \sigma_n}(x)$ where the maximum block is equal to 1.68%, and so there is a 1.68% possibility that a new maximum record of negative performance will be established during the following year.

A similar analysis is possible by considering the GEV distribution adjusted for the quarterly maximum block for the data from the series of stock market returns. The maximum block for the return of this series is $m = 94$. It is observed that estimated standard asymptotic errors are much lower when quarterly blocks are employed. The shape estimator is $\hat{\xi} = 0.268$ ($t_{\hat{\xi}} = 2.031$) and in this case, the asymptotic interval at 95% of confidence for ξ is $[-0.004, 0.532]$ and contains only positive values for the shape parameter, indicating a thick-tailed distribution, with the estimated probability equal to 0.0172. Finally, the estimations of the position and scale parameters are significant: $\hat{\mu}_n = 2.419$, $t_{\hat{\mu}_n} = 9.186$ and $\hat{\sigma}_n = 2.419$, $t_{\hat{\sigma}_n} = 13.705$, respectively.

In Figure 9, the asymmetric form of the asymptotic confidence interval can be observed. Figure 9 allows us to give response to the following question: What is the level of stock market return for the last forty years? The estimated point of the level of return (11.67%) is at the point where the vertical line cuts at the maximum point of the asymmetric curve. The upper extreme point of the confidence interval of 95% is approximately 22%; this point is located where the asymmetrical curve cuts at the straight horizontal line. In addition, Figure 10 shows the estimation of the expected yield level of the negative stock market returns for forty years with a confidence level band of 95% based on the model of GEV for an annual maximum block. In Figure 10, a horizontal line is drawn dividing the graph into two halves and corresponding to the expected level of return (11.67%). In addition, the dot horizontal line below the line for the expected level of return corresponds to the

lowest level of return (9.33%); whereas the dot horizontal line above the expected level of return corresponds to the highest level of return (22.21%). In this Figure, the 24 annual maximum blocks ($m = 24$) obtained from the real data of the stock market returns (point cloud) can also be seen, in which only two points exceed the lower extreme.

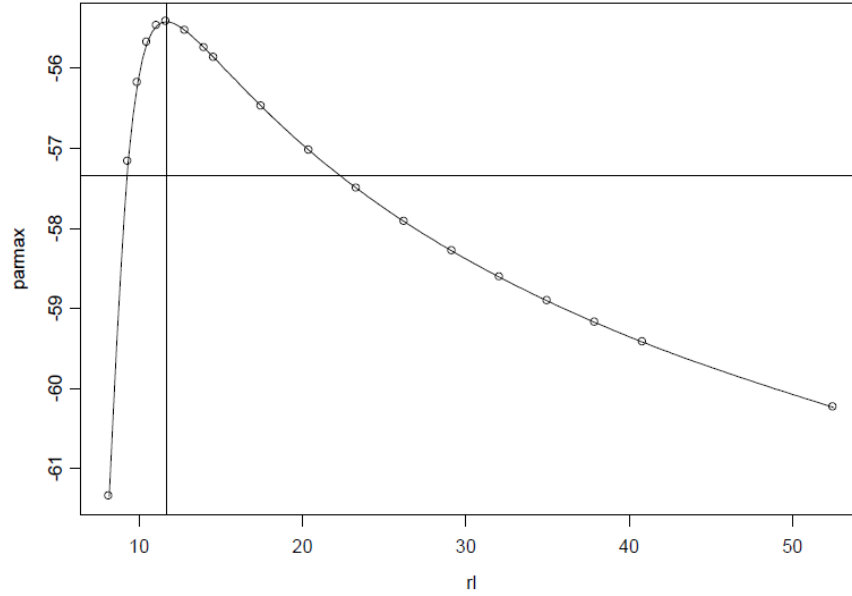


Figure 9: Asymptotic 95% Confidence Interval for the 40 Year Return Level.

Following Zivot and Wang (2006), the 40-year level of return can also be estimated based on the GEV fitted to quarterly periods as a maximum, where forty years correspond to 160 quarters, obtaining the lowest and highest level of return; see Figures 11 and 12. In Figure 12, the horizontal line located on the mean corresponds to the expected level of return (17.18%); the dot horizontal line below the level of expected return corresponds to the lowest level of return (10.88%) and the dotted line above the expected level of return corresponds to the highest level of return, being equal to 40.68%. Figure 12 also shows the 94 ($m = 94$) quarterly maximum blocks obtained from the data on stock market returns (point cloud) below the lower confidence band of the confidence interval, except for two points, which means that the return for these 160 quarters must be above these values.

According to Zivot and Wang (2006), modeling only the maximum block of data is inefficient if there are other data being available on the extreme values. A more efficient, alternative approach that utilizes more observations is to model the behavior of extreme values above a given high threshold. This method is called peaks over threshold (POT). Another advantage of the POT method is that the common risk measurements, such as VaR and ES, can be calculated easily¹³.

To motivate the importance of the foregoing in Figure 13, the calculation of the cumulated

¹³For risk administration purposes, insurance companies may be interested in the frequency of occurrence of a large demand above a certain threshold, as well as the average value of the demand that exceeds the threshold. In addition, they may be interested in the daily VaR and ES. The statistical models for extreme values above a threshold can be used to tackle these questions.

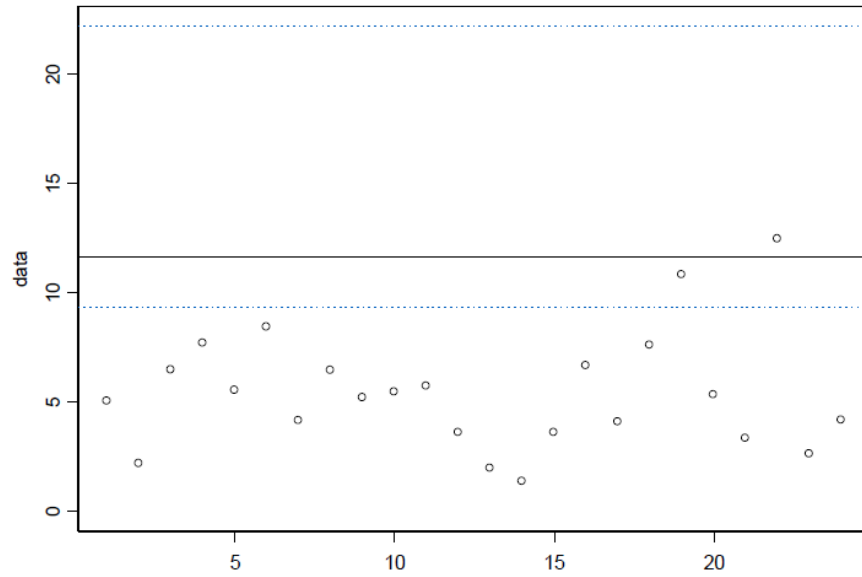


Figure 10: Estimated 40-Year Return Level with 95% Confidence Band for the Stock Daily Returns in Peru.

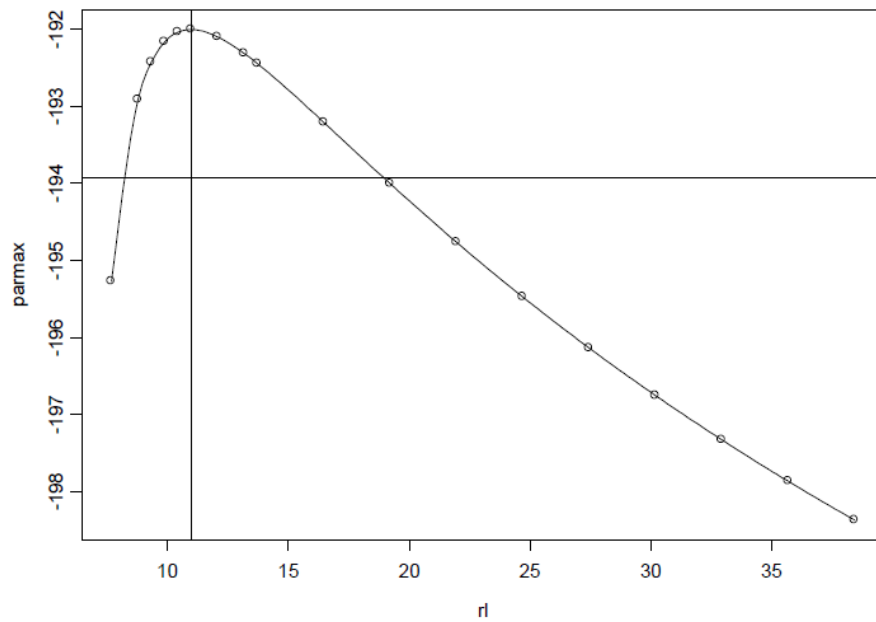


Figure 11: Asymptotic 95% Confidence Interval for the 160-Quarterly Return Level.

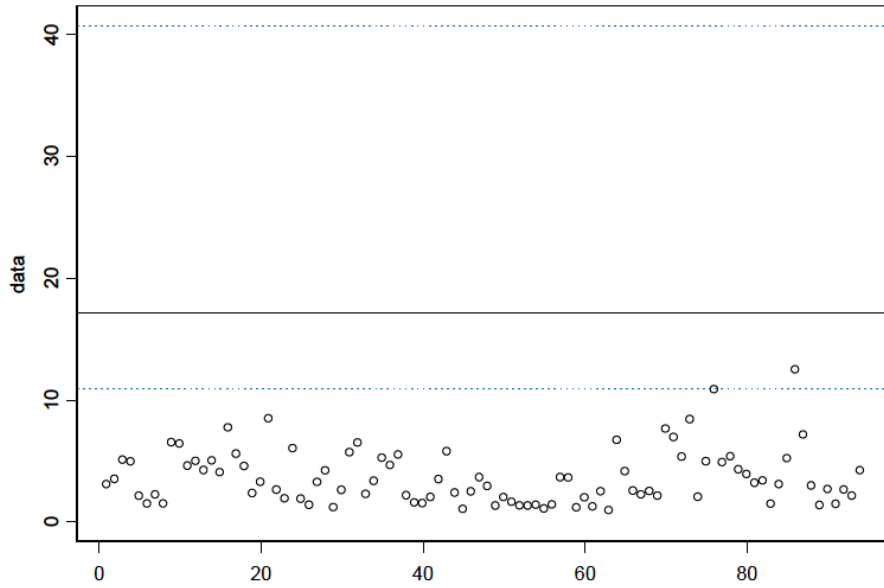


Figure 12: Estimated 160-Quarters Return Level with 95% Confidence Band for the Stock Daily Returns in Peru.

distribution and probability functions are shown with $\beta(\kappa) = 1$ for Pareto ($\xi = 0.5$), exponential ($\xi = 0$), and Pareto type II ($\xi = -0.5$) distributions. The Pareto type II distribution is defined only for $y < 2$. According to Zivot and Wang (2006), to infer the tail behavior of the observed losses, a *q-q plot* is created using the exponential distribution as reference distribution. If the excess on the threshold is a thin-tailed distribution, then the generalized Pareto distribution is exponential with $\xi = 0$ and the *q-q plot* should be linear. Deviations from the linearity in the *q-q plot* indicate thick-tailed behavior ($\xi > 0$) or bounded tails ($\xi < 0$).

In Figure 14, the *q-q plot* is observed for the distribution of negative stock market returns through the threshold when this is equal to one ($\kappa = 1$). The selection of the threshold under this methodology is complicated. Hence, to identify the threshold, there are a number of methodologies, such as parametric and graphic methods¹⁴. Figure 14 shows a slight deviation from the linearity for negative stock market returns, which leads us to conclude that the distribution of negative stock market returns is a thick-tailed distribution.

The main distributional model for excess through the threshold is the GPD. So, when defining the excess function of the empirical sample mean, a graph can be prepared in which the expectation of the values above the threshold κ is represented, once the threshold has been exceeded on the vertical axis associated with each of the thresholds. This is useful for discerning tails of a distribution against the different possible levels of threshold κ on the horizontal axis. Figure 14 must be approximately linear at the level of the selected threshold, and it is possible to determine intervals that allow for selecting the threshold. In general, the thick-tailed distributions give way to a mean excess function that tends toward the infinite for high values of κ and displays a linear form with a

¹⁴One of these methods is the mean excess plot.

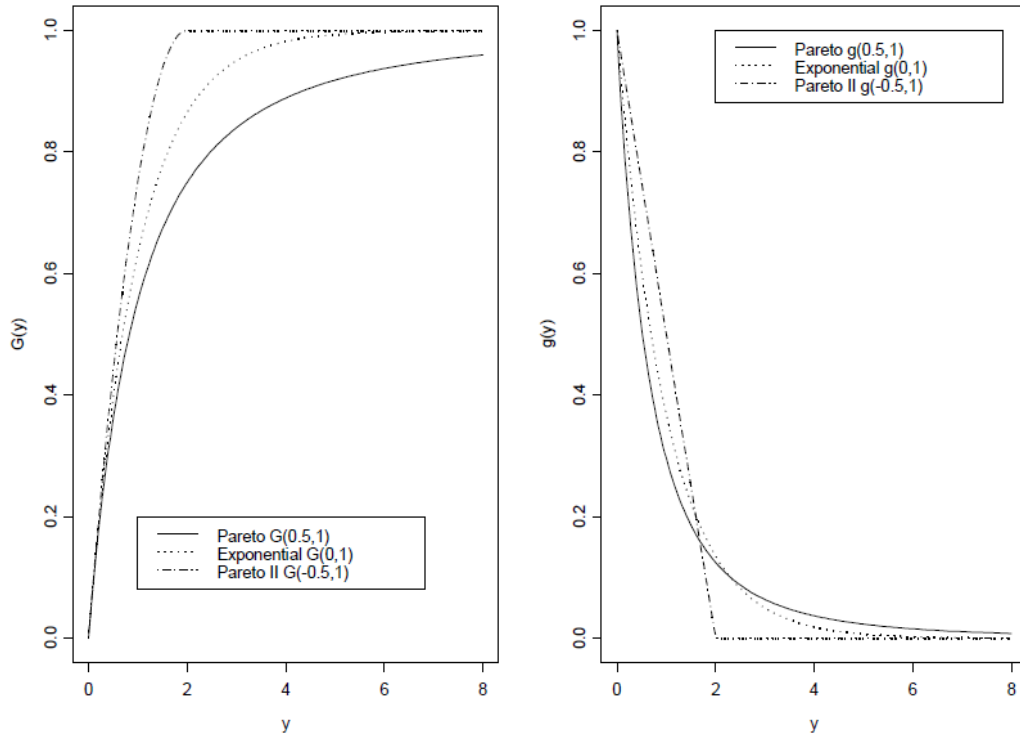


Figure 13: Generalized Pareto CDFs and pdfs for Pareto.

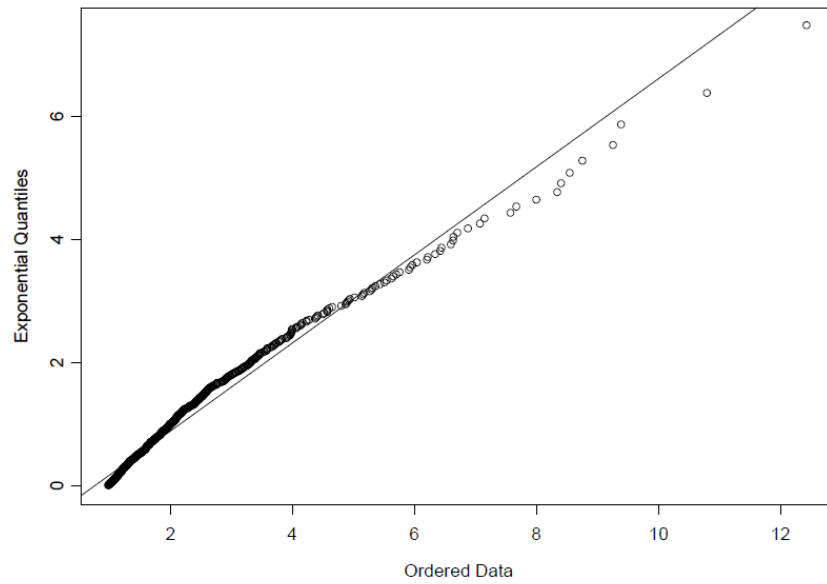


Figure 14: Q-q plot with Exponential Reference Distribution for the Stock Daily Negative Returns over the Threshold $\kappa = 1$.

positive slope. All the above mentioned is shown in Figure 15. On the vertical axis, the empirical mean excess is represented for the series of stock market returns, and on the horizontal axis, the threshold κ is represented. If the points that are represented have an upward trend (upward slope), this indicates thick-tailed behavior in the sample represented, as well as a GPD with positive shape parameter $\xi > 0$. If there is a downward trend (negative slope), this involves the thin-tailed behavior of the GPD with negative parameter $\xi < 0$. Finally, if an approximately linear graph is obtained (tending toward the horizontal axis), this indicates a GPD and the tail behavior is exponential (an exponential excess distribution), with the shape parameter approximately equal to zero ($\xi = 0$). From the observation of Figure 15 on mean excess, a declining trend for the data up to the value of the threshold $\kappa = -1$ is detected, which indicates a thin-tailed distribution therein; but from this value for the threshold, there is an upward trend for the data, indicating the thick-tailed behavior in the sample represented¹⁵.

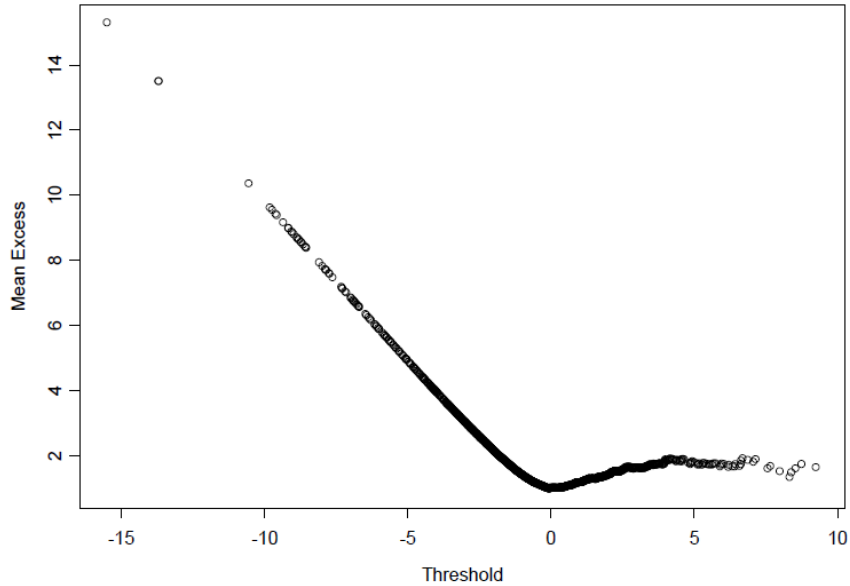


Figure 15: Mean Excess Plot for the Stock Daily Negative Returns.

Once the mean excess function is determined, the tails of the distribution of negative stock market losses are estimated for the period of analysis by way of the maximum likelihood estimation of the parameters $\beta(\kappa)$ and ξ of the GPD. To determine this estimation, a threshold κ must be specified, which must be big enough for the approximation of the GPD to be valid, but must also be small enough so that a sufficient number of observations is available for an exact fit; see Carmona (2004). In Figure 16 on the excess of the mean for stock market returns, it is observed that the threshold has a value of one (that is $\kappa = 1$) and may be appropriate for the GPD to be valid. The estimation of the parameters indicates $\hat{\xi} = 0.185$ ($t_{\hat{\xi}} = 4.463$) and $\hat{\beta}(1) = 0.941$ ($t_{\hat{\beta}(1)} = 18.801$). If the estimated shape parameter for the GPD ($\hat{\xi} = 0.185$) is compared with the GEV estimations of

¹⁵Empirical evidence on different behavior in the tails of the Peruvian stock market returns is also found in Rodríguez (2017) and Lengua Lafosse *et al.* (2014).

the yearly and quarterly maximum blocks, it is seen that this is higher in the case where the analysis is based on quarterly data ($\hat{\xi} = 0.268$), but less if annual data are used ($\hat{\xi} = -0.020$), being close to zero in the latter case. According to Carmona (2004), Figure 16 shows the underlying distribution. In the left-side graph, the survival function $1 - F(x)$ is represented on the vertical axis instead of the cumulative distribution function $F(x)$ and it is seen that the curve moves very close to the horizontal axis, so it is extremely difficult to correctly quantify the quality of fitting. Since this graph is not very useful, the right-side graph represents the survival function in logarithmic scale on the vertical axis, which helps ensure that the fit of the distribution is adequate by taking into account the available data. Observing both graphs in Figure 16, it is concluded that the fitting is good.

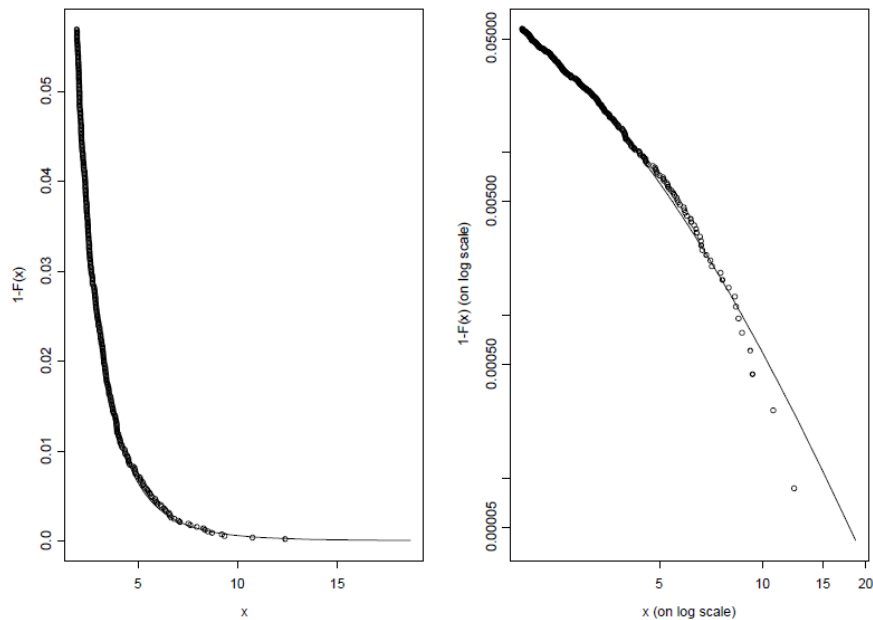


Figure 16: Diagnostic Plots for GPD Fit to Daily Negative Returns on Stock Index.

Changing the value of the threshold brings about changes in the estimation of ξ , so the stability of the shape parameter must be considered. It is optimal not to depend on a procedure that is too sensitive to small changes in the threshold selection. In effect, since there is no clear procedure for the selection of the threshold with a high level of accuracy, the estimation of the shape parameter must remain robust in the face of variations in the errors in the selection of this threshold. The best way to verify the stability of the parameter is through visual inspection. Now, to show how the estimation by maximum likelihood in the shape parameter ξ varies with the threshold which has been selected, we can observe Figure 17 where the lower horizontal axis represents the maximum number of threshold excesses, and it is assumed to be equal to six hundred. On the upper horizontal axis, the threshold is represented, whereas the estimation of the shape parameter with a confidence of 95% is represented on the vertical axis. Figure 17 shows that ξ has a very stable behavior close to 0.185 for threshold values lower than 1.91.

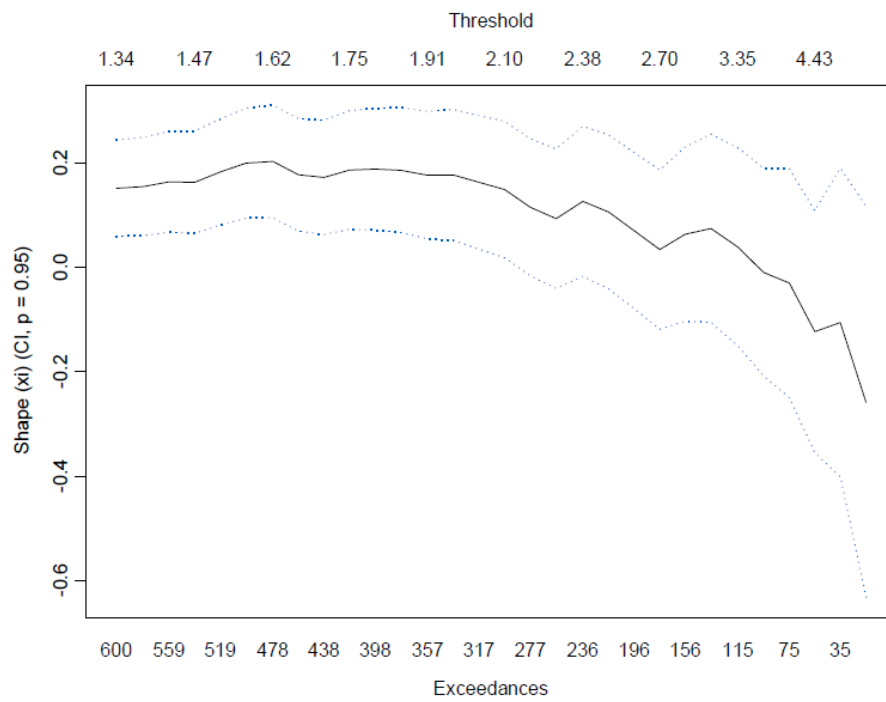


Figure 17: Estimates of the Shape Parameter for the Daily Negative Returns as a Function of the Threshold.

In accordance with the above, Figure 18 shows how the estimator of the GPD shape parameter varies with the threshold, where starting value of the quantile has been specified based on the data equal to 0.9, to be used as a threshold that fits the model. In the upper horizontal axis in Figure 18, the proportion of the points included in the estimation is represented. This information is useful to decide whether it is necessary or not to take seriously some of the estimations of ξ that appear on the left and right extremes of Figure 18. The central part of the graph should essentially be horizontal, though this does not always result in a straight line, when the empirical distribution of the data can be reasonably explained by a GPD. Finally, on the lower horizontal axis in Figure 18, the threshold is represented. The leftmost part of the graph should be ignored because of the following reason: If the threshold is too small, much of the data (that must be included in the center of the distribution) contributes to the estimation of the tail, skewing the result. Analogously, the rightmost part of the graph should also be ignored as few points will contribute to the estimation. This is the case in the current situation, and a value of $\xi = 0.185$ appears to be a reasonable estimation for the intersection of a horizontal line fitted to the central part of the graph.

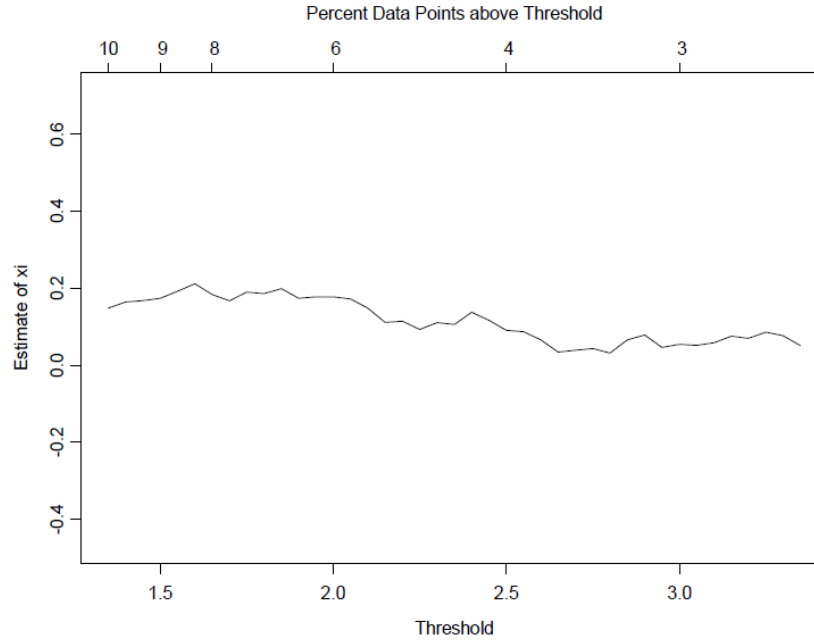


Figure 18: Estimates of the Shape Parameter with Time-Varying Threshold.

On the other hand, according to Zivot and Wang (2006), it is often desirable to estimate the parameters ξ and $\beta(\kappa)$ through the maximum likelihood estimation of the GPD separately for the upper and lower tails of the negative returns (POT analysis). In the analysis of the mean excess through the threshold (see Figure 15), the lower threshold is determined, which is equal to -1 . Analogously, with the help of Figure 15, the upper threshold is selected, which is equal to 1 . The estimations for the lower threshold are $\hat{\xi} = 0.185$ and $\hat{\beta}(\kappa) = 0.912$, while for the upper threshold they are $\hat{\xi} = 0.217$ and $\hat{\beta}(\kappa) = 1.087$. Note that the estimated values of the parameters ξ and $\beta(\kappa)$ are the same as the estimates in the analysis of excess on the previously realized threshold when

the threshold equal to minus one was estimated (see Figure 15).

The next analysis is very similar to the previous one: the difference lies in the presence of two tails instead of one. Figure 19 shows the $q-q$ plots of the excess on the specified threshold versus that of the quantiles of the GPD by employing the estimated shape parameters of the upper and lower tails. In this case, the lower tail (above graph) could start at minus one, and the upper tail (below graph) at one¹⁶. In both representations in Figure 19, it is seen that the point sets form a straight line up to a certain stage, so it is reasonable to assume that a GPD fits the data. Moreover, the two estimations for the shape parameter ξ are not the same based on the particular selections of the upper (0.217) and lower (0.185) thresholds. If the distribution is not symmetrical, there is no special reason for the two values of ξ to be the same; that is, there is no particular reason why, in general, the polynomial decay of the right and left tails must be identical.

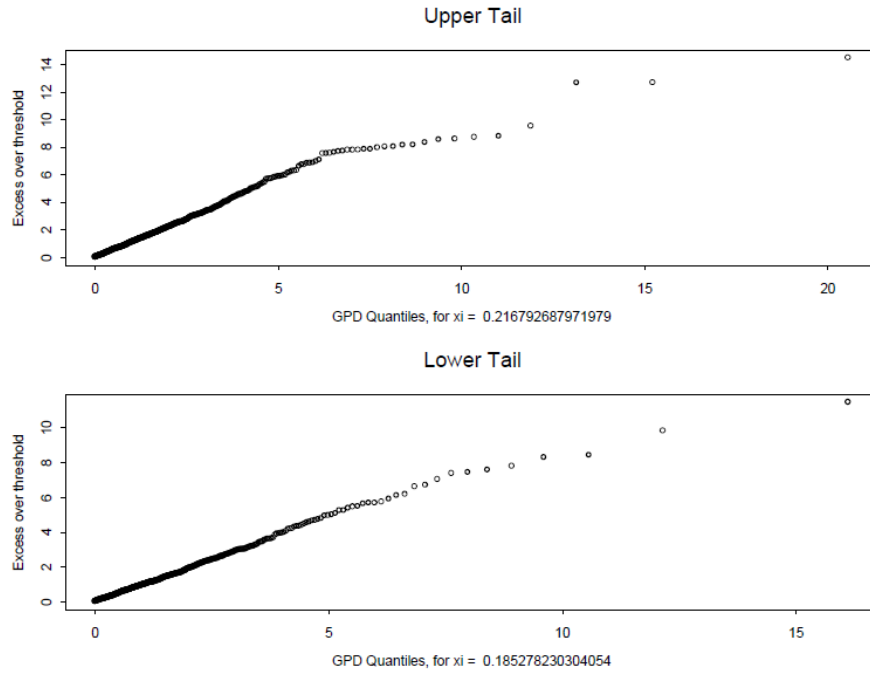


Figure 19: Estimated Tails when Distributions does not have Lower or Upper Limit.

At the beginning of this paper it is held that, for a better understanding of the risk, VaR and ES should be borne in mind to quantify the financial risks. The estimation of these risk measurements is performed for negative stock market returns for the quantiles $q = 0.95$ and $q = 0.99$, which are based on the GPD¹⁷. For the case of the GPD, it is inferred that with probability of 5%, $\widehat{VaR}_{0.95} = -2.146\%$ and, given that the return is less than -2.146% , then $\widehat{ES}_{0.95}$ is -3.562% . Analogously, with probability of 1%, $\widehat{VaR}_{0.99} = -4.309\%$ and $\widehat{ES}_{0.99} = -6.217\%$ given that the

¹⁶It should be recalled that the upper and lower thresholds do not necessarily have to be equal in absolute value, as they are in this case.

¹⁷Under the assumption of normally distributed returns, it is found that $VaR_{0.99} = \mu + \sigma \times q_{0.99}$ and $ES_{0.99} = \mu + \sigma \times \frac{\phi(z)}{1-\Phi(z)}$ for the case of the quantile 0.99.

return is less than -4.309%. Compared with the results obtained utilizing a normal distribution, $\widehat{VaR}_{0.95}$ is less than the estimation of the GPD. Nonetheless, $\widehat{VaR}_{0.99}$ is higher in the case of the GPD in comparison with the normal distribution. In the case of $\widehat{ES}_{0.95}$ and $\widehat{ES}_{0.99}$, both are higher using the GPD approximation. The difference at 99.0% is remarkable and significant (6.217% in the GPD compared with 4.375% for the normal distribution).

Once adjusted to a model of GPD for the excess of stock market returns above a threshold, we proceed to the estimation of valid asymptotic confidence intervals for VaR_q and ES_q .¹⁸ These intervals can be visualized in Figure 20 with the tail estimate $\widehat{F}(x) = 1 - \frac{k}{n} [1 + \widehat{\xi} \times \frac{x-\kappa}{\widehat{\beta}(\kappa)}]$. The confidence intervals for VaR are [2.062, 2.240] and [4.048, 4.643] for 95% and 99%, respectively. With respect to ES, the intervals are [3.358, 3.839] and [5.595, 7.156] for 95% and 99%, respectively.

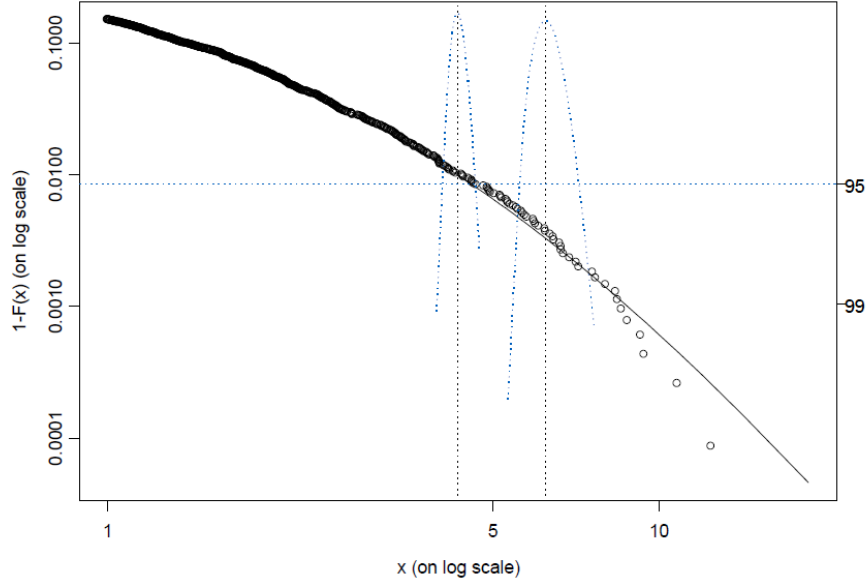


Figure 20: Asymptotic Confidence Intervals for $VaR_{0.99}$ and $ES_{0.99}$ based on the GPD Fit.

Figure 21 allows for an analysis of the sensitivity of VaR_q estimated in response to changes in the threshold κ . It is observed how the estimation by maximum likelihood of the parameter of form ξ varies with the threshold. In Figure 21, it is estimated that the behavior of the shape parameter is very stable and close to the estimated value of the Value at Risk (4.309) for threshold values less than four.

According to McNeil *et al.* (2005), the GPD method is not the only way to estimate the tails of a distribution as has been performed above. The other methodology for the selection of the threshold is based on the Hill estimator, estimating, in a non-parametric way, the Hill tail index $\alpha = 1/\xi$ and the quantile $x_{q,k}$ for the negative stock market returns. This estimator is often a good estimator of α , or its reciprocal ξ . In practice, the general strategy is to graph the Hill estimator for all possible values of k (numbers of excesses through the threshold). Practical experience suggests that the best options for k are relatively small—for example, between 10 and 50 of statistical

¹⁸ VaR_q and ES_q are based on the delta method of the likelihood log function profile.

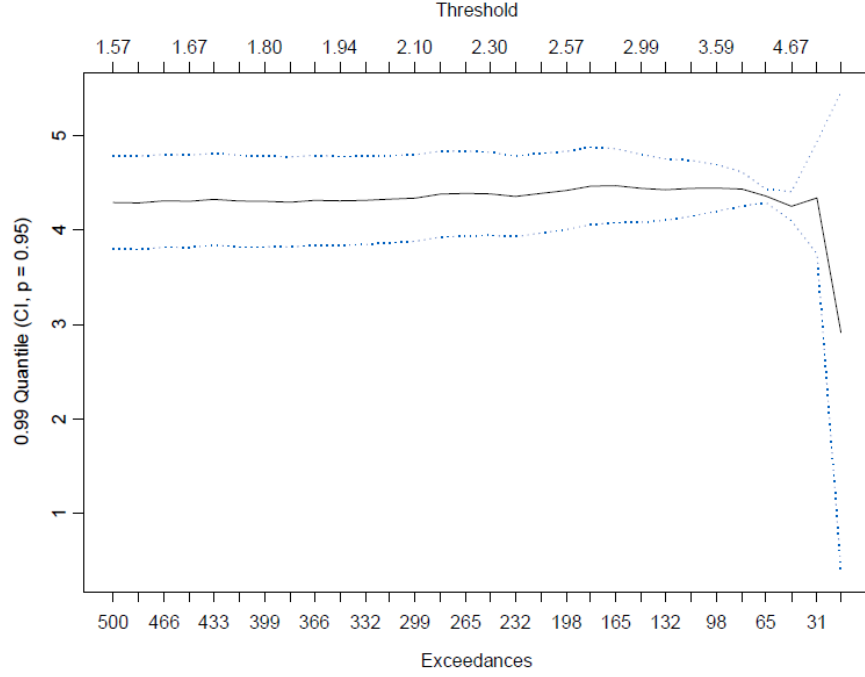


Figure 21: Estimation of $VaR_{0.99}$ as a function of the Threshold.

orders in a sample of size 1000. In Figure 22, the Hill estimator $\{k, \hat{\alpha}_{k,n}^{(H)} : k = 2, \dots, n\}$ is estimated for negative stock market returns of the shape parameter ξ . We expect to find a stable region for the Hill estimator where estimations are constructed based on the different numbers of statistical order. In Figure 22, the upper horizontal axis represents the threshold associated with the possible values of k ; in the lower horizontal axis, the number of observations included in the estimation is represented, and finally the confidence interval is observed at 95% (dotted lines). According to the results, it is observed that the estimation of the shape parameter does not stabilize as the statistical order increases hence, $\hat{\xi}^{Hill}(k)$ is quite unstable. It should be borne in mind that in practice, the ideal situation does not usually occur if the data does not come from a distribution with a tail that changes with regularity. If this occurs, the Hill method is not appropriate. The serial dependence on the data can also impair the performance of the estimator, although this can also be said of the estimator of the GPD.

4 Conclusions

Using daily observations of the index and stock market returns for the Peruvian case from January 3, 1990 to May 31, 2013, this paper models the distribution of daily loss probability, estimates maximum quantiles and tail probabilities of this distribution, and models the extremes through a maximum threshold. This is used to obtain the best measurements of VaR and ES at 95% and 99%.

One of the results on calculating the maximum annual block of the negative stock market returns

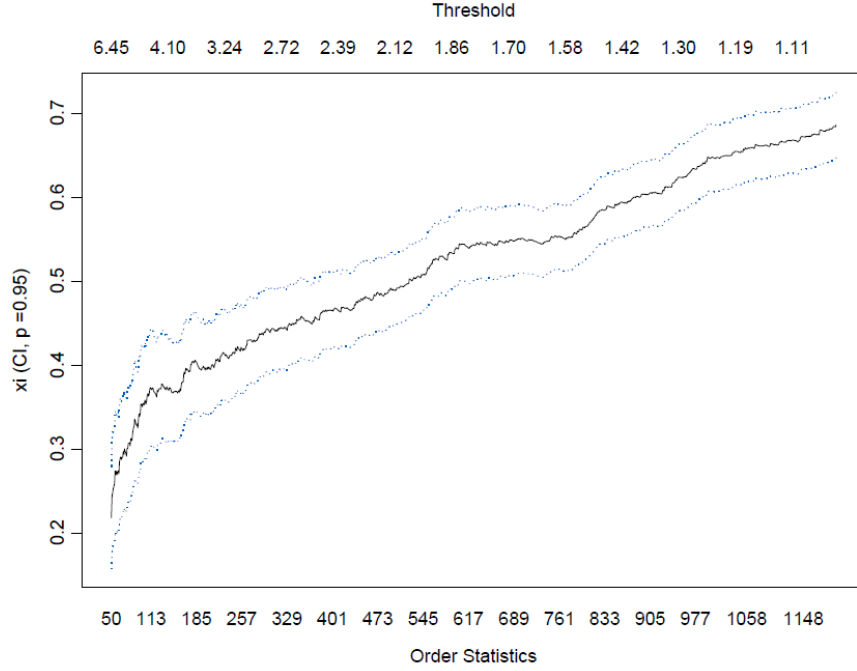


Figure 22: Estimates of the Hill ξ for the Daily Negative Returns.

is the observation of the fact that the largest negative stock market return (daily) is 12.44% in 2011. Moreover, if it is estimated that the probability of the maximum negative annual profitability for the following year exceeds all previous negative returns, turning out equal to 1.68, which means a probability of 1.68% of a negative maximum record of the negative yield being stabilized during the following year.

Then, by way of the estimator of maximum likelihood, the parameter of form and the asymptotic interval are estimated at 95% confidence level thereof for the annual and quarterly maximum block. The results indicate that the shape parameter is equal to -0.020 and 0.268, as well as the asymptotic interval $[-0.337, 0.2968]$ and $[-0.004, 0.532]$ for the maximum annual and quarterly block, respectively. The shape parameter estimation (-0.020) of the calculation of the maximum annual block of negative stock market returns is insignificant, due to which the value of this parameter is equal to zero and determines the tail behavior of the GEV distribution, and it is concluded that the non-degenerate distribution function is Gumbel-type. In the case of the estimation by maximum likelihood for the maximum quarterly block, a positive value was obtained for the shape parameter (0.268), with this being significant, indicating a thick-tailed distribution (Fréchet).

For the case of the GPD, it is inferred that with probability of 5%, the daily return would be as low as -2.146% and, given that the return is less than -2.146%, the average of the value of the return is -3.562%. Analogously, with probability of 1%, the daily returns could be as low as -4.309% with an average return of -6.217%, given that the return is less than -4.309%. Compared with the results obtained utilizing a normal distribution, the $\widehat{VaR}_{0.95}$ is smaller with the estimation of the GPD. Nonetheless, the $\widehat{VaR}_{0.99}$ is higher in the case of the GPD, in comparison with the normal distribution. In the case of $\widehat{ES}_{0.95}$ and $\widehat{ES}_{0.99}$, both are higher using the GPD approximation. The

difference in 99.0% is remarkable and significant (6.217% in the GPD, compared with 4.375% for the normal distribution).

Finally, the non-parametric estimation is performed for the Hill tail-index and the quantile for negative stock market returns, expecting to find a stable region for the Hill estimator. The results related to the estimation of the parameter do not stabilize as the statistical order increases, due to which the estimator of the Hill tail-index is quite unstable. This allows us to infer that the data do not come from a distribution with a tail that regularly changes, where the estimated values of the Hill parameter of form suggest a threshold close to one, according to their respective statistical order.

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