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Mean-Spread-Preserving Transformations

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Mean-Spread-Preserving Transformations

Abstract:
The purpose of this paper is to define various mean-spread-preserving transformations, which can be considered as generalized versions of the mean-Gini-preserving transformation. The mean-Gini-preserving transformation, which was introduced independently by Zoli (1997, 2002) and Aaberge (2000b), is a combination of progressive and regressive transfers that leaves the Gini coefficient unchanged. It will be demonstrated that the various mean-spread-preserving transformations form a useful basis for judging the normative significance of two alternative sequences of nested Lorenz dominance criteria that can be used to rank Lorenz curves in situations where the Lorenz curves intersect. The two alternative sequences of Lorenz dominance criteria suggest two alternative strategies for increasing the number of Lorenz curves that can be strictly ordered; one that places more emphasis on changes that occur in the lower part of the income distribution and the other that places more emphasis on changes that occur in the upper part of the income distribution. Furthermore, it is demonstrated that the sequences of dominance criteria characterize two separate systems of nested subfamilies of inequality measures and thus provide a method for identifying the least restrictive social preferences required to reach an unambiguous ranking of a given set of Lorenz curves.

Scaling up the introduced Lorenz dominance relations of this paper by the mean income μ and replacing the rank-dependent measures of inequality JP with the rank-dependent social welfare functions $W_P = \mu(1- J_P)$, it can be demonstrated that the present results also apply to the generalized Lorenz curve and moreover provide convenient characterizations of the corresponding social welfare orderings.

Keywords: The Lorenz curve, the Gini coefficient, rank-dependent measures of inequality, generalized Gini families of inequality measures, mean-spread-preserving transformations.

JEL classification: D31, D63

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1. Introduction

Adopting the Lorenz curve as a basis for judging the inequality exhibited by income distributions is in line with common practice in applied economics. The strategy for achieving rankings of Lorenz curves by the extent of inequality depends on whether or not the Lorenz curves intersect. In the case of non-intersecting Lorenz curves the higher of the Lorenz curves (first-degree Lorenz dominance) is recognized as the one with lowest inequality. The normative justification of this practice relies on the Pigou-Dalton principle of transfers, which states that an income transfer from a richer to a poorer individual reduces income inequality, provided that their ranks in the income distribution are unchanged. However, when Lorenz curves intersect a weaker ordering criterion than first-degree Lorenz dominance is called for. To this end it is natural to use either second-degree upward or downward Lorenz dominance, i.e. integrating the Lorenz curves from below or above. Mehran (1976) provided a normative justification of second-degree upward Lorenz dominance by demonstrating that an alternative version of Kolm's (1976) principle of diminishing transfers is consistent with second-degree upward Lorenz dominance. The major difference between Kolm's principle of diminishing transfers and Mehran's principle of positional transfer sensitivity[1] is that the former accounts for the difference in incomes between donors and receivers of income transfers, whereas the latter accounts for the difference in the proportion of individuals between donors and receivers.

In order to make clearer the normative properties of the criterion of second-degree upward Lorenz dominance beyond its transfer sensitivity properties Zoli (1997, 2002) and Aaberge (2000b) introduced the (downside) mean-Gini-preserving transformation, which is a combination of progressive and regressive transfers that leaves the Gini coefficient unchanged. By applying a sequence of (downside) mean-Gini-preserving transformations a Lorenz curve L_1 that second-degree upward dominates a Lorenz curve L_2 can be obtained from L_2, provided that L_1 and L_2 have equal Gini coefficients. Thus, the (downside) mean-Gini-preserving transformations transfer inequality from the lower to the higher parts of the Lorenz curve. However, as observed by Aaberge (2000b) a social decision-maker who supports the Pigou-Dalton principle of transfers may alternatively prefer to use a sequence of transformations that transfer inequality from the higher to the lower parts of the income distribution. This property is captured by an upside version of the (downside) mean-Gini-preserving transformation, where the regressive transfer occurs lower down in the income distribution than the progressive transfer. Applying the upside mean-Gini-preserving transformation is demonstrated to be consistent with using the criterion of second-degree downward Lorenz dominance.

[1] See also Zoli (1999).

To judge the normative significance of upward and downward Lorenz dominance of higher degrees than two this paper introduces and justifies the use of various more strict mean-spread-preserving transformations than the mean-Gini-preserving transformation.

As will be demonstrated in Sections 2 and 3, second-degree Lorenz dominance forms a natural basis for the construction of two separate hierarchical sequences of partial orderings (dominance criteria), where one sequence places emphasize on changes that occur in the lower part of the Lorenz curve whereas the other places emphasize on changes that occur in the upper part of the Lorenz curve. The hierarchical and nested structure of the dominance criteria appears to be useful in empirical applications since we are allowed to identify the lowest degree of dominance required to reach unambiguous rankings of Lorenz curves. As the dominance criteria of each sequence are nested these strategies also allow us to identify the value judgments that are needed to reach an unambiguous ranking of Lorenz curves. To this end, appropriate mean-spread-preserving transformations, which can be considered as generalizations of the mean-Gini-preserving transformation, have been introduced. Moreover, Section 3 demonstrates that the two hierarchical sequences of Lorenz dominance criteria divide the family of rank-dependent measures of inequality into two corresponding hierarchical systems of nested subfamilies that offer two different inequality aversion profiles; one exhibits successively higher degrees of downside inequality aversion whereas the other exhibits successively higher degrees of upside inequality aversion. Since the criteria of Lorenz dominance provide convenient computational methods, these results can be used to identify the largest subfamily of the family of rank-dependent measures of inequality and thus the least restrictive social preferences required to reach unambiguous ranking of any given set of Lorenz curves. Section 4 uses these characterization results to arrange the members of two different generalized Gini families of inequality measures into subfamilies according to their relationship to Lorenz dominance of various degrees. Section 5 summarizes the conclusions of the paper and briefly discusses the use of the obtained results as a basis for deriving dominance criteria for generalized Lorenz curves.

2. Lorenz dominance and mean-Gini-preserving transformations

The Lorenz curve L for a cumulative income distribution F with mean μ is defined by

$$L(u) = \frac{1}{\mu}\int_0^u F^{-1}(t)\,dt, \quad 0 \le u \le 1, \tag{1}$$

where $F^{-1}(t) = \inf\{x : F(x) \ge t\}$ is the left inverse of F. Thus, the Lorenz curve L(u) shows the share of total income received by the poorest 100 u per cent of the population. Note that F can either be a

discrete or a continuous distribution function. Although the former is what we actually observe, the latter often allows simpler derivation of theoretical results and is a valid large sample approximation. Thus, in most cases below F will be assumed to be a continuous distribution function, but the assumption of a discrete distribution function will be used where appropriate.

Under the restriction of equal mean incomes the problem of ranking Lorenz curves formally corresponds to the problem of choosing between uncertain prospects. This relationship has been utilized by e.g. Kolm (1969) and Atkinson (1970) to characterize the criterion of non-intersecting Lorenz curves in the case of distributions with equal mean incomes. This was motivated by the fact that in cases of equal mean incomes the criterion of non-intersecting Lorenz curves is equivalent to second-degree stochastic dominance[2], which means that the criterion of non-intersecting Lorenz curves obeys the Pigou-Dalton principle of transfers. *The Pigou-Dalton principle of transfers* states that an income transfer from a richer to a poorer individual reduces income inequality, provided that their ranks in the income distribution are unchanged, and is defined formally by[3]

DEFINITION 2.1. *(**The Pigou-Dalton principle of transfers**.) Consider a discrete income distribution F. A transfer δ from a person with income $F^{-1}(t)$ to a person with income $F^{-1}(s)$, where the transfer is assumed to be rank-preserving, is said to reduce inequality in F when $s < t$ and raise inequality in F when $s > t$. The former transfer will be denoted **a Pigou-Dalton progressive transfer** and the latter transfer **a Pigou-Dalton regressive transfer**.*

To perform inequality comparisons with Lorenz curves we can deal with distributions with equal means, or alternatively simply abandon the assumption of equal means and consider distributions of relative incomes.[4] The latter approach normally forms the basis of empirical studies.

The standard criterion of non-intersecting Lorenz curves, called *first-degree Lorenz dominance*, is based on the following definition[5].

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[2] For a proof see Hardy, Littlewood and Polya (1934).

[3] Note that this definition of the Pigou-Dalton principle of transfers was proposed by Fields and Fei (1978).

[4] The importance of focusing on relative incomes was acknowledged already by Plato who proposed that the ratio of the top income to the bottom should be less than four to one (see Cowell, (1977)). See also Sen's (1992) discussion of relative deprivation and Smith's (1979) discussion of necessities.

[5] Note that most analyses of Lorenz dominance apply a definition that excludes the requirement of strict inequality for some u.

DEFINITION 2.2. *A Lorenz curve L_1 is said to* ***first-degree dominate*** *a Lorenz curve L_2 if*

$$L_1(u) \geq L_2(u) \quad \textit{for all} \quad u \in [0,1]$$

and the inequality holds strictly for some $u \in \langle 0,1 \rangle$.

In order to examine the relationship between various Lorenz dominance criteria and the measurement of inequality we will rely on the *family of rank-dependent measures of inequality*[6] defined by

(2) $$J_P(L) = 1 - \int_0^1 P'(u)\,dL(u) = 1 - \frac{1}{\mu}\int_0^1 P'(u)F^{-1}(u)\,du$$

where L is the Lorenz curve of the income distribution F with mean μ and the weight-function P' is the derivative of a continuous, differentiable and concave function P defined on the unit interval where $P(0) = 0$ and $P(1) = 1$. To ensure that J_P has the unit interval as its range the condition $P'(1) = 0$ is imposed on P. As demonstrated by Yaari (1987,1988) and Aaberge (2001) the J_P-family represents a preference relation defined either on the class of distribution functions (F) or on the class of Lorenz curves (L), where P can be interpreted as a preference function of a social decision-maker. The preference function P assigns weights to the incomes of the individuals in accordance with their rank in the income distribution. Therefore, the functional form of P reveals the attitude towards inequality of a social decision-maker who employs J_P to judge between Lorenz curves. The most well-known member of the J_P-family is the Gini coefficient, which is defined by

(3) $$G(L) = 1 - 2\int_0^1 L(u)\,du = 1 - \frac{2}{\mu}\int_0^1 (1-u)F^{-1}(u)du\,.$$

As demonstrated by Yaari (1988), the J_P-family of inequality measures can be used as a basis for characterizing first-degree Lorenz dominance. For the sake of completeness the characterization result of first-degree Lorenz dominance given by Yaari (1988) is reproduced in Theorem 2.1 below, where **L** denotes the family of Lorenz curves and $\mathbf{P}_1$ is a class of preference functions defined by

[6] Mehran (1976) introduced the J_P-family by relying on descriptive arguments. A slightly different version of J_P was introduced by Piesch (1975), whereas Giaccardi (1950) considered a discrete version of J_P. For alternative normative motivations of the J_P-family and various subfamilies of the J_P-family we refer to Donaldson and Weymark (1980, 1983), Weymark (1981), Yaari (1987, 1988), Ben Porath and Gilboa (1994), Aaberge (2001) and Gajdos (2002).

$$\mathbf{P}_1 = \{P : P' \text{ and } P'' \text{ are continuous on } [0,1],\ P'(t) > 0 \text{ and } P''(t) < 0 \text{ for } t \in \langle 0,1 \rangle, \text{ and } P'(1) = 0\}.$$

THEOREM 2.1. (Fields and Fei (1978) and Yaari (1988)). *Let L_1 and L_2 be members of L. Then the following statements are equivalent,*

(i) *L_1 first-degree dominates L_2*

(ii) *L_1 can be obtained from L_2 by a sequence of Pigou-Dalton progressive transfers*

(iii) *L_2 can be obtained from L_1 by a sequence of Pigou-Dalton regressive transfers*

(iv) $J_P(L_1) < J_P(L_2)$ *for all* $P \in \boldsymbol{P}_1$

We refer to Fields and Fei (1978) for a proof of the equivalence between (i) and (ii) (and (iii))[7] and to Yaari (1988) for a proof of the equivalence between (i) and (iv).

Atkinson (1970) defined inequality aversion as equivalent to risk aversion in the theory of choice under uncertainty. This was motivated by the fact that the Pigou-Dalton transfer principle is identical to the principle of mean-preserving spread introduced by Rothschild and Stiglitz (1970), which is equivalent to the condition of dominating non-intersecting Lorenz curves. Thus, we adopt the following definition.

DEFINITION 2.3. *A social decision-maker that supports the Pigou-Dalton principle of transfers is said to exhibit* ***inequality aversion****.*

A social decision-maker who prefers the dominating one of non-intersecting Lorenz curves favors transfers of incomes which reduce the differences between the income shares of the donor and the recipient, and is therefore said to be *inequality averse*.

The characterization of the condition of first-degree Lorenz dominance provided by Theorem 2.1 shows that non-intersecting Lorenz curves can be ordered without specifying further the functional form of the preference function P other than P being strictly concave. This means that J_P satisfies the Pigou-Dalton principle of transfers for concave P-functions. To deal with situations where Lorenz curves intersect a weaker principle than first-degree Lorenz dominance is called for. To this end we may employ *second-degree upward Lorenz dominance* defined by

[7] See Rothschild and Stiglitz (1973) for a proof of the equivalence between (i), (ii) and (iii) in the case where the rank-preserving condition is abandoned in the definition of the Pigou-Dalton principle of transfers.

DEFINITION 2.4A. *A Lorenz curve L_1 is said to* ***second-degree upward dominate*** *a Lorenz curve L_2 if*

$$\int_0^u L_1(t)\,dt \geq \int_0^u L_2(t)\,dt \quad \textit{for all} \quad u \in [0,1]$$

and the inequality holds strictly for some $u \in \langle 0,1 \rangle$.

The term upward dominance refers to the fact that the Lorenz curves are aggregated from below[8]. The aggregated Lorenz curve can be considered as a sum of weighted income shares, where the weights decrease linearly with increasing rank of the income receiver in the income distribution. Thus, a social decision-maker who prefers the second-degree upward dominating of two intersecting Lorenz curves pays more attention to inequality in the lower than in the upper part of the income distribution. An alternative ranking criterion to second-degree upward Lorenz dominance can be obtained by aggregating the Lorenz curve from above.

DEFINITION 2.4B. *A Lorenz curve L_1 is said to* ***second-degree downward dominate*** *a Lorenz curve L_2 if*

$$\int_u^1 \left(1 - L_2(t)\right) dt \geq \int_u^1 \left(1 - L_1(t)\right) dt \quad \textit{for all} \quad u \in [0,1]$$

and the inequality holds strictly for some $u \in \langle 0,1 \rangle$.

Note that second-degree downward as well as upward Lorenz dominance preserves first-degree Lorenz dominance since first-degree Lorenz dominance implies second-degree upward as well as second-degree downward Lorenz dominance. Consequently, both dominance criteria are consistent with the Pigou-Dalton principle of transfers. The choice between second-degree upward and downward Lorenz dominance clarifies whether or not equalizing transfers between poorer individuals should be considered more important than those between richer individuals. A social decision-maker who favors second-degree upward Lorenz dominance would most likely prefer third-degree upward Lorenz dominance to third-degree downward Lorenz dominance, because third-degree upward Lorenz dominance places the emphasis on equalizing transfers between poorer individuals, whereas third-

[8] Note that second-degree upward Lorenz dominance is equivalent to a normalized version of third-degree inverse stochastic dominance introduced by Muliere and Scarsini (1989).

degree downward Lorenz dominance places the emphasis on equalizing transfers between richer individuals.

As recognized by Muliere and Scarsini (1989) there is no simple relationship between third-degree stochastic dominance and second-degree upward Lorenz dominance, but that second-degree upward Lorenz dominance is equivalent to third-degree upward inverse stochastic dominance. Thus, a general characterization of second-degree Lorenz dominance or third-degree inverse stochastic dominance in terms of ordering conditions for the utilitarian measures introduced by Kolm (1969) and Atkinson (1970) cannot be obtained. As explained in Section 1 the family of rank-dependent measures of inequality (defined by (2)) appears to form a more convenient basis for judging the normative significance of second-degree and higher degrees of Lorenz dominance than the utilitarian families introduced by Kolm (1969) and Atkinson (1970).

To judge the normative significance of criteria for ranking intersecting Lorenz curves, more powerful principles than the Pigou-Dalton principle of transfers are needed. To this end Kolm (1976a,b) introduced *the principle of diminishing transfers*[9], which for a fixed difference in income considers a transfer from a richer to a poorer person to be more equalizing the further down in the income distribution it takes place.[10] As indicated by Shorrocks and Foster (1987) and Muliere and Scarsini (1989) the principle of diminishing transfers is, however, not consistent with second-degree upward Lorenz dominance. Mehran (1976) introduced an alternative version of the principle of diminishing transfers by accounting for the difference in the proportion of individuals between donors and receivers of the income transfers rather than for the difference in income. The principle introduced by Mehran (1976) proves to characterize second-degree upward Lorenz dominance. To provide a formal definition of this principle, called the principle of positional transfer sensitivity by Zoli (1999), let I be an inequality measure and let $\Delta I_t(\delta,h)$ denote the change in I resulting from a transfer δ from a person with income $F^{-1}(t+h)$ to a person with income $F^{-1}(t)$ that leaves their ranks in the income distribution F unchanged, where F is assumed to be a discrete distribution for a finite population. Thus, $\Delta I_t(\delta,h)$ is a negative number[11]. Furthermore, let $\Delta_1 I_{s,t}(\delta,h)$ be defined by

$$\Delta_1 I_{s,t}(\delta,h) = \Delta I_t(\delta,h) - \Delta I_s(\delta,h). \tag{4}$$

As will become evident later it will be convenient to denote Mehran's principle of transfers *the principle of first-degree downside positional transfer sensitivity*.

[9] Denoted aversion to downside inequality by Davis and Hoy (1994, 1995).

[10] For a formal definition, see Kolm (1976a,b).

[11] For convenience the dependence of I on F is suppressed in the notation for I.

DEFINITION 2.5A. *Consider a discrete income distribution F and rank-preserving transfers δ from individuals with ranks $s+h$ and $t+h$ to individuals with ranks s and t in F. Then the inequality measure I is said to satisfy the* ***principle of first-degree downside positional transfer sensitivity (first-degree DPTS)*** *if*

$$\Delta_I I_{s,t}(\delta,h) > 0 \ \ when \ \ s < t .$$

Mehran (1976) demonstrated that J_P defined by (2) satisfies the principle of first-degree positional transfer sensitivity if and only if $P'''(t) > 0$ [12]. Moreover, as stated in Theorem 2.2A below dominance for all J_P that satisfy the Pigou-Dalton principle of transfers and the principle of first-degree downside positional transfer sensitivity proves to be equivalent to the condition of second-degree upward Lorenz dominance.

The condition of second-degree downward Lorenz dominance proves to be equivalent to the condition that $P'''(t) < 0$, when J_P for $P \in \mathbf{P}_1$ is used as a ranking criterion for Lorenz curves. A social decision-maker who employs J_P with $P'''(t) < 0$ considers a given transfer of money from a richer to a poorer person to be more equalizing the higher it occurs in the income distribution, provided that the proportions of the population located between the receivers and the donors are equal. A formal definition of this principle, that we will call *the principle of first-degree upside positional transfer sensitivity (first-degree UPTS)*, is given by

DEFINITION 2.5B. *Consider a discrete distribution F and rank-preserving transfers δ from individuals with ranks $s+h$ and $t+h$ to individuals with ranks s and t in F. Then the inequality measure I is said to satisfy* ***the principle of first-degree upside positional transfer sensitivity (first-degree UPTS)*** *if*

$$\Delta_I I_{s,t}(\delta,h) < 0 \ \ when \ \ s < t .$$

Let $\mathbf{P}_{12}^*$ be a family of preference functions related to J_P and defined by

$$\mathbf{P}_{12}^* = \left\{ P : P \in \mathbf{P}_1,\ P''' \text{ is continuous on } [0,1] \text{ and } P'''(t) > 0 \text{ for } t \in \langle 0,1 \rangle \right\} .$$

[12] Aaberge (2000a) demonstrated that J_P defined by (2) satisfies Kolm's principle of diminishing transfers under conditions that depend on the shape of the preference function P as well as on the shape of the income distribution F.

The following result provides a characterization of second-degree upward Lorenz dominance.[13]

THEOREM 2.2A. *Let L_1 and L_2 be members of* $\boldsymbol{L}$. *Then the following statements are equivalent,*

(i) L_1 *second-degree upward dominates* L_2

(ii) $J_P(L_1) < J_P(L_2)$ *for all* $P \in \boldsymbol{P}_{12}^*$

(iii) $J_P(L_1) < J_P(L_2)$ *for all* $P \in \boldsymbol{P}_1$ *being such that* J_P *obeys the principle of first-degree DPTS.*

(Proof in Appendix).

To ensure equivalence between second-degree upward Lorenz dominance and J_P-measures as decision criteria, Theorem 2.2A shows that it is necessary to restrict the preference functions P to be concave with positive third derivatives. If, by contrast, P has negative third derivative, then Theorem 2.2B yields the downward dominance analogy to Theorem 2.2A.

Let $\mathbf{P}_{22}^*$ be a family of preference functions related to J_P and defined by

$$\mathbf{P}_{22}^* = \left\{ \mathrm{P} : \mathrm{P} \in \mathbf{P}_1,\ \mathrm{P}''' \text{ is continuous on } [0,1] \text{ and } \mathrm{P}'''(\mathrm{t}) < 0 \text{ for } \mathrm{t} \in \langle 0,1 \rangle \right\}.$$

THEOREM 2.2B. Let L_1 and L_2 be members of $\mathbf{L}$. Then the following statements are equivalent,

(i) L_1 *second-degree downward dominates* L_2

(ii) $J_P(L_1) < J_P(L_2)$ *for all* $P \in \boldsymbol{P}_{22}^*$

(iii) $J_P(L_1) < J_P(L_2)$ *for all* $P \in \boldsymbol{P}_1$ *being such that* J_P *obeys the principle of first-degree UPTS.*

(Proof in Appendix).

REMARK. It follows from the proofs of Theorem 2.2A and 2.2B that the condition $\mathrm{P}''(\mathrm{t}) < 0$ for all $\mathrm{t} \in \langle 0,1 \rangle$ can be relaxed and replaced by the conditions $\mathrm{P}''(1) \le 0$ and $\mathrm{P}''(0) \le 0$, respectively. Moreover, note that the equivalence between (ii) and (iii) in Theorem 2.2A means that

[13] Note that a slightly different version of the equivalence between (i) and (ii) in Theorem 2.2A is proved by Zoli (1999). Actually, when we restrict to cases of equal means Proposition 2 of Zoli (1999) and Theorem 2.2 yield identical results.

any J_P with $P'''(t) > 0$ obeys the first-degree DPTS.[14] A similar remark can be made for Theorem 2.2B. Thus, any J_P with $P'''(t) < 0$ obeys the first-degree UPTS. However, the relevance of using measures of inequality that do not obey the Pigou-Dalton of transfers $\left(P''(t) > 0\right)$ may be questioned.

An inequality averse social decision-maker that supports the criterion of second-degree upward Lorenz dominance will act in line with the principle of first-degree DPTS and assign more weight to changes that take place in the lower part of the Lorenz curve than to changes that occur in the upper part of the Lorenz curve. By contrast, the criterion of second-degree downward Lorenz dominance emphasizes changes that occur in the upper part of the Lorenz curve. Thus, an inequality averse social decision-maker that employs the criterion of second-degree downward Lorenz dominance acts in favor of the Pigou-Dalton principle of transfers and the principle of first-degree UPTS. To characterize social preferences of these types we adopt the following definition.

DEFINITION 2.6. *An inequality averse social decision-maker that supports the Pigou-Dalton principle of transfers and the principle of first-degree DPTS (UPTS) is said to exhibit* ***downside (upside) inequality aversion of first-degree***.

Theorems 2.2A and 2.2B demonstrate that the principles of upward and downward Lorenz dominance can be used to divide J_P-measures into wide families of inequality measures that differ in the measures' sensitivity to changes (transfers) in the lower or upper part of the Lorenz curve. Members of the family $\left\{J_P : P \in \mathbf{P}_{12}^*\right\}$ give more weight to changes that take place lower down in the Lorenz curve, whereas members of the family $\left\{J_P : P \in \mathbf{P}_{22}^*\right\}$ give more weight to changes higher up in the Lorenz curve. Note that $P(t) = 2t - t^2$, the P-function that corresponds to the Gini coefficient, is neither included in $\mathbf{P}_{12}^*$ nor in $\mathbf{P}_{22}^*$. Since $P'''(t) = 0$ for all t, the Gini coefficient neither preserves second-degree upward Lorenz dominance nor second-degree downward Lorenz dominance apart from the case when the inequality in (i) of Theorems 2.2A and 2.2B holds strictly for $u = 1$ and $u = 0$, respectively.[15] Thus, the suggestion of Muliere and Scarsini (1989) that the Gini coefficient is coherent with second-degree upward Lorenz dominance requires a definition of second-degree that abandons

[14] See Chateauneuf et al. (2002) for an alternative proof of this result.

[15] Aaberge (2000a) gave an alternative interpretation of this property by demonstrating that the Gini coefficient attaches an equal weight to a given transfer irrespective of where it takes place in the income distribution, as long as the income transfer occurs between individuals with the same difference in ranks.

the condition of strict inequality (for some $u \in \langle 0,1 \rangle$). However, by assuming that the Lorenz curves cross only once the following results hold.[16]

PROPOSITION 2.1A. *Assume that L_1 and L_2 are singly intersecting Lorenz curves and L_1 crosses L_2 initially from above, and let $G(L_1)$ and $G(L_2)$ be the two corresponding Gini coefficients. Then the following statements are equivalent,*

(i) $J_P(L_1) < J_P(L_2)$ *for all* $P \in \boldsymbol{P}_{12}^*$

(ii) $G(L_1) \leq G(L_2)$.

(Proof in Appendix)

PROPOSITION 2.1B. *Assume that L_1 and L_2 are singly intersecting Lorenz curves and L_2 crosses L_1 initially from above, and let $G(L_1)$ and $G(L_2)$ be the two corresponding Gini coefficients. Then the following statements are equivalent,*

(i) $J_P(L_1) < J_P(L_2)$ *for all* $P \in \boldsymbol{P}_{22}^*$

(ii) $G(L_1) \leq G(L_2)$.

The proof of Proposition 2.1B can be achieved by following the line of reasoning used in the proof of Proposition 2.1A. Note that Proposition 2.1A can be considered as a dual version of the results of Shorrocks and Foster (1987) and Dardanoni and Lambert (1988) that clarify the relationship between third-degree (upward) stochastic dominance, ordering conditions for the coefficient of variation and transfer sensitive measures of inequality.

As a preference ordering on **L** the Gini coefficient in general favors neither the lower nor the upper part of the Lorenz curves. Therefore, if we restrict the ranking problem to Lorenz curves with equal Gini coefficients, second-degree upward and downward dominance coincide in the sense that a Lorenz curve L_1 that second-degree upward dominates a Lorenz curve L_2 is always second-degree downward dominated by L_2. Thus, it is clear that L_1 (and the corresponding distribution function) can be obtained from L_2 (the corresponding distribution function) by employing a set of Pigou-Dalton progressive transfers in combination with an equal set of Pigou-Dalton regressive transfers that leaves the Gini mean difference (μG) unchanged, and in which the progressive transfer

[16] Zoli (1999) provided a result similar to Proposition 2.1A for singly intersecting generalized Lorenz curves under the condition of equal means.

of each pair of progressive/regressive transfers occurs lower down in the income distribution than the regressive transfer. We call such a change *a downside mean-Gini-preserving transformation (downside MGPT)*[17]. A formal definition of downside MGPT is given below.

DEFINITION 2.7A. *Consider a discrete distribution F and a Pigou-Dalton progressive transfer* δ_i *from a person with income* $F^{-1}(t_i)$ *to a person with income* $F^{-1}(s_i)$ *and a Pigou-Dalton regressive transfer* γ_i *from a person with income* $F^{-1}(u_i)$ *to a person with income* $F^{-1}(v_i)$ *for* $i = 1,2,...,N$ *. This sequence of combinations of progressive/regressive transfers is a* ***downside mean-Gini-preserving transformation (downside MGPT)*** *provided that* $s_i < t_i < u_i < v_i$ *for* $i = 1,2,...,N$ *and the* (δ_i, γ_i) *pairs are such that the Gini coefficient of the post-transfer distribution is equal to the Gini coefficient of F.*

Thus, by applying the downside MGPT inequality is transferred from lower to higher parts of the Lorenz curve. By contrast, a social decision-maker who favors second-degree downward Lorenz dominance will apply the progressive transfer of each pair of progressive/regressive transfers higher up in the income distribution than the regressive transfer. Such a change will be called *an upside mean-Gini-preserving transformation (upside MGPT)* [18]. In this case inequality is transferred from the higher to the lower parts of the Lorenz curve and the corresponding income distribution.

DEFINITION 2.7B. *Consider a discrete distribution F and a Pigou-Dalton regressive transfer* δ_i *from a person with income* $F^{-1}(s_i)$ *to a person with income* $F^{-1}(t_i)$ *and a Pigou-Dalton progressive transfer* γ_i *from a person with income* $F^{-1}(v_i)$ *to a person with income* $F^{-1}(u_i)$ *for* $i = 1,2,...,N$ *. This sequence of combinations of regressive/progressive transfers is an* ***upside mean-Gini-preserving transformation (upside MGPT)*** *provided that* $s_i < t_i < u_i < v_i$ *for* $i = 1,2,...,N$ *and the* (δ_i, γ_i) *pairs are such that the Gini coefficient of the post-transfer distribution is equal to the Gini coefficient of F.*

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[17] Note that a downside MGPT is equivalent to the favorable composite positional transfer discussed by Zoli (2002).

[18] Note that the MGPT transformations are analogous to the mean-variance-preserving transformation (MVPT) introduced by Menezes et al. (1980) and used by Shorrocks and Foster (1987), Dardanoni and Lambert (1988) and Davis and Hoy (1994) as a basis for analysing the implications of third-degree stochastic dominance on measurement of inequality and social welfare. The major difference between these methods of transformation is that the (downside) MGPT is equivalent to second-degree upward Lorenz dominance (third-degree upward inverse stochastic dominance), whereas the MVPT is equivalent to third degree (upward) stochastic dominance. Moreover, the MGPT relies on the Gini mean difference rather than the variance as a measure of spread. Note that the Gini mean difference was used as an (robust) alternative to the variance as a measure of spread long before Gini introduced it as a measure of inequality (see David (1968)).

The following theorem demonstrates that a Lorenz curve L_1 that second-degree upward dominates a Lorenz curve L_2 can be obtained from L_2 be a sequence of downside MGPTs' and that the Lorenz curve L_2 can be obtained from L_1 by a sequence of upside MGPTs'.

THEOREM 2.3. *Let L_1 and L_2 be Lorenz curves with equal Gini coefficients. Then the following statements are equivalent,*

(i) *L_1 second-degree upward dominates L_2*

(ii) *L_2 second-degree downward dominates L_1*

(iii) *L_1 can be obtained from L_2 by a sequence of downside MGPTs'.*

(iv) *L_2 can be obtained from L_1 by a sequence of upside MGPTs'.*

(Proof in Appendix)[19].

When the ranking problem is restricted to Lorenz curves with equal Gini coefficients then the condition that a downside MGPT (an upside MGPT) reduces inequality is equivalent to the condition for satisfying the principle of first-degree DPTS (UPTS).

3. Lorenz dominance of ith-degree and mean-spread-preserving transformations

Since situations where second-degree (upward or downward) Lorenz dominance does not provide unambiguous ranking of Lorenz curves may arise, it will be useful to introduce weaker dominance criteria than second-degree Lorenz dominance. To this end we will introduce two hierarchical sequences of nested Lorenz dominance criteria; one departs from second-degree upward Lorenz dominance and the other from second-degree downward Lorenz dominance. As explained in Section 2, the choice between second-degree upward and downward Lorenz dominance clarifies whether focus is turned to changes that take place in the lower or upper part of the income distribution. Thus, a person who favors second-degree upward Lorenz dominance would most likely prefer third-degree and higher degrees of upward Lorenz dominance to third-degree and higher degrees of downward Lorenz dominance. Conversely, when the value judgment of a person is consistent with the criterion of second-degree downward Lorenz dominance, higher degrees of downward Lorenz dominance are likely more acceptable than higher degrees of upward Lorenz dominance.

[19] An alternative proof of the equivalence between (i) and (iii) was provided by Zoli (2002).

As will become evident below it is convenient to use the following notation,

(5) $$L^2(u) = \int_0^u L(t)\,dt = \frac{1}{\mu}\int_0^u (u-t)F^{-1}(t)dt, \quad 0 \le u \le 1,$$

$$L^{i+1}(u) = \int_0^u L^i(t)\,dt, \; 0 \le u \le 1, \; i = 2,3,...,$$

and

(6) $$\tilde{L}^2(u) = \int_u^1 (1-L(t))\,dt = \frac{1}{\mu}\int_u^1 (t-u)F^{-1}(t)dt, \; 0 \le u \le 1,$$

$$\tilde{L}^{i+1}(u) = \int_u^1 \tilde{L}^i(t)\,dt, \; 0 \le u \le 1, \; i = 2,3,....$$

Now, using integration by parts, we obtain the following alternative expressions for L^{i+1} and $\tilde{L}^{i+1}$, respectively,

(7) $$L^{i+1}(u) = \frac{1}{(i-1)!}\int_0^u (u-t)^{i-1} L(t)\,dt = \frac{1}{i!\mu}\int_0^u (u-t)^i F^{-1}(t)dt, \quad i = 2,3,...$$

and

(8) $$\tilde{L}^{i+1}(u) = \frac{1}{(i-1)!}\int_u^1 (t-u)^{i-1}(1-L(t))\,dt = \frac{1}{i!\mu}\int_u^1 (t-u)^i F^{-1}(t)dt, \quad i = 2,3,....$$

It is easily verified that $L^{i+1}(1)$ defined by (7) is a linear transformation of a measure of inequality that belongs to *the extended Gini family of inequality measures*[20] $\{G_i : i \ge 1\}$,

(9) $$L^{i+1}(1) = \frac{1}{(i+1)!}(1-G_i(L)), \; i = 1,2,...$$

where

[20] The extended Gini family of inequality measures was introduced by Donaldson and Weymark (1980, 1983) and Yitzhaki (1983).

(10) $$G_i(L) = 1 - i(i+1)\int_0^1 (1-u)^{i-1} L(u)\,du = \frac{1}{\mu}\int_0^\infty (1-F(x))\left(1-(1-F(x))^i\right)dx,\ i \geq 1.$$

Moreover, from the definition (8) of $\tilde{L}$ we get that

(11) $$\tilde{L}^{i+1}(0) = \frac{1}{(i+1)!}\left(i D_i(L) + 1\right),\ i = 1,2,...$$

where

(12) $$D_i(L) = 1 - (i+1)\int_0^1 u^{i-1} L(u)du = \frac{1}{\mu i}\int_0^\infty F(x)\left(1-F^i(x)\right)dx,\ i = 1,2,...$$

and $\{D_i : i = 1,2,...\}$ is an alternative "generalized" Gini family of inequality measures denoted *the Lorenz family of inequality measures*[21], where $D_1 = G_1$ is equal to the Gini coefficient.

As was demonstrated by Aaberge (2000a) there is a one-to-one correspondence between subfamilies of the extended Gini and the Lorenz families of inequality measures shown by the following equation

(13) $$G_i(L) = 1 + (i+1)\sum_{k=1}^{i} (-1)^k \binom{i}{k} \frac{k}{k+1}\left(1 - D_k(L)\right),\quad i = 1,2,....$$

Thus, the extended Gini subfamily $\{G_i(L) : i = 1,2,...,r\}$ is uniquely determined by the corresponding Lorenz subfamily $\{D_i(L) : i = 1,2,...,r\}$ for any integer r.

Expressions (7) and (8) show that L^{i+1} places more weight on changes in the lower and $\tilde{L}^{i+1}$ on changes in the upper part of the Lorenz curve as i increases.

As generalizations of Definitions 2.4A and 2.4B we introduce the notions of ith-degree upward and downward Lorenz dominance[22]. Note that subscripts i and j in the notation L_j^i and $\tilde{L}_j^i$ used below refer to dominance of ith-degree for Lorenz curve L_j and that L_j^1 is the Lorenz curve L_j and $\tilde{L}_j^1 = 1 - L_j$.

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[21] The Lorenz family of inequality measures was introduced by Aaberge (2000a) and proves to be a subclass of the "illfare-ranked single-series Ginis" discussed by Donaldson and Weymark (1980) and Bossert (1990).

[22] A similar definition of ith degree (upward) inverse stochastic dominance was introduced by Muliere and Scarsini (1989). Note that Definitions 3.1A and 3.1B do not require any restrictions on the Lorenz curves (or the distribution functions) and thus differ in this sense from the definitions of stochastic dominance proposed by Whitmore (1970) and Chew (1983).

DEFINITION 3.1A. *A Lorenz curve L_1 is said to* ***i^{th}-degree upward dominate*** *a Lorenz curve L_2 if*

$$L_1^i(u) \geq L_2^i(u) \quad \textit{for all} \quad u \in [0,1]$$

and the inequality holds strictly for some $u \in \langle 0,1 \rangle$.

DEFINITION 3.1B. *A Lorenz curve L_1 is said to* ***i^{th}-degree downward dominate*** *a Lorenz curve L_2 if*

$$\tilde{L}_2^i(u) \geq \tilde{L}_1^i(u) \quad \textit{for all} \quad u \in [0,1]$$

and the inequality holds strictly for some $u \in \langle 0,1 \rangle$.

Note that $(i+1)^{th}$ -degree upward and downward Lorenz dominance are less restrictive dominance criteria than i^{th}-degree upward and downward Lorenz dominance and thus can prove to be useful decision criteria in situations where i^{th}-degree dominance does not yield an unambiguous ranking of Lorenz curves.

It follows from the definitions (7) and (8) of L^i and $\tilde{L}^i$ that

$$L_1^i(u) \geq L_2^i(u) \text{ for all } u \text{ implies } L_1^{i+1}(u) \geq L_2^{i+1}(u) \text{ for all } u,$$

and that

$$\tilde{L}_2^i(u) \geq \tilde{L}_1^i(u) \text{ for all } u \text{ implies } \tilde{L}_2^{i+1}(u) \geq \tilde{L}_1^{i+1}(u) \text{ for all } u,$$

which means that $(i+1)^{th}$ -degree upward Lorenz dominance preserves i^{th}-degree upward Lorenz dominance and that $(i+1)^{th}$ -degree downward Lorenz dominance preserves i^{th}-degree downward Lorenz dominance.

Thus, the various degrees of upward and downward Lorenz dominance form two separate sequences of nested dominance criteria, which turn out to be useful for dividing the J_P-family of inequality measures into nested subfamilies. To this end it will be convenient to introduce the following notation. Let $P^{(j)}$ denote the jth derivative of P and let $\mathbf{P}_{1i}, \mathbf{P}_{1i}^{*}, \mathbf{P}_{1i}^{**}, \mathbf{P}_{1i}^{***}, \mathbf{P}_{2i}, \mathbf{P}_{2i}^{*}, \mathbf{P}_{2i}^{**}, \mathbf{P}_{2i}^{***}$ be families of preference functions defined by

$$\mathbf{P}_{1i} = \left\{ P : P \in \mathbf{P}_1, P^{(i+1)} \text{ is continuous on } [0,1] \text{ and } P^{(j)}(1) = 0,\ j = 2,3,\ldots,i \right\},$$

$$\mathbf{P}_{1i}^{*} = \left\{ P : P \in \mathbf{P}_{1i} \text{ and } (-1)^{j} P^{(j+1)}(t) > 0 \text{ for } t \in \langle 0,1 \rangle,\ j = 1,2,\ldots,i \right\},$$

$$\mathbf{P}_{1i}^{**} = \left\{ P : P \in \mathbf{P}_{1i} \text{ and } (-1)^{i} P^{(i+1)}(t) > 0 \text{ for } t \in \langle 0,1 \rangle \right\},$$

$$\mathbf{P}_{1i}^{***} = \left\{ P : P \in \mathbf{P}_1, P^{(i+1)} \text{ is continuous on } [0,1],\ (-1)^{i} P^{(i+1)}(t) > 0 \text{ for } t \in \langle 0,1 \rangle \text{ and } (-1)^{j-1} P^{(j)}(1) \geq 0,\ j = 2,3,\ldots,i \right\},$$

$$\mathbf{P}_{2i} = \left\{ P : P \in \mathbf{P}_1, P^{(i+1)} \text{ is continuous on } [0,1] \text{ and } P^{(j)}(0) = 0,\ j = 2,3,\ldots,i \right\},$$

$$\mathbf{P}_{2i}^{*} = \left\{ P : P \in \mathbf{P}_{2i} \text{ and } P^{(j+1)}(t) < 0 \text{ for } t \in \langle 0,1 \rangle,\ j = 1,2,\ldots,i \right\},$$

$$\mathbf{P}_{2i}^{**} = \left\{ P : P \in \mathbf{P}_{2i} \text{ and } P^{(i+1)}(t) < 0 \text{ for } t \in \langle 0,1 \rangle \right\},$$

and

$$\mathbf{P}_{2i}^{***} = \left\{ P : P \in \mathbf{P}_1, P^{(i+1)} \text{ is continuous on } [0,1],\ P^{(i+1)}(t) < 0 \text{ for } t \in \langle 0,1 \rangle \text{ and } P^{(j)}(0) \leq 0,\ j = 2,3,\ldots,i \right\},$$

respectively. Note that $\mathbf{P}_{1i}^{*} \subset \mathbf{P}_{1i}^{**} \subset \mathbf{P}_{1i}^{***}$ and $\mathbf{P}_{2i}^{*} \subset \mathbf{P}_{2i}^{**} \subset \mathbf{P}_{2i}^{***}$.

The subfamilies of the J_P-family formed by $\mathbf{P}_{1i}^{**}$ and $\mathbf{P}_{2i}^{**}$ are characterized by the following theorems.

THEOREM 3.1A. *Let L_1 and L_2 be members of **L**. Then the following statements are equivalent,*

(i) L_1 *i^{th}-degree upward dominates* L_2.

(ii) $J_P(L_1) < J_P(L_2)$ *for all* $P \in \boldsymbol{P}_{1i}^{**}$.

(Proof in Appendix).

THEOREM 3.1B. *Let L_1 and L_2 be members of* ***L****. Then the following statements are equivalent,*

(i) L_1 *i^{th}-degree downward dominates* L_2

(ii) $J_P(L_1) < J_P(L_2)$ *for all* $P \in \boldsymbol{P}_{2i}^{**}$.

(Proof in Appendix).

The criteria of Lorenz dominance offer convenient computational methods for applied work. As is demonstrated by Theorems 3.1A and 3.1B this approach is particular attractive since it provides identification of the restrictions on the preference function P that are needed to reach unambiguous rankings of Lorenz curves.[23] As will be demonstrated in Section 4 the extended Gini inequality measure G_k satisfies the conditions $P^j(1) = 0,\ j = 2,3,...,k$, whereas the Lorenz family measure D_k satisfies the conditions $P^{(j)}(0) = 0,\ j = 2,3,...,k$.

To judge the normative significance of ith-degree upward and downward Lorenz dominance, it appears helpful to strengthen the principles of first-degree downside and upside positional transfer sensitivity to be more sensitive to transfers that take place lower down (higher up) in the income distribution.[24] To this end it will be useful to introduce the following notation. Let $\Delta_2 I_{s,t}(\delta, h_1, h_2)$ be defined by

Note that $\Delta_1 I_{s,t} < 0$ for any $s < t$ when I obeys the first-degree UPTS. Since the principle of second-degree UPTS is meant to strengthen the principle of first-degree UPTS, it follows from (14) that this is obtained when $\Delta_2 I_{s,t} > 0$ for $s < t$. Thus, we can only discern between second-degree DPTS and second-degree UPTS if these principles are required to be linked to first-degree DPTS and first-degree UPTS, respectively. Thus, when a sequence of first-degree DPTS (UPTS) transfers is valued more the lower down (higher up) the sequence of the transfers occurs, the sequence of transfers is made in line with *the principle of second-degree downside (upside) positional transfer sensitivity*. To deal with i^{th}-degree Lorenz dominance it is convenient to introduce the notation $\Delta_i I_{s,t}(\delta, h_1, h_2, ..., h_i)$ defined by

(15) $$\Delta_i I_{s,t}(\delta, h_1, h_2, ..., h_i) = \Delta_{i-1} I_{s,t}(\delta, h_1, h_2, ..., h_{i-1}) - \Delta_{i-1} I_{s+h_i, t+h_i}(\delta, h_1, h_2, ..., h_{i-1}),\ i = 3,4,...$$

[23] Note that Muliere and Scarsini (1989) provided a characterization of ith-degree upward Lorenz dominance (inverse stochastic dominance) in terms of order conditions for a subfamily of $\mathbf{P}_{1i}^{*}$.

[24] Note that Fishburn and Willig (1984) introduced an extension of Kolm's principle of diminishing transfers to higher-order transfer principles and demonstrated that these principles are associated to higher orders of upward stochastic dominance.

DEFINITION 3.4A. *Consider a discrete income distribution F, an inequality measure I that obeys the* $(i-1)^{th}$ *-degree DPTS, and rank-preserving transfers* δ *from individuals with ranks* $s+h_1,\ s+h_1+h_2,\ldots,s+h_1+h_i,\ldots,\ s+h_1+h_2+\ldots+h_i,\ t+h_1,\ t+h_1+h_2,\ldots,t+h_1+h_i,\ldots,$ $t+h_1+h_2+\ldots+h_i$ *to individuals with ranks s,* $s+h_2,\ldots,s+h_i,\ldots,s+h_2+h_3+\ldots+h_i,t,t+h_2,\ldots,t+h_i,\ldots,t+h_2+h_3+\ldots+h_i$ *in F. Then I is said to satisfy* ***the principle of ith-degree downside positional transfer sensitivity, the ith-degree DPTS****, if*

$$\Delta_i I_{s,t}\left(\delta,h_1,h_2,\ldots,h_i\right)>0 \ \text{ when } \ s<t\,.$$

DEFINITION 3.4B. *Consider a discrete income distribution F, an inequality measure I that obeys the* $(i-1)^{th}$ *-degree UPTS, and rank-preserving transfers* δ *from individuals with ranks* $s+h_1,\ s+h_1+h_2,\ldots,s+h_1+h_i,\ldots,\ s+h_1+h_2+\ldots+h_i,\ t+h_1,\ t+h_1+h_2,\ldots,t+h_1+h_i,\ldots,$ $t+h_1+h_2+\ldots+h_i$ *to individuals with ranks s,* $s+h_2,\ldots,s+h_i,\ldots,s+h_2+h_3+\ldots+h_i,t,t+h_2,\ldots,t+h_i,\ldots,t+h_2+h_3+\ldots+h_i$ *in F. Then I is said to satisfy* ***the principle of ith-degree upside positional transfer sensitivity, the ith-degree UPTS****, if*

$$\Delta_i I_{s,t}\left(\delta,h_1,h_2,\ldots,h_i\right)<0 \ \text{ when } \ s<t \ \text{ and } \ i=2k-1,\ k=1,2,\ldots,$$

and

$$\Delta_i I_{s,t}\left(\delta,h_1,h_2,\ldots,h_i\right)>0 \ \text{ when } \ s<t \ \text{ and } \ i=2k,\ k=1,2,\ldots.$$

The motivation for introducing the principle of $(\mathrm{i}-1)^{\mathrm{th}}$ -degree DPTS was to strengthen the principle of $(\mathrm{i}-2)^{\mathrm{th}}$ -degree DPTS. A similar consideration is valid for $(\mathrm{i}-1)^{\mathrm{th}}$ -degree UPTS. As stated in Theorems 3.2A and 3.2B support of the Pigou-Dalton principle of transfers and the principles of DPTS of each degree up to $\mathrm{i}-1$ is equivalent to be in favor of the criterion of ith-degree upward Lorenz dominance, whereas support of the Pigou-Dalton principle of transfers and the principles of UPTS of each degree up to $\mathrm{i}-1$ is equivalent to be in favor of ith-degree downward Lorenz dominance.

THEOREM 3.2A. *Let* L_1 *and* L_2 *be members of* ***L****. Then the following statements are equivalent,*

(i) L_1 *i^th^-degree upward dominates* L_2

(ii) $J_P(L_1) < J_P(L_2)$ *for all* $P \in \boldsymbol{P}_{1i}^*$

(iii) $J_P(L_1) < J_P(L_2)$ *for all* $P \in \boldsymbol{P}_{1i}$ *being such that* J_P *obeys the Pigou-Dalton principle of transfers and the principles of DPTS up to and including (i-1)*th*-degree.*

(Proof in Appendix.)

THEOREM 3.2B. *Let* L_1 *and* L_2 *be members of* ***L****. Then the following statements are equivalent,*

(i) L_1 i^{th}*-degree downward dominates* L_2

(ii) $J_P(L_1) < J_P(L_2)$ *for all* $P \in \boldsymbol{P}_{2i}^*$

(iii) $J_P(L_1) < J_P(L_2)$ *for all* $P \in \boldsymbol{P}_{2i}$ *being such that* J_P *obeys the Pigou-Dalton principle of transfers and the principles of UPTS up to and including (i-1)*th*-degree.*

The proof of Theorem 3.2B can be achieved by following the line of reasoning used in the proof of Theorem 3.2.A.

As noted above the motivation for introducing the principles of DPTS (UPTS) was to successively strengthen the emphasis of transfers taking place lower down (higher up) in the income distribution. To characterize social preferences that are consistent with these principles we adopt the following definition.

DEFINITION 3.5. *A social decision-maker that supports the Pigou-Dalton principle of transfers and the principles of DPTS (UPTS) up to and including* i^{th}*-degree is said to exhibit* ***downside (upside) inequality aversion of*** i^{th}***-degree****.*

By adding the condition of dominating extended Gini coefficients G_k for $k = 1, 2, ..., i-1$ to the condition of ith-degree upward Lorenz dominance it follows from the proof of Theorem 3.1A that the conditions $P^{(j)}(1) = 0,\ j = 2, 3, ..., i$ can be replaced by less restrictive conditions for P, which means that the subfamily of J_P-measures that preserves a "restricted" i^{th}-degree upward Lorenz dominance condition is larger than the subfamily of J_P-measures that preserves i^{th}-degree upward Lorenz dominance. Moreover, as indicated above, the latter is a subfamily of the former.[25]

[25] Wang and Young (1998) provide a result similar to Theorem 3.3B for intersecting distribution functions. However, their result relies on the condition of negative derivatives (up to order i) of P whereas the condition for P used in Theorem 3.3B is less strict. Moreover, Wang and Young (1998) don't appear to be aware of the fact that their result concerns downward rather than upward inverse stochastic dominance.

THEOREM 3.3A. *Let L_1 and L_2 be members of* $\boldsymbol{L}$. *Then the following statements are equivalent,*

(i) L_1 *i^{th}-degree upward dominates* L_2 *and* $G_k(L_1) \le G_k(L_2)$ *for* $k = 1,2,\ldots,i-1$

(ii) $J_P(L_1) < J_P(L_2)$ *for all* $P \in \boldsymbol{P}_{1i}^{***}$.

(Proof in Appendix.)

THEOREM 3.3B. *Let L_1 and L_2 be members of* $\boldsymbol{L}$. *Then the following statements are equivalent,*

(i) L_1 *i^{th}-degree downward dominates* L_2 *and* $D_k(L_1) \le D_k(L_2)$ *for* $k = 1,2,\ldots,i-1$

(ii) $J_P(L_1) < J_P(L_2)$ *for all* $P \in \boldsymbol{P}_{2i}^{***}$.

(Proof in Appendix.)

As we demonstrated in Section 2, first-degree DPTS (UPTS) could be given an alternative interpretation in terms of a mean-Gini-preserving transformation. This equivalence arises due to the fact that second-degree upward and downward Lorenz dominance "coincide" when the Gini coefficient (or the mean and the Gini mean difference) of the Lorenz curves are equal. By assuming that $L^3(1)$ defined by (9) is kept fixed, a similar interpretation of second-degree DPTS in terms of a mean-Gini-$L^3(1)$-preserving transformation can be obtained. Requiring $L^3(1)$ to be equal across Lorenz curves is equivalent to require that the extended Gini measure of inequality $G_2(L)$ (or its absolute version and the mean) is kept fixed. Thus, in order to obtain the upward dominating L^2-curve from the dominated L^2-curve we may use combinations of downside and upside MGPTs' that are carried out under the conditions of equal Gini coefficients and equal G_2-coefficients. Similarly, we find that the downward dominating L^2-curve can be obtained from the dominated L^2-curve by combinations of downside and upside MGPTs' provided that $\tilde{L}^3(0)$ is kept fixed, which according to (11) is equivalent to require that the D_2-coefficients are equal. However, when the requirement of equal D_2-coefficients are made in combination with the condition of equal Gini-coefficients, it follows from equation (13) that this is equivalent to require equal G_2-coefficients in combination with equal Gini coefficients. Formal definitions of downside and upside mean-Gini-G_2-preserving (or mean-Gini-$L^3(1)$-preserving) transformations are given by

DEFINITION 3.6. *Consider a discrete distribution F, a downside MGPT of two amounts* (δ_1, γ_1) *and an upside MGPT of two amounts* (δ_2, γ_2). *This pair of transformations is a* ***downside mean-Gini-G_2-preserving transformation (downside MG_2PT)*** *provided that the pairs* (δ_1, γ_1) *and* (δ_2, γ_2) *are such that the G_2-coefficient of the post-transfer distribution is equal to the G_2-coefficient of F, and the downside MGPT occurs lower down in the income distribution than the upside MGPT. By contrast, when the upside MGPT occurs lower than the downside MGPT we say that the pair of transformations is an* ***upside mean-Gini-G_2-preserving transformation (upside MG_2PT)***.

As indicated above there is a similarity between third-degree and second-degree Lorenz dominance in the sense that both dominance criteria can be given a normative characterization in terms of principles for positional transfer sensitivity and mean-"spread"-preserving transformations. Moreover, as for the second-degree dominance criteria, upward and downward third-degree Lorenz dominance require the same measures of "spread" (inequality) to be kept fixed in order to obey the principle of mean-"spread"-preserving transformation[26]. Keeping the Gini coefficient and G_2 fixed is, as noted above, equivalent to keeping the Gini coefficient and D_2 fixed. This means that upside as well as downside MG_2PT relies on fixed G_2- and D_2-coefficients. Observe that the scaled-up (the absolute) version of D_2 (or $\tilde{L}^3(0)$) can be considered as a measure of right spread, i.e. it is more sensitive to changes that affect the spread in the upper part of the income distribution rather than in changes that affect the spread in the lower part of the income distribution. By contrast, the scaled-up version of G_2 can be considered as a measure of left-spread. Accordingly, it is more sensitive to changes in spread that occur in the lower rather in the upper part of the income distribution. Thus, upside and downside MG_2PT place more emphasis on the tails than the upside and downside mean-Gini-preserving transformation. The downside mean-Gini-G_2-preserving transformation reduces inequality (spread) in the lower part of the income distribution at the expense of increased inequality in the middle/upper part of the income distribution, whereas the upside mean-Gini-G_2-preserving transformation reduces inequality (spread) in the upper part of the income distribution at the expense of increased inequality in the middle/lower part of the income distribution. In order to place even stronger emphasis on the tails we may introduce higher-order mean-"spread"-preserving transformations.

[26] In contrast to the absolute Gini-coefficient note that the absolute versions of G_i defined by (10) and D_i defined by (12) do not fulfill the standard conditions of being measures of spread (dispersion) when i>1, but can be used as measures of left and right spread, respectively. See Fernández-Ponce et al. (1998) and Shaked and Shanthikumar (1998) for discussions on measurement of right spread.

DEFINITION 3.7. *Consider a discrete distribution F, a downside $MG_{i-1}PT$ of two amounts (δ_1, γ_1) and an upside $MG_{i-1}PT$ of two amounts (δ_2, γ_2). This pair of transformations is a* ***downside mean-Gini-G_2-…-G_i-preserving transformation (downside MG_iPT)*** *provided that the pairs (δ_1, γ_1) and (δ_2, γ_2) are such that the G_i-coefficient of F, and the downside $MG_{i-1}PT$ occurs lower down in the income distribution than the upside $MG_{i-1}PT$. By contrast, when the upside $MG_{i-1}PT$ occurs lower than the downside $MG_{i-1}PT$ we say that the pair of transformations is an* ***upside mean-Gini-G_2-…-G_i-preserving transformation (upside MG_iPT)****.*

The following theorems demonstrate that a Lorenz curve L_1 that $(i+1)^{th}$ -degree upward (downward) dominates a Lorenz curve L_2 can be obtained from L_2 by a sequence of downside (upside) MG_iPTs'.

THEOREM 3.4A. *Let L_1 and L_2 be Lorenz curves with equal Gini coefficients and $G_j(L_1) = G_j(L_2)$, $j = 2,3,...,i$. Then the following statements are equivalent,*

(i) *L_1 $(i+1)^{th}$-degree upward dominates L_2*

(ii) *$J_P(L_1) < J_P(L_2)$ for all P such that $P^{(i+2)}$ is continuous on $[0,1]$ and*

$(-1)^{i+1} P^{(i+2)}(u) > 0$ for $u \in \langle 0,1 \rangle$

(iii) *L_1 can be obtained from L_2 by a sequence of downside MG_iPTs'*

(Proof in Appendix.)

THEOREM 3.4B. *Let L_1 and L_2 be Lorenz curves with equal Gini coefficients and $G_j(L_1) = G_j(L_2)$, $j = 2,3,...,i$. Then the following statements are equivalent,*

(i) *L_1 $(i+1)^{th}$-degree downward dominates L_2*

(ii) *$J_P(L_1) < J_P(L_2)$ for all P such that $P^{(i+2)}$ is continuous on $[0,1]$ and*

$P^{(i+2)}(u) < 0$ for $u \in \langle 0,1 \rangle$

(iii) *L_1 can be obtained from L_2 by a sequence of upside MG_iPTs'.*

The proof of Theorem 3.4B can be constructed by following exactly the line of reasoning used in the proof of Theorem 3.4A.

Since downside as well as upside MG_iPT relies on equal Gini coefficients and $G_j(L_1) = G_j(L_2),\ j = 2,3,...,i$, it will be of interest to explore the relationship between these two transformation approaches. To this end we draw on Theorems 3.4A and 3.4B, from which we can derive the following results.

COROLLARY 3.1A. *Let L_1 and L_2 be Lorenz curves with equal Gini coefficients and $G_j(L_1) = G_j(L_2),\ j = 2,3,...,i$. Then the following statements are equivalent for $i = 2k,\ k = 1,2,...,$*

(i) L_1 $(i+1)^{th}$*-degree upward dominates* L_2

(ii) L_1 $(i+1)^{th}$*-degree downward dominates* L_2

(iii) $J_P(L_1) < J_P(L_2)$ *when* $P^{(i+2)}(u) < 0$ *for* $u \in \langle 0,1 \rangle$

(iv) L_1 *can be obtained from* L_2 *by a sequence of downside* MG_iPTs'

(v) L_1 *can be obtained from* L_2 *by a sequence of upside* MG_iPTs'.

COROLLARY 3.1B. *Let L_1 and L_2 be Lorenz curves with equal Gini coefficients and $G_j(L_1) = G_j(L_2),\ j = 2,3,...,i$. Then the following statements are equivalent for $i = 2k-1,\ k = 1,2,...,$*

(i) L_1 $(i+1)^{th}$*-degree upward dominates* L_2

(ii) L_2 $(i+1)^{th}$*-degree downward dominates* L_1

(iii) $J_P(L_1) < J_P(L_2)$ *when* $P^{(i+2)}(u) > 0$ *for* $u \in \langle 0,1 \rangle$

(iv) L_1 *can be obtained from* L_2 *by a sequence of downside* MG_iPTs'

(v) L_2 *can be obtained from* L_1 *by a sequence of upside* MG_iPTs'.

REMARK. As demonstrated above the condition of equal Gini coefficients and $G_j(L_1) = G_j(L_2),\ j = 2,3,...,i$ is equivalent to the condition of equal Gini coefficients and $D_j(L_1) = D_j(L_2),\ j = 2,3,...,i$. Thus, the downside and upside mean-Gini-G_2-…-G_i-preserving transformations could alternatively have been denoted the downside and upside mean-Gini-D_2-…-D_i-preserving transformations.

The proposed sequences of dominance criteria along with the results of Theorems 3.1A-3.4A and 3.1B-3.4B suggest two alternative strategies for increasing the number of Lorenz curves that can be strictly ordered by successively narrowing the class of inequality measures under consideration. As the dominance criteria of each sequence are nested these strategies also allow us to identify the

value judgments that are needed to reach an unambiguous ranking of Lorenz curves. It follows from Theorem 3.2A that J_P-measures derived from P-functions with derivatives between second and i^{th} order that alternate in sign $\left((-1)^{j-1} P^{(j)}(t) > 0,\ j = 2,3,\ldots,i\right)$ preserve upward Lorenz dominance of all degrees lower than and equal to i-1. Thus, as demonstrated by Theorem 3.2A their sensitivity to changes that occur in the lower part of the income distribution (and the Lorenz curve) increases as i increases. By contrast, Theorem 3.2B shows that J_P-measures derived from P-functions with negative derivatives of order two and up to i $\left(P^{(j)}(t) < 0,\ j = 2,3,\ldots,i\right)$ preserve downward Lorenz dominance of all degrees lower than and equal to i-1. Theorem 3.2B demonstrates that this means that they increase their sensitivity to changes that occur in the upper part of the Lorenz curve as i increases. Note that the theorems, propositions and corollaries introduced above are only valid for finite i. At the extreme, as $i \to \infty$, observe that

(16) $$(i+1)!L^{i+1}(u) \to \begin{cases} 0, & 0 \le u < 1 \\ \dfrac{F^{-1}(0+)}{\mu}, & u = 1 \end{cases}$$

and

(17) $$(i+1)!\tilde{L}^{i+1}(u) \to \begin{cases} \dfrac{F^{-1}(1)}{\mu}, & u = 0 \\ 0, & 0 < u \le 1, \end{cases}$$

where $F^{-1}(0+)$ and $F^{-1}(1)$ denote the lowest and highest income, respectively. Hence, at the limit upward and downward Lorenz dominance solely depend on the income share of the worst-off and best-off income recipient, respectively. At the extreme upward Lorenz dominance is solely concerned with transfers that benefit the poorest unit. By contrast, downward Lorenz dominance solely focuses on transferring money from the richest to anyone else.

REMARK. Restricting the comparisons of Lorenz curves to distributions with equal means the various dominance results of Sections 2 and 3 are valid for generalized Lorenz curves and also apply to the so-called dual theory representation for choice under uncertainty introduced by Yaari (1987, 1988).

4. The relationship between downward and upward Lorenz dominance and generalized Gini families of inequality measures

The dominance results in Sections 2 and 3 show that application of the criteria of upward Lorenz dominance requires a higher degree of aversion to downside inequality the higher is the degree of upward Lorenz dominance. A similar relationship holds between downward Lorenz dominance and aversion to upside inequality aversion. As suggested in Section 3 the highest degree of downside inequality aversion is achieved when focus is exclusively turned to the situation of the worst-off income recipient. Thus, the most downside inequality averse J_P-measure that is obtained as the preference function approaches

(18) $$P_d(t) = \begin{cases} 0, & t = 0 \\ 1, & 0 < t \leq 1, \end{cases}$$

can be considered as the J_P-measure that exhibits the highest degree of downside inequality aversion. As P_d is not differentiable, it is not a member of the family $\mathbf{P}_1$ of inequality averse preference functions, but it is recognizable as the upper limit of inequality aversion for members of $\mathbf{P}_1$. Inserting (18) in (2) yields

(19) $$J_{P_d}(L) = 1 - \frac{F^{-1}(0+)}{\mu}.$$

Hence, the inequality measure J_{P_d} corresponds to *the Rawlsian maximin criterion*. Since J_{P_d} is compatible with the limiting case of upward Lorenz dominance the Rawlsian (relative) maximin criterion preserves all degrees of upward Lorenz dominance and rejects downward Lorenz dominance.

By contrast, the J_P-measure that is obtained as P approaches[27]

(20) $$P_u(t) = \begin{cases} 1, & 0 \leq t < 1 \\ 0, & t = 1, \end{cases}$$

exhibits the highest degree of upside inequality aversion. Inserting (20) in (2) yields

(21) $$J_{P_u}(L) = 1 + \frac{F^{-1}(1)}{\mu}.$$

Thus, J_{P_u}, which we will denote *the relative minimax criterion*, is "dual" to the Rawlsian (relative) maximin criterion in the sense that it is compatible with the limiting case of downward Lorenz

[27] Note that the normalization condition P'(0)=1 is ignored in this case.

dominance. When the comparison of Lorenz curves is based on the relative minimax criterion the Lorenz curve for which the largest relative income is smaller is preferred, regardless of all other differences. The only transfer which decreases inequality is a transfer from the richest unit to anyone else.

Based on the results in Sections 2 and 3, we shall now demonstrate how the above Lorenz dominance results can be applied to evaluate the ranking properties of the Lorenz and the extended Gini families of inequality measures. The extended Gini family is defined by (10). Note that $\{G_k : k > 0\}$ is a subfamily of $\{J_P : P \in \mathbf{P}_1\}$ formed by the following family of P-functions,

(22) $$P_{1k}(t) = 1 - (1-t)^{k+1},\ k \geq 0.$$

Differentiating P_{1k} defined by (22), we find that

(23) $$P_{1k}^{(j)}(t) = \begin{cases} (-1)^{j-1}\dfrac{(k+1)!}{(k-j+1)!}(1-t)^{k-j+1}, & j = 1,2,...,k+1 \\ 0 & ,\ j = k+2, k+3,... \end{cases}$$

As can be observed from (23) the weight function P_{1k} of the extended Gini family of inequality measures is a member of P_{1i}^{*} (and P_{1i}^{**}) for $k = i, i+1, \ldots$. Thus, we get the following result from Theorem 3.2A.[28]

COROLLARY 4.1A. *Let L_1 and L_2 be members of* ***L****. Then*

(i) *L_1 i^{th}-degree upward dominates L_2*

implies

(ii) $G_k(L_1) < G_k(L_2)$ *for* $k = i, i+1, i+2, \ldots$

Equation (23) implies that $P_{1k}''(t) < 0$ for all $t \in \langle 0,1 \rangle$ when $k > 0$ and thus that the G_k-measures satisfy the Pigou-Dalton principle of transfers for $k > 0$. Moreover, $P_{1k}'''(t) > 0$ for all $t \in \langle 0,1 \rangle$ when $k > 1$. Hence all G_k for $k > 1$ preserve second-degree upward Lorenz dominance. Moreover, the derivatives of P_k alternate in sign up to the $(k+1)^{th}$ derivative and $P_{1k}^{(j)}(1) = 0$ for all $j \leq k$. Thus, it follows from Theorem 3.2A that G_k preserves upward Lorenz dominance of degree k and obeys the principles of DPTS up to and including $(k-1)^{th}$-degree. The highest degree of downside inequality averse behavior occurs as $k \to \infty$, which corresponds to the inequality averse

[28] Muliere and Scarsini (1989) gave an alternative proof of Corollary 4.1A

behavior of the Rawlsian (relative) maximin criterion. Thus, G_k satisfies all degrees of upward Lorenz dominance as $k \to \infty$. At the other extreme, as $k = 0$, the preference function $P_0(t) = t$, which means that J_{P_0} does not obey the Pigou-Dalton principle of transfers or any principle of DPTS. The stated properties of the G_k-measures are summarized in the following corollary,

COROLLARY 4.2A. *The extended Gini family of inequality measures defined by (10) has the following properties,*

(i) G_k *preserves upward Lorenz dominance of degree k and all degrees lower than k,*

(ii) G_k *obeys the principle of transfers for* $k > 0$,

(iii) G_k *obeys the Pigou-Dalton principle of transfers and the principles of DPTS up to and including (k-1)th degree,*

(iv) *If* $G_k(L_1) < G_k(L_2)$ *then* $G_k(L_1)$ *can be obtained from* $G_k(L_2)$ *by a sequence of downside* $MG_{k-1}PTs'$.

(v) *The sequence* $\{G_k\}$ *approaches 0 as* $k \to 0$.

(vi) *The sequence* $\{G_k\}$ *approaches the Rawlsian relative maximin criterion as* $k \to \infty$.

Note that P_{1k} has negative derivatives (of any order) when $0 < k < 1$. Thus, G_k for $0 < k < 1$ preserves downward Lorenz dominance of all degrees.

As demonstrated by Aaberge (2000a) the Lorenz family of inequality measures is a subfamily of $\{J_P : P \in \mathbf{P}_1\}$ formed by the following family of P-functions,

$$P_{2k}(t) = \frac{1}{k}\left((k+1)t - t^{k+1}\right), k = 1, 2, \ldots \tag{24}$$

Differentiating P_{2k} defined by (24) yields

$$P_{2k}^{(j)}(t) = \begin{cases} -(k+1)(k-1)(k-2)\ldots(k-j+2)t^{k-j+1}, & j = 2, 3, \ldots, k+1 \\ 0 & , j = k+2, k+3, \ldots \end{cases} \tag{25}$$

By noting from (25) that the weight-function P_{2k} of the Lorenz family of inequality measures is a member of $\mathbf{P}_{2i}^*$ for $k = i, i+1, \ldots,$ we obtain the following result from Theorem 3.2B.

COROLLARY 4.1B. *Let L_1 and L_2 be members of* ***L****. Then*

(i) L_1 *i^{th}-degree downward dominates* L_2

implies

(ii) $D_k(L_1) < D_k(L_2)$ *for* $k = i, i+1, i+2, \ldots.$

and $\dfrac{F_1^{-1}(1)}{\mu_1} < \dfrac{F_2^{-1}(1)}{\mu_2}.$

The results of a similar evaluation of the Lorenz family of inequality measures as that carried out for the extended Gini family are summarized in the following corollary.

COROLLARY 4.2B. *The Lorenz family of inequality measures defined by (12) has the following properties,*

(i) D_k *preserves downward Lorenz dominance of degree k and all degrees lower than k,*

(ii) D_k *obeys the principle of transfers for* $k > 0$,

(iii) D_k *obeys the Pigou-Dalton principle of transfers and the principles of UPTS up to and including* $(k\text{-}1)^{th}$ *degree,*

(iv) *If* $D_k(L_1) < D_k(L_2)$ *then* $D_k(L_1)$ *can be obtained from* $D_k(L_2)$ *by e sequence of upside* $DG_{k-1}PTs'$.

(v) *The sequence* $\{D_k\}$ *approaches 0 as* $k \to \infty$.

(vi) *The sequence* $\{kD_k + 1\}$ *approaches the relative minimax criterion as* $k \to \infty$.

Note that the derivatives of P_{2k} alternate in sign when $-1 < k < 1$. Thus, D_k for $-1 < k < 1$ preserves upward Lorenz dominance of all degrees and approaches the Rawlsian relative maximin as k approaches -1. As demonstrated by Aaberge (2000a) D_k approaches the Bonferroni coefficient as $k \to 0$.

Corollary 4.1A shows that the various degrees of upward Lorenz dominance is preserved by sub-families of the extended Gini measures of inequality, which divide the integer subscript subclass of the extended Gini family into nested subfamilies. Thus, the hierarchical sequence of nested upward Lorenz dominance criteria offers a convenient computational method for identifying the largest subfamily of the integer subscript extended Gini family of inequality measures that is consistent with the actual ranking of Lorenz curves. As demonstrated by Corollary 4.1B the various

degrees of downward Lorenz dominance divide the Lorenz family of inequality measures into a similar sequence of nested subfamilies.

5. Summary and discussion

This paper introduces two sequences of partial orderings for achieving complete rankings of Lorenz curves. In particular, we have examined situations where Lorenz curves intersect by introducing ranking criteria that are weaker than non-intersecting dominance (first-degree Lorenz dominance) and stronger than single measures of inequality. The proposed dominance criteria are shown to characterize nested subsets of the families of inequality measures defined by $\int P'(u)\,d\,L(u)$ where P' is the derivative of a function P that defines the inequality aversion profile of the inequality measure. The condition of first-degree Lorenz dominance corresponds to concave P-functions. By introducing higher degrees of dominance, this paper provides a method for identifying the lowest degree of dominance and the weakest restriction on the functional form of the preference function P that is needed to reach unambiguous rankings of Lorenz curves, irrespective of whether one's social preferences is consistent with downside or upside inequality aversion. To judge the normative significance of the sequences of dominance criteria, appropriate principles of transfers and mean-"spread"-preserving transformations have been introduced. The criteria of Lorenz dominance provide convenient computational methods for ranking a set of Lorenz curves and for exploring how robust the attained ranking would be with respect to choice of rank-dependent measures of inequality. Thus, in applied work the ranking obtained by applying this approach should in general have a wider degree of support than that obtained by applying arbitrarily chosen summary measures of inequality.

To deal with the mean income income inequality trade-off, in cases where they conflict, Shorrocks (1983) introduced the "generalized Lorenz curve", defined as a mean scaled-up version of the Lorenz curve. Moreover, Shorrocks (1983) obtained characterizations of social welfare functions based on first-degree dominance relations between generalized Lorenz curves. Thus, scaling up the introduced Lorenz dominance relations of this paper by the mean income (μ) and replacing the rank-dependent measures of inequality J_P defined by (2) with the rank-dependent social welfare functions $W_P = \mu(1 - J_P)$, it can be demonstrated that the present results also apply to the generalized Lorenz curve and moreover provide convenient characterizations of the corresponding social welfare orderings.

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Appendix

Proofs of Dominance Results

LEMMA 1. *Let H be the family of bounded, continuous and non-negative functions on [0,1] which are positive on* $\langle 0,1\rangle$ *and let g be an arbitrary bounded and continuous function on [0,1]. Then*

$$\int g(t)\, h(t)\, dt > 0 \quad \text{for all} \quad h \in H$$

implies

$$g(t) \geq 0 \quad \text{for all} \quad t \in [0,1]$$

and the inequality holds strictly for at least one $t \in \langle 0,1\rangle$.

The proof of Lemma 1 is known from mathematical textbooks.

The proof of the equivalence between (i) and (ii) in Theorem 2.2A is analogous to the proof for stochastic dominance in Hadar and Russel (1969) but is included below for the sake of completeness.

Proof of Theorem 2.2A. Using integration by parts we have that

$$J_P(L_2) - J_P(L_1) = -P''(1)\int_0^1 \left(L_1(u) - L_2(u)\right)du + \int_0^1 P'''(u)\int_0^u \left(L_1(t) - L_2(t)\right)dt\,du.$$

Thus, if (i) holds then $J_P(L_2) > J_P(L_1)$ for all $P \in \mathbf{P}_2$.

To prove the converse statement we restrict to preference functions $P \in \mathbf{P}_2$ for which $P''(1) = 0$. Hence,

$$J_P(L_2) - J_P(L_1) = \int_0^1 P'''(u)\int_0^u \left(L_1(t) - L_2(t)\right)dt\,du$$

and the desired result it obtained by applying Lemma 1.

To prove the equivalence between (ii) and (iii) consider a case where we transfer a small amount γ from persons with incomes $F^{-1}(s+h_1)$ and $F^{-1}(t+h_1)$ to persons with incomes $F^{-1}(s)$ and

$F^{-1}(t)$, respectively, where t is assumed to be larger than s. Then J_P defined by (2) obeys the first-degree DPTS if and only if

$$P'(s) - P'(s + h_1) > P'(t) - P'(t + h_1)$$

which for small h_1 is equivalent to

$$P''(t) - P'(s) > 0 .$$

Next, inserting for $t = s + h_2$, we find, for small h_2, that this is equivalent to $P'''(s) > 0$.

The proof of Theorem 2.2B is analogous to the proof of Theorem 2.2A and is based on the expression

$$J_P(L_2) - J_P(L_1) = -P''(0)\int_0^1 (L_1(t) - L_2(t))dt - \int_0^1 P'''(u)\int_u^1 (L_1(t) - L_2(t))dt\,du$$

which is obtained by using integration by parts. Thus, by arguments like those in the proof of Theorem 2.2A the results of Theorem 2.2B are obtained.

Proof of Proposition 2.1A. The statement (i) implies (ii) follows from Theorem 2.2A.

To prove the converse statement assume that (ii) holds and that L_1 and L_2 cross at $u = a$. Then the following inequalities hold,

(A1) $$\int_0^a (L_1(u) - L_2(u))du > 0$$

and

(A2) $$\int_0^1 (L_1(u) - L_2(u))du = \frac{1}{2}(G_2 - G_1) \geq 0 .$$

Since L_1 and L_2 cross only once (A1) and (A2) imply that

$$\int_0^u (L_1(u) - L_2(u))du \geq 0 \text{ for all } u \in [0,1]$$

and the inequality holds strictly for some u, and the desired result is obtained by applying Theorem 2.2A.

Proof of Theorem 2.3. Assume that

(i) $$\int_0^u L_1(t)\,dt \geq \int_0^u L_2(t)\,dt \quad \text{for all } u \in [0,1]$$

and the inequality holds strictly for some $u \in \langle 0,1 \rangle$ and that L_1 and L_2 intersect m times, where m is an arbitrary integer. Since $\int_0^1 L_1(t)\,dt = \int_0^1 L_2(t)\,dt$ and L_1 and L_2 intersect m times, then (i) implies that $L_1(u) \geq L_2(u)$ for $u \in [0, a_1]$ and $L_1(u) \leq L_2(u)$ for $u \in [a_m, 1]$ where a_1 is the first intersection point and a_m is the last intersection point. Accordingly, m has to be odd. For convenience we let $m = 2n-1$, where n is an arbitrary integer, and let $a_i, i = 1,2,...,2n-1$ denote the $2n-1$ u-values where L_1 and L_2 intersect, i.e., $L_1(a_j) = L_2(a_j)$ for $j = 1,2,...,2n-1$. Thus, we have that

(A3) $$L_1(u) \begin{cases} \geq L_2(u) & \text{for} \quad u \in [a_{2j-2}, a_{2j-1}] \\ \leq L_2(u) & \text{for} \quad u \in [a_{2j-1}, a_{2j}] \end{cases}$$

for $j = 1,2,...,n$ where $0 < a_1 < a_2 < ... < a_{2n-1} < 1$, $a_0 = 0$ and $a_{2n} = 1$. Furthermore, let L_{1j} and L_{2j} be the Lorenz curves defined by

(A4) $$L_{1j}(u) = \begin{cases} L_1(u) & \text{for} \quad u \in [0, a_{2j-2}] \quad \text{and} \quad u \in [a_{2j-1}, 1] \\ L_2(u) & \text{for} \quad u \in [a_{2j-2}, a_{2j-1}] \end{cases}$$

and

(A5) $$L_{2j}(u) = \begin{cases} L_1(u) & \text{for} \quad u \in [0, a_{2j-1}] \quad \text{and} \quad u \in [a_{2j}, 1] \\ L_2(u) & \text{for} \quad u \in [a_{2j-1}, a_{2j}] \end{cases}$$

for $j = 1,2,...,n$.

Then it follows from (A3) that

$$L_{1j}(u) \leq L_1(u) \quad \text{for all } u \in [0,1]$$

and

$$L_{2j}(u) \geq L_1(u) \quad \text{for all } u \in [0,1]$$

for $j = 1,2,...,n$.

By applying Theorem 2.1 we get that

$$L_{1j}(u) \le L_1(u) \text{ for all } u \in [0,1]$$

if and only if L_1 can be obtained from L_{1j} by a sequence of Pigou-Dalton progressive transfers, and

$$L_{2j}(u) \ge L_1(u) \text{ for all } u \in [0,1]$$

if and only if $L_1(u)$ can be obtained from L_{2j} by a sequence of Pigou-Dalton regressive transfers.

Next, by noting that

(A6) $$L_1(u) - L_2(u) = \sum_{i=1}^{2} \sum_{j=1}^{n} \left(L_1(u) - L_{ij}(u) \right)$$

we then have that L_1 can be obtained from L_2 by sequences of Pigou-Dalton progressive and regressive transfers.

It follows from (A3) that the 2n segments formed by the $2n-1$ intersections may be arranged in n pairs where $L_1(u)$ dominates $L_2(u)$ when $u \in \left[a_{2j-2}, a_{2j-1} \right]$ and $L_2(u)$ dominates $L_1(u)$ when $u \in \left[a_{2j-1}, a_{2j} \right]$, $j = 1, 2, ..., n$. Thus, since $a_{2j-2} < a_{2j-1} < a_{2j}$ it follows for each pair of segments that the progressive transfers occur for lower income levels than the regressive transfers. Moreover, since

$$\int_0^1 \sum_{i=1}^{2} \sum_{j=1}^{n} \left(L_1(u) - L_{ij}(u) \right) du =$$
$$\sum_{j=1}^{n} \left[\int_{a_{2j-2}}^{a_{2j-1}} \left(L_1(u) - L_2(u) \right) du + \int_{a_{2j-1}}^{a_{2j}} \left(L_1(u) - L_2(u) \right) du \right] =$$
$$\int_0^1 \left(L_1(u) - L_2(u) \right) du = 0$$

we get that

(A7) $$\sum_{j=1}^{n} \int_{a_{2j-2}}^{a_{2j-1}} \left(L_1(u) - L_2(u) \right) du = \sum_{j=1}^{n} \int_{a_{2j-1}}^{a_{2j}} \left(L_2(u) - L_1(u) \right) du .$$

Thus, in order to fulfill the condition of equal Gini coefficients the sequence of Pigou-Dalton progressive transfers captured by the left side of equation (A7) has to be matched by a corresponding sequence of Pigou-Dalton regressive transfers captured by the right side of equation (A7).

Accordingly, we have found that L_1 can be obtained from L_2 by a downside mean-Gini-preserving transformation.

To prove that (iii) implies (i) we will rely on Theorem 2.2A and follow the line of reasoning used by Zoli (1999) for the proof of the "only if part" of his Proposition 3.

Consider the family J_P defined by (2) for $P \in \mathbf{P}_1$, i.e. for increasing concave P, let F_1 and F_2 be discrete distributions with Lorenz curves L_1 and L_2, and assume that L_1 can be obtained from L_2 by a downside mean-Gini-preserving transformation where $N = 1$. Thus we consider a Pigou-Dalton progressive transfer δ from a person with income $F_2^{-1}(s+h)$ to a person with income $F_2^{-1}(s)$ and a Pigou-Dalton regressive transfer γ from a person with income $F_2^{-1}(t)$ to a person with income $F_2^{-1}\left(t+\tilde{h}\right)$, where $s < t$. Since these transfers are assumed to leave the Gini coefficient unchanged it follows from (3) that

$$\delta\left[(1-s)-(1-s-h)\right] = \gamma\left[(1-t)-\left(1-t-\tilde{h}\right)\right]$$

which is equivalent to

(A8) $$\delta h = \gamma\tilde{h}.$$

Furthermore, from (2) we have that a downside MGPT reduces inequality, i.e. $J_P(L_1) < J_P(L_2)$, if and only if

(A9) $$\delta\left[P'(s)-P'(s+h)\right] > \gamma\left[P'(t)-P'\left(t+\tilde{h}\right)\right].$$

Inserting for (A8) in (A9) yields

$$\frac{P'(s)-P'(s+h)}{h} > \frac{P'(t)-P'\left(t+\tilde{h}\right)}{\tilde{h}}$$

which for small h and $\tilde{h}$ is equivalent to

$$P''(t)-P''(s) > 0.$$

Next, inserting for $t = s + h_1$, we find, for small h_1, that this inequality is equivalent to $P'''(s) > 0$. Then it follows from Theorem 2.2A that L_1 second-degree upward dominates L_2.

The equivalence between (i) and (ii) follows by noting that

$$\int_u^1 \left[\left(1-L_1(t)\right)-\left(1-L_2(t)\right)\right]dt = \int_u^1 \left(L_2(t)-L_1(t)\right)dt =$$
$$\int_0^1 \left(L_2(t)-L_1(t)\right)dt + \int_0^u \left(L_1(t)-L_2(t)\right)dt = \int_0^u \left(L_1(t)-L_2(t)\right)dt$$

when L_1 and L_2 have equal Gini coefficients.

To prove that (ii) implies (iv) we follow the line of reasoning used for proving that (i) implies (iii). To this end it is convenient to introduce the Lorenz curves $\tilde{L}_{1i}$ and $\tilde{L}_{2i}$ defined by

(A10) $$\tilde{L}_{1i}(u) = \begin{cases} L_2(u) & \text{for} \quad u \in \left[0, a_{2i-2}\right] \quad \text{and} \quad u \in \left[a_{2i-1}, 1\right] \\ L_1(u) & \text{for} \quad u \in \left[a_{2i-2}, a_{2i-1}\right] \end{cases}$$

and

(A11) $$\tilde{L}_{2i}(u) = \begin{cases} L_2(u) & \text{for} \quad u \in \left[0, a_{2i-1}\right] \quad \text{and} \quad u \in \left[a_{2i}, 1\right] \\ L_1(u) & \text{for} \quad u \in \left[a_{2i-1}, a_{2i}\right] \end{cases}$$

for $i = 1, 2, ..., n$.

Then it follows from (A3) that

$$\tilde{L}_{1j}(u) \geq L_2(u) \text{ for all } u \in \left[0,1\right]$$

and

$$\tilde{L}_{2j}(u) \leq L_2(u) \text{ for all } u \in \left[0,1\right]$$

for $j = 1, 2, ..., n$.

By applying Theorem 2.1 we thus get that L_2 can be obtained from $\tilde{L}_{1j}$ by a sequence of Pigou-Dalton regressive transfers and from $\tilde{L}_{2j}$ by a sequence of Pigou-Dalton progressive transfers. Moreover, by observing that

(A12) $$L_2(u) - L_1(u) = \sum_{i=1}^{2} \sum_{j=1}^{n} \left(L_2(u) - \tilde{L}_{ij}(u)\right)$$

we thus have that L_2 can be obtained from L_1 by sequences of Pigou-Dalton regressive and progressive transfers. As noted above $L_2(u)$ dominates $L_1(u)$ when $u \in \left[a_{2j-1}, a_{2j}\right]$ and $L_2(u)$ is dominated by $L_1(u)$ when $u \in \left[a_{2j-2}, a_{2j-1}\right]$, $j = 1, 2, ..., n$. Thus, by arranging the adjacent segments $\left[a_{2j-2}, a_{2j-1}\right]$ and $\left[a_{2j-1}, a_{2j}\right]$ in n pairs we find for each pair of segments that the regressive transfers occur for lower income levels than the progressive transfers. Moreover, since

$$\int_0^1 \sum_{i=1}^{2} \sum_{j=1}^{n} \left(L_2(u) - \tilde{L}_{ij}(u)\right) du = \sum_{j=1}^{n} \left[\int_{a_{2j-2}}^{a_{2j-1}} \left(L_2(u) - L_1(u)\right) du + \int_{a_{2j-1}}^{a_{2j}} \left(L_2(u) - L_1(u)\right) du \right] =$$

$$\int_0^1 \left(L_2(u) - L_1(u)\right) du = 0$$

we get that

$$\text{(A13)} \qquad \sum_{j=1}^{n} \int_{a_{2j-2}}^{a_{2j-1}} \left(L_1(u) - L_2(u)\right) du = \sum_{j=1}^{n} \int_{a_{2j-1}}^{a_{2j}} \left(L_2(u) - L_1(u)\right) du .$$

Thus, in order to fulfill the condition of equal Gini coefficients the sequence of Pigou-Dalton regressive transfers captured by the left side of equation (A13) has to be matched by a corresponding sequence of Pigou-Dalton progressive transfers captured by the right side of equation (A13).

To prove that (iv) implies (ii) we consider a Pigou-Dalton regressive transfer δ from a person with income $F_1^{-1}(s)$ to a person with income $F_1^{-1}(s+h)$ and a Pigou-Dalton progressive transfer γ from a person with income $F_1^{-1}\left(t+\tilde{h}\right)$ to a person with income $F_1^{-1}(t)$ where $s < t$. Since these transfers are assumed to leave the Gini coefficient unchanged it follows that the condition (A8) has to be fulfilled. Furthermore, it follows from (2) that an upside MGPT reduces inequality, i.e. $J_P\left(L_2\right) < J_P\left(L_1\right)$, if and only if

$$\gamma\left[P'(t) - P'\left(t+\tilde{h}\right)\right] > \delta\left[P'(s) - P'\left(s+h\right)\right]$$

which by inserting for the condition (A8) is equivalent to

$$\frac{P'(t) - P'\left(t+\tilde{h}\right)}{\tilde{h}} > \frac{P'(s) - P'\left(s+h\right)}{h}$$

which for small $\tilde{h}$ and h is equivalent to

$$P''(t) - P''(s) < 0 .$$

Next, inserting for $t = s + h_1$, we find, for small h_1, that this inequality is equivalent to $P'''(s) < 0$. Then it follows from Theorem 2.2B that L_2 second-degree downward dominates L_1.

Proof of Theorem 3.1A. To examine the case of ith-degree upward Lorenz dominance we integrate $J_P(L_2) - J_P(L_1)$ by parts i times,

$$J_P(L_2) - J_P(L_1) = \sum_{j=2}^{i} (-1)^{j-1} P^{(j)}(1)\left(L_1^j(1) - L_2^j(1)\right) + (-1)^i \int_0^1 P^{(i+1)}(u)\left(L_1^i(u) - L_2^i(u)\right)du \quad \text{(A14)}$$

and use this expression in constructing the proof of the equivalence between (i) and (ii).

Assume first that (i) in Theorem 3.1A is true, i.e.

$$L_1^i(u) - L_2^i(u) \geq 0 \text{ for all } u \in [0,1]$$

and > holds for at least one $u \in \langle 0,1 \rangle$.

Then $J_P(L_2) > J_P(L_1)$ for all $P \in \mathbf{P}_{1i}^{**}$.

Conversely, assume that

$$J_P(L_2) > J_P(L_1) \text{ for all } P \in \mathbf{P}_{1i}^{**} .$$

For this family of preference functions we have that

$$J_P(L_2) - J_P(L_1) = (-1)^i \int_0^1 P^{(i+1)}(u)\left(L_1^i(u) - L_2^i(u)\right)du .$$

Then, as demonstrated by Lemma 1, the desired result can be obtained by a suitable choice of $P \in \mathbf{P}_{1i}^{**}$.

Proof of Theorem 3.2A. Assume that (i) in Theorem 3.2A is true, i.e.

$$L_1^i(u) - L_2^i(u) \geq 0 \text{ for all } u \in [0,1]$$

and > holds for at least one $u \in \langle 0,1 \rangle$. Then it follows from Theorem 3.1A that $J_P(L_2) > J_P(L_1)$ for all $P \in \mathbf{P}_{i1}$ such that $(-1)^j P^{(j+1)}(t) > 0$ for $t \in \langle 0,1 \rangle$, $j = 1,2,...,i$ since this family of P-functions is a subfamily of $\mathbf{P}_{1i}^{**}$.

Conversely, assume that $J_P(L_2) > J_P(L_1)$ for all $P \in \mathbf{P}_{1i}$ such that $(-1)^j P^{(j+1)}(t) > 0$ for $t \in \langle 0,1 \rangle$, $j = 1,2,...,i$. For this family of P-functions we have that

$$J_P(L_2) - J_P(L_1) = (-1)^i \int_0^1 P^{(i+1)}(u)\left(L_1^i(u) - L_2^i(u)\right)du$$

and the desired result is obtained by applying Lemma 1.

A proof by mathematical induction will be used to prove the equivalence between (ii) and (iii). To this end it is convenient to introduce the following notation. Let H_1, H_2 and H_{j+1} be defined by

(A15) $$H_1(v,h_1) = P'(v) - P'(v+h_1),$$

(A16) $$H_2(s,t,h_1) = H_1(s,h_1) - H_1(t,h_1)$$

and

(A17) $$H_{j+1}(s,t,h_1,h_2,...,h_j) = H_j(s,t,h_1,h_2,...,h_{j-1}) - H_j(s+h_j,t+h_j,h_1,h_2,...,h_{j-1}),\ j = 2,3,...\ .$$

Moreover, let

(A18) $$H_2^{(1)}(s,t) = \lim_{h_1 \to 0} \frac{1}{h_1} H_2(s,t,h_1)$$

and

(A19) $$H_{j+1}^{(j)}(s,t) = \lim_{h_j \to 0} ... \lim_{h_1 \to 0} \frac{1}{\prod_{k=1}^{j} h_k} H_{j+1}(s,t,h_1,h_2,...,h_j).$$

It follows from Theorems 2.1 and 2.2A that J_P obeys the Pigou-Dalton principle of transfers and the first-degree DPTS iff $P''(t) < 0$ and $P'''(t) > 0$. From (14), definition (2) of J_P and (A15)-(A19) we then get that J_P obeys the second-degree DPTS iff

(A20) $$H_3^{(2)}(s,t) > 0 \text{ for } s < t\ .$$

Inserting for (A17), (A16) and (A15) in (A19) for $j = 2$ yields

$$H_3^{(2)}(s,t) = \lim_{h_2\to 0}\lim_{h_1\to 0}\frac{1}{h_1h_2}H_3(s,t,h_1,h_2) =$$

$$\lim_{h_2\to 0}\lim_{h_1\to 0}\frac{1}{h_1h_2}\left(H_2(s,t,h_1) - H_2(s+h_2,t+h_2,h_1)\right) =$$

$$\lim_{h_2\to 0}\frac{1}{h_2}\left(H_2^{(1)}(s,t) - H_2^{(1)}(s+h_2,t+h_2)\right) =$$

$$\lim_{h_2\to 0}\frac{1}{h_2}\lim\frac{1}{h_1}\Big\{P'(s) - P'(s+h_1) - \left(P'(t) - P'(t+h_1)\right) -$$

$$\left[P'(s+h_2) - P'(s+h_1+h_2) - \left(P'(t+h_2) - P'(t+h_1+h_2)\right)\right]\Big\} =$$

$$\lim_{h_2\to 0}\left[-P''(s) + P''(s+h_2) - \left(-P''(t) - P''(t+h_2)\right)\right] = P^{(3)}(s) - P^{(3)}(t).$$

Inserting for $t = s + h$, we find, for small h, that this is equivalent to $P^{(4)}(s) < 0$.

Next, assume that

(A21) $$H_j^{(j-1)}(s,t) = (-1)^{j-1}\left(P^{(j)}(s) - P^{(j)}(t)\right).$$

It follows from Theorem 2.2A and the proof above that (A21) is true for j equal to 2 and 3.

Inserting for (A17) in (A19) we get

$$H_{j+1}^{(j)}(s,t) = 1\lim_{h_j\to 0}\ldots\lim_{h_1\to 0}\frac{1}{\prod_{k=1}^{j}h_k}\left(H_j(s,t,h_1,h_2,\ldots,h_{j-1}) - H_j(s+h_j,t+h_j,h_1,h_2,\ldots,h_{j-1})\right) =$$

$$\lim_{h_j\to 0}\ldots\lim_{h_2\to 0}\frac{1}{\prod_{k=2}^{j}h_k}\left(H_j^{(1)}(s,t,h_2,h_3,\ldots,h_{j-1}) - H^{(1)}(s+h_j,t+h_j,h_2,h_3,\ldots,h_{j-1})\right) =$$

$$\lim_{h_j\to 0}\frac{1}{h_j}\left(H_j^{(j-1)}(s,t) - H_j^{(j-1)}(s+h_j,t+h_j)\right),$$

which by inserting for (A21) yields

$$H_{j+1}^{(j)}(s,t) = (-1)^{j}\left(P^{(j+1)}(s) - P^{(j+1)}(t)\right).$$

Thus, (A21) is proved to be true by induction.

Since J_P defined by (2) obeys the $(i-1)^{th}$-degree DPTS if and only if

$$H_i^{(i-1)}(s,t) > 0 \text{ for } s < t$$

we get from (A21) that this condition is equivalent to

$$(-1)^i P^{(i+1)}(s) > 0 .$$

Proof of Theorem 3.3A. By inserting for (9) in (A14) we get that

(A22) $$J_P(L_2) - J_P(L_1) = \sum_{j=2}^{i} (-1)^{j-1} P^{(j)}(1)\left(G_{j-1}(L_2) - G_{j-1}(L_1)\right) + (-1)^i \int_0^1 P^{(i+1)}(u)\left(L_1^i(u) - L_2^i(u)\right)du .$$

Assume first that (i) of Theorem 3.3A is true. Then $J_P(L_2) > J_P(L_1)$ for all $P \in \mathbf{P}_{1i}^{***}$.

Conversely, assume that

$$J_P(L_2) > J_P(L_1) \text{ for all } P \in \mathbf{P}_{1i}^{***} .$$

Then this statement holds for the subfamily of $\mathbf{P}_{1i}^{***}$ for which $P^{(j)}(1) = 0$ for $j = 2,3,...,i$. For this particular family of preference functions we get that

$$J_P(L_2) - J_P(L_1) = (-1)^i \int_0^1 P^{(i+1)}(u)\left(L_1^i(u) - L_2^i(u)\right)du .$$

By applying Lemma 1 we get that L_1 i^{th}-degree upward dominates L_2.

Next, consider the subfamily of preference functions defined by

(A23) $$P_k(t) = 1 - (1-t)^{k+1}, \quad k = 1,2,...,i-1 .$$

By observing that $P_k \in \mathbf{P}_{1i}^{***}$ we find by inserting for (A23) in J_P that

$$0 < J_{P_k}(L_2) - J_{P_k}(L_1) = G_k(L_2) - G_k(L_1)$$

for $k = 1,2,...,i-1$.

The proofs of Theorems 3.1B, 3.2B and 3.3B can be constructed by following exactly the line of reasoning used in the proofs of Theorems 3.1A and 3.3A. The proofs use the following expression,

(A24) $$J_P(L_2) - J_p(L_1) = -\sum_{j=2}^{i} P^{(j)}(0)\left(\tilde{L}_2^j(0) - \tilde{L}_1^j(0)\right) - \int_0^1 P^{(i+1)}(u)\left(\tilde{L}_2^i(u) - \tilde{L}_1^i(u)\right)du ,$$

which is obtained by using integration by parts i times.

Proof of Theorem 3.4A.

To prove the equivalence between (i) and (ii) we draw on the proof of Theorem 3.3A. Under the condition of equal Gini coefficients and $G_j(L_1) = G_j(L_2),\ j = 2,3,...,i$ we get from (A22) that

(A25) $$J_P(L_2) - J_P(L_1) = (-1)^{i+1} \int_0^1 P^{(i+2)}(u)\left(L_1^{i+1}(u) - L_2^{i+1}(u)\right)du\,,$$

and the desired result is obtained by applying Lemma 1.

Next, we will prove that (i) implies (iii). Assume that

(i) $$\int_0^u L_1^i(t)\,dt \geq \int_0^u L_2^i(t)\,dt \quad \text{for all } u \in [0,1]$$

and the inequality holds strictly for some $u \in \langle 0,1 \rangle$ and L_1^i and L_2^i intersect m times, where m is an arbitrary integer. Since $G_i(L_1) = G_i(L_2)$ is equivalent to $\int_0^1 L_1^i(t)\,dt = \int_0^1 L_2^i(t)\,dt$ and L_1^i and L_2^i intersect m times, then $L_1^i(u) \geq L_2^i(u)$ for $u \in [0, b_1]$ and $L_1^i(u) \leq L_2^i(u)$ for $u \in [b_m, 1]$ where b_1 is the first intersection point and b_m is the last intersection point. Accordingly m have to be odd. For convenience we let $m = 2n-1$, where n is an arbitrary integer, and let $b_i,\ i = 1,2,...,2n-1$ denote the $2n-1$ u-values where L_1^i and L_2^i intersect, i.e. $L_1^i(b_j) = L_2^i(b_j)$ for $j = 1,2,...,2n-1$. Thus, we have that

(A26) $$L_1^i(u) \begin{cases} \geq L_2^i(u) & \text{for} \quad u \in [b_{2j-2}, b_{2j-1}] \\ \leq L_2^i(u) & \text{for} \quad u \in [b_{2j-1}, b_{2j}] \end{cases}$$

for $j = 1,2,...,n$, where $0 < b_1 < b_2 < ... < b_{2n-1} < 1$, $b_0 = 0$ and $b_{2n} = 1$. Furthermore, let L_{1j}^i and L_{2j}^i be the L^i-curves defined by

(A27) $$L_{1j}^i(u) = \begin{cases} L_1^i(u) & \text{for} \quad u \in [0, b_{2j-2}] \text{ and } u \in [b_{2j-1}, 1] \\ L_2^i(u) & \text{for} \quad u \in [b_{2j-2}, b_{2j-1}] \end{cases}$$

and

(A28) $$L^i_{2j}(u) = \begin{cases} L^i_1(u) & \text{for} \quad u \in \left[0, b_{2j-1}\right] \text{ and } u \in \left[b_{2j}, 1\right] \\ L^i_2(u) & \text{for} \quad u \in \left[b_{2j-1}, b_{2j}\right] \end{cases}$$

for $j = 1, 2, ..., n$.

Then it follows from (A26) that

$$L^i_{1j}(u) \le L^i_1(u) \text{ for all } u \in [0,1]$$

and

$$L^i_{2j}(u) \ge L^i_1(u) \text{ for all } u \in [0,1]$$

for $j = 1, 2, ..., n$.

Let us first consider the case where $i = 2$, i.e. third degree upward Lorenz dominance. By applying Theorem 2.3 we have that

$$L^2_{1j}(u) \le L^2_1(u) \text{ for all } u \in [0,1]$$

if and only if L_1 can be obtained from L_{1j} by a downside mean-Gini-preserving transformation (downside MGPT), and

$$L^2_{2j}(u) \ge L^2_1(u) \text{ for all } u \in [0,1]$$

if and only if $L_1(u)$ can be obtained from L_{2j} by an upside MGPT.

Next, by noting that

(A29) $$L^2_1(u) - L^2_2(u) = \sum_{k=1}^{2} \sum_{j=1}^{n} \left(L^2_1(u) - L^2_{kj}(u)\right)$$

we then have that L_1 can be obtained from L_2 by a combination of a downside and an upside MGPT. It follows from (A26) for $i = 2$ that the 2n segments formed by the $2n - 1$ intersections can be arranged in n pairs where $L_1(u)$ second-degree upward dominates $L_2(u)$ when $u \in \left[b_{2j-2}, b_{2j-1}\right]$ and $L_1(u)$ second-degree downward dominates $L_2(u)$ when $u \in \left[b_{2j-1}, b_{2j}\right]$. Thus, since $b_{2j-2} < b_{2j-1} < b_{2j}$ it follows for each pair of segments that the downside MGPT occurs for lower income levels than the upside MGPT. Moreover, since

$$\int_0^1 \sum_{k=1}^{2} \sum_{j=1}^{n} \left(L_1^2(u) - L_{kj}^2(u)\right) du =$$

$$\sum_{j=1}^{n} \left[\int_{b_{2j-2}}^{b_{2j-1}} \left(L_1^2(u) - L_2^2(u)\right) du + \int_{b_{2j-1}}^{b_{2j}} \left(L_1^2(u) - L_2^2(u)\right) du \right] =$$

$$\int_0^1 \left(L_1^2(u) - L_2^2(u)\right) du = G_2\left(L_2\right) - G_2\left(L_1\right) = 0$$

we get that

(A30) $$\sum_{j=1}^{n} \int_{b_{2j-2}}^{b_{2j-1}} \left(L_1^2(u) - L_2^2(u)\right) du = \sum_{j=1}^{n} \int_{b_{2j-1}}^{b_{2j}} \left(L_2^2(u) - L_1^2(u)\right) du .$$

Thus, in oder to fulfill the condition of equal G_2-coefficients the sequence of downside MGPTs' captured by the left side of equation (A30) has to be matched by a corresponding sequence of upside MGPTs' captures by the right side of equation (A30). Accordingly, we have found that L_1 can be obtained from L_2 by a downside MG_2PT.

By following the line of reasoning used to prove that third-degree upward Lorenz dominance implies that the dominating Lorenz curve can be obtained from the dominated Lorenz curve by a downside MG_2PT we assume that a Lorenz curve L_1 that i^{th} -degree upward dominates a Lorenz curve L_2 can be obtained from L_2 by a combination of a downside and an upside $MG_{i-1}PT$. Then it follows from (A26) that the 2n segments formed by the $2n-1$ intersections of L_1^i and L_2^i can be arranged in n pairs where $L_1(u)$ i^{th}-degree upward dominates $L_2(u)$ when $u \in \left[b_{2j-2}, b_{2j-1}\right]$ and $L_1(u)$ i^{th}-degree downward dominates $L_2(u)$ when $u \in \left[b_{2j-1}, b_{2j}\right]$. Thus, since $b_{2j-2} < b_{2j-1} < b_{2j}$ it follows for each pair of segments that the downside $MG_{i-1}PT$ occurs for lower income levels than the upside $MG_{i-1}PT$. Moreover, since

$$\int_0^1 \sum_{k=1}^{2} \sum_{j=1}^{n} \left(L_1^i(u) - L_{kj}^i(u)\right) du =$$

$$\sum_{j=1}^{n} \left[\int_{b_{2j-2}}^{b_{2j-1}} \left(L_1^i(u) - L_2^i(u)\right) du + \int_{b_{2j-1}}^{b_{2j}} \left(L_1^i(u) - L_2^i(u)\right) du \right] =$$

$$\int_0^1 \left(L_1^i(u) - L_2^i(u)\right) du = G_i\left(L_2\right) - G_i\left(L_1\right) = 0$$

we get that

(A31) $$\sum_{j=1}^{n}\int_{b_{2j-2}}^{b_{2j-1}}\left(L_1^i(u)-L_2^i(u)\right)du=\sum_{j=1}^{n}\int_{b_{2j-1}}^{b_{2j}}\left(L_2^i(u)-L_1^i(u)\right)du\,.$$

Thus, in order to fulfill the condition of equal G_i-coefficients the sequence of downside $MG_{i-1}PTs'$ captured by the left side of equation (A31) has to be matched by a corresponding sequence of upside $MG_{i-1}PTs'$ captured by the right side of equation (A31). Accordingly, we have found that L_1 can be obtained from L_2 by a downside MG_iPT.

A proof by mathematical induction will be used to prove that (iii) implies (i). To this end it is convenient to introduce the following notation.

Let

(A32) $$A_2(j,k)=\delta_j\left(\left(1-s_j\right)^k-\left(1-s_j-h_j\right)^k\right)-\gamma_j\left(\left(1-t_j\right)^k-\left(1-t_j-\tilde{h}_j\right)^k\right),$$

(A33) $$A_i(j,k)=A_{i-1}(j,k+1)-A_{i-1}(j+a(i),k+1)\,,$$

(A34) $$T_P(2,j)=\delta_j\left(P'\left(s_j\right)-P'\left(s_j+h_j\right)\right)-\gamma_j\left(P'\left(t_j\right)-P'\left(t_j+\tilde{h}_j\right)\right)$$

and

(A35) $$T_P(i+1,j)=T_P(i,j)-T_P\left(i,j+a(i+1)\right)$$

where

(A36) $$a(i)=2^{i-3},\ i=3,4,\ldots\,.$$

Consider the family J_P defined by (2) where F_1 and F_2 are discrete distributions with Lorenz curves L_1 and L_2, and assume that L_1 can be obtained from L_2 by a sequence of downside MG_iPTs'. Thus, we consider pairs $\left(\delta_j,\gamma_j\right)$ of Pigou-Dalton progressive/regressive transfers δ_j from a person with income $F_2^{-1}\left(s_j+h_j\right)$ to a person with income $F_2^{-1}\left(s_j\right)$ and γ_j from a person with income $F_2^{-1}\left(t_j\right)$ to a person with income $F_2^{-1}\left(t_j+\tilde{h}_j\right)$, where $s_j<t_j$, $s_j<s_{j+1}$ and $t_j<t_{j+1}$. Since the transfers are assumed to leave the Gini coefficient unchanged it follows from (3) that

$$A_2(j,1)=0\,,$$

which from (A32) is found to be equivalent to

(A37) $$\delta_j h_j = \gamma_j \tilde{h}_j .$$

The condition of fixed G_2-coefficients is equivalent to the following requirement

$$A_3(j,1) = A_2(j,2) - A_2(j+1,2) = 0 ,$$

which by inserting for (A37) is equivalent to

$$\delta_j h_j \left(\frac{(1-s_j)^2 - (1-s_j-h_j)^2}{h_j} - \frac{(1-t_j)^2 - (1-t_j-\tilde{h}_j)^2}{\tilde{h}_j} \right) =$$
$$\delta_{j+1} h_{j+1} \left(\frac{(1-s_{j+1})^2 - (1-s_{j+1}-h_{j+1})^2}{h_{j+1}} - \frac{(1-t_{j+1})^2 - (1-t_{j+1}-\tilde{h}_{j+1})^2}{\tilde{h}_{j+1}} \right)$$

which for small $h_j, \tilde{h}_j, h_{j+1}$ and $\tilde{h}_{j+1}$ is found to be equivalent to

(A38) $$\delta_j h_j (t_j - s_j) = \delta_{j+1} h_{j+1} (t_{j+1} - s_{j+1}) .$$

By proceeding for $i = 4$ we get that the condition of fixed G_3-coefficients is equal to the requirement

$$A_4(j,1) = A_3(j,2) - A_3(j+2,2) = 0 ,$$

which by inserting for (A37) and (A38) (for small $h_j, \tilde{h}_j, h_{j+1}, \tilde{h}_{j+1}, h_{j+2}, \tilde{h}_{j+2}, h_{j+3}$ and $\tilde{h}_{j+3}$) is equivalent to

$$\delta_j h_j (t_j - s_j) \left[\frac{(1-s_j)^2 - (1-t_j)^2}{t_j - s_j} - \frac{(1-s_{j+1})^2 - (1-t_{j+1})^2}{t_{j+1} - s_{j+1}} \right] =$$
$$\delta_{j+2} h_{j+2} (t_{j+2} - s_{j+2}) \left[\frac{(1-s_{j+2})^2 - (1-t_{j+2})^2}{t_{j+2} - s_{j+2}} - \frac{(1-s_{j+3})^2 - (1-t_{j+3})^2}{t_{j+3} - s_{j+3}} \right],$$

which for small $t_{j+t} - s_{j+t}$ for $t = 0,1,2,3$ is found to be equivalent to

$$\delta_j h_j (t_j - s_j)(s_{j+1} - s_j) = \delta_{j+2} h_{j+2} (t_{j+2} - s_{j+2})(s_{j+3} - s_{j+2}) .$$

Similarly, the condition of unchanged G_i-coefficients is captured by the requirement

$$A_{i+1}(j,1) = A_i(j,2) - A_i(j + a(i+1), 2) = 0,$$

which by inserting for the conditions of equal G_j-coefficients, $j = 1,2,...,i-1$ for small $s_{j+t} - s_{j+t-1}$ is found to be equivalent to

$$\text{(A39)} \qquad \delta_j h_j (t_j - s_j) \prod_{t=1}^{i-2} (s_{j+t} - s_j) = \delta_{j+a(i+1)} h_{j+a(i+1)} (t_{j+a(i+1)} - s_{j+a(i+1)}) \prod_{t=1}^{i-2} (s_{j+a(i+1)+t} - s_{j+a(i+1)}).$$

The case for $i = 1$ (second-degree Lorenz dominance) was proved for Theorem 2.3. Thus, let us firstly consider the case $i = 2$. Inserting for (A37) in $T_P(3,1)$ defined by (A35) we find that a downside MG_2PT reduces inequality, i.e. $J_P(L_1) < J_P(L_2)$, if and only if

$$T_P(3,1) = \delta_1 h_1 (-P''(s_1) + P''(t_1)) - \delta_2 h_2 (-P''(s_2) - P''(t_2)) > 0,$$

which by inserting for (A38) is equivalent to

$$\frac{P''(t_1) - P''(s_1)}{t_1 - s_1} > \frac{P''(t_2) - P''(s_2)}{t_2 - s_2},$$

which for small $t_1 - s_1$ and $t_2 - s_2$ is equivalent to

$$P'''(s_2) - P'''(s_1) < 0,$$

which for small $s_2 - s_1$ is equivalent to

$$P^{(4)}(s_1) < 0.$$

Next, we have that a downside MG_iPT reduces inequality if and only if

$$T_P(i+1,1) = T_P(i,1) - T_P(i, a(i+1)+1) > 0,$$

which by inserting for (A37) in $T_P(2,1)$, (A38) in $T_P(3,1)$ and so forth is found to be equivalent to

$$\begin{aligned}
&\delta_1 h_1 (t_1 - s_1) \prod_{t=1}^{i-3} (s_{t+1} - s_1) \left[(-1)^i \left(P^{(i)}(s_{i-1}) - P^{(i)}(s_1) \right) \right] > \\
&\delta_{a(i+1)+1} h_{a(i+1)+1} (t_{a(i+1)+1} - s_{a(i+1)+1}) \cdot \\
&\prod_{t=1}^{i-3} (s_{a(i+1)+1+t} - s_{a(i+1)+1}) \left[(-1)^i \left(P^{(i)}(s_{a(i+1)+i-1}) - P^{(i)}(s_{a(i+1)+1}) \right) \right]
\end{aligned}$$

which by inserting for (A39) is equivalent to

$$\delta_1 h_1\left(t_1 - s_1\right)\prod_{t=1}^{i-2}\left(s_{t+1} - s_1\right)\left[\left(-1\right)^i\left(\frac{P^{(i)}\left(s_{i-1}\right) - P^{(i)}\left(s_1\right)}{s_{i-1} - s_1} - \frac{P^{(i)}\left(s_{a(i+1)+i-1}\right) - P^{(i)}\left(s_{a(i+1)+1}\right)}{s_{a(i+1)+i-1} - s_{a(i+1)+1}}\right)\right] > 0,$$

which for small $s_{i-2} - s_1$ and $s_{a(i+1)+i-1} - s_{a(i+1)+1}$ is equivalent to

$$\delta_1 h_1\left(t_1 - s_1\right)\prod_{t=1}^{i-2}\left(s_{t+1} - s_1\right)\left[\left(-1\right)^{(i)}\left(P^{(i+1)}\left(s_1\right) - P^{(i+1)}\left(s_{a(i+1)+1}\right)\right)\right] > 0$$

which for small $s_{a(i+1)+1} - s_1$ is equivalent to

$$\left(-1\right)^{i+1} P^{(i+2)}\left(s_1\right) > 0 .$$

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