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Masters and Slaves – A Matching Approach with Heterogeneous Workers

von

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Masters and Slaves – A Matching Approach with Heterogeneous Workers

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Abstract
At present, most countries have officially ratified the ILO Convention concerning forced or compulsory labor; however, serfdom is still present in the twenty-first century. This paper addresses the questions of how situations of modern slavery arise and how oppressors select their victims. The analytical framework is a labor-market model in which masters and slaves are matched via a matching function. In contrast to the standard matching model, not the workers exert effort to find jobs but the employers exert effort to find and hire slaves. Workers are heterogeneous regarding their “slavability”, which is ex-ante unknown to the potential employers. Employers exert effort to recruit slaves. The employer's decision whether and to what extent to engage in forced labor depends on governmental labor protection and on the probability of detection. Moreover, the model includes the possibility of bribery such that an employer can avoid sanctions if illicit behavior is detected. The model is solved and the impact of policy variables and other exogenous parameters on the firms’ activities are investigated.

Keywords: Coerced Labor, Modern Slavery, Matching

JEL codes: J23, J47, J71

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1. Introduction and motivation

Slavery was officially abolished 1981, when Mauritania as the last country outlawed forced labor. However, in practice, coerced labor still persists in various forms (The Economist 2018). Results from earlier studies of the International Labour Organisation (ILO) suggest, that there are “at least 2 victims of forced labor per 1000 inhabitants and in relation to the total world labor force the minimum estimate corresponds to about 4 persons in forced labor per 1000 workers” (Belser, Cock, and Mehran 2005). As abusive labor practices often happen in legal gray areas or are concealed, the real extent of forced labor remains unknown.

1.1. What is forced labor?

According to the ILO Convention No. 29 (Art. 2.1), adopted in 1930, forced or compulsory labor is defined as “all work or service, which is exacted from any person under the menace of any penalty and for which the said person has not offered himself voluntarily” (International Labor Organisation 2018). The key-criteria in the definition of forced labor are “penalty” and “involuntariness” as victims are forced to work on threat of physical and psychological penalties. Coerced labor is not equal to poor working conditions as it implies a lack of freedom and physical movement (Belser, Cock, and Mehran 2005).

By which means do victims end up in the hands of slaveholders? Not all victims of force labor are recruited from their exploiters in the same way. According to Kelly and Regan (2000) four different approaches of “victimization” are existent. Simply abducting an individual is considered “complete coercion”. When victims are promised jobs in the legitimate economy, but are forced into servitude later on, the authors speak of “deception”. Sometimes individuals are only told “half-truths”, where they are made to believe, that they will be working for example as exotic dancers but are forced into (sexual) servitude instead. In other cases, victims willingly become bonded laborers, but are unaware of the extent of intimidation, exploitation and indebtedness of the “contract”.

Although uneducated or handicapped people from impoverished areas and ethnic minorities are the most vulnerable, individuals from developed countries fall prey to slaveholders as well (Fletcher, Bales, and Stover 2005).
After release, recue or escape most forced labor survivors suffer from physical or psychological problems and social stigmata.

Most forced labor cases can be attributed to the following economic sectors: prostitution, domestic service, manual labor in agriculture, fishing, mining, the hotel and restaurant industry, production (e.g. carpet factories), and in the entertainment industry. Forced labor is highly profitable and often tied to organized (international) crime and driven by the demand for cheap (sex) services. As most coerced labor takes place in the absence of labor force protection, e.g. in (remote) areas where enforcement of existing rules is weak, “employers” often get away unpunished easily. Even if the illicit behavior is detected, punishment is seldom harsh (Fletcher, Bales, and Stover 2005). Victims scarcely report misconduct because of fear (of painful punishment or deportation) or lack of physical mobility (e.g. to a police station) (U.S. Department of State 2001-17).

2. Literature review

Preliminary works in the field of slavery focused primarily on the development of theoretical models in a historical context (Findlay 1975; Chwe 1990; Ergin and Sayan 1998; Eltis, Lewis, and Richardson 2005). One of the first empirical investigations into slavery was performed in 1988 by Field (1988). Scheidel (2010) discusses slavery in the Roman economy.

More recent studies focus on the transition from a slave economy to a free-labor system and vice versa (Lagerlöf 2009) and on the use of punishment to achieve labor coercion (Acemoglu and Wolitzky 2011).

Research concerning modern slavery mostly focusses on the empirical aspects of human trafficking. Some of these studies focus on the effects of prostitution (Cho, Dreher, and Neumayer 2013; Jakobsson and Kotsadam 2013), others on ethnic fragmentation and conflicts (Akee et al. 2010) or determinants of human trafficking in general (Danailova-Trainor and Belser 2006; Frank 2013; Cho 2015).

A substantial body of research focusses on child labor, which is a subset of forced labor. Various approaches have been put forward to account for the influence of the parents on child labor (Ranjan 2001; Weinberg 2001; Basu and Chau 2004; Dessy and Pallage 2005; Rogers and Swinnerton 2007; Strulik 2008). Moreover, child labor has been identified as being a substitute to low-skilled labor (Dinopoulos and Zhao 2006). Noteworthy studies
of child soldiering and child trafficking were carried out by (Beber and Blattman 2010) and (Dessy, Mbi ekop, and Pallage 2005) respectively. Links between international labor laws and child labor has been discussed by Basu (1999).

Very little has been written on forced labor of adults. Fletcher, Bales, and Stover (2005) and Belser, Cock, and Mehran (2005) attempt to measure the extent of forced labor in the United States and worldwide respectively.

The review of the literature indicates that research has been limited to slavery in a historical context, human trafficking (mostly sex trafficking) and child labor. Theoretical approaches examining the mechanisms inducing forced labor have received very little attention in the literature. Only Acemoglu and Wolitzky (2011) model an environment of labor coercion. In their model, firm and worker are randomly matched, and the worker chooses her/his effort according to the severity of punishment inflicted.

This paper, to my knowledge, is the first to use a matching approach to examine how perpetrators find their heterogeneous victims. Potential slaves are heterogeneous regarding their individual vulnerability and protection against labor exploitation These individual characteristics are summarized as a single parameter called “slavability” and include education, age, gender, phenotype, and endowments. A higher level of “slavability” indicate a higher risk of being enslaved. Moreover, the employer’s effort to recruit slaves and her/his expenditure on bribes is influenced by government activities (labor protection, prosecution). We solve the model to determine the impact of policy parameters on slavery and derive comparative-static results.

The remainder of this paper is organized as follows. The next section outlines the theoretical model and its major assumptions. In subsection 3.4, a model with heterogeneous slaves is developed using a matching approach. Subsection 3.7 and 3.8 provide a detailed explanation of all inner solutions and the boundary case, respectively. Section four concludes. All mathematical derivations of results can be found in the appendix.
3. The model

We start by outlining the characteristics of workers, employers and the governmental protection against slavery used in the forthcoming models. Building on an idea by Dessy, Mbiekop, and Pallage (2005) used in their model of child labor, we argue that there are private and public measures that protect individual laborers from slavery. Private measures are incorporated in the variable termed “slavability”, public measures take the form of governmental labor protection, and both of them influence the decisions of potential slaveholders.

3.1. Workers

There is a continuum of workers with mass \( N \). Each individual has her own individual protection \( v_i \) against slavery in form of education, external characteristics (age, gender, race) and endowments, where \( v_i \in \{0,1\} \). As mentioned above, potential slaves are assumed to be heterogeneous with respect to “slavability” and \( v_i \) is assumed to be equally distributed in the population. The degree of an individual’s “slavability” is unknown to the employer, who is assumed to be risk-neutral. However, the density function of \( v_i \) is known. Individual \( V \) is the cut-off individual, i.e. all individuals with \( v_i \geq V \) become slaves and all individuals with \( v_i < V \) are “normal” workers. Thus, \( V \cdot N \) is the number of slaves in society and correspondingly \((1-V) \cdot N\) will denote the number of “normal” workers.

3.2. Employers

Only employers willing to engage in slavery are considered. The mass of all homogeneous firms sums up to 1 (prices are given), subsequently only the actions of a representative firm active in the slave-sector will be discussed. An employer \( k \), who wants to engage in slavery chooses the level of effort, \( e \), necessary in order to break down the protective barrier. Employers know the distribution of the \( v_i \) in the society.

3.3. Government-protection against slavery

Workers are protected against slavery by government policy, e.g. labor protection laws and their enforcement, Let its strictness being denoted by \( g \). Governmental protection is exogenous to the firm and in practice country-specific. E.g., \( g \) is likely to be small in corrupt states.
3.4. Matching

The following matching model is based on Cahuc, Carcillo, and Zylberberg (2014, 583–87) and Pissarides (2000). Slaves and slave-masters are matched randomly. They are selected from the workers available in society, \(N\), and from the employers willing to engage in forced labor \(K\), respectively.

In order to recruit a slave, an employer \(j\) chooses a fixed effort level \(e_j\) beforehand. Since the employer does not know the “slavability” of the potential victim, the match (actual enslavement) is successful only with a certain probability \(p\). The probability of success depends on the number of available workers, \(N\), and the chosen effort level \(e_j\). The continuum of potential slaves is represented by the integral from zero to \(N\). Further, the individual firm is too small to exert any monopolistic power.

The number of matches in the economy is:

\[
M = M(N,e) = \int_0^N \left[ 1 - \prod_{j=1}^K \left( 1 - \frac{e_j \nu_i}{N} \right) \right] di,
\]

(1)

where \(1 - \frac{e_j \nu_i}{N}\) is the probability that an individual \(i\) is not “offered” a job in slavery by a particular employer. Aggregating over all firms \(K\), \(1 - \prod_{j=1}^K \left( 1 - \frac{e_j \nu_i}{N} \right)\) is the probability that an individual \(i\) does not receive an “offer” to be enslaved by any firm.

Summing up over all workers, we get the number of matches, which equals the number of slaves in the society.

It is assumed that \(N\) is large compared to \(e\); therefore \(1 - \frac{e_j \nu_i}{N}\) can be approximated by \(exp[-\left(\frac{e_j \nu_i}{N}\right)]\) and the matching function is:

\[
M = \int_0^N \left[ 1 - \exp \left( - \frac{eK \nu_i}{N} \right) \right] di,
\]

(1')

where multiplying over all employers leads to the emergence of \(K\) in the numerator and \(e\) is the average effort to recruit a slave.
Solving the integral, substituting \( v_i = \frac{i}{N} \) and inserting upper and lower bounds yields\(^1\):

\[
M = N \left[ 1 - \frac{N}{e^K} \left( 1 - \exp \left( -\frac{e^K}{N} \right) \right) \right].
\] (2)

Let us define \( m \left( \frac{e^K}{N} \right) \equiv \left[ 1 - \frac{N}{e^K} \left( 1 - \exp \left( -\frac{e^K}{N} \right) \right) \right] \) as the matching function measuring the average probability of an individual to be enslaved by some firm. It is monotonously increasing and strictly concave\(^2\). The total amount of effort of all employers is \( e \cdot K \). Thus, the probability of an employer \( j \) to recruit a slave is \( \frac{e_j M}{e^K} \). Therefore, an employer has a greater chance to enslave an individual, the greater her or his level of relative effort \( \frac{e_j}{e} \) is.

If the relative effort of a firm goes towards infinity, all individuals will be enslaved; whereas at an effort level of \( e = 0 \) no one has to endure slavery. For the remainder of the paper, we use \( m(\cdot) \) instead of the explicit function derived above, i.e.

\[
M = N m \left( \frac{e^K}{N} \right).
\] (3)

3.5. The decision of the firm

Compared to hiring workers on the legal market, the additional revenue generated by slaves for a representative firm is:

\[
Y = S \pi,
\] (4)

where the number of slaves per firm is

\[
S = \frac{N}{K} m \left( \frac{e^K}{N} \right) = \frac{m(\sigma)}{\sigma},
\] (5)

where \( \frac{K}{N} = \sigma \) represents the ratio of slaveholders to the available workforce.

The extra profit derived from employing a slave instead of a “normal” worker is \( \pi \) (as slaves earn lower wages and suffer from lower working conditions). Equation (5) shows that the effect of \( \sigma \) on the number of slaves are ambiguous. The probability of an individual being enslaved depends positively on the ratio of slaveholders to workers and the number of slaves an individual firm gets depends positively on its inverse.

\(^1\) A detailed derivation of the matching function can be found in the Appendix.

\(^2\) For proof, see Appendix.
3.6. Costs

The costs accruing to a representative slaveholder are threefold. First, costs for dealing with legal institutions (executive, judiciary) arise in the form of bribery $B$. Second, costs $C$, which emerge upon detection, arise\(^3\). On the one hand, these costs increase with stricter law enforcement, $g$, and a higher number of firms engaging in slavery, $S$. On the other hand, $C$ is decreasing with higher $B$. Third, recruiting slaves with effort $e$ is costly as well as it is measured in monetary units. The probability of detection is denoted by $q$.

$$B + qC(B, S, g) + e.$$  

(6)

A representative employer maximizes profits,

$$\Pi = S\pi - B - qC(B, S, g) - e,$$  

(7)

with respect to $B$ and $e$ subjects to the constraints $B \geq 0$, $e \geq 0$, $\pi \geq 0$ and $\Pi \geq 0$.

The first-order conditions are:

$$\frac{\partial \Pi}{\partial B} = 0: qC_B = 1$$  

(8)

$$\frac{\partial \Pi}{\partial e} = 0: m'\pi = qC_S + 1$$  

(9)

Equation (8) states that the expected marginal benefit of bribery (the reduction in the expected cost of punishment) equals marginal cost, which is unity in this model. In equation (9), the marginal additional profits generated by slaves equal the marginal costs of acquiring slaves times the probability of detection plus the marginal cost of effort, which is unity here.

\(^3\) $C_B < 0$, $C_S > 0$, $C_g > 0$, $C_{BB} > 0$, $C_{SS} > 0$, $C_{Bg} < 0$, $C_{BS} < 0$, $C_{Sg} > 0$, subscripts representing the respective partial first, second, and cross derivatives.
3.7. Results

Based on the first-order conditions (8) and (9), we obtain the following comparative-static results. Table 1 shows that all results are ambiguous (proof in the Appendix).

Table 1: Comparative Statics

<table>
<thead>
<tr>
<th>on</th>
<th>$g$</th>
<th>$\sigma$</th>
<th>$q$</th>
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<tbody>
<tr>
<td>$B$</td>
<td>$+/-$</td>
<td>$+/-$</td>
<td>$+/-$</td>
</tr>
<tr>
<td>$e$</td>
<td>$+/-$</td>
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</tbody>
</table>

The results can be interpreted as follows:

Bribery is affected by exogenous parameters and variables as follows:

- **Strength of governmental labor protection and probability of detection.**

  As shown in the appendix, the partial derivatives of the first-order conditions with respect to $g$ and $q$ have the same signs. Thus, the impacts of these variables on bribery can be discussed jointly. The direct effect of an increase of governmental protection or of the probability of detection on bribery is determined by equation (8). An increase in $g$ or $q$ strengthens the marginal effect of bribery on punishment reduction and thus induces increased bribery. The indirect effect (how a change in effort affects bribery) is determined by equation (9). An increase in $g$ or $q$ raises the marginal costs of having additional slaves, which in turn reduces the effort of the slaveholder. Lower effort leads to a lower number of slaves and the marginal effect of bribery is reduced, thus a lower amount of bribes is paid. It should be noted, that the total effect of $g$ and $q$ may differ in their signs as the direct and indirect effects, albeit having the same signs, may differ in magnitude.

- **Ratio of slaveholders to workers.**

  As shown earlier, the effect of $\sigma$ on the number of slaves is unclear. If $\sigma$ is large, the probability of being enslaved, $m$, increases (thus the number of slaves increases). Simultaneously, the probability that a specific firm can hire a slave is small. Hence, the effect of $\sigma$ on bribery is ambiguous as well. A large number of
slaves strengthens the marginal effect of bribery on punishment reduction and thus induces increased bribery (direct effect). The indirect effect is ambiguous as well, because it consists of two parts. First, if $\sigma$ increases, a large number of employed slaves increases the marginal costs of punishment and effort is decreased. Second, if $\sigma$ increases, the marginal profit of exploiting slaves decreases, effort is reduced as well as the amount of bribes.

The effort to find and hire slaves is affected as follows:

- **Strength of governmental labor protection and probability of detection.**
  As in the case of bribery the effect of $g$ and $q$ can be discussed jointly. On the one hand, an increase in $g$ or $q$ increases the marginal costs to acquire slaves and thus decreases the effort to recruit slaves (direct effect). On the other hand, an increase in $g$ or $q$ requires more bribes in order to avoid punishment, therefore if the amount of bribes increases, effort increases as well (indirect effect). Again, the total effect of $g$ and $q$ may differ in their signs as the direct and indirect effects, albeit having the same signs, may differ in magnitude.

- **Ratio of slaveholders to workers.**
  As the effect of $\sigma$ on the number of slaves is unclear, the effect of $\sigma$ on effort is ambiguous as well. A high $\sigma$ increases the marginal costs to obtain slaves and thus effort decreases (direct effect). A high number of slaves employed strengthens the marginal effect of bribery on punishment reduction, which increases bribery and in turn leads to increased effort (indirect effect).

### 3.8. Negative profits and the prevention of slavery

All previous results require that the maximized profits are strictly positive. However, if the economic conditions are unfavorable for bribery, profits can be negative even in the optimum. In this case, the firms will abstain from slavery and the corresponding effort will be zero. In order to analyze how firms determine the impact of exogenous parameters on their maximized profits, the boundary solution where the profits of the slaveholder become zero, needs to be examined. The maximized profits are differentiated with respect to the exogenous parameters, $g$, $K_S$, $N$ and $q$. 
Differentiating equation (7) with respect to $g$ yields:

$$\frac{d\Pi}{dg} = \frac{\partial \Pi}{\partial g} + \frac{de}{dg} \frac{\partial \Pi}{\partial e} + \frac{dB}{dg} \frac{\partial \Pi}{\partial B}$$

(10)

Due to the envelope theorem (first-order conditions of the decision maker), the last two terms cancel out. Thus,

$$\frac{\partial \Pi}{\partial g} = -qC_g \cdot < 0$$

(11)

The same applies to the comparative statics with respect to the other parameters:

$$\frac{\partial \Pi}{\partial K} = Nm'e \frac{e}{N} \pi - qC_S e < 0$$

(12)

$$\frac{\partial \Pi}{\partial N} = m'eK \pi - qC_N eK > 0$$

(13)

$$\frac{\partial \Pi}{\partial q} = -C(B, S, g) < 0$$

(14)

The signs of the comparative statics can be interpreted as follows: Stricter governmental law enforcement $g$, a higher probability of detection $q$, and more firms engaging in slavery $K$ decrease the profits of the individual slaveholder $\Pi$. A large working population $N$ indicates a higher number of potential slaves and increases the profits of the slaveholder. Strict governmental labor regulations and a high probability of detection are complements. A larger working population implies more firms engaging in slavery, thus $N$ and $K$ are complementary. The probability of detection and the number of firms engaging in slavery are substitutes.
Figure 1 shows the impact of governmental labor protection $g$, on the effort $e_i$, and the profits $\Pi_j$, of the slaveholder. Stricter governmental labor protection requires higher $e$ in order to recruit slaves successfully. Accordingly, effort increases with higher $g$ until the critical value $g^*$, where further effort is not profitable anymore. Profits however, decrease with stricter labor protection levels. When profits become zero or negative, slaveholders will choose an effort of zero and will not engage in slavery anymore. Thus, if labor protection is stricter than $g^*$, employers will abstain from slavery.

*Figure 1: Impact of governmental labor protection on effort and profits*
4. Conclusion

This paper has gone some way towards understanding the mechanisms associated with labor coercion. The main objective was to find out how situations of slavery and suppression arise and how oppressors select their victims.

Some results confirm the a priori expectations; for example, a higher probability of detection increases the amount of bribes. Some results are less intuitive, but can be explained. For example, the presence of stricter labor force protection induces more effort of the slaveholder to acquire slaves. At the same time, however strict labor protection reduces profits. If profits become zero or even negative, employers will abstain from recruiting slaves. Thus, the effort is increasing for low values of governmental labor protection and then drops to zero if a critical level of government protection of labor is reached. At this point, the number of slaves drops from strictly positive to zero.

Future research may address how a “slaveholder society” can evolve into a “slave-free society” and vice versa. It is to be expected that hysteresis plays a major role here, i.e. that large changes in parameters are necessary to move from one regime to the other. Novel modelling strategies going beyond the simple model presented here are needed to address these discontinuities.

Future work may also look at welfare implications of different policies. In addition, one could model heterogeneous slaveholders who differ with regard to their degree of risk aversion or unscrupulousness. Furthermore, a dataset to look into labor coercion from an empirical point of view is already in preparation. An aggregate version of the model analyzed in section 3 could be used to address slavery on a country level with maximum likelihood methods, which are the appropriate way to deal with the discontinuities identified in the theoretical model.
References


Appendix – mathematical derivations

Calculating the matching function

The matching function is (equation (1’) in the text):

\[ M = \int_0^N \left[ 1 - \exp \left( - \frac{e^{K+v_i}}{N} \right) \right] di. \]

(A.1)

In a next step we insert \( v_i = \frac{i}{N} \), which gives

\[ M = N - \int_0^N \exp \left( - \frac{e^{K+i}}{N^2} \right) di. \]

(A.1.a)

Solving the integral yields:

\[ M = N + \left[ \frac{N^2}{eK} \star \exp \left( - \frac{e^{K+i}}{N^2} \right) \right]_0^N. \]

(A.1.b)

Inserting upper and lower bounds:

\[ M = N - \frac{N^2}{eK} + \frac{N^2}{eK} \star \exp \left( - \frac{e^{K}}{N^2} \right), \]

(A.1.c)

which yields after rearranging:

\[ M = N \left[ 1 - \frac{N}{eK} \left( 1 - \exp \left( - \frac{e^{K}}{N} \right) \right) \right] \equiv Nm \left( \frac{eK}{N} \right). \]

(A.1.d)

Properties of the matching function

The matching function \( m(\cdot) \) has a positive slope with \( m'(\cdot) > 0 \) and \( m''(\cdot) < 0 \) and its values range from 0 to 1.

Figure 2: Matching Function converges to \( N \).

Proof: For \( m(\cdot) \lim_{x \to \infty} \frac{1-exp^{-x}}{x} = 0 \) and applying de l’Hospital’s rule \( \lim_{x \to 0} \frac{1-exp^{-x}}{x} = 1 \).

To save on notation, let us replace \( x \equiv \frac{eK}{N} \).
The first derivative of $f(x) = 1 - \frac{1-e^{-x}}{x}$ is:

$$f'(x) = -\frac{(1-e^{-x})-xe^{-x}}{x^2} = \frac{-((1-e^{-x})+xe^{-x})}{x^2} = \frac{(x+1)e^{-x} - 1}{x^2}. \quad \text{(A.2)}$$

where $(x + 1)e^{-x} - 1 > 0$ and $x + 1 > e^{-x}$.

Calculating the second derivative of $f(x)$ yields:

$$f''(x) = \frac{2x - x^3e^{-x} - xe^{-x} - e^{-x}}{x^3}. \quad \text{(A.2.a)}$$

Comparative Statics

Total differentiation of equations (8) and (9) yields:

$$\begin{pmatrix} -qC_{BB} & -qC_{BS}m' \\ -qC_{BS} & m''(\sigma)\pi - qC_{SS}m' \end{pmatrix} \ast \begin{pmatrix} dB \\ de \end{pmatrix} = \begin{pmatrix} qC_{Bg} & qC_{BS} \left( \frac{m'(e\sigma) - me}{\sigma^2} \right) \\ qC_{Sg} & -m''(e\sigma)e^2\pi + qC_{SS} \left( \frac{m'(e\sigma) - me}{\sigma^2} \right) \end{pmatrix} \begin{pmatrix} C_B \\ C_S \end{pmatrix} \ast \begin{pmatrix} dg \\ d\sigma \\ dq \end{pmatrix} \quad \text{(A.3)}$$

Combining the determinant of the matrix on the left-hand side of (A.3), $D$, \footnote{$D \equiv -\sigma m''\pi qC_{BB} - m'qC_{SS} - q^2C_{BS}^2m'$} with the corresponding adjoints leads to:

$$\frac{dB}{dg} = \frac{1}{D} \left( m''\pi qC_{BB}\sigma - m'qC_{SS} + q^2C_{Sg}C_{BS}m' \right) \quad \text{(A.4.a)}$$

$$\frac{de}{dg} = \frac{1}{D} \left( -q^2C_{BB}C_{Sg} + q^2C_{Bg}C_{BS} \right) \quad \text{(A.4.b)}$$

$$\frac{dB}{d\sigma} = \frac{1}{D} \left( q\left( m'''\pi C_{BS}[\pi e^{-x} - m'\pi C_{BB}e^{-x} - 1]m'C_{SS}\sigma \right) + \frac{m''q^2C_{BS}C_{SS}[e^{-x} - m^2qC_{BS}C_{SS}e^{-x} - m''\pi e^3]}{\sigma^2} \right) \quad \text{(A.4.c)}$$

$$\frac{de}{d\sigma} = \frac{1}{D} \left( q(e(m'''\pi C_{BB}\sigma e^{-x} + m'C_{SS}\pi C_{BB}e^{-x} + mC_{SS}) + \frac{q^2m'\pi e^{C_{BS}(\sigma - 1)}}{\sigma^2} \right) \quad \text{(A.4.d)}$$

$$\frac{dB}{dq} = \frac{1}{D} \left( m''\pi C_{BB}\sigma - m'qC_{SS} + qC_{Sg}qC_{BS}m' \right) \quad \text{(A.4.e)}$$

$$\frac{de}{dq} = \frac{1}{D} \left( -qC_{BB}C_{Sg} + qC_{BB}C_{BS} \right) \quad \text{(A.4.f)}$$