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Attention please!

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Attention Please!

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Attention Please!*

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Abstract

We study the impact of manipulating the attention of a decision-maker who learns sequentially about a number of items before making a choice. Under natural assumptions on the decision-maker's strategy, forcing attention toward one item increases its likelihood of being chosen.

keywords: sequential sampling, marketing, persuasion, attention allocation

JEL codes: D8, D91

1 Introduction

The struggle for attention is a pervasive phenomenon. Its importance has been documented in the context of advertising at least since Fogg-Meade (1901). More broadly, attention-seeking behavior plays an important role in marketing, finance, industrial organization, psychology, and biology.[1] The main message is consistent across fields: drawing attention toward an item increases its demand.

The existing literature provides two main explanations for how attention-grabbing advertising and marketing influence demand. One is that they directly affect preferences. While difficult to disprove, a theory of changing preferences offers limited predictive power and makes welfare analysis challenging. The other major explanation is that advertising conveys information—either directly or through signaling—and thereby changes beliefs. But this second channel alone does not

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[1]See, e.g., Lou (2014) for finance, Eliaz and Spiegler (2011) for industrial organization, Orquin and Loose (2013) for psychology, and Dukas (2002) for biology.

suffice to explain the empirical evidence. In fact, there is a sizable body of evidence showing that manipulating attention has a direct influence on demand even when devoid of information.[2]

We identify a mechanism through which grabbing attention increases demand without influencing preferences or conveying information. In our model, a decision-maker learns sequentially about the quality of a number of items by dynamically switching attention among them before making a choice. Paying attention to an item generates a noisy signal about its value. Due to cognitive limitations, the decision-maker can focus only on one item at a time; while she pays attention to a given item, her belief about its value evolves stochastically, while her beliefs about the other items remain the same. Starting from a given strategy governing the decision-maker's attention, we introduce an attention-grabbing manipulation that forces the decision-maker to focus on one "target" item for a fixed duration. We show that, under general conditions, such a manipulation increases demand for the target item.

For the most part, we focus on a setting with binary values, in which each item is either good or bad; we extend our results to allow for more than two values in Section 5.[3] The decision-maker can choose one of the items or an outside option of known value that she prefers to a bad item but not to a good one. We impose a simple stopping rule: the decision-maker ceases to learn about an item once she is sufficiently certain of its value. If she believes an item is sufficiently likely to be good, she stops learning and chooses that item. If she believes an item is sufficiently likely to be bad, she continues to learn about the other items until she finds one that is likely to be good or determines that all items are likely to be bad (in which case she chooses the outside option).

The decision-maker's learning is governed by an *attention strategy* that maps beliefs in any given time period to a (possibly random) item of focus. An attention strategy generates, for each profile of values of the items, a stochastic process over beliefs and items of attention, and a probability that each item is chosen. We refer to these probabilities as *interim demands* for the items. The impact of attention manipulation is captured by the difference in interim demands under the *baseline* and *manipulated* attention strategies, where the baseline strategy is the one the decision-maker employs in the absence of manipulation, while the manipulated strategy forces the decision-maker to focus on a *target* item for a fixed duration, after which she returns to her baseline strategy.

We show that manipulation of attention increases demand and decreases the time to decision in favor of the target item. Also, when the manipulated strategy leads to the choice of an item other than the target, it takes longer to do so than does the baseline strategy. These results hold for any realization of the items' values. In particular, forced attention increases demand even if the target item is bad.

The key to understanding the effect of manipulation is to consider the path of learning for each possible realization of the sequence of signals for each item. Given such a realization, we can view an attention strategy as selecting, in each period, an item for which to uncover one more step along the sequence. The choice of item at the end of the process can be thought of as resulting from a kind

[2]See, e.g., Chandon et al. (2009), Krajbich and Rangel (2011), or, for a survey, Orquin and Loose (2013).

[3]Binary-value models are common in the literature on sequential sampling; see, e.g., Wald (1945) or, for more recent work, Che and Mierendorff (2017) and Morris and Strack (2017).

of approval contest: the decision-maker continues to learn until she approves of one of the items, or until she finds all of them to be unworthy of approval. For a given realization of signals, there may be multiple items the decision-maker would approve of were she to pay enough attention to them. The choice then comes down to which of these items she approves of first. Forcing attention toward one of these items accelerates the process of approval for this item while slowing it down for the other items. Consequently, the likelihood that the target item is chosen increases.

This simple intuition ignores the significant complication that manipulating attention generally affects future attention choice. It could happen, then, that manipulation toward a target item leads to a path along which the decision-maker pays much less attention to the target item afterwards, more than compensating for the direct effect of increased attention. If there are only two items (not including the outside option), then this cannot happen: our results hold regardless of the baseline attention strategy. With more than two items, we require two additional assumptions. First, the attention strategy should be stationary: focus in each period must depend only on the current beliefs, not on the current time. The second assumption is a form of *independence of irrelevant alternatives* (IIA): conditional on not focusing on an item i, the probability of focusing on each other item is independent of the belief about the value of item i (though it may depend on the beliefs about items other than i). Together, these two assumptions allow us to consider learning about the target item separately from learning among the remaining items, effectively reducing the problem to one with two items.

To formalize these intuitions, we rely on a technique known in probability theory as *coupling*. In short, we construct a joint probability space in which we fix, for each item, the outcome of the learning process that would arise if the decision-maker focused only on that item. We refer to a profile of realizations of these learning processes across items as a *draw*. We show that, for *every* draw, forced attention toward an item both increases demand for that item and decreases decision time.

Both stationarity and IIA are needed for our results in the sense that their conclusions do not hold if we dispense with either assumption; we provide counterexamples in Section 4. Both assumptions, though restrictive, are automatically satisfied if the attention strategy is optimized given the stopping rules in our model: we prove that strategies that minimize a general class of expected attention costs have a (stationary) Gittins index structure as in the theory of multi-armed bandits. It follows that these strategies are stationary and satisfy IIA.

The presence of an outside option plays an important role in our analysis. Without it, the decision-maker could choose by a process of elimination rather than approval; that is, she could seek to eliminate items that she believes to be bad and ultimately choose an item—the last one remaining—with little knowledge of its value. In this case, manipulating attention toward an item may increase the chance that it is eliminated before the other items, thereby decreasing the demand for it.

When each item can take on more than two values, the presence of an outside option is no longer sufficient to generate our results. For example, with two items, it could be that the decision-maker

is confident both are better than the outside option, making the problem effectively the same as one with no outside option. However, our results go through as long as the decision-maker stops and chooses an item only when her belief about that item falls within a given set. We think of this set as consisting of those beliefs at which the decision-maker is sufficiently certain about the item's value. Requiring a degree of certainty about the chosen item is natural if its value affects subsequent decisions. In the binary values case, then, the key role played by the outside option is that it makes the decision-maker choose an item only when she is confident that it is good.

Our result is robust to many aspects of the learning process. The information structure for each item is general, allowing for any number of signal realizations and dependence on the current belief about the item. The decision-maker need not be Bayesian; we can, for instance, reinterpret her beliefs as intensities of accumulated neural stimuli in favor of each item, which can evolve according to an arbitrary stationary Markov process (not necessarily corresponding to Bayesian updating). We also allow for attention strategies that are not optimal, making our results robust with respect to the structure of the attention costs if these strategies were chosen optimally.

Related literature Evidence that increased attention boosts demand comes from several fields. In marketing, Chandon et al. (2009) show that drawing attention to products—for instance, with large displays or placement at eye level—increases demand. In finance, Seasholes and Wu (2007) show that attention-grabbing events about individual stocks increase demand for them. In biology, Yorzinski et al. (2013) study the display strategies through which peacocks grab and retain the attention of peahens during courtship.

When applied to advertising, our approach finds support in Fogg-Meade (1901) who notes that "successful advertisement is obtrusive. It continually forces itself upon the attention." Bagwell (2007) surveys the economics literature on advertising and divides it according to whether advertisements are treated as *persuasive* or *informative*. In the persuasive approach, advertising directly influences customers' preferences. In the informative approach, advertisements inform consumers about product availability, prices, or characteristics. Relative to these approaches, we show how advertising can boost demand with stable preferences even when advertisements convey no information.

Our assumptions on attention allocation are rooted in psychology. Though humans are able to pay attention to multiple stimuli simultaneously, such division of attention is difficult, especially when the stimuli are similar to each other (e.g., Spelke, Hirst, and Neisser, 1976). Psychologists distinguish between exogenous and endogenous attention, where the first is beyond the decision-maker's control and is triggered by sudden movements, bright colors and such, while endogenous attention shifts are controlled by the decision-maker (Mayer et al., 2004). We can interpret attention allocation during our manipulation window as being exogenous, whereas our baseline attention strategy fits the endogenous attention interpretation.

Our model builds on a long tradition in statistics and economic theory originating in Wald (1945), who proposed a theory of optimal sequential learning about a single binary state. A grow-

ing literature studies optimal sequential learning about several options when attention must focus on one item at a time (Mandelbaum, Shepp, and Vanderbei, 1990; Ke, Shen, and Villas-Boas, 2016; Ke and Villas-Boas, 2017; Nikandrova and Pancs, 2018; Austen-Smith and Martinelli, 2018). The structure of the optimal learning strategy varies depending on the costs and information structure. Our results on the impact of attention manipulation are independent of these considerations; however, relative to this literature, we make simplifying assumptions on the rules that govern termination of learning. In a different vein, Che and Mierendorff (2017) study sequential allocation of attention between two Poisson signals about a binary state. In contrast, in our model, the decision-maker chooses among signals about multiple independent states.

In the drift-diffusion model of Ratcliff (1978), a decision-maker accumulates an internal signal based on the difference in the values of two actions, making a choice when the signal becomes sufficiently strong.[4] Krajbich, Armel, and Rangel (2010) explicitly incorporate attention choice in this model, and introduce an exogenous bias in the accumulated signal toward the item on which the decision-maker is currently focusing. This extended drift-diffusion model accommodates empirical findings showing that exogenous shifts in attention tend to bias choice (see, e.g., Armel, Beaumel, and Rangel, 2008; Milosavljevic et al., 2012). Relative to this literature, whose primary modeling goal is to fit choice data, we focus on foundations for the mechanism by which attention affects demand.

Optimal sequential learning about several items is related to the theory of multi-armed bandits (Gittins and Jones, 1974). We exploit this connection to show that optimal attention strategies satisfy IIA by using the Gittins index characterization.

2 Main result in a simple setting

In this section, we show in the simplest possible setting that a temporary forced attention to an item increases the probability that the decision-maker (DM) chooses it.

The DM chooses one among two items $i \in \{1, 2\}$ of unknown values $v^i \in \{0, 1\}$ or an outside option with a known value $z \in (0, 1)$. The two values v^i are independent ex ante, and each is equal to 1 with prior probability p_0^i. The DM learns sequentially about each item, and can vary the focus of her learning as specified below. She chooses when to stop learning, at which point she selects an item or the outside option based on whichever one has the highest posterior expected value.

Let p_t^i denote the DM's belief about each item i at the beginning of the period t and write $\mathbf{p}_t$ for the pair of beliefs (p_t^1, p_t^2). At the beginning of each period $t = 0, 1, \ldots$, the DM chooses an item ι_t on which to focus in period t. She receives a signal that is informative about the value of item ι_t and independent of the value of the other item. In this section, the signal takes on values 0 and 1, and for each v, $\Pr(x_t = v^{\iota_t} \mid v^{\iota_t} = v) = \lambda$ and $\Pr(x_t = 1 - v^{\iota_t} \mid v^{\iota_t} = v) = 1 - \lambda$ for some $\lambda > 1/2$. Upon observing a signal realization, the DM updates her belief according to Bayes' rule.

[4] When the signal is interpreted as a belief and the stopping rule is optimized, Ratcliff's model is essentially equivalent to a sequential sampling model in the style of Wald (1945).

In particular, the DM's belief about the item she is focusing on changes while her belief about the other item remains fixed. Thus, letting

$$p[+] = \frac{\lambda p}{\lambda p + (1-\lambda)(1-p)} \tag{1}$$
$$\text{and} \quad p[-] = \frac{(1-\lambda)p}{(1-\lambda)p + \lambda(1-p)}$$

for each $p \in (0,1)$, we have $p^{\iota_t}_{t+1} = p^{\iota_t}_t[+]$ or $p^{\iota_t}_t[-]$ according to whether $x_t = 1$ or 0, and $p^i_{t+1} = p^i_t$ for item $i \neq \iota^t$. Attention allocation is governed by a *(pure) attention strategy* $\alpha : [0,1]^2 \longrightarrow \{1,2\}$ that specifies the item of focus $\iota^t = \alpha(\mathbf{p}_t)$ of a DM with beliefs $\mathbf{p}_t$.

The DM stops and makes a choice once she is sufficiently sure that either (i) one of the items is of high value or (ii) both items are of low value. Accordingly, we introduce thresholds $\underline{p}$ and $\overline{p}$ such that $\underline{p} < z, p^i_0 < \overline{p}$. We define stopping regions $F^i = \{\mathbf{p} : p^i \geq \overline{p}\}$ for $i = 1, 2$, $F^{oo} = \{\mathbf{p} : p^1, p^2 \leq \underline{p}\}$ and $F = F^1 \cup F^2 \cup F^{oo}$. Learning stops in the period $\tau = \min\{t : \mathbf{p}_t \in F\}$ with the DM choosing item i if $\mathbf{p}_\tau \in F^i$ and the outside option if $\mathbf{p}_\tau \in F^{oo}$. We let τ^i denote the period in which the DM chooses item i; that is, $\tau^i = \tau$ if item i is chosen and $\tau^i = \infty$ otherwise. Note that τ and τ^i depend on the attention strategy; accordingly, we sometimes write $\tau(\alpha)$ and $\tau^i(\alpha)$ if the attention strategy α is not otherwise clear from the context.

The above rules specify, for any given pair of values $\mathbf{v} = (v^1, v^2)$, the joint stochastic process of beliefs and focus of attention $(\mathbf{p}_t, \iota_t)_t$, with the joint law denoted by $P^{\mathbf{v}}_\alpha$. For any strategy α and pair of values $\mathbf{v}$, we let the *interim demand* for item i,

$$D^i(\mathbf{v}; \alpha) = P^{\mathbf{v}}_\alpha \left(\mathbf{p}_\tau \in F^i\right),$$

be the probability that the DM stops with the choice of i when the true values are $\mathbf{v}$. (Stopping in F^1 and F^2 are mutually exclusive.)

We are interested in how manipulation of the DM's attention strategy affects her choice. To this end, given a baseline strategy α, we introduce a *manipulated strategy* β constructed from α by forcing the DM to focus on item 1 in the initial period and then returning to α in all subsequent periods. That is, the item of focus $\beta(\mathbf{p}, t)$ in period t for beliefs $\mathbf{p}$ is given by

$$\beta(\mathbf{p}, t) = \begin{cases} 1 & \text{if } t = 0, \\ \alpha(\mathbf{p}) & \text{if } t > 0. \end{cases}$$

We say that an attention strategy α is *non-wasteful* if $\alpha(\mathbf{p}) \neq i$ for any $\mathbf{p}$ such that $p^i \leq \underline{p}$. Non-wasteful strategies do not focus on an item that the DM deems to have low value.

Proposition 1. *Suppose that the baseline attention strategy α is non-wasteful. Forced focus on item 1 in the first period*

1. *(weakly) increases the demand for item 1 and decreases the demand for item 2; that is,*

$$\begin{aligned} & D^1(\mathbf{v};\beta) \geq D^1(\mathbf{v};\alpha) \\ \textit{and} \quad & D^2(\mathbf{v};\beta) \leq D^2(\mathbf{v};\alpha) \end{aligned}$$

for all pairs of values $\mathbf{v} \in \{0,1\}^2$*; and*

2. *accelerates the choice of item 1 and decelerates the choice of item 2; that is,*

$$\begin{aligned} & P^{\mathbf{v}}_{\alpha}(\tau^1 \geq t) \geq P^{\mathbf{v}}_{\beta}(\tau^1 \geq t) \\ \textit{and} \quad & P^{\mathbf{v}}_{\alpha}(\tau^2 \geq t) \leq P^{\mathbf{v}}_{\beta}(\tau^2 \geq t) \end{aligned}$$

for all pairs of values $\mathbf{v} \in \{0,1\}^2$ *and all* $t \in \mathbb{N}$.

When an item has low value, the DM's belief about it tends to drift downward whenever she focuses on it. Yet, perhaps surprisingly, the proposition indicates that forced focus on an item boosts its demand even in this case.

The result is a special case of Theorem 1, and thus we provide only an informal proof here. Imagine that there is a large (countably infinite) deck of cards for each item, with each card showing a signal realization of 0 or 1. In each period t, the attention strategy chooses a deck $\alpha(\mathbf{p}_t)$ from which to draw the next card, and then the DM updates the relevant belief based on the signal shown on that card. Now consider the effect of manipulation on choice for a given ordering of each deck of cards, where manipulation forces the first card to come from the deck for item 1. To avoid trivialities, focus on the case in which, absent manipulation, the DM first draws from deck 2.

The DM chooses item 1 if her belief p^1_t reaches (at least) $\overline{p}$ before p^2_t does. Intuitively, forcing the DM to draw first from deck 1 should only cause p^1_t to reach $\overline{p}$ sooner. There is a complication, however, insofar as manipulation can cause the order of subsequent draws to change since the DM may reach pairs of beliefs that she would not have reached otherwise. The key observation is that, once we have fixed the ordering of the cards, we only need to keep track of how many cards the baseline and manipulated strategies have drawn from each deck. At the end of the first period, compared to the baseline strategy, the manipulated strategy is further ahead with deck 1 in the sense that more cards have been drawn from deck 1. Correspondingly, the baseline strategy is further ahead with deck 2. In each subsequent period, either the manipulated strategy remains ahead with deck 1 (perhaps pulling even further ahead) and the baseline strategy remains ahead with deck 2, or the numbers of draws from both decks under the baseline strategy "meet" the numbers under the manipulated strategy. In the latter case, the beliefs under the two processes coincide after the period in which they meet (since beliefs are independent of the order in which signals are received). Therefore, the manipulation has no effect on choice if the two processes meet. Accordingly, consider the case in which the two processes do not meet before one of them stops.

Suppose that the baseline strategy leads to the choice of item 1; that is, the belief about item 1 reaches $\overline{p}$ in some period τ before the belief about item 2 reaches $\overline{p}$. Since the manipulated strategy

is further ahead with deck 1 and behind with deck 2 in each period $t < \tau$, it must be that under the manipulated strategy, the belief about item 1 reaches $\overline{p}$ before the belief about item 2 does, and does so no later than period τ. Therefore, the manipulated strategy also leads to the choice of item 1, with this choice occurring no later than under the baseline strategy.

The argument for the statements about item 2 is symmetric. Suppose that the manipulated strategy leads to the choice of item 2; that is, under the manipulated strategy, the belief about item 2 reaches $\overline{p}$ in some period $\hat{\tau}$ before the belief about item 1 does. Since the baseline strategy is further ahead with deck 2 and behind with deck 1 in each period $t < \hat{\tau}$, under the baseline strategy, the belief about item 2 reaches $\overline{p}$ before the belief about item 1 does, and does so no later than period $\hat{\tau}$.

This argument shows that the statements in the proposition hold for each draw; it follows that they also hold when averaging across draws.

3 General result for binary values

We now extend the setting to allow for more than two items, general signal structures, stochastic attention strategies, and arbitrary length of the manipulation window. At the end of the learning process, the DM chooses one item from the set $\mathcal{I} = \{1, \ldots, I\}$ or an outside option with known value $z \in (0, 1)$. Each item $i \in \mathcal{I}$ has an uncertain value $v^i \in \{0, 1\}$, and again, $\mathbf{v} = (v^1, \ldots, v^I)$.

In each period $t = 0, 1, \ldots$, the DM focuses on a single item. Her belief in period t that item i is of high value ($v^i = 1$) is denoted p_t^i, and we write $\mathbf{p}_t = (p_t^i)_i$ for the *belief vector* in period t.[5] A *(stochastic) attention strategy* is a function $\alpha\colon (\Delta(\{0,1\}))^I \times \mathbb{N} \longrightarrow \Delta(\mathcal{I})$ that specifies a probability distribution over items of focus as a function of the belief vector at the beginning of a period together with the current time. We can think of $\alpha(\mathbf{p}, t)$ as a vector $(\alpha^i(\mathbf{p}, t))_i$, where $\alpha^i(\mathbf{p}, t)$ is the probability with which the DM focuses on item i in period t. An attention strategy is *stationary* if it does not depend on time (i.e., if $\alpha(\mathbf{p}, t) = \alpha(\mathbf{p}, t')$ for all $\mathbf{p}$, t, and t'), in which case we simply write $\alpha(\mathbf{p})$ for $\alpha(\mathbf{p}, t)$. If, for every $\mathbf{p}$ and t, an attention strategy $\alpha(\mathbf{p}, t)$ assigns probability one to a single item, then we say that it is a *pure strategy*, and abuse notation slightly by writing $\alpha(\mathbf{p}, t)$ for the item of focus.

The evolution of beliefs as a function of the attention strategy is as follows. The DM begins with a prior belief vector $\mathbf{p}_0 = (p_0^1, \ldots, p_0^I)$. If the beliefs at the beginning of period t are $\mathbf{p}_t$ and the DM focuses on item i in period t, then the belief about item i follows a stochastic transition $\phi^i : \Delta(\{0,1\}) \times \{0,1\} \longrightarrow \Delta\left(\Delta(\{0,1\})\right)$ that depends on p_t^i and on v^i, while the beliefs about all other items remain unchanged; that is, conditional on $\mathbf{p}_t$, $\mathbf{p}_{t+1}$ is a random variable such that p_{t+1}^i is distributed according to $\phi^i(p_t^i, v^i)$ and $p_{t+1}^j = p_t^j$ for all $j \neq i$. By fixing the DM's beliefs about items she is not currently focusing on, we are implicitly assuming that she treats the items' values as independent. One special case of this setting is when the DM receives a (serially conditionally independent) signal about the item she focuses on and updates her belief according to Bayes' rule.

[5] We identify the probability $p_t^i \in [0, 1]$ attached to $v^i = 1$ with the corresponding belief in $\Delta(\{0, 1\})$.

However, we also allow for non-Bayesian processes; for example, the "belief" p_t^i could alternatively represent the strength of a mental impulse towards choosing item i, or the DM could under- or over-react to new information relative to Bayes' law.

An attention strategy α naturally induces a process of items of focus $(\iota_t)_t$ and a process of belief vectors $(\mathbf{p}_t)_t$. The law of ι_t conditional on $(\mathbf{p}_0, \iota_0, \ldots, \mathbf{p}_{t-1}, \iota_{t-1}, \mathbf{p}_t)$ is $\alpha(\mathbf{p}_t, t)$, and the belief vector $\mathbf{p}_{t+1}$ is drawn as described above, conditional on $(\mathbf{p}_0, \iota_0, \ldots, \mathbf{p}_{t-1}, \iota_{t-1}, \mathbf{p}_t, \iota_t)$. Given any vector of values $\mathbf{v}$, we let $P_\alpha^{\mathbf{v}}$ denote the joint law of the process $(\mathbf{p}_t, \iota_t)_t$.

The time at which the DM stops learning is governed by thresholds $\underline{p}^i$ and $\overline{p}^i$ for each i satisfying $0 \leq \underline{p}^i < p_0^i, z < \overline{p}^i \leq 1$. If the DM's belief p_t^i satisfies $p_t^i \leq \underline{p}^i$ then we say that she is *sufficiently certain* that $v^i = 0$, and similarly, if $p_t^i \geq \overline{p}^i$ then she is sufficiently certain that $v^i = 1$. The DM learns until she is sufficiently certain of an optimal choice. We thus define the stopping region F according to $F = \left(\bigcup_i F^i\right) \cup F^{oo}$, where $F^i = \{\mathbf{p} : p^i \geq \overline{p}^i\}$ and $F^{oo} = \{\mathbf{p} : p^i \leq \underline{p}^i \text{ for all } i\}$. The DM makes her choice at the stopping time $\tau = \min\{t : \mathbf{p}_t \in F\}$: either $\mathbf{p}_\tau \in F^i$ for some i, in which case this i is chosen, or $\mathbf{p}_\tau \in F^{oo}$, in which case the outside option is chosen (these cases are mutually exclusive). For any item i, let the *stopping time τ^i for i* be equal to τ if i is chosen and ∞ otherwise. (We allow for the possibility that learning does not stop, in which case $\tau = \tau^i = \infty$ for all i.)

An attention strategy α is *non-wasteful* if $\alpha^i(\mathbf{p}, t) = 0$ for all $\mathbf{p}$ such that $p^i \leq \underline{p}^i$. A non-wasteful strategy never focuses on an item that the DM is sufficiently certain is of low value.

A special case of particular interest is when the thresholds are given by $\underline{p}^i = 0$ and $\overline{p}^i = 1$ for all i. These thresholds ensure that the DM learns until she is certain that she can make an optimal choice (at which point she stops immediately). One type of learning that eventually leads to certainty is that with Poisson information in which the DM is a Bayesian who receives a signal about the item she focuses on that perfectly reveals its value with positive probability. (This probability may depend on the item and its value.) Under such a learning process, thresholds of 0 and 1 are optimal for a DM who incurs a time cost of learning, but lexicographically prioritizes the value of her choice above this cost. More generally, however, we do not endogenize the stopping regions as resulting from optimization in a costly learning process; doing so would make for a very challenging problem involving tradeoffs that are orthogonal to the effect we identify. The key difference between our setting and one with endogenous boundaries is that we require the thresholds $\underline{p}^i$ and $\overline{p}^i$ to be independent of the DM's beliefs about other items $j \neq i$. When the thresholds are interior, we interpret them as capturing bounded rationality that is particularly natural when the cost of learning is low and the thresholds are close to 0 and 1 (in which case the additional gain from precisely tailoring the thresholds is small). We conjecture that, in this case, our results approximate those arising from optimal strategies in Bayesian models with a small cost of learning.

An attention strategy α satisfies *Independence of Irrelevant Alternative i* (IIAi) if, conditional on not focusing on item i, the probabilities of focusing on each item $j \neq i$ are independent of p^i. Formally, for every t and $\mathbf{p}, \mathbf{q} \in (\Delta(\{0,1\}))^I$ such that $p^j = q^j$ for all $j \neq i$, $\alpha^i(\mathbf{p}, t), \alpha^i(\mathbf{q}, t) \neq 1$

imply that, for every $j \neq i$,

$$\frac{\alpha^j(\mathbf{p},t)}{1-\alpha^i(\mathbf{p},t)} = \frac{\alpha^j(\mathbf{q},t)}{1-\alpha^i(\mathbf{q},t)}.$$

We say that α satisfies *Independence of Irrelevant Alternatives* (IIA) if it satisfies IIAi for all items i. Note that IIA is automatically satisfied if there are only two items.

We define the *interim demand for item i* as

$$D^i(\mathbf{v};\alpha) = P^{\mathbf{v}}_{\alpha}\left(\mathbf{p}_\tau \in F^i\right);$$

this is the probability, under strategy α, that the DM chooses item i when the vector of values is $\mathbf{v}$.

We compare the interim demand under a baseline attention strategy to that under a manipulated strategy capturing the effect of forced attention. Given a *baseline strategy* α, a *target item* i, and a *manipulation length* $m \geq 1$, the *manipulated attention strategy (in favor of i)* is

$$\beta^j[\alpha,i,m](\mathbf{p},t) = \begin{cases} 1_{j=i} & \text{if } t \leq m-1, \text{ and } p^i > \underline{p}^i, \\ \alpha(\mathbf{p},t) & \text{otherwise.} \end{cases}$$

Thus, under the manipulated attention strategy, the DM focuses on item i in the first m periods unless she is sufficiently certain that i is of low value, and then follows her baseline strategy in every subsequent period.[6]

The following proposition states that manipulation in favor of an item both increases and accelerates demand for this item, and decreases and decelerates demand for each other item. (The comparison of timing here is in the sense of first-order stochastic dominance.) The result holds regardless of the underlying values: even if the target item is worse than other items, drawing attention to it is never detrimental to the likelihood that it is chosen.

Theorem 1. *If an attention strategy α is stationary, satisfies IIAi, and is non-wasteful, then for every $\mathbf{v}$ and manipulation length $m \geq 1$,*

$$\begin{aligned} & D^i(\mathbf{v};\beta[\alpha,i,m]) \geq D^i(\mathbf{v};\alpha) \\ \textit{and} \quad & D^j(\mathbf{v};\beta[\alpha,i,m]) \leq D^j(\mathbf{v};\alpha) \quad \textit{for every } j \neq i. \end{aligned}$$

Moreover, for every $t \geq 0$,

$$\begin{aligned} & P^{\mathbf{v}}_{\alpha}(\tau^i \geq t) \geq P^{\mathbf{v}}_{\beta[\alpha,i,m]}(\tau^i \geq t) \\ \textit{and} \quad & P^{\mathbf{v}}_{\alpha}(\tau^j \geq t) \leq P^{\mathbf{v}}_{\beta[\alpha,i,m]}(\tau^j \geq t) \quad \textit{for every } j \neq i. \end{aligned}$$

[6] Since Theorem 1 holds for any manipulation length, m can be randomized without affecting the conclusion.

3.1 Proof of Theorem 1

The proof relies on a technique known as "coupling" (see, e.g., Lindvall, 1992): we fix the vector of values $\mathbf{v}$ and construct a common probability space on which we can compare the process of beliefs and items of focus $(\mathbf{p}_t, \iota_t)_t$ under the baseline attention strategy α with the process $(\hat{\mathbf{p}}_t, \hat{\iota}_t)_t$ under the manipulated strategy $\beta = \beta[\alpha, i, m]$. We construct this space in such a way that the law of $(\mathbf{p}_t, \iota_t)_t$ is $P^{\mathbf{v}}_{\alpha}$, while the law of $(\hat{\mathbf{p}}_t, \hat{\iota}_t)_t$ is $P^{\mathbf{v}}_{\beta}$.

We present here a coupling construction that suffices to prove the proposition for pure attention strategies. In Appendix A.1, we extend the construction to stochastic attention strategies, in which case the coupling argument is significantly more complex.

The probability space consists of realizations of a *learning process* $\pi = (\pi^j)_{j=1}^I$. The process π is a family of independent learning processes $\pi^j = (\pi^j_\kappa)_{\kappa=0,1,\ldots}$ for each item j, where π^j is a Markov process starting at p^j_0 with transitions $\phi^j(\cdot, v^j)$. The κth term π^j_κ of the learning process for item j specifies the belief about item j after κ periods of focus on that item. A *learning draw* is a realization of the learning process π.

We now construct, for each pure strategy $\gamma \in \{\alpha, \beta\}$, a realization of the process $(\mathbf{p}_t(\gamma), \iota_t(\gamma))_t$ as a function of the learning draw. Recursively define $(\mathbf{p}_t(\gamma), \iota_t(\gamma))_t$ as follows. For $t \geq 0$, and given the process $(\iota_s(\gamma))_{s<t}$, for each item j, let $k(j, t; \gamma)$ denote the *cumulative focus* of strategy γ on item j before period t; that is,

$$k(j, t; \gamma) = |\{s < t : \iota_s(\gamma) = j\}|.$$

Similarly, let $k(-i, t; \gamma) = t - k(i, t; \gamma)$ denote the number of periods of focus on items other than i. Set $p^j_t(\gamma) = \pi^j_{k(j,t;\gamma)}$ for every j; i.e., set the belief about each item j after $k(j, t; \gamma)$ periods of focus on this item to be the $k(j, t; \gamma)$-th value of the learning process π^j. Given γ, let the focus in period t be $\iota_t(\gamma) = \gamma(\mathbf{p}_t)$. By construction, the law of the process $(\mathbf{p}_t(\gamma), \iota_t(\gamma))_t$ is $P^{\mathbf{v}}_{\gamma}$, as needed.

For notational purposes, it is convenient to extend the process $(\mathbf{p}_t(\gamma), \iota_t(\gamma))_t$ beyond the stopping time τ. To this end, if $\mathbf{p}_t(\gamma) \in F$, then we set $\iota_t(\gamma) = \emptyset$ and $\mathbf{p}_{t+1}(\gamma) = \mathbf{p}_t(\gamma)$. From this point forward, we write $(\mathbf{p}_t, \iota_t)$ for $(\mathbf{p}_t(\alpha), \iota_t(\alpha))$, $(\hat{\mathbf{p}}_t, \hat{\iota}_t)$ for $(\mathbf{p}_t(\beta), \iota_t(\beta))$, and similarly for k, and τ. We fix a learning draw and compare the two corresponding realizations of the processes $(\mathbf{p}_t, \iota_t)_t$ and $(\hat{\mathbf{p}}_t, \hat{\iota}_t)_t$.

For any $n \geq 0$, let $\mathfrak{t}(n)$ be the n-th period such that $\iota_t \neq i$; that is, $\mathfrak{t}(n) = \min\{t : k(-i, t) \geq n\}$, where potentially $\mathfrak{t}(n) = \infty$, and similarly, $\hat{\mathfrak{t}}(n) = \min\{t : \hat{k}(-i, t) \geq n\}$.

Lemma 1 (Coupling Lemma)**.** *The baseline and manipulated processes coincide when restricted to periods of focus on items other than the target i. That is, for every n,*

$$\left(\mathbf{p}^{-i}_{\mathfrak{t}(0)}, \iota_{\mathfrak{t}(0)}, \ldots, \mathbf{p}^{-i}_{\mathfrak{t}(n)}, \iota_{\mathfrak{t}(n)}\right) = \left(\hat{\mathbf{p}}^{-i}_{\hat{\mathfrak{t}}(0)}, \hat{\iota}_{\hat{\mathfrak{t}}(0)}, \ldots, \hat{\mathbf{p}}^{-i}_{\hat{\mathfrak{t}}(n)}, \hat{\iota}_{\hat{\mathfrak{t}}(n)}\right).$$

Proofs omitted in the main text can be found in the appendix.

For $t \geq m$, we say that the baseline and the manipulated processes *meet (in period t)* if

the cumulative focus on the target item i up to time t is the same for both processes, i.e. if $k(i,t) = \hat{k}(i,t)$. Note that for the two processes to meet, the definition considers only the cumulative focus on the *target* item. In general, this does not imply that the cumulative focus on any other item coincides under the two processes at time t (see counterexamples 4.3 and 4.4). The previous result shows that when stationarity and IIAi hold, meeting of the two processes implies that the cumulative focus is the same for *every* item. According to the next result, the processes then coincide in every subsequent period.

Lemma 2 (Meeting Lemma)**.** *If the baseline and manipulated processes meet in period t, then $(\mathbf{p}_s, \iota_s) = (\hat{\mathbf{p}}_s, \hat{\iota}_s)$ for all $s \geq t$.*

The Meeting Lemma implies the next result.

Lemma 3 (Attention Lemma)**.** *In every period, the cumulative focus on the target item i is at least as large under the manipulated process as under the baseline process, and the cumulative focus on any $j \neq i$ is at least as large under the baseline process as under the manipulated process. That is, for every $t \geq 1$ and $j \neq i$,*

$$\begin{aligned} &\hat{k}(i,t) \geq k(i,t) \\ \textit{and} \quad &k(j,t) \geq \hat{k}(j,t). \end{aligned}$$

For any attention strategy γ, let $\tau^{oo}(\gamma) = \tau(\gamma)$ if $\mathbf{p}_{\tau(\gamma)} \in F^{oo}$ (and thus the outside option is chosen) and $\tau^{oo}(\gamma) = \infty$ (indicating that the outside option is not chosen) otherwise.

Lemma 4 (Outside Option Lemma)**.** *For any two non-wasteful attention strategies γ and γ' and any learning draw, $\tau^{oo}(\gamma) = \tau^{oo}(\gamma')$.*

The Outside Option Lemma implies in particular that the outside option is chosen under the baseline process if and only if it is chosen under the manipulated process. Thus attention manipulation merely shifts demand within $\mathcal{I}$ and does not affect the total demand across all of the items.

Lemma 5 (Choice Lemma)**.** *For any learning draw,*

1. *if the target item i is chosen under the baseline process, then i is also chosen under the manipulated process, and no later than under the baseline process; and*

2. *if some item $j \neq i$ is chosen under the manipulated process, then j is also chosen under the baseline process, and no later than under the manipulated process.*

Proof. Statement 1: Consider any learning draw such that the target item i is chosen in period τ under the baseline process. Then, by the Outside Option Lemma, the outside option is not chosen under the manipulated process since it is not chosen under the baseline process. Suppose for contradiction that $j \neq i$ is chosen at some time $\hat{\tau} \leq \tau$ under the manipulated process. Then

$\hat{p}^j_{\hat{\tau}} \geq \overline{p}^j$. By the Attention Lemma, the cumulative focus on items $-i$ by period $\hat{\tau}$ under the baseline process is at least as large as that under the manipulated process; that is, $k(-i, \hat{\tau}) \geq \hat{k}(-i, \hat{\tau})$. By the Coupling Lemma, there exists a period $t \leq \hat{\tau}$ such that $\mathbf{p}^{-i}_t = \hat{\mathbf{p}}^{-i}_{\hat{\tau}}$, and hence the baseline process stops with the choice of j in period t, which establishes the contradiction since stopping in F^i and F^j are mutually exclusive for $j \neq i$. Therefore, it cannot be that, under the manipulated process, an item $j \neq i$ is chosen at a time $\hat{\tau} \leq \tau$. By the Attention Lemma, $\hat{k}(i, t) \geq k(i, t)$ for all t. Hence, there exists $\hat{\tau} \leq \tau$ such that $\hat{p}^i_{\hat{\tau}} = p^i_\tau \geq \overline{p}^i$ (since the manipulated process does not stop with the choice of $j \neq i$ or the outside option before $\hat{\tau}$). Thus, the manipulated process stops at time $\hat{\tau} \leq \tau$ with the choice of i, as needed.

Statement 2: The proof of the second statement is symmetric to that of the first. Accordingly, consider any draw such that, under the manipulated process, an item $j \neq i$ is chosen in period $\hat{\tau}$. Then, by the Outside Option Lemma, the baseline process does not choose the outside option. Suppose for contradiction that, under the baseline process, the target item i is chosen in some period $\tau \leq \hat{\tau}$. Then $p^i_\tau \geq \overline{p}^i$. By the Attention Lemma, $\hat{k}(i, \tau) \geq k(i, \tau)$. Thus, there exists a period $t \leq \tau$ such $\hat{p}^i_t = p^i_\tau \geq \overline{p}^i$ and hence under the manipulated process, i is chosen in period $t \leq \hat{\tau}$, which establishes the contradiction. Therefore, it cannot be that, under the baseline process, i is chosen at a time $\tau \leq \hat{\tau}$. By the Attention Lemma, $k(-i, t) \geq \hat{k}(-i, t)$ for all t. By the Coupling Lemma, the beliefs $\mathbf{p}^{-i}$ and $\hat{\mathbf{p}}^{-i}$ coincide when restricted to the periods of focus on items $-i$. Hence, there exists $\tau \leq \hat{\tau}$ such that $\mathbf{p}^{-i}_\tau = \hat{\mathbf{p}}^{-i}_{\hat{\tau}}$, and the baseline process stops in period τ with the choice of item j, as needed. □

Theorem 1 follows from the Choice Lemma by taking expectations across learning draws.

3.2 IIA and stationarity

We now provide an argument in support of the IIA and stationarity assumptions based on optimization of the attention strategy. We fix, for each item i, the belief-updating process ϕ^i and the stopping thresholds $\underline{p}^i$ and $\overline{p}^i$, and let the DM control her attention strategy α. We assume in this section that the beliefs follow a Markov process (unconditional on $\mathbf{v}$), as is the case if they are obtained through Bayesian updating based on observed signals. Until she stops learning, the DM pays a flow cost $0 \leq c\left(p^{\iota_t}_t, \iota_t\right) \leq \overline{c}$ in each period t, where this flow cost may depend on the item ι_t of current focus and on the belief $p^{\iota_t}_t$ in the current period; $\overline{c}$ is a finite upper bound on the flow cost. The DM chooses a strategy α to minimize the expected discounted flow cost

$$C(\alpha) = \mathrm{E} \sum_{t=0}^{\tau} \delta^t c\left(p^{\iota_t}_t, \iota_t\right), \tag{2}$$

where $\delta \in (0, 1)$ is a discount factor. Note that such a cost-minimizing strategy is necessarily non-wasteful.

We rely here on the theory of multi-armed bandits to show that a Gittins index strategy is optimal: for each item i, there exists a Gittins index function $G^i(p^i_t)$ that depends only on the

belief about item i, such that the optimal strategy consists in each period of focusing on an item with the highest Gittins index. When ties are broken with uniform randomization, such a strategy satisfies IIA and stationarity.

Proposition 2. *There exists a non-wasteful strategy that minimizes the objective* (2) *and satisfies IIA and stationarity.*

The main challenge in proving this result is that the cost in (2) exhibits interdependence across items because whether the decision process stops with the DM choosing the outside option depends on the whole profile of beliefs. Since the theory of multi-armed bandits applies to problems with flow payoffs that are independent across objects, we need to construct an auxiliary multi-armed bandit problem with this property. The construction is based on Lemma 4 (the Outside Option Lemma), which states that, conditional on the outside option being chosen, the stopping time is independent of the attention strategy. This construction applies because the stopping thresholds are independent of the DM's beliefs about the other items; if both the attention strategy and the stopping region are chosen jointly to maximize the expected value of the choice less the cost of learning as in Nikandrova and Pancs (2018) and Ke and Villas-Boas (2017), then there need not exist an optimal Gittins index strategy.

4 Examples and counterexamples

4.1 Example: the fastest strategy

To illustrate the quantitative impact of manipulating attention, we return to the example from Section 2 and examine the fastest attention strategy (for the given stopping thresholds). This strategy has a simple form that allows for analytical computation. Whenever manipulation changes the DM's focus in the first period, we show that it has a nonzero impact on demand.

Recall from Section 2 that if the DM has beliefs $\mathbf{p}_t$ and focuses on item $j \in \{1, 2\}$ in period t, she updates her belief about j to $p_t^j[-]$ with probability $(1-\lambda)v^j + \lambda(1-v^j)$ and to $p_t^j[+]$ with probability $\lambda v^j + (1-\lambda)(1-v^j)$, where $p[-]$ and $p[+]$ are specified in (1). For simplicity, assume that the stopping thresholds $\overline{p}$ and $\underline{p}$ are the same for the two items, and that each of these thresholds can be reached exactly through some sequence of signals.[7] Thus, the set of attainable beliefs takes the form $\{\underline{p}, \underline{p}[+], \ldots, \overline{p}[-], \overline{p}\}$, which is the same for both items.

Recall that $P_\alpha^{\mathbf{v}}$ denotes the interim law of the learning process for given values $\mathbf{v}$ and strategy α. Let $P_\alpha^{\text{ea}} = \mathrm{E}\, P_\alpha^{\mathbf{v}}$ be the ex ante law, where the expectation is with respect to $\mathbf{v}$, distributed according to the prior belief vector $\mathbf{p}_0$; that is, $P_\alpha^{\text{ea}} = \sum_{\mathbf{v}} \left(\prod_j p_0^j(v^j)\right) P_\alpha^{\mathbf{v}}$. The stopping time $\tau^{\text{ea}}(\alpha)$ for strategy α is the minimal time t at which $\mathbf{p}_t(\alpha) \in F$ under the law P_α^{ea}.

[7] On its own, the latter assumption is without loss of generality since moving a threshold within a region between attainable beliefs has no effect. When combined with the commonality of the thresholds, this assumption places a restriction on the prior belief vector.

The strategy α^* depicted in Figure 1A focuses on whichever item the DM views as more promising. Accordingly, for each item j and $i \neq j$,

$$\alpha^{*j}(\mathbf{p}) = \begin{cases} 1 & \text{if } p^j > p^i, \\ 1/2 & \text{if } p^j = p^i, \\ 0 & \text{otherwise.} \end{cases} \tag{3}$$

The next result states that α^* is the fastest attention strategy in this environment. Hence α^* is optimal for a DM who, given F, minimizes a monotone time cost.

Proposition 3. *For any strategy α, $\tau^{\text{ea}}(\alpha)$ weakly first-order stochastically dominates $\tau^{\text{ea}}(\alpha^*)$.*

As with our main results, the proof makes use of coupling, although the particular construction is distinct from our main one. A different coupling construction is necessary because the strategy α^* is not the fastest one in every learning draw: there exist draws in which focusing on the more promising item leads to a long sequence of contradictory signals.

The effect of manipulation identified in Theorem 1 is strict when the DM employs α^* provided the manipulation is nontrivial. Consider a manipulated strategy $\beta = \beta[\alpha^*, i, 1]$ that targets item i for one period. Assume that $p_0^j \geq p_0^i$ for $j \neq i$ (otherwise, α^* focuses on i initially and the baseline and manipulated processes trivially coincide). This manipulation generates a strict increase in the demand for the target item; that is,

$$D^i(\mathbf{v}; \beta) > D^i(\mathbf{v}; \alpha^*)$$

for all $\mathbf{v}$. To see this, recall from the Choice Lemma that in any draw, if the target item is chosen under the baseline process, then it is also chosen under the manipulated one. Hence, the impact of manipulation is strict if there exists a positive measure of draws in which β chooses i and α^* does not. We construct such draws as follows.

Let the learning draw π_κ^i for the target item i be an increasing sequence until it reaches $\overline{p}$. Thus, i is chosen by either strategy $\gamma \in \{\alpha^*, \beta\}$ if $p_t^i(\gamma) > p_t^j(\gamma)$ in some period $t \geq 1$, since the DM focuses on i thereafter until she chooses it. For item $j \neq i$, recalling that $p_0^j \geq p_0^i$, let π_κ^j decrease until it reaches p_0^i, and increase thereafter until it reaches the threshold $\overline{p}$. The decision process stops in finite time in this learning draw. Consider draws such that, when the tie between the two beliefs occurs, the DM focuses on j. Such draws have a nonzero probability. By construction, in these draws, the strategy α^* leads to the choice of item j, while β eventually switches focus from j to i and then continues to learn about i until it is chosen.

How much does the manipulation of attention affect demand? To quantify the effect, define the *ex ante demand* by

$$D_{\text{ea}}^i(\mathbf{p}_0; \alpha) = P_\alpha^{\text{ea}}\left(\mathbf{p}_\tau \in F^i\right) = \mathrm{E}_{\mathbf{p}_0} D^i(\mathbf{v}; \alpha). \tag{4}$$

Thus $D_{\text{ea}}^i(\mathbf{p}_0; \alpha)$ is the (ex ante) probability that a DM with prior beliefs $\mathbf{p}_0$ chooses item i. The

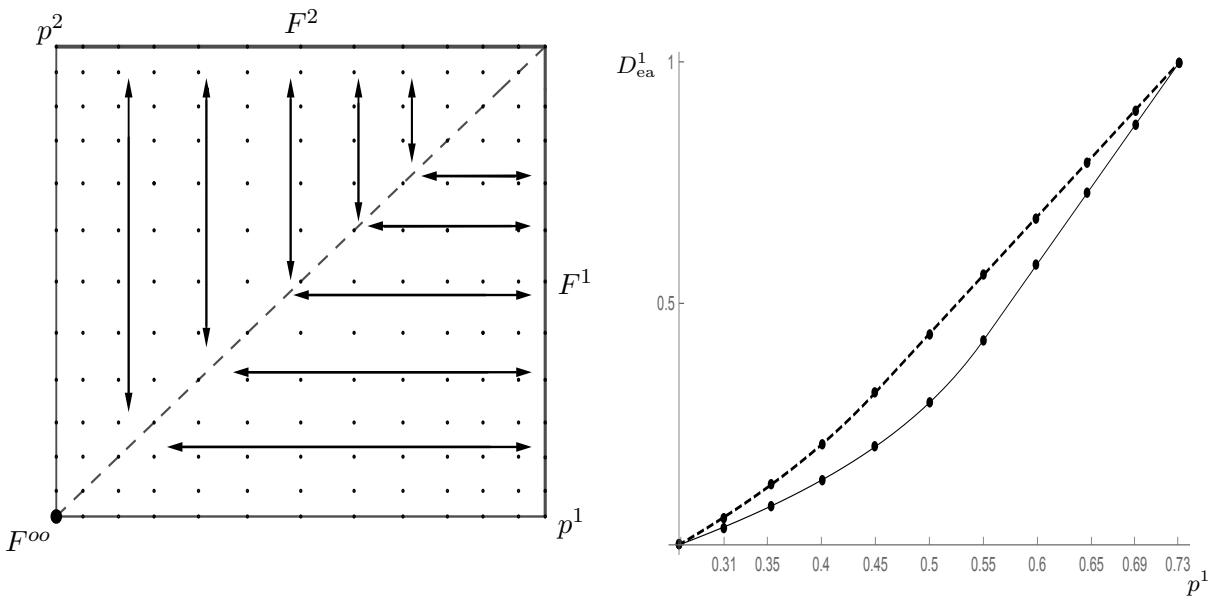

Figure 1: A: attention strategy α^*. B: ex ante demand $D^1_{\text{ea}}(p^1, p^2; \alpha^*)$ for item 1 as a function of p^1. The belief p^2 is fixed at .55 (solid curve), or .45 (dashed curve), respectively. The stopping thresholds are $\underline{p} = .27$ and $\overline{p} = .73$. The values depicted on the horizontal axis are attainable transient beliefs for the given information structure.

change in the ex ante demand for the target item i resulting from a single-period manipulation is

$$\mathrm{E}\, D^i_{\text{ea}}\left(\tilde{p}^i, p^{-i}; \alpha\right) - D^i_{\text{ea}}(p^i, p^{-i}; \alpha),$$

where $\tilde{p}^i$ is the belief resulting from a one-period update of p^i, and the expectation is over the possible values $p^i[+]$ and $p^i[-]$ of $\tilde{p}^i$. The magnitude of the effect is therefore determined by the curvature of $D^i_{\text{ea}}(p^{-i}, p^i; \alpha)$ with respect to p^i around $\mathbf{p}$.

Appendix A.4 provides an explicit expression for $D^i_{\text{ea}}(\mathbf{p}; \alpha^*)$ under the strategy α^*. Figure 1B depicts the ex ante demand as a function of p^1 for two values of the belief p^2. The curvature of the demand function, and hence the strength of the manipulation effect, is large when both beliefs are high (and when $p^1 \leq p^2$, which is necessary for the two processes to differ). This is because the DM's choice can be manipulated only in learning draws $(\pi^i_\kappa)^i_\kappa$ in which both π^1_κ and π^2_κ reach the high threshold $\overline{p}$ before they reach $\underline{p}$, since the choices under the two strategies coincide in all other draws. Thus the manipulation is more likely to have an effect when p^1 and p^2 are high.

4.2 Counterexample: no outside option

To illustrate the role of the outside option in our main result, we now consider a variant of the example from Section 2 in which there is no outside option. The main result does not go through: manipulating attention toward an item may decrease its demand even if the attention strategy is

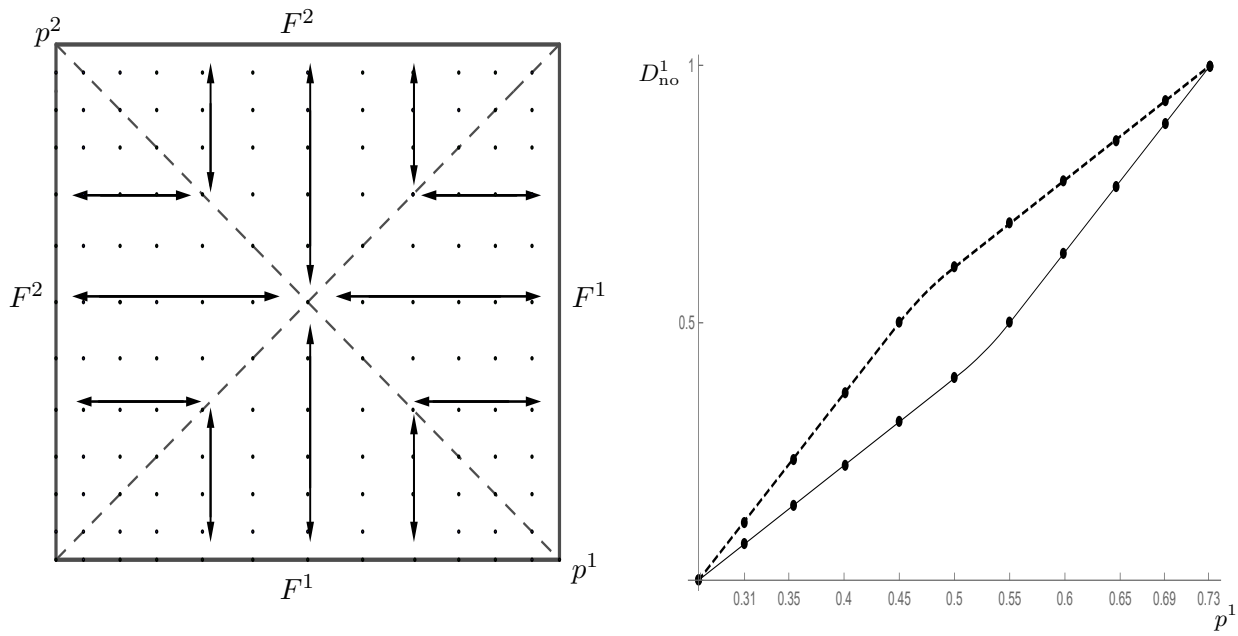

Figure 2: A: attention strategy α^{**}. B: demand $D^1_{\text{no}}(p^1, p^2; \alpha^{**})$ for item 1 as a function of p^1. The belief p^2 is fixed at .55 (solid curve), or .45 (dashed curve), respectively. The stopping thresholds are $\underline{p} = .27$ and $\overline{p} = .73$. The values depicted on the horizontal axis are attainable transient beliefs for the given information structure.

optimized.

The DM must choose one of two items $i \in \{1, 2\}$ with values $v^i \in \{0, 1\}$. She stops learning as soon as she is sufficiently certain about the value of one of the items: if she is sufficiently certain that an item has high value then she chooses it, and if she is sufficiently certain that an item has low value then she chooses the other item. Let F^i_{no} be the set of beliefs at which the DM stops learning and chooses i, that is,

$$F^i_{\text{no}} = \left\{\mathbf{p} : p^i \geq \overline{p}^i \text{ or } p^{-i} \leq \underline{p}^{-i}\right\},$$

and let $F_{\text{no}} = F^1_{\text{no}} \cup F^2_{\text{no}}$. Define the stopping time under the law P^{ea}_{α} to be $\tau^{\text{ea}}_{\text{no}}(\alpha) = \min\{t : \mathbf{p}_t \in F_{\text{no}}\}$; to simplify notation, we write τ in place of $\tau^{\text{ea}}_{\text{no}}(\alpha)$ when the meaning is clear from the context. Relative to the setting with an outside option, the current stopping rule differs in that the DM stops as soon as she is sufficiently certain that she has identified the better item, even if she has little certainty about whether that item has high value.

For simplicity, assume that (i) the threshold beliefs are the same for the two items, (ii) $\overline{p} = 1 - \underline{p}$, and (iii) each of these thresholds can be reached exactly through some sequence of signals. Thus, the set of attainable beliefs for each item takes the form $\{\underline{p}, \underline{p}[+], \ldots, \overline{p}[-], \overline{p}\}$ and is symmetric around 1/2.

Let $\alpha^{**j}(\mathbf{p})$ be the strategy that focuses on the item about which the DM is more certain; that is, for $j \neq i$, let

$$\alpha^{**j}(\mathbf{p}) = \begin{cases} 1 & \text{if } |p^j - 1/2| > |p^i - 1/2|, \\ 1/2 & \text{if } |p^j - 1/2| = |p^i - 1/2|, \\ 0 & \text{otherwise.} \end{cases} \tag{5}$$

Figure 2A depicts this strategy, which, as the following result indicates, is the fastest attention strategy in this setting.

Proposition 4. *For any strategy α, $\tau_{\text{no}}^{\text{ea}}(\alpha)$ weakly first-order stochastically dominates $\tau_{\text{no}}^{\text{ea}}(\alpha^{**})$.*

For any strategy α, let $D_{\text{no}}^i(\mathbf{p}_0; \alpha)$ be the ex ante demand for item i; that is,

$$D_{\text{no}}^i(\mathbf{p}_0; \alpha) = P_\alpha^{\text{ea}}\left(\mathbf{p}_\tau \in F_{\text{no}}^i\right).$$

We analytically characterize $D_{\text{no}}^i(\mathbf{p}; \alpha^{**})$ in Appendix A.5 and plot it in Figure 2B as a function of p^1. When p^2 is less than 1/2, the demand is concave in p^1, which indicates that manipulating attention toward item 1 decreases its demand. To understand why, consider a learning draw in which both π_κ^1 and π_κ^2 reach $\underline{p}$ before $\overline{p}$. In any such draw, the DM eliminates the first item for which her belief reaches $\underline{p}$, and thus manipulating attention toward an item increases its chance of being eliminated.[8]

The reversal of our result in this example arises because of the change in the stopping region, not because of the strategy α^{**}. Since Theorem 1 applies to *any* attention strategy, if the DM employs α^{**} in the setting with an outside option, manipulation (weakly) increases the demand for the target item.

The presence of an outside option affects demand in two different ways: (i) directly, by allowing the DM not to choose either item, and (ii) indirectly, by affecting the stopping region. It turns out that (i) is irrelevant for the direction of the manipulation effect; the difference between the two settings is driven by (ii).

To disentangle the two channels, consider an alternative model in which there is no outside option, and the stopping region is that from our main setting. Thus learning stops with the choice of item i whenever $\mathbf{p}_\tau \in F^i = \{\mathbf{p} : p^i \geq \overline{p}\}$ for some i, and with an equal probability of choosing either item whenever $\mathbf{p}_\tau \in F^{oo} = \{\mathbf{p} : p^1, p^2 \leq \underline{p}\}$. The (ex ante) demand for item i is therefore

$$\begin{aligned} D_{\text{alt}}^i(\mathbf{p}_0; \alpha) &= P_\alpha^{\text{ea}}\left(\mathbf{p}_\tau \in F^i\right) + \frac{1}{2} P_\alpha^{\text{ea}}\left(\mathbf{p}_\tau \in F^{oo}\right) \\ &= D_{\text{ea}}^i(\mathbf{p}_0; \alpha) + \frac{1}{2} P_\alpha^{\text{ea}}\left(\mathbf{p}_\tau \in F^{oo}\right) \\ &= D_{\text{ea}}^i(\mathbf{p}_0; \alpha) + \frac{1}{2} \frac{\overline{p} - p_0^1}{\overline{p} - \underline{p}} \frac{\overline{p} - p_0^2}{\overline{p} - \underline{p}}, \end{aligned} \tag{6}$$

[8] When p_2 is greater than 1/2, the demand in Figure 2B is convex in p^1, and thus manipulating attention toward item 1 boosts its demand. This effect arises due to draws in which both π_κ^1 and π_κ^2 reach $\overline{p}$ before $\underline{p}$.

where $D^i_{\text{ea}}(\mathbf{p};\alpha)$ is the demand defined in (4) from the example with an outside option. The second summand in (6) is linear in p^i_0 for each i. Thus the impact of a single-period manipulation toward target i is

$$\mathrm{E}\, D^i_{\text{alt}}\left(\tilde{p}^i, p^{-i};\alpha\right) - D^i_{\text{alt}}(\mathbf{p};\alpha) = \mathrm{E}\, D^i_{\text{ea}}\left(\tilde{p}^i, p^{-i};\alpha\right) - D^i_{\text{ea}}(\mathbf{p};\alpha),$$

where $\tilde{p}^i$ is the belief resulting from a one-period update of p^i, and the expectation is over the possible values $p^i[+]$ and $p^i[-]$ of $\tilde{p}^i$. Therefore, the impact of manipulation in this alternative model is the same as in our main model with an outside option; the presence of the outside option affects the impact of manipulation only through the stopping region.[9]

4.3 Counterexample: failure of IIA

The result in Theorem 1 does not generally hold if the attention strategy does not satisfy IIA. To illustrate, consider an example with three items and a Bayesian DM with prior belief $p^i_0 = 1/2$ for each item i. If the DM's belief about an item i is p^i_t in period t, focusing on i in t leads to belief $p^i_{t+1} = p^i_t[-]$ or $p^i_t[+]$ as in Section 2. The stopping boundaries are $\underline{p} = \frac{1}{2}[-][-]$ and $\overline{p} = \frac{1}{2}[+][+]$ for each item. Let α be a stationary pure attention strategy satisfying $\alpha(\mathbf{p}_0) = 2$, $\alpha(\mathbf{p}) = 1$ when $p^2 \neq p^2_0$ and $p^1 > \frac{1}{2}[-][-]$, and $\alpha(p^1_0[-], p^2_0, p^3) = \alpha(p^1_0[+], p^2_0, p^3) = 3$ for all $p^3 > \frac{1}{2}[-][-]$. When following such a strategy, the DM first focuses on item 2, and then, from the second period onwards, focuses on item 1 until p^1 reaches $\underline{p}$ or $\overline{p}$. This leads to item 1 being chosen with probability 1/2.

Now consider the manipulated strategy $\beta = \beta[\alpha, 1, 1]$ obtained by forcing the DM to focus on item 1 in the first period. From the second period onwards, the DM focuses on item 3 until p^3 reaches $\underline{p}$ or $\overline{p}$, and thus β chooses item 1 only if p^3 reaches $\underline{p}$ *and* p^1 reaches $\overline{p}$. Therefore, the manipulated DM chooses the target item 1 with probability at most 1/4.

The strategy α violates the IIAi assumption for $i = 1$ since the allocation of attention between items 2 and 3 at $\mathbf{p}^{-1} = (p^2_0, p^3_0)$ depends on the belief about item 1. A failure of IIA can cause the Meeting Lemma to be violated: the baseline and manipulated processes may meet but not coincide thereafter. Recall that two processes meet at some time if, by that time, they focus on the target item for the same number of periods. In this example, the baseline and manipulated processes meet after the first two periods since both focus on the target item (item 1) once. Yet the baseline beliefs $\mathbf{p}^{-1}_2$ and the manipulated beliefs $\hat{\mathbf{p}}^{-1}_2$ about the non-target items differ, and hence the continuations of the processes differ as well.

4.4 Counterexample (failure of stationarity)

We now show that the result of Theorem 1 can fail if the attention strategy satisfies IIA but is non-stationary. Suppose there are three items with prior beliefs $p^i_0 = 1/2$ for each i. For items 1 and 2, the learning process is the same as in the previous counterexample. For item 3, the value is

[9] In Section 5, we extend the model to allow for multiple values. There, we abstract from the presence of the outside option, and instead assume that the DM wishes to learn the value of the chosen item. The demand defined in Section 5 is thus an extension of that in the alternative model described here.

perfectly revealed whenever the DM focuses on it for a single period. The stopping thresholds are $\frac{1}{2}[-][-]$ and $\frac{1}{2}[+][+]$ for each item.

Consider a non-stationary pure attention strategy satisfying the following conditions whenever $p^1 > \frac{1}{2}[-][-]$:

$$\alpha(\mathbf{p}, t) = \begin{cases} 2 & \text{if } t = 0, \\ 3 & \text{if } t = 1,\ p^1 \neq p_0^1, \text{ and } p^3 > \frac{1}{2}[-][-], \\ 1 & \text{otherwise.} \end{cases}$$

Under this strategy, the DM first focuses on item 2, and then focuses on item 1 until p^1 reaches $\underline{p}$ or $\overline{p}$. Hence she chooses item 1 with probability 1/2. Under the manipulated strategy $\beta = \beta[\alpha, 1, 1]$, the DM first focuses on item 1, and then focuses on item 3—thereby learning its value—in the second period. Thus the DM chooses item 1 only if item 3 has value 0 *and* p^1 reaches $\overline{p}$, which occurs with probability 1/4.

As in the previous counterexample, Theorem 1 does not apply because the Meeting Lemma fails. The baseline and manipulated processes meet after two periods since each focuses on item 1 for exactly one of those periods. However, the beliefs about items 2 and 3 differ between these two processes at $t = 2$, which causes the continuation of the processes to differ. When IIAi and stationarity are satisfied, the Meeting Lemma follows from the fact that, for each draw, the accumulated focus $k(i, t)$ on the target item i is a sufficient statistic for the distribution of beliefs $\mathbf{p}_t$ about all items in period t.

5 Multiple values

We now extend the model from Section 3 to allow for the values of the items to lie in a finite set $V \subset \mathbb{R}$. Instead of comparing items to an outside option, we assume that the DM chooses an item i only if the belief p^i lies in a fixed set. Our main interpretation is that the DM wants to be informed about the chosen item, which may be useful for subsequent decisions in which the optimal action depends on the chosen item's value; accordingly, she stops learning only once she is sufficiently certain of this value. For example, one can think of the previous setup as involving a choice among items followed by a subsequent decision of whether to trade the chosen item for the outside option. Even in the absence of an outside option, an investor choosing among projects may not be content merely to learn that one project is likely to be better than the alternatives if information about the chosen project will help with other investment decisions.

As before, an attention strategy $\alpha(\mathbf{p}, t)$ specifies the probability distribution over the focus in period t at beliefs $\mathbf{p} \in (\Delta(V))^I$. For $p^j \in \Delta(V)$, $\phi^j(p^j, v^j) \in \Delta(\Delta(V))$ describes the distribution over beliefs about item j after one period of focus on this item starting from belief p^j when the true value is v^j. We extend the stopping rule from Section 3 as follows. For each item j and value v, there is a nonempty *sufficient certainty region* $C^j(v) \subset \Delta(V)$. When $p^j \in C^j(v)$, we say that the DM is *sufficiently certain* that item j has value v. Let $C^j = \bigcup_{v \in V} C^j(v)$ denote the set of beliefs at

which the DM is sufficiently certain of the value of item j. Similarly, for each j and v, there exists a nonempty *dominated region* $L^j(v) \subset \Delta(V)$. When $p^j \in L^j(v)$, we say that item j is *v-dominated*, with the interpretation that the DM is sufficiently certain that the value of item j is at most v. If for some v and i, $p^j \in L^j(v)$ and $p^i \in C^i(v)$, we say that j is *dominated*.

For any $\mathcal{I}^* \subseteq \mathcal{I}$ and any v, define the *stopping region* to be

$$F(\mathcal{I}^*, v) = \left\{\mathbf{p} : p^j \in C^j(v) \text{ for all } j \in \mathcal{I}^* \text{ and } p^j \in L^j(v) \text{ for all } j \in \mathcal{I} \setminus \mathcal{I}^*\right\}.$$

Thus $F(\mathcal{I}^*, v)$ consists of those beliefs at which the DM is sufficiently certain that items in $\mathcal{I}^*$ have value v and all other items are v-dominated. Let $F^{\mathcal{I}^*} = \bigcup_v F(\mathcal{I}^*, v)$; whenever $\mathbf{p} \in F^{\mathcal{I}^*}$, we say that the DM *deems items in $\mathcal{I}^*$ optimal*. Finally, let $F = \bigcup_{\mathcal{I}^*} F^{\mathcal{I}^*}$. The DM stops learning and makes a choice as soon as $\mathbf{p}_t \in F$; accordingly, $\tau = \min\{t : \mathbf{p}_t \in F\}$ is the stopping time for the learning process. (Again, we allow for the possibility that the process does not stop.) This stopping rule formalizes the assumption that the DM only stops when she is sufficiently certain of the value of the chosen item (and that it is optimal).

One example of particular interest is that of a Bayesian DM who receives signals that eventually, with some probability, perfectly reveal the value of the item she focuses on (as with Poisson learning). The DM may then always choose an optimal item, and stop learning as soon as she is certain of an optimal choice and of its value. In this case, the sets $C^i(v)$ are singletons consisting of the belief that attaches probability one to item i having value v, and the sets $L^i(v)$ consist of those beliefs that attach probability one to the event that $v^i \leq v$.

We impose two restrictions on the attention strategy α:

1. *non-wastefulness I*: if $\mathbf{p} \notin F$ and $p^j \in C^j$, then $\alpha^j(\mathbf{p}, t) = 0$; and
2. *non-wastefulness II*: if $\mathbf{p} \notin F$, and, for some v, $p^j \in L^j(v)$ and $p^{j'} \in C^{j'}(v)$ for some j', then $\alpha^j(\mathbf{p}, t) = 0$.

The first of these properties states that the DM does not focus on an item if she is sufficiently certain of its value. The second states that she does not focus on any item that is dominated. For example, in the case with perfectly revealing signals described above, these two properties are satisfied if she always focuses on an item for which the maximum of the support of the belief is the highest. One such strategy, which also satisfies stationarity and IIA, is when $\alpha(\mathbf{p})$ selects uniformly from $\arg\max_j \overline{v}(p^j)$, where $\overline{v}(p^j) = \max\{v \in V : p^j(v) > 0\}$.[10]

Unlike the setting with binary values, with multiple values, the DM may deem more than one item optimal at the stopping time. Her choice then depends on a tie-breaking rule. Accordingly, let $\sigma(j, \mathcal{I}^*)$ be the probability that the DM chooses item j if she deems the items in $\mathcal{I}^*$ optimal. Assume that (i) $\sigma(j, \mathcal{I}^*) = 0$ if $j \notin \mathcal{I}^*$ (*optimality*), and (ii) $\sigma(j, \mathcal{I}^*) \geq \sigma(j, \mathcal{J}^*)$ if $\mathcal{I}^* \subseteq \mathcal{J}^*$ (*monotonicity*). For instance, these two properties hold if the DM selects uniformly from $\mathcal{I}^*$. The interim demand

[10] In particular, non-wastefulness I holds because if the DM is certain of an item in $\arg\max_j \overline{v}(p^j)$ then the process stops.

for item j under attention strategy α is

$$D^j(\mathbf{v};\alpha) = \sum_{\mathcal{I}^*} P_\alpha^{\mathbf{v}}\left(\mathbf{p}_\tau \in F^{\mathcal{I}^*}\right)\sigma(j,\mathcal{I}^*).$$

As before, to compare the stopping times under the baseline and manipulated processes, let $\tau^j = \tau$ if item j is chosen, and $\tau^j = \infty$ otherwise. Given any α and $\mathbf{v}$, τ^j has distribution function

$$H^j(t;\mathbf{v},\alpha) = \sum_{\tau' \leq t, v, \mathcal{I}^*} P_\alpha^{\mathbf{v}}\left(\mathbf{p}_{\tau'} \in F^{\mathcal{I}^*}\right)\sigma\left(j,\mathcal{I}^*\right).$$

Let $\beta[\alpha,i,m] = \beta$ denote the manipulated strategy constructed from the baseline strategy α by forcing the DM to focus on the target item i in periods $t = 0,\ldots,m-1$ whenever the DM is not sufficiently certain about the value of i and i is not dominated; that is, $\beta^j(\mathbf{p},t) = 1_{j=i}$ if $t \leq m-1$, $p^i \notin C^i$, and there is no item j and value v such that $p^j \in C^j(v)$ and $p^i \in L^i(v)$, and $\beta(\mathbf{p},t) = \alpha(\mathbf{p},t)$ otherwise. Note that β inherits non-wastefulness I and II from α.

We conclude with the following generalization of Theorem 1.

Theorem 2. *If an attention strategy α satisfies stationarity, IIAi, and non-wastefulness properties I and II, then for every $\mathbf{v}$ and $m \geq 1$,*

$$D^i(\mathbf{v};\beta[\alpha,i,m]) \geq D^i(\mathbf{v};\alpha), \tag{7}$$

$$D^j(\mathbf{v};\beta[\alpha,i,m]) \leq D^j(\mathbf{v};\alpha) \text{ for all } j \neq i, \tag{8}$$

and, for every $t \geq 0$,

$$H^i(t;\mathbf{v},\alpha) \leq H^i(t;\mathbf{v},\beta) \tag{9}$$

$$\text{and} \quad H^j(t;\mathbf{v},\alpha) \geq H^j(t;\mathbf{v},\beta) \text{ for all } j \neq i. \tag{10}$$

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A Proofs

A.1 Proofs for Section 3

To accommodate stochastic attention strategies, we introduce the *attention process* describing, for each time t, (i) for each belief vector $\mathbf{p} = (p^i, \mathbf{p}^{-i})$, whether the DM focuses on the target item i, and (ii) for each $\mathbf{p}^{-i}$, which item she focuses on when she does not focus on i. The attention process is given by a family of random variables $(a_{\mathbf{p},t})_{\mathbf{p},t}, (b_{\mathbf{p}^{-i},t})_{\mathbf{p}^{-i},t}$ where all draws are independent and independent of the learning process. The random variable $a_{\mathbf{p},t}$ takes values in $\{0,1\}$; the probability that it takes the value 1 is $\alpha^i(\mathbf{p})$. The random variable $b_{\mathbf{p}^{-i},t}$ takes values in $\mathcal{I} \setminus \{i\}$. For a fixed value of $\mathbf{p}^{-i}$, if there exists p^i such that $\alpha^i(p^i, \mathbf{p}^{-i}) \neq 1$, then the probability that $b_{\mathbf{p}^{-i},t}$ takes the value $j \neq i$ is $\alpha^j\left(p^i, \mathbf{p}^{-i}\right) / \left(1 - \alpha^i\left(p^i, \mathbf{p}^{-i}\right)\right)$, where we note that the particular value of p^i in the formula is irrelevant since α satisfies IIAi.[11] A realization of the attention process is called an *attention draw*. We refer to the pair of the learning and attention draws simply as a *draw*.

We now recursively construct, for stochastic attention strategies $\gamma \in \{\alpha, \beta\}$, the processes $(\mathbf{p}_t(\gamma), \iota_t(\gamma))_t$ as functions of the learning and attention processes. As in the construction for the pure strategies in subsection 3.1, let the belief $p_t^j(\gamma) = \pi^j_{k(j,t;\gamma)}$ where $k(j,t;\gamma)$ is the cumulative focus on item j in periods $0, \ldots, t-1$. We proceed to construct the focus $\iota_t(\gamma)$. Suppose that

[11] The specification of $b_{\mathbf{p}^{-i},t}$ when no such p^i exists is immaterial for our purposes.

$\mathbf{p}_t \notin F$. Given a sequence of beliefs and focus items $(\mathbf{p}_s(\gamma), \iota_s(\gamma))_{s<t}$ and any vector $\mathbf{p}^{-i}$ of beliefs about $j \neq i$, we let $\mu\left(\mathbf{p}^{-i}, t; \gamma\right)$ be the total number of periods starting with belief $\mathbf{p}^{-i}$ about items $j \neq i$ in which the DM has not focused on the target item i under the strategy γ before time t; that is,

$$\mu(\mathbf{p}^{-i}, t; \gamma) = |\{s < t : \mathbf{p}_s^{-i}(\gamma) = \mathbf{p}^{-i}, \iota_s(\gamma) \neq i\}|.$$

Let $\iota_t(\gamma) = i$ if $a_{\mathbf{p}_t(\gamma),t} = 1$ and $\iota_t(\gamma) = b_{\mathbf{p}_t^{-i}(\gamma),\mu\left(\mathbf{p}_t^{-i}(\gamma),t;\gamma\right)}$ otherwise. Note a subtlety in the construction: the element $a_{\mathbf{p},t}$ of the attention draw deciding whether the focus at $\mathbf{p}$ is on the target item i depends on t, while the element $b_{\mathbf{p}^{-i},\mu(\mathbf{p}^{-i},t;\gamma)}$ of the draw deciding the item of focus conditional on its *not* being i depends on $\mu(\mathbf{p}^{-i}, t; \gamma)$. This asymmetry in the construction is exploited in the proof of Lemma 2.

By construction, conditional on $(\mathbf{p}_0(\gamma), \iota_0(\gamma), \ldots, \mathbf{p}_{t-1}(\gamma), \iota_{t-1}(\gamma), \mathbf{p}_t(\gamma))$, $\iota_t(\gamma)$ is distributed according to $\alpha(\mathbf{p}_t(\gamma))$. Conditional on $(\mathbf{p}_0(\gamma), \iota_0(\gamma), \ldots, \mathbf{p}_t(\gamma), \iota_t(\gamma))$, $p_{t+1}^j(\gamma) = p_t^j(\gamma)$ for all $j \neq \iota_t(\gamma)$, and $p_{t+1}^{\iota_t(\gamma)}(\gamma)$ is distributed according to the transition probability $\phi^{\iota_t(\gamma)}\left(p_t^{\iota_t(\gamma)}(\gamma), v^{\iota_t(\gamma)}\right)$. Therefore, the law of the process $(\mathbf{p}_t(\gamma), \iota_t(\gamma))_t$ is $P_\gamma^{\mathbf{v}}$, as needed.

Proof of Lemma 1 (Coupling Lemma). We prove the result by induction. The property holds for $n = 0$ since $\mathbf{p}_{\mathfrak{t}(0)}^{-i} = \hat{\mathbf{p}}_{\hat{\mathfrak{t}}(0)}^{-i} = \mathbf{p}_0^{-i}$. Assume it holds for n. For item $j = \iota_{\mathfrak{t}(n)} = \hat{\iota}_{\hat{\mathfrak{t}}(n)}$, we have $p_{\mathfrak{t}(n+1)}^j = \pi_{k(j,\mathfrak{t}(n))+1}^j = \pi_{\hat{k}(j,\hat{\mathfrak{t}}(n))+1}^j = \hat{p}_{\hat{\mathfrak{t}}(n+1)}^j$. For items $j \neq i, \iota_{\mathfrak{t}(n)}$ that do not receive attention in periods $\mathfrak{t}(n)$ and $\hat{\mathfrak{t}}(n)$, respectively, we have $p_{\mathfrak{t}(n+1)}^j = p_{\mathfrak{t}(n)}^j = \hat{p}_{\mathfrak{t}(n)}^j = \hat{p}_{\hat{\mathfrak{t}}(n+1)}^j$. We thus have $\mathbf{p}_{\mathfrak{t}(n+1)}^{-i} = \hat{\mathbf{p}}_{\hat{\mathfrak{t}}(n+1)}^{-i}$.

It remains to show that the items of attention in period $\mathfrak{t}(n+1)$ of the baseline process and in period $\hat{\mathfrak{t}}(n+1)$ of the manipulated process coincide. By the definitions of the attention processes, we have

$$\iota_{\mathfrak{t}(n+1)} = b_{\mathbf{p}_{\mathfrak{t}(n+1)}^{-i},\mu\left(\mathbf{p}_{\mathfrak{t}(n+1)}^{-i},\mathfrak{t}(n+1)\right)} = b_{\hat{\mathbf{p}}_{\hat{\mathfrak{t}}(n+1)}^{-i},\mu\left(\hat{\mathbf{p}}_{\hat{\mathfrak{t}}(n+1)}^{-i},\mathfrak{t}(n+1)\right)} = b_{\hat{\mathbf{p}}_{\hat{\mathfrak{t}}(n+1)}^{-i},\hat{\mu}\left(\hat{\mathbf{p}}_{\hat{\mathfrak{t}}(n+1)}^{-i},\hat{\mathfrak{t}}(n+1)\right)} = \hat{\iota}_{\hat{\mathfrak{t}}(n+1)}$$

since, by the induction hypothesis, $\mu(\hat{\mathbf{p}}_{\mathfrak{t}(n+1)}^{-i}, \mathfrak{t}(n+1)) = \hat{\mu}(\hat{\mathbf{p}}_{\hat{\mathfrak{t}}(n+1)}^{-i}, \hat{\mathfrak{t}}(n+1))$. □

The next proof exploits the subtlety in the coupling construction mentioned above. The subtlety ensures that if the processes meet in period t, they coincide thereafter. The event that they meet in period t implies that the two processes have visited each vector $\mathbf{p}^{-i}$ in the same number of periods, but it does not ensure that the number of periods they visited each vector $\mathbf{p}$ is the same. To this end, the draw a_t in the coupling construction does not depend on the number of periods that $\mathbf{p}$ was visited (as it could be different for the two processes), but rather on t.

Proof of Lemma 2 (Meeting Lemma). Suppose the processes meet in period t, and thus $k(-i,t) = \hat{k}(-i,t)$. Hence, by the Coupling Lemma,

$$\left(\mathbf{p}_{\mathfrak{t}(0)}^{-i}, \iota_{\mathfrak{t}(0)} \ldots, \mathbf{p}_{\mathfrak{t}(k(-i,t))}^{-i}, \iota_{\mathfrak{t}(k(-i,t))}\right) = \left(\hat{\mathbf{p}}_{\hat{\mathfrak{t}}(0)}^{-i}, \hat{\iota}_{\hat{\mathfrak{t}}(0)}, \ldots, \hat{\mathbf{p}}_{\hat{\mathfrak{t}}(\hat{k}(-i,t))}^{-i}, \hat{\iota}_{\hat{\mathfrak{t}}(\hat{k}(-i,t))}\right).$$

Therefore, for every $j \in \mathcal{I}$, $k(j,t) = \hat{k}(j,t)$, so that the cumulative focus on each item before time t is the same in the two processes, which implies that $\mathbf{p}_t = \hat{\mathbf{p}}_t$. The baseline process focuses on the target item i at time t if and only if the manipulated process focuses on i at t, since both processes focus on i at t if $a_{\mathbf{p}_t,t} = a_{\hat{\mathbf{p}}_t,t} = 1$. The baseline process focuses on item $j \neq i$ at time t if and only if the manipulated process focuses on j at t, since the baseline process focuses on j at t if $b_{\mathbf{p}_t^{-i},\mu(\mathbf{p}_t^{-i},t)} = j$, the manipulated process focuses on j in t if $b_{\hat{\mathbf{p}}_t^{-i},\hat{\mu}(\hat{\mathbf{p}}_t^{-i},t)} = j$, and $\mu(\mathbf{p}_t^{-i},t) = \hat{\mu}(\hat{\mathbf{p}}_t^{-i},t)$ by the Coupling Lemma. Thus $\iota_t = \hat{\iota}_t$.

Note that $k(i,t+1) = \hat{k}(i,t+1)$. Therefore, if the two processes meet in period $t \geq m$, they also meet in period $t+1$, and hence in every period $s \geq t$. Thus, $(\mathbf{p}_t, \iota_s) = (\hat{\mathbf{p}}_t, \hat{\iota}_s)$ for all $s \geq t$. □

Proof of Lemma 3 (Attention Lemma). The statement obviously holds for every $t \leq m$. Since $\hat{k}(i,m) = m \geq k(i,m)$, and $\hat{k}(i,t) = k(i,t)$ implies $\hat{k}(i,s) = k(i,s)$ for every $s > t$, we have $\hat{k}(i,t) \geq k(i,t)$ and $\hat{k}(-i,t) \leq k(-i,t)$ for every $t > m$. The Coupling Lemma implies that items $j \neq i$ are explored in the same order under both processes, which in turn implies that, for every t and every $j \neq i$, $\hat{k}(j,t) \leq k(j,t)$, as needed. □

Proof of Lemma 4 (Outside Option Lemma). Fix a learning draw such that $\tau^{oo}(\alpha)$ is finite; that is, strategy α leads to the outside option being chosen. For each item i, let $\kappa^i = \min\{\kappa : \pi^i_\kappa \leq \underline{p}^i\}$ be the number of steps needed for the learning process π^i to reach the threshold $\underline{p}^i$. Since α stops with the choice of the outside option once all beliefs reach their respective lower thresholds, $\tau^{oo}(\alpha) = \sum_i \kappa^i$, which is independent of α. □

Proof of Proposition 2. Let $\tilde{\phi}^i\left(p^i_{t+1} \mid p^i_t\right)$ denote the ex ante Markov transition probabilities of the DM's belief about item i when she focuses on i in period t. Consider the following bandit problem. In each period $t = 0, 1, \ldots$, the DM chooses an item $\iota_t \in \mathcal{I}$ and pays a flow cost

$$\gamma\left(p^{\iota_t}_t, \iota_t\right) = \begin{cases} c\left(p^{\iota_t}_t, \iota_t\right) & \text{if } \underline{p}^{\iota_t} < p^{\iota_t}_t < \overline{p}^{\iota_t}, \\ 0 & \text{if } p^{\iota_t}_t \geq \overline{p}^{\iota_t}, \\ \overline{c} & \text{otherwise.} \end{cases}$$

Interpret p^i_t as the state of item i. In each period t, the item ι_t chosen by the DM transitions to a new state $p^{\iota_t}_{t+1}$ according to the Markov process $\tilde{\phi}^i$, and the states of all other items remain unaltered; that is, $p^j_{t+1} = p^j_t$ for $j \neq \iota_t$. The bandit problem is to choose a non-wasteful strategy α to minimize

$$\Gamma(\alpha) = \mathrm{E} \sum_{t=0}^{\infty} \delta^t \gamma\left(p^{\iota_t}_t, \iota_t\right). \tag{11}$$

This problem differs from the original problem in that the flow cost from focusing on item i in period t depends only on p^i_t and is independent of p^j_t for $j \neq i$. The flow payoffs in the original problem are interdependent in that whether the DM stops upon reaching $p^i_t \leq \underline{p}^i$ depends on whether $p^j_t \leq \underline{p}^j$ for all other items $j \neq i$. This difference allows us to apply the following standard textbook result from the theory of multi-armed bandits (see e.g. Weber (1992) or Whittle (1980)): an optimal solution

to the bandit problem (11) is a Gittins index strategy where the index for an item depends only on the current belief. In particular, when ties for the highest Gittins index are broken by uniform randomization, this describes an optimal strategy that satisfies IIA and stationarity.

It remains to connect the bandit problem (11) to the original problem (2). Note that any solution to the bandit problem (i) always chooses an item i such that $p_t^i \geq \overline{p}$ if such an item exists, and (ii) always chooses an item i such that $p_t^i > \underline{p}$ if such an item exists and no item satisfying condition (i) exists.

Note that $\Gamma(\alpha) = C(\alpha) + \overline{c}\,\mathrm{E}\sum_{t=1}^{\infty} 1_{t \geq \tau^{oo}(\alpha)}$, where the expectation is with respect to $\tau^{oo}(\alpha)$. By the Outside Option Lemma, $\tau^{oo}(\alpha)$ is identical for all α satisfying (i) and (ii). Therefore, there exists a constant K_δ such that

$$\Gamma(\alpha) = C(\alpha) + K_\delta. \tag{12}$$

Therefore, a non-wasteful strategy α solves the auxiliary bandit problem if and only if it solves the original problem, as needed. □

A.2 Proofs for Section 4

Proof of Proposition 3. This proof makes use of coupling. We again construct a common probability space on which we can compare the processes of beliefs $\mathbf{p}_t(\alpha)$ and $\mathbf{p}_t(\alpha^*)$, where, by construction, the beliefs under strategy α follow the law P_α^{ea} while those under α^* follow $P_{\alpha^*}^{\mathrm{ea}}$. However, the particular construction differs from that in the proof of Theorem 1.

Let $\Pi = \{\underline{p}[+], \underline{p}[+][+], \ldots, \overline{p}[-][-], \overline{p}[-]\}$, which is the set of transient beliefs that the DM may attain for either item. For each $\pi \in \Pi$ and $\kappa = 0, 1, \ldots$, let $\ell(\pi, \kappa)$ be an i.i.d. random variable that attains values $\pi[+]$ and $\pi[-]$ with probabilities $\frac{\pi - \pi[-]}{\pi[+] - \pi[-]}$ and $\frac{\pi[+] - \pi}{\pi[+] - \pi[-]}$, respectively. An *updating draw* is a collection $(\ell(\pi, \kappa))_{\pi,\kappa}$ of realizations of $\ell(\pi, \kappa)$, one for each pair $(\pi, \kappa) \in \Pi \times \mathbb{N}$.

We interpret $\ell(\pi, \kappa)$ as the updated belief of a DM who learns for one period about an item i starting at the belief $p^i = \pi$ where κ is a counter indicating the total number of times the DM has focused on an item with associated belief π. We now construct, for each fixed updating draw and any attention strategy γ, the process of beliefs $\mathbf{p}_t(\gamma)$. In this construction, we use an auxiliary counter $\mathbf{k}_t(\gamma)$ that takes values in $\mathbb{N}^{|\Pi|}$. Define the joint process of $\mathbf{p}_t(\gamma)$ and $\mathbf{k}_t(\gamma)$ as follows. Let $k_0^\pi(\gamma) = 0$ for all π and $\mathbf{p}_0(\gamma) = \mathbf{p}_0$. In each period t, the focus of attention ι_t in t is chosen according to the attention strategy $\gamma(\mathbf{p}_t, t)$. Recursively define

$$k_{t+1}^\pi(\gamma) = \begin{cases} k_t^\pi(\gamma) + 1 & \text{if } \pi = p_t^{\iota_t}, \\ k_t^\pi(\gamma) & \text{otherwise} \end{cases}$$

and

$$p_{t+1}^j(\gamma) = \begin{cases} \ell\left(\pi, k_t^{p_t^j(\gamma)}(\gamma)\right) & \text{if } j = \iota_t, \\ p_t^j(\gamma) & \text{otherwise.} \end{cases}$$

By construction, for each strategy γ, the beliefs $\mathbf{p}_t(\gamma)$ follow the law P_γ^{ea}.

For each transient belief $\pi \in \Pi$, we introduce the belief process $a_t(\pi)$ that would result from learning about a single item starting from the prior belief π, making use of a counter $h_t^{\tilde{\pi}}$. Formally, let $a_0(\pi) = \pi$, $h_0^{\tilde{\pi}} = 0$ for all $\tilde{\pi} \in \Pi$, and recursively define $a_{t+1}(\pi) = \ell\left(a_t(\pi), h_t^{a_t(\pi)}\right)$, and, for each $\tilde{\pi} \in \Pi$,

$$h_{t+1}^{\tilde{\pi}} = \begin{cases} h_t^{\tilde{\pi}} + 1 & \text{if } \tilde{\pi} = a_t(\pi), \\ h_t^{\tilde{\pi}} & \text{otherwise.} \end{cases}$$

For each strategy γ, let $M_t(\gamma) = \max_{i=1,2} p_t^i(\gamma)$, $\overline{M}_t(\gamma) = \max_{s=0,\ldots,t} M_s(\gamma)$, and $\underline{M}_t(\gamma) = \min_{s=0,\ldots,t} M_s(\gamma)$. The strategy γ stops by period t, i.e., $\tau^{\text{ea}}(\gamma) \leq t$, if

$$\underline{p} = \underline{M}_t(\gamma) \quad \text{or} \quad \overline{M}_t(\gamma) = \overline{p}.$$

We will prove that, for every prior, in each updating draw, if a strategy α stops by t, then the strategy α^* also stops by t. We proceed by induction on t. To see that the statement holds for $t = 1$, note that if $p_0^1 \neq p_0^2$, then $M_0(\alpha) = M_0(\alpha^*) = \max\{p_0^1, p_0^2\}$ and $M_1(\alpha^*) = a_1\left(\max\{p_0^1, p_0^2\}\right)$ while $M_1(\alpha)$ equals $a_1\left(\max\{p_0^1, p_0^2\}\right)$ or $\max\{p_0^1, p_0^2\}$. (The latter case arises when α focuses on the item with the lower belief in period 0.) Thus

$$\underline{M}_1(\alpha^*) \leq \underline{M}_1(\alpha) \leq \overline{M}_1(\alpha) \leq \overline{M}_1(\alpha^*),$$

as needed.

Suppose the statement holds for $t-1$. If a strategy α stops by period t then, since the induction hypothesis applies regardless of the prior, the strategy $\tilde{\alpha}$ stops by t, where $\tilde{\alpha}(\mathbf{p}, 0) = \alpha(\mathbf{p}, 0)$ and $\tilde{\alpha}(\mathbf{p}, t) = \alpha^*(\mathbf{p})$ for $t > 0$. Therefore, to close the induction step, it suffices to prove that if $\tilde{\alpha}$ stops by t then α^* stops by t. This is immediate if $p_0^1 = p_0^2$. Accordingly, suppose that $p_0^1 \neq p_0^2$ and, without loss of generality, take $p_0^1 < p_0^2$. If $\tilde{\alpha}$ focuses on item 2 in period 0, then the belief processes are the same under $\tilde{\alpha}$ and α^*. Thus it suffices to show that if $\beta = \beta[\alpha^*, 1, 1]$ stops by t then α^* stops by t.[12]

To prove the last implication, we distinguish two sets of updating draws. The first set consists of those for which $a_s(p_0^2) > p_0^1$ in every period $s = 0, 1, \ldots, t-1$. For any (ordered) belief pair $\mathbf{p} = (p^1, p^2)$, write $\langle \mathbf{p} \rangle$ for the unordered pair $\{p^1, p^2\}$. (By considering the unordered pairs of beliefs we eliminate the need to keep track of which item has the higher belief and which is randomly chosen at a tie.) For each updating draw in this first set, for each $s = 1, 2, \ldots, t$,

$$\langle \mathbf{p}_s(\alpha^*) \rangle = \left\{p_0^1, a_s\left(p_0^2\right)\right\} \quad \text{and} \quad M_s(\alpha^*) = a_s\left(p_0^2\right),$$

and

$$\langle \mathbf{p}_s(\beta) \rangle = \left\{a_1\left(p_0^1\right), a_{s-1}\left(p_0^2\right)\right\} \quad \text{and} \quad M_s(\beta) = a_{s-1}\left(p_0^2\right)$$

[12] Recall that $\beta[\alpha^*, 1, 1]$ is the strategy constructed from α^* by forcing the DM to focus on item 1 at $t = 0$ and follow α^* for every $t > 0$.

since $a_s(p_0^2) > p_0^1$ and $a_s(p_0^2) \geq a_1(p_0^1)$ for all $s = 1, 2, \ldots, t-1$. Therefore, α^* updates the belief $a_s(p_0^2)$ in all periods $s = 0, \ldots, t$, and β updates the belief $a_{s-1}(p_0^2)$ in all periods $s = 1, \ldots, t$. Thus, for each updating draw in this first set,

$$\underline{M}_t(\alpha^*) \leq \underline{M}_t(\beta) \leq \overline{M}_t(\beta) \leq \overline{M}_t(\alpha^*),$$

and the induction step holds.

The second set of updating draws consists of those for which there exists a period $s \in \{1, 2, \ldots, t-1\}$ in which $a_s(p_0^2) = p_0^1$. (Note that the second set is complementary to the first set.) Let s^* be the minimal such period. For each draw in this second set, we have

$$\langle \mathbf{p}_{s^*+1}(\alpha^*) \rangle = \{p_0^1, a_1(p_0^1)\}$$

since $\langle \mathbf{p}_{s^*}(\alpha^*) \rangle = \{p_0^1, p_0^1\}$ and the belief p_0^1 is updated once by s^*. We also have

$$\langle \mathbf{p}_{s^*+1}(\beta) \rangle = \{a_1(p_0^1), a_{s^*}(p_0^2)\}$$

since the belief p_0^1 is updated once in period 0 and in each period $s = 1, \ldots, s^*$, $\langle \mathbf{p}_s(\beta) \rangle = \{a_1(p_0^1), a_{s-1}(p_0^2)\}$ and $a_{s-1}(p_0^2) \geq a_1(p_0^1)$. Thus the strategy β updates the belief $a_{s-1}(p_0^2)$ in all periods $s = 1, \ldots, s^*$. Since $a_{s^*}(p_0^2) = p_0^1$, we have

$$\langle \mathbf{p}_{s^*+1}(\beta) \rangle = \{a_1(p_0^1), p_0^1\} = \langle \mathbf{p}_{s^*+1}(\alpha^*) \rangle .$$

Therefore, in each updating draw from the second set, $\langle \mathbf{p}_s(\alpha^*) \rangle = \langle \mathbf{p}_s(\beta) \rangle$ for all $s \geq s^* + 1$. In particular, α^* and β stop in the same period, concluding the proof of the induction step. □

Proof of Proposition 4. The proof is similar to that of Proposition 3. The coupling construction exploits the symmetry with respect to belief 1/2 as follows. Let

$$\overline{\Pi} = \{\underline{p}[+], \underline{p}[+][+], \ldots, \overline{p}[-][-], \overline{p}[-]\} \cap [1/2, 1)$$

and

$$\underline{\Pi} = \{\underline{p}[+], \underline{p}[+][+], \ldots, \overline{p}[-][-], \overline{p}[-]\} \cap (0, 1/2).$$

An updating draw is a collection $(\ell(\pi, \kappa))_{(\pi,\kappa) \in \overline{\Pi} \times \mathbb{N}}$, where $\ell(\pi, \kappa)$ is an i.i.d. random variable that attains values $\pi[+]$ and $\pi[-]$ with probabilities $\frac{\pi - \pi[-]}{\pi[+] - \pi[-]}$ and $\frac{\pi[+] - \pi}{\pi[+] - \pi[-]}$, respectively.

For any attention strategy γ, we construct the process of beliefs $\mathbf{p}_t(\gamma)$ in each updating draw as follows. Let $k_0^\pi(\gamma) = 0$ for all $\pi \in \overline{\Pi}$ and $\mathbf{p}_0(\gamma) = \mathbf{p}_0$. In each period t, the focus of attention ι_t in t is chosen according to $\gamma(\mathbf{p}_t, t)$. We distinguish two cases: $p^{\iota_t} \geq 1/2$ and $p^{\iota_t} < 1/2$. If $p_t^{\iota_t} \geq 1/2$, then

$$k_{t+1}^\pi(\gamma) = \begin{cases} k_t^\pi(\gamma) + 1 & \text{if } \pi = p_t^{\iota_t}, \\ k_t^\pi(\gamma) & \text{if } \pi \in \overline{\Pi} \setminus \{p_t^{\iota_t}\}. \end{cases}$$

and

$$p_{t+1}^j(\gamma) = \begin{cases} \ell\left(p_t^j(\gamma), k_t^{p_t^j(\gamma)}(\gamma)\right) & \text{if } j = \iota_t, \\ p_t^j(\gamma) & \text{otherwise.} \end{cases}$$

If $p_t^{\iota_t} < 1/2$, then

$$k_{t+1}^\pi(\gamma) = \begin{cases} k_t^\pi(\gamma) + 1 & \text{if } \pi = 1 - p_t^{\iota_t}, \\ k_t^\pi(\gamma) & \text{if } \pi \in \overline{\Pi} \setminus \{1 - p_t^{\iota_t}\}, \end{cases}$$

and

$$p_{t+1}^j(\gamma) = \begin{cases} 1 - \ell\left(1 - p_t^j(\gamma), k_t^{1-p_t^j(\gamma)}(\gamma)\right) & \text{if } j = \iota_t, \\ p_t^j(\gamma) & \text{otherwise.} \end{cases}$$

By construction, for each strategy γ, the beliefs $\mathbf{p}_t(\gamma)$ follow the law P_γ^{ea}.

We again define a belief process that would result from learning about a single item. For any belief $\pi \in \underline{\Pi} \cup \overline{\Pi}$, let $\hat{\pi} = \max\{\pi, 1 - \pi\}$. Define the belief process $a_t(\pi)$ as follows: $a_0(\pi) = \hat{\pi}$ and $h_0^{\tilde{\pi}} = 0$ for all $\tilde{\pi} \in \overline{\Pi}$, and for $t > 0$, recursively define

$$a_{t+1}(\pi) = \max\left\{\ell\left(a_t(\pi), h_t^{a_t(\pi)}\right), 1 - \ell\left(a_t(\pi), h_t^{a_t(\pi)}\right)\right\}$$

and, for each $\tilde{\pi} \in \overline{\Pi}$,

$$h_{t+1}^{\tilde{\pi}} = \begin{cases} h_t^{\tilde{\pi}} + 1 & \text{if } \tilde{\pi} = a_t(\pi), \\ h_t^{\tilde{\pi}} & \text{otherwise.} \end{cases}$$

For any attention strategy γ, let $M_t(\gamma) = \max_{i=1,2} \hat{p}_t^i(\gamma)$ and $\overline{M}_t(\gamma) = \max_{s=0,\ldots,t} M_s(\gamma)$. The strategy γ stops by period t, i.e., $\tau_{\text{no}}^{\text{ea}}(\gamma) \leq t$, if $\overline{M}_t(\gamma) = \overline{p}$. We will prove by induction on t that, for every prior, in each updating draw, if a strategy α stops by t, then the strategy α^{**} also stops by t.

To see that the statement holds for $t = 1$, note that if $p_0^1 \neq p_0^2$, then $M_0(\alpha) = M_0(\alpha^{**}) = \max\left\{\hat{p}_0^1, \hat{p}_0^2\right\}$ and $M_1(\alpha^{**}) = a_1\left(\max\left\{\hat{p}_0^1, \hat{p}_0^2\right\}\right)$ while $M_1(\alpha)$ equals $a_1\left(\max\left\{\hat{p}_0^1, \hat{p}_0^2\right\}\right)$ or $\max\left\{\hat{p}_0^1, \hat{p}_0^2\right\}$. (The latter case arises when α focuses on the less certain item in period 0.) Thus

$$\overline{M}_1(\alpha) \leq \overline{M}_1(\alpha^{**}),$$

as needed.

Suppose the statement holds for $t-1$. If α stops by t then, since the induction hypothesis holds for every prior, strategy $\tilde{\alpha}$ also stops by t, where $\tilde{\alpha}(\mathbf{p}, 0) = \alpha(\mathbf{p}, 0)$ and $\tilde{\alpha}(\mathbf{p}, t) = \alpha^{**}(\mathbf{p})$ for $t > 0$. Therefore, to close the induction step, it suffices to prove that if $\tilde{\alpha}$ stops by t then α^{**} stops by t. This is immediate if $\hat{p}_0^1 = \hat{p}_0^2$ since then the two belief processes coincide. Accordingly, suppose that $p_0^1 \neq p_0^2$ and, without loss of generality, take $\hat{p}_0^1 < \hat{p}_0^2$. If $\tilde{\alpha}$ focuses on item 2 in period 0 then the belief processes coincide under $\tilde{\alpha}$ and α^{**}. Thus it suffices to show that if $\beta = \beta[\alpha^{**}, 1, 1]$ stops by t then α^{**} stops by t.

To prove the last implication, we distinguish two sets of updating draws. The first set consists of those for which $a_s(\hat{p}_0^2) > \hat{p}_0^1$ in every period $s = 0, 1, \ldots, t-1$. For each draw in this first set and each $s = 1, 2, \ldots, t$,

$$\left\{\hat{p}_s^1(\alpha^{**}), \hat{p}_s^2(\alpha^{**})\right\} = \left\{\hat{p}_0^1, a_s\left(p_0^2\right)\right\} \quad \text{and} \quad M_s(\alpha^{**}) = a_s\left(p_0^2\right),$$

and

$$\left\{\hat{p}_s^1(\beta), \hat{p}_s^2(\beta)\right\} = \left\{a_1\left(p_0^1\right), a_{s-1}\left(p_0^2\right)\right\} \quad \text{and} \quad M_s(\beta) = a_{s-1}\left(p_0^2\right)$$

since $a_s(p_0^2) > \hat{p}_0^1$ and $a_s(p_0^2) \geq a_1(p_0^1)$ for all $s = 1, 2, \ldots, t-1$. Therefore, α^{**} updates $a_s(p_0^2)$ in all periods $s = 0, \ldots, t$, and β updates $a_{s-1}(p_0^2)$ in all periods $s = 1, \ldots, t$. Thus, for each updating draw in this first set,

$$\overline{M}_t(\beta) \leq \overline{M}_t(\alpha^{**}),$$

and the induction step holds.

The second set of updating draws consists of those for which there exists a period $s \in \{1, 2, \ldots, t-1\}$ in which $a_s(p_0^2) = \hat{p}_0^1$. (Note that the second set is complementary to the first.) Let s^* be the minimal such period and observe that

$$\left\{\hat{p}_{s^*+1}^1(\alpha^{**}), \hat{p}_{s^*+1}^2(\alpha^{**})\right\} = \left\{\hat{p}_0^1, a_1(\hat{p}_0^1)\right\}$$

since $\left\{\hat{p}_{s^*}^1(\alpha^{**}), \hat{p}_{s^*}^2(\alpha^{**})\right\} = \left\{\hat{p}_0^1, \hat{p}_0^1\right\}$ and the belief with value $\hat{p}_0^1$ or $1 - \hat{p}_0^1$ is updated once by s^*. Also,

$$\left\{\hat{p}_{s^*+1}^1(\beta), \hat{p}_{s^*+1}^2(\beta)\right\} = \left\{a_1(p_0^1), a_{s^*}(p_0^2)\right\}$$

since the belief p_0^1 is updated once in period 0 and, for each period $s = 1, \ldots, s^*$, $\left\{\hat{p}_s^1(\beta), \hat{p}_s^2(\beta)\right\} = \left\{a_1(p_0^1), a_{s-1}(p_0^2)\right\}$ and $a_{s-1}(p_0^2) \geq a_1(p_0^1)$. Since $a_{s^*}(p_0^2) = \hat{p}_0^1$, we have

$$\left\{\hat{p}_{s^*+1}^1(\beta), \hat{p}_{s^*+1}^2(\beta)\right\} = \left\{a_1(p_0^1), \hat{p}_0^1\right\} = \left\{\hat{p}_{s^*+1}^1(\alpha^{**}), \hat{p}_{s^*+1}^2(\alpha^{**})\right\}.$$

Therefore, in each updating draw from the second set, $\left\{\hat{p}_s^1(\beta), \hat{p}_s^2(\beta)\right\} = \left\{\hat{p}_s^1(\alpha^{**}), \hat{p}_s^2(\alpha^{**})\right\}$ for all $s \geq s^* + 1$. In particular, α^{**} and β stop in the same period, concluding the proof of the induction step. □

A.3 Proof of Theorem 2

As in the proof of Theorem 1, for any variable η in the baseline process, let $\hat{\eta}$ denote its counterpart in the manipulated process. We construct the probability space and the baseline and manipulated processes $(\mathbf{p}_t, \iota_t)_t$ as in the proof of Theorem 1. Note that Lemmata 1, 2, and 3 extend verbatim to the current setting.

Given any draw, let $\mathcal{I}^*$ denote the set of items deemed optimal in the baseline process and $\hat{\mathcal{I}}^*$ the corresponding set in the manipulated process.

Lemma 6 (Choice Lemma for Multiple Values)**.** *Let i be the target item. For any draw,*

1. *if $i \in \mathcal{I}^*$, then (i) $i \in \hat{\mathcal{I}}^*$, (ii) $\hat{\mathcal{I}}^* \subseteq \mathcal{I}^*$, and (iii) $\hat{\tau} \leq \tau$.*

2. *for any $j \neq i$, if $j \in \hat{\mathcal{I}}^*$, (i) $j \in \mathcal{I}^*$, (ii) $\mathcal{I}^* \subseteq \hat{\mathcal{I}}^*$, and (iii) $\tau \leq \hat{\tau}$.*

Proof. Statement 1: Consider any draw in which the baseline process deems the target item i optimal (i.e., $i \in \mathcal{I}^*$) with stopping time τ. First suppose $\hat{\tau} \geq \tau$. By the attention lemma, $\hat{k}(i,\tau) \geq k(i,\tau)$. Since α deems i optimal and stops at τ, the DM is sufficiently certain of i's value at τ. Hence, by non-wastefulness I, $\hat{k}(i,\tau) = k(i,\tau)$, and the two processes meet at τ. Thus $\hat{\mathcal{I}}^* = \mathcal{I}^*$, $\hat{\tau} = \tau$, and (i–iii) hold.

Now consider any draw in which $\hat{\tau} < \tau$. We first show that, under the manipulated process, the DM does not deem any item $j \neq i$ optimal. Suppose for contradiction that $j \in \hat{\mathcal{I}}^*$. Then there exists v such that $\hat{p}^j_{\hat{\tau}} \in C^j(v)$ and for all $j' \neq j$, we have $\hat{p}^{j'}_{\hat{\tau}} \in C^{j'}(v)$ or $\hat{p}^{j'}_{\hat{\tau}} \in L^{j'}(v)$. By the Attention Lemma, $k(j'',\hat{\tau}) \geq \hat{k}(j'',\hat{\tau})$ for all $j'' \neq i$. By the Coupling Lemma, there exists a period $t \leq \hat{\tau}$ such $\mathbf{p}^{-i}_t = \hat{\mathbf{p}}^{-i}_{\hat{\tau}}$ and $k(-i,t) = \hat{k}(-i,\hat{\tau})$. Non-wastefulness I and II imply that the baseline strategy does not focus on items $j'' \neq i$ in periods $s > t$, and thus $k(-i,\hat{\tau}) = k(-i,t) = \hat{k}(-i,\hat{\tau})$. Therefore, the baseline and manipulated processes meet at $\hat{\tau}$. By the Meeting Lemma, $\mathbf{p}_{\hat{\tau}} = \hat{\mathbf{p}}_{\hat{\tau}}$, which implies that $\tau \leq \hat{\tau}$, contradicting the assumption that $\hat{\tau} < \tau$. Therefore, if the manipulated process stops at $\hat{\tau} < \tau$, then $\hat{\mathcal{I}}^* = \{i\}$ and properties (i–iii) again hold.

Statement 2: Consider any draw in which the manipulated process deems an item $j \neq i$ optimal (i.e., $j \in \hat{\mathcal{I}}^*$) with stopping time $\hat{\tau}$. First suppose $\tau \geq \hat{\tau}$. By the Attention Lemma, $k(-i,\hat{\tau}) \geq \hat{k}(-i,\hat{\tau})$. By the Coupling Lemma, there exists $t \leq \hat{\tau}$ such that $k(-i,t) = \hat{k}(-i,\hat{\tau})$ and $\mathbf{p}^{-i}_t = \mathbf{p}^{-i}_{\hat{\tau}}$. Since, under the manipulated strategy, the process stops at $\hat{\tau}$ and the DM deems j optimal, it must be that for some v, $p^j_{\hat{\tau}} \in C^j(v)$ and for each $j' \neq j$, $p^{j'}_{\hat{\tau}} \in C^{j'}$ or $p^{j'}_{\hat{\tau}} \in L^{j'}(v)$. Thus, by non-wastefulness I and II, the baseline strategy focuses only on item i in each period $s \geq t$, and hence $k(-i,\hat{\tau}) = k(-i,t) = \hat{k}(-i,\hat{\tau})$. Therefore, the two processes meet at $\hat{\tau}$, $\hat{\mathcal{I}}^* = \mathcal{I}^*$, $\hat{\tau} = \tau$, and (i–iii) hold.

Now consider any draw in which $\tau < \hat{\tau}$. We first show that, under the baseline process, the DM does not deem item i optimal. Suppose for contradiction that $i \in \mathcal{I}^*$. Then, under the baseline process, in period τ the DM is sufficiently certain of the value of i. By the Attention Lemma, $\hat{k}(i,\tau) \geq k(i,\tau)$. Non-wastefulness I implies that $\hat{k}(i,\tau) = k(i,\tau)$. Therefore, the two processes meet at τ. By the Meeting Lemma, $\mathbf{p}_\tau = \hat{\mathbf{p}}_\tau$, which implies that $\hat{\tau} \leq \tau$, contradicting the assumption that $\tau < \hat{\tau}$. Therefore, if the baseline process stops at $\tau < \hat{\tau}$, then $i \notin \mathcal{I}^*$.

Next, observe that $k(-i,\tau) = \hat{k}(-i,\hat{\tau})$. Otherwise, one of the two processes $\gamma \in \{\alpha,\beta\}$ focuses on items other than i in fewer periods by $\tau(\gamma)$ than the other process γ' does by $\tau(\gamma')$; that is, $k(-i,\tau(\gamma'),\gamma') > k(-i,\tau(\gamma),\gamma)$. By the Coupling Lemma, there exists a period t such that the process γ' does not stop by t, $k(-i,t;\gamma') = k(-i,\tau(\gamma);\gamma)$ and $\mathbf{p}^{-i}_t(\gamma') = \mathbf{p}^{-i}_{\tau(\gamma)}(\gamma)$. Since γ stops at $\tau(\gamma)$ and the DM deems $j \neq i$ optimal, it must be that, for some v, $p^j_{\tau(\gamma)} \in C^j(v)$ and for each item $j' \neq i$, either $p^{j'}_{\tau(\gamma)} \in C^{j'}$ or $p^{j'}_{\tau(\gamma)} \in L^{j'}(v)$. Therefore, non-wastefulness I and II imply that

the strategy γ' focuses only on item i in each period $s > t$. Hence $k(-i, \tau(\gamma'), \gamma') = k(-i, t; \gamma') = \hat{k}(-i, \tau(\gamma), \gamma)$.

Therefore, $\mathbf{p}_\tau^{-i} = \hat{\mathbf{p}}_{\hat{\tau}}^{-i}$. Since i is not the unique optimal item under either process, there exists some v and a nonempty set $\mathcal{I}^{-i,*} \subseteq \mathcal{I} \setminus \{i\}$ such that $p_\tau^j = \hat{p}_{\hat{\tau}}^j \in C^j(v)$ for each $j \in \mathcal{I}^{-i,*}$, and $p_\tau^j = \hat{p}_{\hat{\tau}}^j \in L^j(v)$ for each $j \neq i$ such that $j \notin \mathcal{I}^{-i,*}$. Since the DM does not deem i optimal under the baseline process, $p_\tau^i \in L^i(v)$ and $\mathcal{I}^* = \mathcal{I}^{-i,*}$. The DM may deem i optimal under the manipulated process, and hence either $\hat{\mathcal{I}}^* = \mathcal{I}^{-i,*}$ or $\hat{\mathcal{I}}^* = \mathcal{I}^{-i,*} \cup \{i\}$. Therefore, either $\hat{\mathcal{I}}^* = \mathcal{I}^*$ or $\hat{\mathcal{I}}^* = \mathcal{I}^* \cup \{i\}$, and properties (i-iii) hold. □

For (7) and (8), observe that optimality and monotonicity of σ and statements 1 and 2 of Lemma 6 imply, respectively, that $\sigma(i, \hat{\mathcal{I}}^*) \geq \sigma(i, \mathcal{I}^*)$ and $\sigma(j, \hat{\mathcal{I}}^*) \leq \sigma(j, \mathcal{I}^*)$ in each draw. For (9) and (10), let $h^j(t) = 1_{\tau \leq t} \sigma(j, \mathcal{I}^*)$ and $\hat{h}^j(t) = 1_{\hat{\tau} \leq t} \sigma(j, \hat{\mathcal{I}}^*)$. Statement 1 of Lemma 6 implies that $h^i(t) \leq \hat{h}^i(t)$ for the target item i and any period t. Statement 2 of the lemma implies that $h^j(t) \geq \hat{h}^j(t)$ for all items $j \neq i$ and any period t. Taking expectations across draws yields the result.

A.4 Computation of the ex ante demand from Example 4.1

We characterize D_{ea}^i for $i = 1$; the case for $i = 2$ is symmetric. The following computation shows that

$$D_{\text{ea}}^1(\mathbf{p}; \alpha^*) = \begin{cases} \frac{\overline{p}-p^1}{2(\overline{p}-p^2)} \left(1 - \frac{(\overline{p}-p^2)^2}{(\overline{p}-\underline{p})^2}\right) + \frac{p^1-p^2}{\overline{p}-p^2} & \text{if } p^1 \geq p^2, \\ \frac{\overline{p}-p^2}{2(\overline{p}-p^1)} \left(1 - \frac{(\overline{p}-p^1)^2}{(\overline{p}-\underline{p})^2}\right) & \text{if } p^1 \leq p^2. \end{cases} \tag{13}$$

If $p_0^1 = p_0^2 = p$, then α^* stops with beliefs $(p_\tau^1, p_\tau^2) = (\underline{p}, \underline{p})$ (and hence the outside option is chosen) with probability $\left(\frac{\overline{p}-p}{\overline{p}-\underline{p}}\right)^2$. By symmetry, conditional on not choosing the outside option, the DM chooses item 1 with probability 1/2. Thus

$$D_{\text{ea}}^1(p, p; \alpha^*) = \frac{1}{2}\left(1 - \left(\frac{\overline{p}-p}{\overline{p}-\underline{p}}\right)^2\right).$$

Now consider prior beliefs such that $p_0^1 > p_0^2$. The strategy α^* initially focuses on item 1 until $p_t^1 = \overline{p}$ or $p_t^1 = p_0^2$. In the former case, which occurs with probability $\frac{p_0^1-p_0^2}{\overline{p}-p_0^2}$, the DM chooses item 1. In the latter case, which occurs with probability $\frac{\overline{p}-p_0^1}{\overline{p}-p_0^2}$, the DM chooses item 1 with probability $D_{\text{ea}}^1(p_0^2, p_0^2; \alpha^*)$. Therefore, for $p^1 > p^2$,

$$D_{\text{ea}}^1(p^1, p^2; \alpha^*) = \frac{p^1-p^2}{\overline{p}-p^2} + \frac{\overline{p}-p^1}{\overline{p}-p^2} D_{\text{ea}}^1(p^2, p^2; \alpha^*),$$

in agreement with (13).

Finally, consider prior beliefs such that $p_0^1 < p_0^2$. The strategy α^* initially focuses on item 2 until $p_t^2 = \overline{p}$ or $p_t^2 = p_0^1$. In the former case, the DM chooses item 2. In the latter case, which occurs

with probability $\frac{\overline{p}-p_0^2}{\overline{p}-p_0^1}$, the DM chooses item 1 with probability $D^1_{\text{ea}}(p_0^1, p_0^1; \alpha^*)$. Thus, for $p^1 < p^2$ we have

$$D^1_{\text{ea}}(p^1, p^2; \alpha^*) = \frac{\overline{p} - p^2}{\overline{p} - p^1} D^1_{\text{ea}}(p^1, p^1; \alpha^*), \tag{14}$$

as needed.

A.5 Computation of the ex ante demand from Example 4.2

We characterize D^i_{no} for $i = 1$; the case of $i = 2$ is symmetric. By symmetry of α^{**}, $D^1_{\text{no}}(p^1, p^2; \alpha^{**}) = D^2_{\text{no}}(p^2, p^1; \alpha^{**}) = 1 - D^1_{\text{no}}(p^2, p^1; \alpha^{**})$, and thus it suffices to compute $D^1_{\text{no}}(p^1, p^2; \alpha^{**})$ for $p_1 \geq p_2$ only. Demand D^1_{no} for this set of beliefs is characterized in (15)—(18) below.

Consider beliefs on the two main diagonals. First, symmetry implies that $D^1_{\text{no}}(p, p; \alpha^{**}) = 1/2$ whenever $\underline{p} < p < \overline{p}$. Second, consider prior beliefs of the form $(p_0^1, p_0^2) = (p, 1-p)$, where $1/2 < p < \overline{p}$. Starting from such a prior, if the strategy α^{**} never leads to beliefs $(1/2, 1/2)$, then it stops either with $p_\tau^2 = \underline{p}$ or with $p_\tau^1 = \overline{p}$; in either case, the DM chooses item 1. If α^{**} does lead to beliefs $(1/2, 1/2)$, then the DM chooses item 1 with probability $1/2$. Therefore, if $1/2 < p < \overline{p}$, $D^1_{\text{no}}(p, 1-p; \alpha^{**}) = 1 - x + x/2$ where x is the probability that the DM's beliefs reach $(1/2, 1/2)$.

The value of x may be computed as follows. Consider the event that π_κ^1 stops at $\underline{p}$ and π_κ^2 stops at $\overline{p}$. The ex ante probability of this event is

$$\frac{\overline{p} - p}{\overline{p} - \underline{p}} \frac{1 - p - \underline{p}}{\overline{p} - \underline{p}} = \left(\frac{\overline{p} - p}{\overline{p} - \underline{p}}\right)^2,$$

where the equality follows from $\underline{p} = 1 - \overline{p}$. Since this event can occur only if the DM's beliefs reach $(1/2, 1/2)$, in which case it occurs with probability $1/4$, we have

$$\left(\frac{\overline{p} - p}{\overline{p} - \underline{p}}\right)^2 = \frac{x}{4}$$

and

$$D^1_{\text{no}}(p, 1-p; \alpha^{**}) = 1 - 2\left(\frac{\overline{p} - p}{\overline{p} - \underline{p}}\right)^2$$

whenever $1/2 < p < \overline{p}$.

Now consider beliefs in the set $\{\mathbf{p} : p_1 \geq p_2\}$. We partition this set into four disjoint subsets:

1. For $\{\mathbf{p} : p^1 \in (\underline{p}, 1/2], p^2 \in (\underline{p}, p^1]\}$, α^{**} initially focuses on item 2 until p_t^2 reaches $\underline{p}$ or p^1. Thus, for $\mathbf{p}$ in this subset,

$$D^1_{\text{no}}(\mathbf{p}; \alpha^{**}) = \frac{p^1 - p^2}{p^1 - \underline{p}} + \frac{p^2 - \underline{p}}{p^1 - \underline{p}} D^1_{\text{no}}(p^1, p^1; \alpha^{**}) = \frac{p^1 - p^2/2 - \underline{p}/2}{p^1 - \underline{p}}. \tag{15}$$

2. For $\{\mathbf{p} : p^1 \in (1/2, \overline{p}], p^2 \in (\underline{p}, 1 - p^1]\}$, α^{**} initially focuses on item 2 until p_t^2 reaches $\underline{p}$ or

$1-p^1$. Thus, for $\mathbf{p}$ in this subset,

$$D^1_{\text{no}}(\mathbf{p};\alpha^{**}) = \frac{1-p^1-p^2}{1-p^1-\underline{p}} + \frac{p^2-\underline{p}}{1-p^1-\underline{p}} D^1_{\text{no}}(p^1, 1-p^1; \alpha^{**}) = 1 - 2\frac{(\overline{p}-p^1)(p^2-\underline{p})}{(\overline{p}-\underline{p})^2}. \quad (16)$$

3. For $\{\mathbf{p} : p^1 \in (1/2, \overline{p}], p^2 \in (1-p^1, 1/2]\}$, α^{**} initially focuses on item 1 until p^1_t reaches $1-p^2$ or $\overline{p}$. Thus, for $\mathbf{p}$ in this subset,

$$D^1_{\text{no}}(\mathbf{p};\alpha^{**}) = \frac{p^1-1+p^2}{\overline{p}-1+p^2} + \frac{\overline{p}-p^1}{\overline{p}-1+p^2} D^1_{\text{no}}(1-p^2, p^2; \alpha^{**}) = 1 - 2\frac{(\overline{p}-p^1)(p^2-\underline{p})}{(\overline{p}-\underline{p})^2}. \quad (17)$$

4. For $\{\mathbf{p} : p^1 \in (1/2, \overline{p}], p^2 \in (1/2, p^1]\}$, α^{**} initially focuses on item 1 until p^1_t reaches p^2 or $\overline{p}$. Thus, for $\mathbf{p}$ in this subset,

$$D^1_{\text{no}}(\mathbf{p};\alpha^{**}) = \frac{p^1-p^2}{\overline{p}-p^2} + \frac{\overline{p}-p^1}{\overline{p}-p^2} D^1_{\text{no}}(p^2, p^2; \alpha^{**}) = \frac{p^1/2+\overline{p}/2-p^2}{\overline{p}-p^2}. \quad (18)$$