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## UMP Unbiased Tests for Multi-parameter Testing Problems with Restricted Alternatives

**Abstract:**

Applications of the standard theory of UMP unbiased tests depends on conditions which in general are difficult to verify. In the present paper, however, we suggest more simple rules for applying this theory for regular exponential families of distributions. This approach leads to UMP unbiased tests for various multiparameter testing problems with restricted alternatives, and is shown to give justification for conditional tests of various test problems for contingency tables. The derived tests are shown to possess attractive small sample properties.

**Keywords:** Exponential families of distributions, UMP unbiased tests, exact conditional tests, contingency.

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# 1. Introduction

Assume the distribution  $P_{\theta}$  of the random vector  $X$  belongs to a regular exponential family  $\wp$ , i.e. the members of  $\wp$  has the form

$$(1.1) \quad dP_{\theta} = c(\theta) \exp\left(\sum_{i=1}^h \theta_i T_i(x)\right) dP_0$$

where  $P_0$  is an arbitrary element of  $\wp$ ,  $\theta = (\theta_1, \dots, \theta_h)$  is a vector of minimal canonical parameters from an open domain  $\Theta$  in  $\mathbb{R}^h$  (c.f. Barndorff-Nielsen, 1978).

The problem of testing hypotheses related to one of the parameters  $\theta_1, \dots, \theta_h$  or a linear transformation of the minimal canonical parameters has been studied by Lehmann (1986). For the particular problem of testing  $H_0 : \theta_1 = \theta_0$  against  $H_1 : \theta_1 > \theta_0$ , the uniformly most powerful (UMP) unbiased test is obtained by considering the null distribution of  $T_1(X)$  conditional on the minimal sufficient statistics  $T_2(X), \dots, T_h(X)$  (when  $\theta_0$  is known).

Most test-problems are much more complicated than the problem mentioned above. Often more than one parameter restriction is needed to specify many important hypotheses. Since, in general, it is impossible to find a test which is UMP unbiased against all alternatives, we shall develop tests which are UMP unbiased against a subset of the alternative space. Schaafsma (1966) follows a somewhat similar approach by introducing most stringent somewhere most powerful test.

The first step in obtaining a UMP unbiased test is to verify that the hypothesis can be formulated within the class of regular models. Thus, our strategy aims at introducing a regular subfamily of  $\wp$ , in which all regular hypotheses can be expressed through a single parameter. Within this subfamily the standard Neyman-Pearson theory can be applied to construct UMP unbiased tests.

When analysing contingency tables the multinomial model is a commonly used probability model. This model is a typical example of a regular exponential family. The problem of testing the hypothesis of independence in a two-dimensional contingency table (rxs-table) involves  $(r-1)(s-1)$  parameter restrictions, since the dimension of the parameter space is reduced from  $rs-1$  to  $r+s-2$  under the hypothesis of independence. For applying the results in Section 2 in this or other test-problems for regular exponential families we simply have to make sure that the hypothesis restrictions can be expressed as linear functions of minimal canonical parameters. Note that this condition is much easier to verify than the usual conditions appearing in the theory of UMP unbiased tests (c.f. Lehmann, 1986).

In Section 3 the results are demonstrated on four test-problems. Among these the problem of testing the hypothesis of independence in a two-dimensional contingency table is studied. The derived tests are exact in the sense that their sampling distributions are independent of the nuisance parameters which make it unnecessary to use large-sample approximations to sampling distributions.

## 2. The existence and construction of UMP unbiased tests for multiparameter exponential families

Several well-known hypotheses for the exponential family of distributions can be formulated as linear restrictions on minimal cononical parameters. For hypotheses of this kind we shall establish a class of tests which are UMP unbiased against certain restricted alternatives. To this end the following results of Barndorff-Nielsen (1978) will be employed.

*Definition 2.1.*  $L$  is an affine set if  $c_1 \mathbf{l}_1 + c_2 \mathbf{l}_2 \in L$  for every  $\mathbf{l}_i \in L$ ,  $i = 1, 2$ , and for every  $c_1$  and  $c_2$  such that  $c_1 + c_2 = 1$ .

*Definition 2.2.* Let  $\{P_{\boldsymbol{\theta}} : \boldsymbol{\theta} \in \Theta\}$  be a canonical parametrization of the exponential family  $\wp$ . A subset  $\wp_0$  of  $\wp$  is said to be affine if  $\wp_0$  is of the form  $\wp_0 = \{P_{\boldsymbol{\theta}} : \boldsymbol{\theta} \in \Theta \cap L\}$  where  $L$  is an affine subset of  $\mathbb{R}^h$ .

*Lemma 2.1.* Let  $\wp$  be a regular exponential family and let  $\wp_0$  be a subset of  $\wp$ . Then the two following statements are equivalent:

- (i)  $\wp_0$  is regular
- (ii)  $\wp_0$  is affine.

As will be demonstrated below Lemma 2.1 allows an alternative formulation of the conditions appearing in the theory of UMP tests which makes it convenient to express hypotheses for exponential families as regular subfamilies of  $\wp$  defined by (1.1). More specifically, this is done formally by introducing a real parameter  $\gamma$  which is used to generate various subspaces of the parameter space.

Define  $\wp_{\gamma, B} \subset \wp$  by

$$(2.1) \quad \wp_{\gamma, B} = \left\{ P_{\boldsymbol{\theta}} \in \wp : \mathbf{B}\boldsymbol{\theta} = \mathbf{v} + \mathbf{U} \begin{pmatrix} \gamma \\ \boldsymbol{\xi} \end{pmatrix} \right\}$$

where  $\boldsymbol{\theta} \in \Theta$ ,  $\gamma \in \mathbb{R}$ ,  $\boldsymbol{\xi}$  is an unspecified  $d_0$ -dimensional vector,  $\mathbf{B}$  is a specified  $d_1 \times h$ -matrix with rank  $d_1$ ,  $\mathbf{U}$  is a specified  $d_1 \times (d_0 + 1)$ -matrix with rank  $d_0 + 1$ ,  $\mathbf{v}$  is a specified  $d_1$ -dimensional vector,  $d_0 = 0, 1, \dots, d_1 - 1$ ,  $d_1 = 1, 2, \dots, h$  and  $h = \dim \Theta$ .

According to (1.1) and (2.1) a distribution from  $\mathcal{P}_{\gamma, \mathbf{B}}$  can be written on the form

$$(2.2) \quad dP_{\boldsymbol{\theta}} = \left[ c(\boldsymbol{\theta}) \exp \left( \gamma Y(\mathbf{x}) + \sum_{i=1}^{h-d} \tau_i Z_i(\mathbf{x}) \right) \right] dP_0$$

where  $d = d_1 - d_0$ ,  $\tau_1, \dots, \tau_{h-d}$  are nuisance parameters and  $Y, Z_1, \dots, Z_{h-d}$  are sufficient statistics.

Observe that  $\mathcal{P}_{\gamma, \mathbf{B}}$  is a subfamily of the regular exponential family of distributions where the parameter of interest  $\gamma$  is specified as a linear combination of minimal canonical parameters. Hence, for a fixed value  $\gamma_0$  of  $\gamma$ ,  $\mathcal{P}_{\gamma_0, \mathbf{B}}$  represents the class of hypotheses constituting linear restrictions on the minimal canonical parameter space. The elements of  $\mathbf{B}$ ,  $\mathbf{U}$  and  $\mathbf{v}$  in the definition of  $\mathcal{P}_{\gamma, \mathbf{B}}$  are fixed real numbers which of course depend on the actual application.

As an immediate consequence of (2.2), Lemma 2.1 and Theorem 4.1, Lehmann (1986) we get

*Theorem 2.1.* Let  $X$  be a vector of random variables with distribution  $P_{\boldsymbol{\theta}} \in \mathcal{P}_{\gamma, \mathbf{B}}$  defined by (2.1) where  $\gamma$  is a real parameter. For testing the hypothesis  $H: \gamma = \gamma_0$  against the alternatives  $A: \gamma > \gamma_0$  there exist a UMP unbiased level  $\varepsilon$  test given by

$$\delta(\mathbf{x}) = \begin{cases} 1 & \text{when } Y(\mathbf{x}) > k(z_1, z_2, \dots, z_{h-d}) \\ \mu & \text{when } Y(\mathbf{x}) = k(z_1, z_2, \dots, z_{h-d}) \\ 0 & \text{when } Y(\mathbf{x}) < k(z_1, z_2, \dots, z_{h-d}) \end{cases}$$

where  $k$  and  $\mu$  are determined by

$$E_{\gamma_0} (\delta(X) | z_1, z_2, \dots, z_{h-d}) = \varepsilon.$$

*Proof.* Let  $\mathbf{Q}_B = \left\{ \mathbf{q} \in \mathbb{R}^h : \mathbf{B}\mathbf{q} = \mathbf{v} + \mathbf{U} \begin{pmatrix} \gamma \\ \boldsymbol{\xi} \end{pmatrix} \right\}$  where  $\gamma, \boldsymbol{\xi}, \mathbf{v}, \mathbf{U}$  and  $\mathbf{B}$  are defined in (2.1).

Since

$$\begin{aligned}
\mathbf{B}(c_1 \mathbf{q}_1 + c_2 \mathbf{q}_2) &= c_1 \mathbf{B} \mathbf{q}_1 + c_2 \mathbf{B} \mathbf{q}_2 \\
&= c_1 \left( \mathbf{v} + \mathbf{U} \begin{pmatrix} \gamma \\ \xi \end{pmatrix} \right) + c_2 \left( \mathbf{v} + \mathbf{U} \begin{pmatrix} \gamma \\ \xi \end{pmatrix} \right) \\
&= \mathbf{v} + \mathbf{U} \begin{pmatrix} \gamma \\ \xi \end{pmatrix}
\end{aligned}$$

for every  $\mathbf{q}_1, \mathbf{q}_2 \in \mathbf{Q}_B$  and for every real  $c_1$  and  $c_2$  such that  $c_1 + c_2 = 1$ , the set  $\mathbf{Q}_B$  satisfies Definition 2.1 which according to Definition 2.2 and Lemma 2.1 make  $\mathcal{P}_{\gamma, B}$  affine and regular. Hence, the conditions of Theorem 4.3, Lehmann (1986) are fulfilled. Application of that theorem completes the proof.

In the following, attention will be confined to the test-problem

$H: \gamma = \gamma_0$  against  $A: \gamma > \gamma_0$ .

*Remark.* Assume  $d = \text{rank } \mathbf{B} - \text{rank } \mathbf{U} = \dim \Theta$  which means that (2.2) has the form

$$(2.3) \quad dP_{\Theta} = C(\Theta) \exp(\gamma Y(x)) dP_0.$$

In this case we see that the nuisance parameters vanish and Theorem 2.1 therefore yields a UMP level  $\varepsilon$  test for  $H$  against  $A$ .

For Theorem 2.1 to be interesting we must verify if the model under the null hypothesis belongs to the family (2.1). As stated above, it is sufficient to consider hypotheses expressible as linear restrictions of minimal canonical parameters. The elements of  $\mathbf{v}$  and  $\mathbf{B}$  are determined from the form of the restrictions defining the hypothesis. Next, the elements of  $\mathbf{U}$  must be specified. This has to be done on the basis of assumptions or theories about the particular problem under investigation. If a subset of the original alternative space is of particular interest, this may guide us in the specification of  $\mathbf{U}$ .

Observe that the model (2.1) represents no restrictions on the parameter space under hypotheses which constitute linear restrictions on minimal canonical parameters. Hence, the tests derived in Theorem 2.1 have level  $\varepsilon$  for hypotheses of this type.

Note that two-sided test problems can be handled similarly to the one-sided test problem discussed above.

### 3. Four test-problems

By using the present approach the first test-problem illustrates the simplicity of detecting the Behrens-Fisher model as non-regular. In the second test-problem we derive an exact test for the problem of testing Hardy-Weinberg equilibrium. The remaining test-problems are chosen from Aaberge (1980), each illustrates various aspects of the application of Theorem 2.1.

#### 3.1. Behrens-Fisher problem

Let  $X_1, \dots, X_m; Y_1, \dots, Y_n$  be independent  $N(\xi, \sigma_1^2)$  and  $N(\xi, \sigma_2^2)$ , respectively. We want to test the hypothesis  $H: \xi = \eta$  against  $A: \xi > \eta$ . The a priori model  $P_\theta$  is a familiar member of the family of regular exponential models and is given by

$$(3.1) \quad dP_\theta = \left[ C(\theta) \exp \left( \theta_1 \sum x_i + \theta_2 \sum y_i + \theta_3 \sum x_i^2 + \theta_4 \sum y_i^2 \right) \right] dP_0$$

where  $P_0$  is the distribution corresponding to  $\xi = \eta = 0, \sigma_1 = \sigma_2 = 1$ , and  $\theta_1, \theta_2, \theta_3, \theta_4$  defined by  $\theta_1 = \xi/\sigma_1^2, \theta_2 = \eta/\sigma_2^2, \theta_3 = -\sigma_1^2/2, \theta_4 = -\sigma_2^2/2$  are minimal canonical parameters.

The hypothesis of equal means  $\xi = \eta$  can be expressed by the minimal canonical parameters as

$$H: \theta_1 \theta_3 = \theta_2 \theta_4$$

which is a non-linear restriction of the minimal canonical parameter space. Accordingly, the model under  $H$  is not contained in (2.1), e.g., it is non-regular and the Neyman-Pearson theory is not applicable.

#### 3.2. Testing for Hardy-Weinberg equilibrium

Let  $N$  be the size of a random sample from a diploid population. Suppose that a certain locus carries two allelic genes  $A$  and  $a$  and let  $p_1, p_2, p_3$  denote the proportions of individuals in the population having genotypes  $AA, Aa$  and  $aa$ , respectively. The probability distribution of the random vector  $X = (X_1, X_2, X_3)$  of the observed numbers of individuals of genotype  $AA, Aa, aa$  is given by the multinomial distribution

$$(3.2) \quad \frac{N!}{x_1! x_2! x_3!} p_1^{x_1} p_2^{x_2} p_3^{x_3}.$$

Let  $\wp$  be the exponential family given by (3.2).  $\wp$  is regular and can also be expressed as



$$(3.3) \quad dP_{\theta} = \left[ C(\theta) \exp(\theta_1 x_1 + \theta_2 x_2) \right] dP_0$$

where  $\theta_i = \log(p_i/p_3)$ ,  $i = 1, 2$ , constitute a set of minimal canonical parameters,  $P_0$  is the multinomial distribution corresponding to  $p_1 = p_2 = p_3 = 1/3$  and

$$C(\theta) = 3^N (1 + \exp(\theta_1 + \theta_2))^{-N}.$$

According to genetic theory, the hypothesis

$$H: (p_1, p_2, p_3) = (p^2, 2p(1-p), (1-p)^2)$$

for arbitrary  $p$  defines the Hardy-Weinberg equilibrium state.

It can easily be verified that the restriction

$$p_2 = 2(p_1 p_3)^{1/2}$$

is an equivalent expression for  $H$ . Consequently,  $H$  is also equivalent to

$$H: 2\theta_2 - \theta_1 = 2 \log 2$$

which is a linear restriction of the minimal canonical parameter space. Hence, the Hardy-Weinberg equilibrium state yields a model contained in the family (2.1); the model under the hypothesis is regular.

Let  $\gamma$  be a real parameter. By substituting

$$(3.4) \quad \theta_1 = \gamma + 2\theta_2 - 2 \log 2$$

in (3.3) we get

$$(3.5) \quad dP_{\theta} = \left[ C(\theta) 2^{-2x_1} \exp(\gamma x_1 + \theta_2 (2x_1 + x_2)) \right] dP_0$$

as an alternative expression of the multinomial family (3.2). By (3.4), the hypothesis

$$H: \gamma = 0$$

is equivalent to the hypothesis of Hardy-Weinberg equilibrium.

By Theorem 2.1, a UMP unbiased test for  $H:\gamma=0$  against  $A:\gamma>0$  is obtained by considering the null distribution of  $X_1$  conditional on the statistic  $Z=2X_1+X_2$ .

By using the relation

$$\sum_{z=0}^{2N} \binom{2N}{z} a^z b^{2N-z} = \sum_{z=0}^{2N} \left[ 2^z \sum_{x_1 \in S} \frac{N!}{x_1! x_2! x_3!} a^{x_1} b^{2N-x_1} \right] a^z b^{2N-z}$$

where  $S = \{x_1 : x_1 + x_2 + x_3 = N, 2x_1 + x_2 = z\}$ , we obtain the following expression for the null distribution

$$\begin{aligned} \Pr_H(X_1 = x_1 | Z = z) &= \frac{N!}{x_1!(z-2x_1)!(N-z+x_1)!} \left[ \binom{2N}{N} \right]^{-1} 2^{z-2x_1} \\ &= \frac{\binom{N}{x_1} \binom{N-x_1}{z-2x_1}}{\binom{2N}{z}} 2^{z-2x_1}, \quad 0 \leq x_1 \leq \frac{z}{2} \end{aligned}$$

Note that Haldane (1954) derives this test by using combinatorial arguments.

Because of its exact properties the present conditional test is particularly useful for testing the hypothesis of Hardy-Weinberg equilibrium in small samples.

### 3.3. Testing for diagonals-parameter symmetry in a 3×3 contingency table

For the analysis of square contingency tables having ordered categories, Goodman (1979a) introduced the diagonals-parameter symmetry model. The purpose of this example is to derive an UMP unbiased test for the hypothesis of diagonals-parameter symmetry in a 3×3-table.

Consider a two-dimensional contingency table with 3 rows, 3 columns and cell probabilities  $p_{ij}$  satisfying

$$p_{ij} > 0, \quad i, j = 1, 2, 3 \quad \text{and} \quad \sum_i \sum_j p_{ij} = 1.$$

The probability distribution of the random vector  $X = (X_{11}, X_{12}, \dots, X_{33})$  of the observed cell numbers is given by the multinomial distribution

$$(3.6) \quad \frac{N!}{\prod_i \prod_j x_{ij}!} \prod_i \prod_j p_{ij}^{x_{ij}}$$

where  $\sum_i \sum_j x_{ij} = N$  is the size of the random sample.

Let  $\mathcal{P}$  be the exponential family given by (3.6). By introducing the minimal canonical parameters

$$\theta_{ij} = \log(p_{ij}/p_{33}), i, j = 1, 2, 3, (i, j) \neq (3, 3)$$

a distribution  $P_{\theta} \in \mathcal{P}$  can be written on the form

$$(3.7) \quad dP_{\theta} = \left[ C(\theta) \exp \left( \sum_{(i,j) \neq (3,3)} \theta_{ij} x_{ij} \right) \right] dP_0$$

where

$$C(\theta) = 9^N \left( 1 + \sum_{(i,j) \neq (3,3)} \exp(\theta_{ij}) \right)^{-N}$$

and  $P_0$  is the element of  $\mathcal{P}$  corresponding to  $p_{ij} = 1/9, i, j = 1, 2, 3$ .

The hypothesis of diagonals-parameter symmetry is defined by

$$(3.8) \quad \frac{p_{ij}}{p_{ji}} = \delta_k, k = j - i, i < j$$

where  $\delta_k$  denotes a parameter connected to the cells  $(i, j)$  for which  $j - i = k, k = 1, 2$ .

The parameter  $\delta_k$  in (3.8) is simply the odds that an observation will fall in one of the cells  $(i, j)$  where  $j - i = k$ , rather than in one of the cells where  $j - i = -k, k = 1, 2$ .

It is easily seen that (3.8) is equivalent to

$$H: \theta_{ij} - \theta_{ji} = \xi_{j-i}, i < j$$

where

$$\xi_k = \log \delta_k, k = 1, 2.$$

Consequently, the diagonals-parameter symmetry model is contained in the family (2.1).

By substituting

$$(3.9) \quad \theta_{ij} - \theta_{ji} = \begin{cases} \xi_{j-i} + \gamma, & i = 2, j = 3 \\ \xi_{j-i}, & i = 1, j = 2, 3 \end{cases}$$

in (3.7) we obtain the following alternative expression of the multinomial family (3.6)

$$(3.10) \quad dP_{\theta} = \left[ C(\theta) \exp \left( \gamma x_{23} + \sum_{i=1}^2 \theta_{ii} x_{ii} \sum_{i < j} \theta_{ji} (x_{ij} + x_{ji}) + \sum_{k=1}^2 \xi_k \sum_{i=1}^{3-k} x_{ii+k} \right) \right] dP_0.$$

According to (3.9), the hypothesis

$$H: \gamma = 0$$

is equivalent to the hypothesis of diagonals-parameter symmetry. The test which rejects

$$H: \gamma = 0 \text{ against } A: \gamma > 0$$

when

$$X_{23} > k(V, W)$$

where

$$V = (V_1, V_2), \quad W = (W_{12}, W_{13}, W_{23})$$

$$V_k = \sum_{i=1}^{3-k} X_{ii+k} \quad \text{and} \quad W_{ij} = X_{ij} + X_{ji},$$

is UMP unbiased. A randomization is needed in order to fulfill the optimum property. For practical reasons, however, this randomization will be ignored.  $k$  is chosen to satisfy the given significance level,  $\varepsilon$ , conditional upon  $V$  and  $W$ . To obtain this test, we have made use of Theorem 2.1 and the fact that the null-distribution of  $X_{23}$  conditional on the statistics  $V_1, V_2, W_{12}, W_{23}, X_{11}$  and  $X_{22}$  is independent of  $X_{11}$  and  $X_{22}$ .

We have the following expression for the null-distribution (see Aaberge, 1980)

$$\Pr_H (X_{23} = x_{23} \mid V = v, W = w) = \frac{\binom{w_{23}}{x_{23}} \binom{w_{12}}{v_1 - x_{23}}}{\binom{w_{12} + w_{23}}{v_1}}.$$

Consequently, the null-distribution of the UMP unbiased test is hypergeometric. Thus, no computational problem is involved in applying the test.

The UMP unbiased level  $\varepsilon$  test for the hypothesis of diagonals-parameter symmetry in  $r \times r$ -tables with ordered categories was derived by Aaberge (1980). It is a generalization of the present test result for  $3 \times 3$ -tables and is shown to possess convenient distributional properties which allows exact inference.

### 3.4. A small sample test for independence in two-way contingency tables

Consider a two-dimensional contingency table with  $r$  rows and  $s$  columns. The observed number in cell  $(i,j)$  is denoted by  $x_{ij}$  and the corresponding random variables is  $X_{ij}$ . Assume that the probability distribution of the random vector  $X = (X_{11}, \dots, X_{rs})$  is given by the multinomial distribution

$$(3.11) \quad \frac{N!}{\prod_i \prod_j x_{ij}!} \prod_i \prod_j p_{ij}^{x_{ij}}$$

where  $\sum_i \sum_j x_{ij} = N$  is the size of the sample,  $p_{11}, \dots, p_{rs}$  denote cell probabilities and

$$\sum_i \sum_j p_{ij} = 1.$$

Let  $\wp$  be the exponential family given by (3.11). By introducing the minimal canonical parameters

$$\theta_{ij} = \log p_{ij} / p_{rs}, \quad i = 1, \dots, r; \quad j = 1, \dots, s$$

we obtain the following alternative expression for  $P_{\theta} \in \wp$

$$(3.12) \quad dP_{\theta} \left[ C(\theta) \exp \left( \sum_{(i,j) \neq (r,s)} \exp(\theta_{ij} x_{ij}) \right) \right] dP_0$$

where

$$C(\theta) = (rs)^N \left( 1 + \sum_{(i,j) \neq (r,s)} \exp(\theta_{ij}) \right)^{-N}$$

and  $P_0$  is the element in  $\wp$  corresponding to  $p_{ij} = 1/rs$  for  $i = 1, \dots, r; j = 1, \dots, s$ .

The hypothesis of independence is defined by

$$H: p_{ij} = p_{i+} p_{+j}, \quad i=1, \dots, r; \quad j=1, \dots, s$$

where

$$p_{i+} = \sum_{j=1}^s p_{ij} \quad \text{and} \quad p_{+j} = \sum_{i=1}^r p_{ij}$$

are marginal probabilities.

Alternatively, the hypothesis of independence can be expressed as

$$H: \theta_{ij} - \theta_{is} - \theta_{rj} = 0, \quad i=1, \dots, r-1; \quad j=1, \dots, s-1$$

i.e. the hypothesis can be formulated as linear restrictions of the minimal canonical parameter space.

According to the results in Section 2 we can therefore establish a UMP unbiased  $\varepsilon$ -level test for independence. Consequently, let us confine our attention to the class of alternatives defined by the following a priori restrictions

$$(3.13) \quad \theta_{ij} - \theta_{is} - \theta_{rj} = a_{ij} \gamma, \quad i=1, \dots, r-1; \quad j=1, \dots, s-1$$

where  $\gamma$  is unspecified and  $a_{11}, \dots, a_{r-1, s-1}$  are predetermined values, i.e. we will construct a test for

$$H: \gamma = 0 \quad \text{against} \quad A: \gamma > 0$$

under the model  $\wp_{\gamma, B}$  given by (3.12) and (2.1), where  $\mathbf{v} = 0$ ,

$\mathbf{U} = (a_{11}, \dots, a_{1s-1}, a_{21}, \dots, a_{2s-1}, \dots, a_{r-1,1}, \dots, a_{r-1, s-1})'$ ,  $h = rs - 1$ ,  $d_1 = (r-1)(s-1)$  and  $\mathbf{B}$  is a  $d \times h$ -matrix with rank  $d_1$  and elements equal to 0, 1 and -1.

From (3.12) and (3.13) we see that every distribution  $P_{\theta}$  given by

$$(3.14) \quad dP_{\theta} = \left[ C(\theta) \exp \left( \gamma \sum_{i=1}^{r-1} \sum_{j=1}^{s-1} a_{ij} x_{ij} + \sum_{i=1}^{r-1} \theta_{is} x_{i+} + \sum_{j=1}^{s-1} \theta_{rj} x_{+j} \right) \right] dP_0$$

where

$$x_{i+} = \sum_{j=1}^s x_{ij} \quad \text{and} \quad x_{+j} = \sum_{i=1}^r x_{ij}$$

is an element of the family  $\wp_{\gamma, B}$ .

Hence, by Theorem 2.1, an UMP unbiased test of  $H:\gamma=0$  against  $A:\gamma>0$  is obtained

by considering the null-distribution of  $Y = \sum_{i=1}^{r-1} \sum_{j=1}^{s-1} a_{ij} X_{ij}$  conditional on

$$X_{1+}, \dots, X_{r-1+}, X_{+1}, \dots, X_{+s-1}.$$

Note that this particular class of tests is UMP unbiased for the hypothesis of independence against the alternative class of dependence defined by (3.13).

The null-distribution is easily found from the generalized multivariate hypergeometric distribution

$$(3.15) \quad \begin{aligned} & \Pr_H \left( X_{11} = x_{11}, \dots, X_{r-1s-1} = x_{r-1s-1} \mid X_{1+} = x_{1+}, \dots, X_{+1} = x_{+1}, \dots \right) \\ &= \frac{\binom{x_{1+}}{x_{11}, \dots, x_{1s}} \binom{x_{2+}}{x_{21}, \dots, x_{2s}} \cdots \binom{x_{r+}}{x_{r1}, \dots, x_{rs}}}{\binom{N}{x_{+1}, \dots, x_{+s}}} \end{aligned}$$

where

$$x_{rj} = x_{+j} - \sum_{i=1}^{r-1} x_{ij}, \quad j = 1, \dots, s$$

and

$$x_{is} = x_{i+} - \sum_{j=1}^{s-1} x_{ij}, \quad i = 1, \dots, r.$$

In the particular case where all  $a_{ij} = 1$  it can be shown that the null distribution reduces to the hypergeometric distribution

$$\Pr_H \left( Y = y \mid X_{1+} = x_{1+}, \dots, X_{+1} = x_{+1}, \dots \right) = \frac{\binom{\sum_{j=1}^{s-1} x_{+j}}{y} \binom{N - \sum_{j=1}^{s-1} x_{+j}}{\sum_{i=1}^{r-1} x_{i+} - y}}{\binom{N}{\sum_{i=1}^{r-1} x_{i+}}}.$$

The model for which all  $a_{ij}$  are equal to 1 is known as the uniform association model and is also considered by Goodman (1979b). When  $\gamma = 0$ , we have the null association model, which is the usual model of statistical independence.

To decide upon an appropriate test (i.e. predetermination of the values for all  $a_{ij}$ ), some particularly interesting alternatives against which the hypothesis is to be tested may guide us in choosing a certain set of values of  $a_{ij}$ ,  $i = 1, \dots, r - 1$ ,  $j = 1, \dots, s - 1$ . In the case when no a priori information is available, we suggest the test for which  $a_{ij} = 1$  for all  $i$  and  $j$ .

When  $r = s = 2$  the null-distribution (3.15) reduces to the univariate hypergeometric distribution and the test shows to be the well-known Fischer-Irwin's exact test for independence in  $2 \times 2$ -tables (Fischer, 1934; Irwin, 1935; Lehmann, 1986).

## 4. Summary and discussion

The present paper provides a simplified strategy for applying the theory of UMP unbiased tests for regular exponential families of distributions. This strategy aims at introducing a regular subfamily of the a priori model, in which all regular hypotheses can be expressed through a single parameter.

Within this subfamily the standard Neyman-Pearson theory can be applied to construct UMP unbiased tests. This approach leads to UMP unbiased tests for various multiparameter testing problems with restricted alternatives and is inter alia used to construct a UMP unbiased test for the hypothesis of independence in two-way tables against a restricted alternative class of dependence. Moreover, the obtained test proves to be exact since we can use the exact null-distribution rather than its large sample approximation. An interesting question is how this test performs when the a priori restrictions are not quite correct. By comparing the power of the UMP unbiased test to the power of some reasonable competitors, information about that problem is obtained.

For the particular problem of comparison of proportions in several  $2 \times 2$  tables, Aaberge (1983) gives asymptotic Pitman efficiencies of a UMP unbiased test and some competitors and shows that the comparisons favor use of the UMP unbiased test when reasonable a priori information is available. This result suggests that the present test principle is attractive even when the restricted a priori models are not quite correct.



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