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Optimal sup-norm rates and uniform inference on nonlinear functionals of nonparametric IV regression

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Optimal Sup-norm Rates and Uniform Inference on Nonlinear Functionals of Nonparametric IV Regression*

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Abstract

This paper makes several important contributions to the literature about nonparametric instrumental variables (NPIV) estimation and inference on a structural function h_0 and its functionals. First, we derive sup-norm convergence rates for computationally simple sieve NPIV (series 2SLS) estimators of h_0 and its derivatives. Second, we derive a lower bound that describes the best possible (minimax) sup-norm rates of estimating h_0 and its derivatives, and show that the sieve NPIV estimator can attain the minimax rates when h_0 is approximated via a spline or wavelet sieve. Our optimal sup-norm rates surprisingly coincide with the optimal root-mean-squared rates for severely ill-posed problems, and are only a logarithmic factor slower than the optimal root-mean-squared rates for mildly ill-posed problems. Third, we use our sup-norm rates to establish the uniform Gaussian process strong approximations and the score bootstrap uniform confidence bands (UCBs) for collections of nonlinear functionals of h_0 under primitive conditions, allowing for mildly and severely ill-posed problems. Fourth, as applications, we obtain the first asymptotic pointwise and uniform inference results for plug-in sieve t-statistics of exact consumer surplus (CS) and dead-weight loss (DL) welfare functionals under low-level conditions when demand is estimated via sieve NPIV. Empiricists could read our real data application of UCBs for exact CS and DL functionals of gasoline demand that reveals interesting patterns and is applicable to other markets.

Keywords: Series 2SLS; Optimal sup-norm convergence rates; Uniform Gaussian process strong approximation; Score bootstrap uniform confidence bands; Nonlinear welfare functionals; Nonparametric demand with endogeneity.

JEL codes: C13, C14, C36

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1 Introduction

Well founded empirical evaluation of economic policy is often based upon inference on nonlinear welfare functionals of nonparametric or semiparametric structural models. This paper makes several important contributions to estimation and inference on a flexible (i.e. nonparametric) structural function h_0 and nonlinear functionals of h_0 within the framework of a nonparametric instrumental variables (NPIV) model:

$$Y_i = h_0(X_i) + u_i \quad E[u_i|W_i] = 0 \quad (1)$$

where h_0 is an unknown function, X_i is a vector of continuous *endogenous* regressors, W_i is a vector of (conditional) instrumental variables, and the conditional distribution of X_i given W_i is unspecified.

Given a random sample $\{(Y_i, X_i, W_i)\}_{i=1}^n$ (of size n) from the NPIV model (1), our first two main theoretical results address how well one may estimate h_0 and its derivatives simultaneously in sup-norm loss, i.e. we bound

$$\sup_x \left| \widehat{h}(x) - h_0(x) \right| \quad \text{and} \quad \sup_x \left| \partial^k \widehat{h}(x) - \partial^k h_0(x) \right|$$

for estimators \widehat{h} of h_0 , where $\partial^k h(x)$ denotes k -th partial derivatives of h with respect to components of x . We first provide upper bounds on sup-norm convergence rates for the computationally simple sieve NPIV (i.e., series two stage least squares (2SLS)) estimators (Newey and Powell, 2003; Ai and Chen, 2003; Blundell, Chen, and Kristensen, 2007). We then derive a lower bound that describes the best possible (i.e., minimax) sup-norm convergence rates among all estimators for h_0 and its derivatives, and show that the sieve NPIV estimator can attain the minimax lower bound when spline or wavelet base is used to approximate h_0 .¹ Next, we apply our sup-norm rate results to establish the uniform Gaussian process strong approximation and the validity of score bootstrap uniform confidence bands (UCBs) for collections of possibly nonlinear functionals of h_0 under primitive conditions.² This includes valid score bootstrap UCBs for h_0 and its derivatives as special cases. Finally, as important applications, we establish first pointwise and uniform inference results for two leading nonlinear welfare functionals of a nonparametric demand function h_0 estimated via sieve NPIV, namely the exact consumer surplus (CS) and deadweight loss (DL) arising from price changes at different income levels when prices (and possibly income) are endogenous.³ We present two real data applications to illustrate the easy implementation and usefulness of the score bootstrap UCBs based on sieve NPIV estimators. The first is to nonparametric exact CS and DL functionals of gasoline demand and the second is to nonparametric Engel curves and their derivatives. The UCBs reveal new interesting and sensible patterns in both data applications. We note that the score bootstrap UCBs for exact CS and DL nonlinear functionals are new to the literature even when the prices might be exogenous. Empiricists could jump to Section

¹The optimal sup-norm rates for estimating h_0 were in the first version (Chen and Christensen, 2013). The optimal sup-norm rates for estimating derivatives of h_0 were obtained in the second version (Chen and Christensen, 2015a).

²The uniform strong approximation and the score bootstrap UCBs results were in the second version (Chen and Christensen, 2015a).

³The pointwise inference results on exact CS and DL were in the second version (Chen and Christensen, 2015a).

2 to read the sieve score bootstrap UCBs procedure and these real data applications without the need to read the rest of more theoretical sections.

Regardless of whether the regressor X_i is endogenous or not, sup-norm convergence rates provide sharper measures of how well h_0 and its derivatives can be estimated nonparametrically than the usual L^2 -norm (i.e., root-mean-squared) convergence rates. This is also why, in the existing literature on nonparametric models without endogeneity, consistent specification tests in sup-norm (i.e., Kolmogorov-Smirnov type statistics) are widely used. Further, sup-norm rates are particularly useful for controlling nonlinearity bias when conducting inference on *highly nonlinear* (i.e., beyond quadratic) functionals of h_0 . In addition to being useful in constructing pointwise and uniform confidence bands for nonlinear functionals of h_0 via plug-in estimators, the sup-norm convergence rates for estimating h_0 are also useful in semiparametric two-step procedures when h_0 enters the second-stage moment conditions (equalities or inequalities) nonlinearly.

Despite the usefulness of sup-norm convergence rates in nonparametric estimation and inference, as yet there are no published results on optimal sup-norm convergence rates for estimating h_0 or its derivatives in the NPIV model (1). This is because, unlike nonparametric least squares (LS) regression (i.e. estimation of $h_0(x) = E[Y_i|X_i = x]$ when X_i is exogenous), estimation of h_0 in the NPIV model (1) is a difficult ill-posed inverse problem with an unknown operator (Newey and Powell, 2003; Carrasco, Florens, and Renault, 2007). Intuitively, h_0 in model (1) is identified by the integral equation

$$E[Y_i|W_i = w] = Th_0(w) := \int h_0(x)f_{X|W}(x|w) dx$$

where T must be inverted to obtain h_0 . Since integration smoothes out features of h_0 , a small error in estimating $E[Y_i|W_i = w]$ using the data $\{(Y_i, X_i, W_i)\}_{i=1}^n$ may lead to a large error in estimating h_0 . In addition, the conditional density $f_{X|W}$ and hence the operator T is generally unknown, so T must be also estimated from the data. Due to the difficult ill-posed inverse nature, even the L^2 -norm convergence rates for estimating h_0 in model (1) have not been established until recently.⁴ In particular, Hall and Horowitz (2005) derived minimax L^2 -norm convergence rates for mildly ill-posed NPIV models and showed that their estimators can attain the optimal L^2 -norm rates for h_0 . Chen and Reiss (2011) derived minimax L^2 -norm convergence rates for mildly and severely ill-posed NPIV models and showed that sieve NPIV estimators can attain the optimal rates.⁵ Moreover, it is generally much harder to obtain optimal nonparametric convergence rates in sup-norm than in L^2 -norm.⁶

In this paper, we derive the best possible (i.e., minimax) sup-norm convergence rates of any estimator of h_0 and its derivatives in mildly and severely ill-posed NPIV models. Surprisingly, the optimal sup-

⁴See e.g., Hall and Horowitz (2005); Blundell et al. (2007); Chen and Reiss (2011); Darolles, Fan, Florens, and Renault (2011); Horowitz (2011); Chen and Pouzo (2012); Gagliardini and Scaillet (2012); Florens and Simoni (2012); Kato (2013) and references therein.

⁵Appendix B extends the results in Chen and Reiss (2011) to L^2 -norm optimality for estimating derivatives of h_0 .

⁶Even for the simple nonparametric LS regression of h_0 (without endogeneity), the optimal sup-norm rates for series LS estimators of h_0 were not obtained till recently in Cattaneo and Farrell (2013) for locally partitioning series LS, Belloni, Chernozhukov, Chetverikov, and Kato (2015) for spline LS and Chen and Christensen (2015b) for wavelet LS.

norm convergence rates for estimating h_0 and its derivatives coincide with the optimal L^2 -norm rates for severely ill-posed problems and are only a power of $\log n$ slower than optimal L^2 -norm rates for mildly ill-posed problems. We also obtain sup-norm convergence rates for sieve NPIV estimators of h_0 and its derivatives. We show that a sieve NPIV estimator using a spline or wavelet basis to approximate h_0 can attain the minimax sup-norm rates for estimating both h_0 and its derivatives. When specializing to series LS regression (without endogeneity), our results automatically imply that spline and wavelet series LS estimators will also achieve the optimal sup-norm rates of [Stone \(1982\)](#) for estimating the derivatives of a nonparametric LS regression function, which strengthen the recent sup-norm optimality results in [Belloni et al. \(2015\)](#) and [Chen and Christensen \(2015b\)](#) for estimating regression function h_0 itself. We focus on the sieve NPIV estimator because it has been used in empirical work, can be implemented as easily as 2SLS, and can reduce to simple series LS when the regressor X_i is exogenous. Moreover, both h_0 and its derivatives may be simultaneously estimated at their respectively optimal convergence rates via a sieve NPIV estimator when the same sieve dimension is used to approximate h_0 . This is a desirable property to practitioners. In addition, the sieve NPIV estimator for h_0 in model (1) and our proof of its sup-norm rates could be easily extended to estimating unknown functions in other semiparametric models with nonparametric endogeneity, such as a system of shape-invariant Engel curve IV regression model ([Blundell et al., 2007](#)).

We provide two important applications of our results on sup-norm convergence rates in details; both are about inferences on nonlinear functionals of h_0 based on plug-in sieve NPIV estimators; see Section 6 for discussions of additional applications. Inference on highly nonlinear (i.e., beyond quadratic) functionals of h_0 in a NPIV model is very difficult because of the combined effects of nonlinearity bias and the slow convergence rates (in sup-norm and L^2 -norm) of any estimators. Indeed, our minimax rate results show that *any* estimator of h_0 in an ill-posed NPIV model must necessarily converge slower than their nonparametric LS counterpart. For example, the optimal sup- and L^2 -norm rates for estimating h_0 in a severely ill-posed NPIV model is $(\log n)^{-\gamma}$ for some $\gamma > 0$. It is well-known that a plug-in series LS estimate of a weighted quadratic functional could be root- n consistent. But, a plug-in sieve NPIV estimate of a weighted quadratic functional of h_0 in a severely ill-posed NPIV model will fail to be root- n consistent; see Appendix C and [Chen and Pouzo \(2015\)](#).

In the first application, we extend the seminal work of [Hausman and Newey \(1995\)](#) about pointwise inference on exact CS and DL functionals in nonparametric demand estimation without endogeneity to allow for prices, and possibly incomes, to be endogenous. According to [Hausman \(1981\)](#) and [Hausman and Newey \(1995, 2016, 2017\)](#), exact CS and DL functionals are the most widely used welfare and economic efficiency measures. Exact CS is a leading example of a complicated nonlinear functional of h_0 , which is defined as the solution to a differential equation involving a demand function ([Hausman, 1981](#)). [Hausman and Newey \(1995\)](#) were the first to establish the pointwise asymptotic normality of plug-in kernel estimators of *exact* CS and DL functionals of a nonparametric demand without endogeneity. [Vanhems \(2010\)](#) was the first to estimate exact CS via the plug-in [Hall and Horowitz \(2005\)](#) kernel NPIV estimator of h_0 when price is endogenous, and derived its convergence rate in L^2 -norm for the mildly ill-posed case, but did not establish any inference results (such as the

pointwise asymptotic normality). Our paper is the first to provide low-level sufficient conditions to establish inference results for plug-in (spline and wavelet) sieve NPIV estimators of exact CS and DL functionals, allowing for both mildly and severely ill-posed NPIV models. Precisely, we use our sup-norm convergence rates for sieve NPIV estimators of h_0 and its derivatives to locally linearize plug-in estimators of exact CS and DL, which then leads to asymptotic normality of sieve t -statistics for exact CS and DL under primitive sufficient conditions.⁷ Recently, [Chen and Pouzo \(2015\)](#) presented a set of high-level conditions for the pointwise asymptotic normality of sieve t -statistics of possibly nonlinear functionals of h_0 in a general class of nonparametric conditional moment restriction models (including the NPIV model as a special case). They verified their high-level conditions for pointwise asymptotic normality of sieve t -statistics for linear and quadratic functionals. But, without sup-norm convergence rate result, [Chen and Pouzo \(2015\)](#) were unable to provide low-level sufficient conditions for pointwise asymptotic normality of plug-in sieve NPIV estimators for complicated nonlinear (beyond quadratic) functionals such as the exact CS functional. This was actually the original motivation for us to derive sup-norm convergence rates for sieve NPIV estimators of h_0 and its derivatives.

In the second important application of our sup-norm rate results, we establish the uniform Gaussian process strong approximation and the validity of score bootstrap uniform confidence bands (UCBs) for collections of possibly nonlinear functionals of h_0 , under primitive sufficient conditions that allow for mildly and severely ill-posed NPIV models. The low-level sufficient conditions for UCBs are applicable to complicated nonlinear functionals such as collections of exact CS and DL functionals of nonparametric demand with endogenous price (and possibly income). When specializing to collections of linear functionals of the NPIV function h_0 , our score bootstrap UCBs for h_0 and its derivatives are valid under mild conditions. When specializing further to score bootstrap UCBs for a NPIV function h_0 of a scalar regressor, our sufficient conditions are comparable to those in [Horowitz and Lee \(2012\)](#) for their notion of UCBs with a growing number of grid points by interpolation for h_0 estimated via the modified orthogonal series NPIV estimator of [Horowitz \(2011\)](#).⁸ When specializing to a nonparametric LS regression (with exogenous X_i), our result on the validity of score bootstrap UCBs for collections of nonlinear functionals of h_0 is still new to the literature and complements the recent UCB results in [Chernozhukov, Lee, and Rosen \(2013\)](#) for h_0 and [Belloni et al. \(2015\)](#) for linear functionals of h_0 estimated via series LS.

Our sieve score bootstrap UCBs procedure is extremely easy to implement since it computes the sieve NPIV estimator only once using the data, and then perturbs the sieve score statistics by random weights that are mean zero and independent of the data. So it should be very useful to empirical researchers who conduct nonparametric estimation and inference on structural functions with endogeneity in diverse subfields of applied economics, such as consumer theory, IO, labor economics, public

⁷We also establish the asymptotic normality of plug-in sieve NPIV t -statistic for an *approximate* CS functional, extending [Newey \(1997\)](#)'s result from nonparametric exogenous demand to endogenous demand. Without nonparametric endogeneity, our sharp bounds on sup-norm rates of spline and wavelet series LS estimation of h_0 and its derivative lead to asymptotic normality of sieve t -statistics for exact CS and DL, and approximate CS functionals under conditions that are weaker than those in the existing literature.

⁸They provided bootstrap estimated critical values over fixed finite number of grid points.

finance, health economics, development and trade, to name only a few. Two real data illustrations are presented in Section 2. In the first, we construct UCBs for exact CS and DL welfare functionals for a range of gasoline taxes at different income levels. For this illustration, we use the same data set as in Blundell, Horowitz, and Parey (2012, 2016) and estimate household gasoline demand via spline sieve NPIV (other data sets and other goods could be used). Despite the slow convergence rates of NPIV estimators, the UCBs for exact CS are particularly informative. In the second empirical illustration, we use the same data set as in Blundell et al. (2007) to estimate Engel curves for households with kids via a spline sieve NPIV and construct UCBs for Engel curves and their derivatives for various categories of household expenditure.

The rest of the paper is organized as follows. Section 2 presents the sieve NPIV estimator, the score bootstrap UCBs procedure and two real-data applications. This section aims at empirical researchers. Section 3 establishes the minimax optimal sup-norm rates for estimating a NPIV function h_0 and its derivatives. Section 4 presents low-level sufficient conditions for the uniform Gaussian process strong approximation and sieve score bootstrap UCBs for collections of general nonlinear functionals of a NPIV function. Section 5 deals with pointwise and uniform inferences on exact CS and DL, and approximate CS functionals in nonparametric demand estimation with endogeneity. Section 6 concludes with discussions of additional applications of the sup-norm rates of sieve NPIV estimators. Appendix A contains additional results on sup-norm convergence rates. Appendix B presents optimal L^2 -norm rates for estimating derivatives of a NPIV function under extremely weak conditions. Appendix C establishes the minimax lower bounds for estimating quadratic functionals of a NPIV function. The online supplementary appendix contains pointwise normality of sieve t statistics for nonlinear functionals of NPIV under lower-level sufficient conditions than those in Chen and Pouzo (2015) (Appendix D); background material on B-spline and wavelet sieves (Appendix E); useful lemmas on random matrices (Appendix F); and additional technical lemmas and all of the proofs (Appendix G).

2 Estimator and motivating applications to UCBs

This section describes the sieve NPIV estimator and a score bootstrap UCBs procedure for collections of functionals of the NPIV function. It mentions intuitively why sup-norm convergence rates of a sieve NPIV estimator are needed to formally justify the validity of the computationally simple score bootstrap UCBs procedure. It then present two real data applications of uniform inferences on functionals of a NPIV function: UCBs for exact CS and DL functionals of nonparametric demand with endogenous price, and UCBs for nonparametric Engel curves and their derivatives when the total expenditure is endogenous. This section is presented to practitioners.

Sieve NPIV estimators. Let $\{(Y_i, X_i, W_i)\}_{i=1}^n$ denote a random sample from the NPIV model (1). The sieve NPIV estimator \hat{h} of h_0 is simply the 2SLS estimator applied to some basis functions of X_i

(the endogenous regressors) and W_i (the conditioning variables), namely

$$\widehat{h}(x) = \psi^J(x)' \widehat{c} \quad \text{with} \quad \widehat{c} = [\Psi' B (B' B)^{-1} B' \Psi]^{-1} \Psi' B (B' B)^{-1} B' Y \quad (2)$$

where $Y = (Y_1, \dots, Y_n)'$,

$$\psi^J(x) = (\psi_{J1}(x), \dots, \psi_{JJ}(x))' \quad \Psi = (\psi^J(X_1), \dots, \psi^J(X_n))' \quad (3)$$

$$b^K(w) = (b_{K1}(w), \dots, b_{KK}(w))' \quad B = (b^K(W_1), \dots, b^K(W_n))' \quad (4)$$

and $\{\psi_{J1}, \dots, \psi_{JJ}\}$ and $\{b_{K1}, \dots, b_{KK}\}$ are collections of basis functions of dimension J and K for approximating h_0 and the instrument space, respectively (Blundell et al., 2007; Chen and Pouzo, 2012; Newey, 2013). The *regularization* parameter J is the dimension of the sieve for approximating h_0 . The *smoothing* parameter K is the dimension of the instrument sieve. From the analogy with 2SLS, it is clear that we need $K \geq J$. Blundell et al. (2007); Chen and Reiss (2011); Chen and Pouzo (2012) have previously shown that $\lim_J(K/J) = c \in [1, \infty)$ can lead to the optimal L^2 -norm convergence rate for sieve NPIV estimator. Thus we assume that K grows to infinity at the same rate as that of J , say $J \leq K \leq cJ$ for some finite $c > 1$ for simplicity.⁹ When $K = J$ and $b^K = \psi^J$ being an orthogonal series basis, the sieve NPIV estimator becomes Horowitz (2011)'s modified orthogonal series NPIV estimator. Note that the sieve NPIV estimator (2) reduces to a series LS estimator $\widehat{h}(x) = \psi^J(x)' [\Psi' \Psi]^{-1} \Psi' Y$ when $X_i = W_i$ is exogenous, $J = K$ and $\psi^J(x) = b^K(w)$ (Newey, 1997; Huang, 1998).

2.1 Uniform confidence bands for nonlinear functionals

One important motivating application is to *uniform inference* on a collection of nonlinear functionals $\{f_t(h_0) : t \in \mathcal{T}\}$ where \mathcal{T} is an index set (e.g. an interval). Uniform inference may be performed via uniform confidence bands (UCBs) that contain the function $t \mapsto f_t(h_0)$ with prescribed coverage probability. UCBs for h_0 (or its derivatives) are obtained as a special case with $\mathcal{T} = \mathcal{X}$ (support of X_i) and $f_t(h_0) = h_0(t)$ (or $f_t(h_0) = \partial^k h_0(t)$ for k -th derivative). We present applications below to uniform inference on exact CS and DL functionals over a range of price changes as well as UCBs for Engel curves and their derivatives.

A $100(1 - \alpha)\%$ bootstrap-based UCB for $\{f_t(h_0) : t \in \mathcal{T}\}$ is constructed as

$$t \mapsto \left[f_t(\widehat{h}) - z_{1-\alpha}^* \frac{\widehat{\sigma}(f_t)}{\sqrt{n}}, f_t(\widehat{h}) + z_{1-\alpha}^* \frac{\widehat{\sigma}(f_t)}{\sqrt{n}} \right]. \quad (5)$$

In this display $f_t(\widehat{h})$ is the plug-in sieve NPIV estimator of $f_t(h_0)$, $\widehat{\sigma}^2(f_t)$ is a *sieve variance estimator* for $f_t(\widehat{h})$, and $z_{1-\alpha}^*$ is a bootstrap-based critical value to be defined below.

⁹Monte Carlo evidences in (Blundell et al., 2007; Chen and Pouzo, 2015) and others suggest that sieve NPIV estimators often perform better with $K > J$ than with $K = J$, and that the regularization parameter J is important for finite sample performance while the parameter K is not as important as long as it is larger than J . See our second version (Chen and Christensen, 2015a) for data-driven choice of J .

To compute the sieve variance estimator for $f_t(\hat{h})$ with $\hat{h}(x) = \psi^J(x)' \hat{c}$ given in (2), one would first compute the 2SLS covariance matrix estimator (but applied to basis functions) for \hat{c} :

$$\hat{\mathcal{U}} = [\hat{S}' \hat{G}_b^{-1} \hat{S}]^{-1} \hat{S}' \hat{G}_b^{-1} \hat{\Omega} \hat{G}_b^{-1} \hat{S} [\hat{S}' \hat{G}_b^{-1} \hat{S}]^{-1} \quad (6)$$

where $\hat{S} = B' \Psi / n$, $\hat{G}_b = B' B / n$, $\hat{\Omega} = n^{-1} \sum_{i=1}^n \hat{u}_i^2 b^K(W_i) b^K(W_i)'$ and $\hat{u}_i = Y_i - \hat{h}(X_i)$. One then compute a “delta-method” correction term, a $J \times 1$ vector $Df_t(\hat{h})[\psi^J] := (Df_t(\hat{h})[\psi_{J1}], \dots, Df_t(\hat{h})[\psi_{JJ}])'$, by calculating $Df_t(\hat{h})[v] = \lim_{\delta \rightarrow 0^+} [\delta^{-1} f_t(\hat{h} + \delta v)]$, which is the (functional directional) derivative of f_t at \hat{h} in direction v , for $v = \psi_{J1}, \dots, \psi_{JJ}$. The sieve variance estimator for $f_t(\hat{h})$ is then

$$\hat{\sigma}^2(f_t) = (Df_t(\hat{h})[\psi^J])' \hat{\mathcal{U}} (Df_t(\hat{h})[\psi^J]). \quad (7)$$

We use the following *sieve score bootstrap* procedure to calculate the critical value $z_{1-\alpha}^*$. Let $\varpi_1, \dots, \varpi_n$ be IID random variables independent of the data with mean zero, unit variance and finite 3rd moment, e.g. $N(0, 1)$.¹⁰ We define the bootstrap sieve t -statistic process $\{\mathbb{Z}_n^*(t) : t \in \mathcal{T}\}$ as

$$\mathbb{Z}_n^*(t) := \frac{(Df_t(\hat{h})[\psi^J])' [\hat{S}' \hat{G}_b^{-1} \hat{S}]^{-1} \hat{S}' \hat{G}_b^{-1}}{\hat{\sigma}(f_t)} \left(\frac{1}{\sqrt{n}} \sum_{i=1}^n b^K(W_i) \hat{u}_i \varpi_i \right) \quad \text{for each } t \in \mathcal{T}. \quad (8)$$

To compute $z_{1-\alpha}^*$, one would calculate $\sup_{t \in \mathcal{T}} |\mathbb{Z}_n^*(t)|$ for a large number of independent draws of $\varpi_1, \dots, \varpi_n$. The critical value $z_{1-\alpha}^*$ is the $(1 - \alpha)$ quantile of $\sup_{t \in \mathcal{T}} |\mathbb{Z}_n^*(t)|$ over the draws. Note that this sieve score bootstrap procedure is different from the usual nonparametric bootstrap (based on resampling the data, then recomputing the estimator): here we only compute the estimator once, and then perturb the sieve t -statistic process by the innovations $\varpi_1, \dots, \varpi_n$.

An intuitive description of why sup-norm rates are very useful to justify this procedure is as follows. Under regularity conditions, the sieve t -statistic for an individual functional $f_t(h_0)$ admits an expansion

$$\frac{\sqrt{n}(f_t(\hat{h}) - f_t(h_0))}{\hat{\sigma}(f_t)} = \hat{\mathbb{Z}}_n(t) + \text{nonlinear remainder term} \quad (9)$$

(see Section 4 for the definition of $\hat{\mathbb{Z}}_n(t)$). The term $\hat{\mathbb{Z}}_n(t)$ is a CLT term, i.e. $\hat{\mathbb{Z}}_n(t) \rightarrow_d N(0, 1)$ for each fixed $t \in \mathcal{T}$. Therefore, the sieve t -statistic for $f_t(h_0)$ also converges to a $N(0, 1)$ random variable provided that the “nonlinear remainder term” is asymptotically negligible (i.e. $o_p(1)$) (see Assumption 3.5 in Chen and Pouzo (2015)). Our sup-norm rates are very useful for providing weak regularity conditions under which the remainder is $o_p(1)$ for fixed t .¹¹ This justifies constructing confidence intervals for *individual* functionals $f_t(h_0)$ for any fixed $t \in \mathcal{T}$ by inverting the sieve t -statistic (on

¹⁰Other examples of distributions with these properties include the re-centered exponential (i.e. $\varpi_i = \text{Exp}(1) - 1$), Rademacher (i.e. ± 1 each with probability $\frac{1}{2}$), or the two-point distribution of Mammen (1993) (i.e. $(1 - \sqrt{5})/2$ with probability $(\sqrt{5} + 1)/(2\sqrt{5})$ and $(\sqrt{5} + 1)/\sqrt{2}$ with remaining probability).

¹¹Chen and Pouzo (2015) verified their high-level Assumption 3.5 for a plug-in sieve estimator of a weighted quadratic functional example. Without sup-norm convergence rates, it is difficult to verify their Assumption 3.5 for nonlinear functionals (such as the exact CS) that are more complicated than quadratic functionals.

the left-hand side of display (9)) and using $N(0, 1)$ critical values. However, for *uniform* inference the usual $N(0, 1)$ critical values are no longer appropriate as we need to consider the sampling error in estimating the whole process $t \mapsto f_t(h_0)$. For this purpose, display (9) is strengthened to be valid uniformly in $t \in \mathcal{T}$ (see Lemma 4.1). Under some regularity conditions, $\sup_{t \in \mathcal{T}} |\widehat{Z}_n(t)|$ converges in distribution to the supremum of a (non-pivotal) Gaussian process. As its critical values are generally not available, we use the sieve score bootstrap procedure to estimate its critical values.

Section 4 formally justifies the use of this procedure for constructing UCBs for $\{f_t(h_0) : t \in \mathcal{T}\}$. The sup-norm rates are useful for controlling the nonlinear remainder terms for UCBs for collections of nonlinear functionals. Theorem 4.1 appears to be the first to establish the consistency of sieve score bootstrap UCBs for general nonlinear functionals of NPIV under low-level conditions, allowing for mildly and severely ill-posed problems. It includes as special cases the score bootstrap UCBs for nonlinear functionals of h_0 under exogeneity when h_0 is estimated via series LS, and the score bootstrap UCBs for the NPIV function h_0 and its derivatives.¹² Theorem 4.1 is applied in Section 5 to formally justify the validity of score bootstrap UCBs for exact CS and DL functionals over a range of price changes when demand is estimated nonparametrically via sieve NPIV.

2.2 Empirical application 1: UCBs for nonparametric exact CS and DL functionals

Here we apply our methodology to study the effect of gasoline price changes on household welfare. We extend the important work by Hausman and Newey (1995) on pointwise confidence bands for exact CS and DL of demand without endogeneity to UCBs for exact CS and DL of demand with endogeneity.

Let demand of consumer i be

$$Q_i = h_0(P_i, Y_i) + u_i$$

where Q_i is quantity, P_i is price, which may be endogenous, Y_i is income of consumer i , and u_i is an error term.¹³ Hausman (1981) shows that the exact CS from a price change from \mathbf{p}^0 to \mathbf{p}^1 at income level y , denoted $S_y(\mathbf{p}^0)$, solves

$$\begin{aligned} \frac{\partial S_y(\mathbf{p}(u))}{\partial u} &= -h_0(\mathbf{p}(u), y - S_y(\mathbf{p}(u))) \frac{d\mathbf{p}(u)}{du} \\ S_y(\mathbf{p}(1)) &= 0 \end{aligned} \tag{10}$$

where $\mathbf{p} : [0, 1] \rightarrow \mathbb{R}$ is a twice continuously differentiable path with $\mathbf{p}(0) = \mathbf{p}^0$ and $\mathbf{p}(1) = \mathbf{p}^1$. The corresponding DL functional $D_y(\mathbf{p}^0)$ is

$$D_y(\mathbf{p}^0) = S_y(\mathbf{p}^0) - (\mathbf{p}^1 - \mathbf{p}^0)h_0(\mathbf{p}^1, y). \tag{11}$$

¹²One also needs to use sup-norm convergence rates of \widehat{h} to h_0 to build a valid UCB for $\{h_0(t) : t \in \mathcal{X}\}$.

¹³Endogeneity may also be an issue in the estimation of static models of labor supply, in which Q_i represents hours worked, P_i is the wage, and Y_i is other income. In this setting it is reasonable to allow for endogeneity of both P_i and Y_i (see Blundell, Duncan, and Meghir (1998), Blundell, MaCurdy, and Meghir (2007), and references therein).

	Quantity (gal)	Price (\$/gal)	Income (\$)
mean	1455	1.33	58307
25th %	871	1.28	42500
median	1269	1.32	57500
75th %	1813	1.40	72500
std dev	894	0.07	19584

Table 1: Summary statistics for gasoline demand data.

As is evident from (10) and (11), exact CS and DL are (typically nonlinear) functionals of h_0 . An exception is when demand is independent of income, in which case exact CS and DL are linear functionals of h_0 . Let $t = (\mathbf{p}^0, \mathbf{p}^1, y)$ index the initial price, final price, and income level and let $\mathcal{T} \subseteq [\underline{\mathbf{p}}^0, \bar{\mathbf{p}}^0] \times [\underline{\mathbf{p}}^1, \bar{\mathbf{p}}^1] \times [\underline{y}, \bar{y}]$ denote a range of price changes and/or incomes over which inference is to be performed. To denote dependence on h_0 , we use the notation

$$f_{CS,t}(h) = \text{solution to (10) with } h \text{ in place of } h_0 \quad (12)$$

$$f_{DL,t}(h) = f_{CS,t}(h) - (\mathbf{p}^1 - \mathbf{p}^0)h(\mathbf{p}^1, y) \quad (13)$$

so $S_y(\mathbf{p}^0) = f_{CS,t}(h_0)$ and $D_y(\mathbf{p}^0) = f_{DL,t}(h_0)$.

We estimate exact CS and DL using the plug-in estimators $f_{CS,t}(\hat{h})$ and $f_{DL,t}(\hat{h})$. The sieve variance estimators $\hat{\sigma}^2(f_{CS,t})$ and $\hat{\sigma}^2(f_{DL,t})$ are as described in (7) with the delta-method correction terms

$$Df_{CS,t}(\hat{h})[\psi^J] = \int_0^1 \psi^J(\mathbf{p}(u), \mathbf{y} - \hat{S}_y(\mathbf{p}(u))) e^{-\int_0^u \partial_2 \hat{h}(\mathbf{p}(v), \mathbf{y} - \hat{S}_y(\mathbf{p}(v))) \mathbf{p}'(v) dv} \mathbf{p}'(u) du \quad (14)$$

$$Df_{DL,t}(\hat{h})[\psi^J] = Df_{CS,t}(\hat{h})[\psi^J] - (\mathbf{p}^1 - \mathbf{p}^0)\psi^J(\mathbf{p}^1, y) \quad (15)$$

where $\mathbf{p}'(u) = \frac{d\mathbf{p}(u)}{du}$, $\partial_2 h$ denotes the partial derivative of h with respect to its second argument and $\hat{S}_y(\mathbf{p}(u))$ denotes the solution to (10) with \hat{h} in place of h_0 .

We use the 2001 National Household Travel Survey gasoline demand data from [Blundell et al. \(2012, 2016\)](#).¹⁴ The main variables are annual household gasoline consumption (in gallons), average price (in dollars per gallon) in the county in which the household is located, household income, and distance from the Gulf coast to the capital of the state in which the household is located. Due to censoring, we consider the subset of households with incomes less than \$100,000 per year. To keep households somewhat homogeneous, we select household with incomes above \$25,000 per year (the 8th percentile), with at most 6 inhabitants, and 1 or 2 drivers. The resulting sample has size $n = 2753$.¹⁵ Table 1 presents summary statistics.

We estimate the household gasoline demand function in levels via sieve NPIV using distance as instru-

¹⁴We are grateful to Matthias Pary for sharing the dataset with us. We refer the reader to section 3 of [Blundell et al. \(2012\)](#) for a detailed description of the data.

¹⁵We also exclude one household that reports 14,635 gallons; the next largest is 8089 gallons. Similar results are obtained using the full set of $n = 4811$ observations.

ment for price. To implement the estimator, we form Ψ_J by taking a tensor product of quartic B-spline bases of dimension 5 for both price and income (so $J = 25$) and B_K by taking a tensor product of quartic B-spline bases of dimension 8 for distance and 5 for income (so $K = 40$) with interior knots spaced evenly at quantiles.

We consider exact CS and DL resulting from price increases from $\mathbf{p}^0 \in [\$1.20, \$1.40]$ to $\mathbf{p}^1 = \$1.40$ at income levels of $\mathbf{y} = \$42,500$ (low) and $\mathbf{y} = \$72,500$ (high). We estimate exact CS at each initial price level by solving the ODE (10) by backward differences. We construct UCBs for exact CS as described above by setting $\mathcal{T} = [\$1.20, \$1.40] \times \{\$1.40\} \times \{\$42,500\}$ for the low-income group and $\mathcal{T} = [\$1.20, \$1.40] \times \{\$1.40\} \times \{\$72,500\}$ for the high-income group, $f_t(h) = f_{CS,t}(h)$ from display (12), and $Df_t(\hat{h})[\psi^J] = Df_{CS,t}(\hat{h})[\psi^J]$ from display (14). The ODE (10) is solved numerically by backward differences and the integrals in (14) are computed numerically. UCBs for DL are formed similarly, $f_t(h) = f_{DL,t}(h)$ from display (13), and $Df_t(\hat{h})[\psi^J] = Df_{DL,t}(\hat{h})[\psi^J]$ from display (15). We draw the bootstrap innovations ϖ_i from Mammen’s two-point distribution with 1000 bootstrap replications.

The exact CS and DL estimates are presented in Figure 1 together with their UCBs. It is clear that exact CS is much more precisely estimated than DL. This is to be expected, since exact CS is computed by essentially integrating over one argument of the estimated demand function and is therefore smoother than the DL functional, which depends on h_0 estimated at the point $(\mathbf{p}^1, \mathbf{y})$. In fact, even though the sieve NPIV \hat{h} itself converges slowly, the UCBs for exact CS are still quite informative. At their widest point (with initial price \$1.20), the 95% UCBs for exact CS for low-income households are [\$259, \$314]. In terms of comparison across high- and low-income households, the exact CS estimates are higher for the high-income households whereas DL estimates are higher for the low-income households.

Figure 2 displays estimates obtained when we treat price as exogenous and estimate demand (h_0) by series LS regression. This is a special case of the preceding analysis with $X_i = W_i = (\mathbf{P}_i, \mathbf{Y}_i)'$, $K = J$ and $\psi^J = b^K$. These estimates display several notable features. First, the exact CS estimates are very similar whether demand is estimated via series LS or via sieve NPIV. Second, the UCBs for exact CS estimates are of a similar width to those obtained when demand was estimated via sieve NPIV, even though NPIV is an ill-posed inverse problem whereas nonparametric LS regression is not. Third, the UCBs for DL are noticeably narrower when demand is estimated via series LS than when demand is estimated via sieve NPIV. Fourth, the DL estimates for LS and sieve NPIV are similar for high income households but quite different for low income households. This is consistent with Blundell et al. (2016), who find some evidence of endogeneity in gasoline prices for low income groups.

2.3 Empirical application 2: UCBs for Engel curves and their derivatives

Engel curves describe the household budget share for expenditure categories as a function of total household expenditure. Following Blundell et al. (2007), we use sieve NPIV to estimate Engel curves, taking log total household income as an instrument for log total household expenditure. We use data

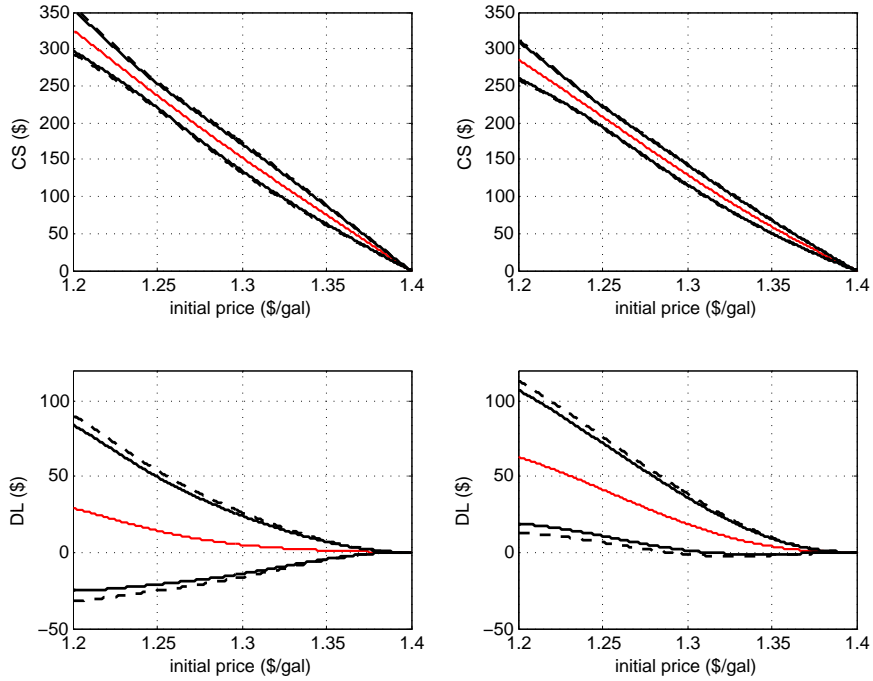


Figure 1: Estimated CS and DL from a price increase to \$1.40/gal (solid red line) and their bootstrap UCBs (solid black lines are 90%, dashed black lines are 95%) when demand is estimated via NPIV. Left panels are for household income of \$72,500; right panels are for household income of \$42,500.

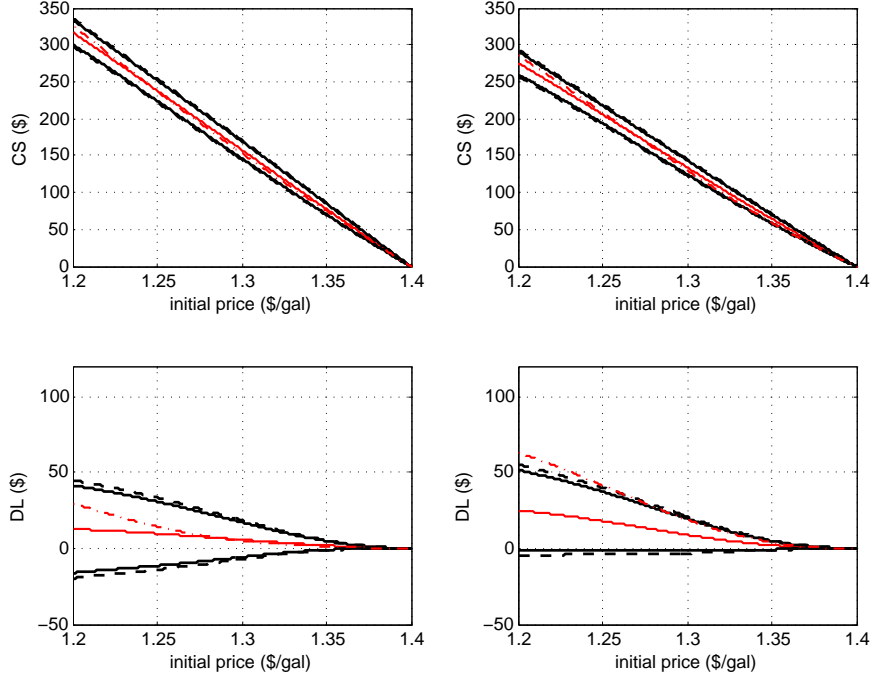


Figure 2: Estimated CS and DL from a price increase to \$1.40/gal (solid red lines) and their bootstrap UCBs (solid black lines are 90%, dashed black lines are 95%) when demand is estimated via series LS. CS and DL when demand is estimated via NPIV are also shown (red dash-dot lines). Left panels are for household income of \$72,500; right panels are for household income of \$42,500.

from the 1995 British Family Expenditure Survey, focusing on the subset of married or cohabitating couples with one or two children, with the head of household aged between 20 and 55 and in work. This leaves a sample of size $n = 1027$. We consider six categories of nondurables and services expenditure: food in, food out, alcohol, fuel, travel, and leisure.

We construct UCBs for Engel curves as described above by setting $\mathcal{T} = [4.75, 6.25]$ (approximately the 5th to 95th percentile of log expenditure), $f_t(h) = h(t)$, and $Df_t(\widehat{h})[\psi^J] = \psi^J(t)$. We also construct UCBs for derivatives of the Engel curves by setting $\mathcal{T} = [4.75, 6.25]$, $f_t(h)$ to be the derivative of h evaluated at t , and $Df_t(\widehat{h})[\psi^J]$ to be the vector formed by taking derivatives of $\psi_{J1}, \dots, \psi_{JJ}$ evaluated at t . For both constructions, we use a quartic B-spline basis of dimension $J = 5$ for Ψ_J and a quartic B-spline basis of dimension $K = 9$ for B_K , with interior knots evenly spaced at quantiles (an important feature of sieve estimators is that the same sieve dimension can be used for optimal estimation of the function and its derivatives; this is not the case for kernel-based estimators). We draw the bootstrap innovations ϖ_i from Mammen’s two-point distribution with 1000 bootstrap replications.

The Engel curves presented in Figure 3 and their derivatives presented in Figure 4 exhibit several interesting features. The curves for food-in and fuel (necessary goods) are both downward sloping, with the curve for fuel exhibiting a pronounced downward slope at lower income levels. The derivative of the curve for fuel is negative, though the UCBs are positive at the extremities. In contrast, the curve for leisure expenditure (luxury good) is strongly upwards sloping and its derivative is positive except at low income levels. Remaining curves for food-out, alcohol and travel appear to be non-monotonic.

3 Optimal sup-norm convergence rates

This section presents several results on sup-norm convergence rates. Subsection 3.1 presents upper bounds on sup-norm convergence rates of NPIV estimators of h_0 and its derivatives. Subsection 3.2 presents (minimax) lower bounds. Subsection 3.3 considers NPIV models with endogenous and exogenous regressors that are useful in empirical studies.

Notation: We work on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. \mathcal{A}^c denotes the complement of an even $\mathcal{A} \in \mathcal{F}$. We abbreviate “with probability approaching one” to “wpa1”, and say that a sequence of events $\{\mathcal{A}_n\} \subset \mathcal{F}$ holds wpa1 if $\mathbb{P}(\mathcal{A}_n^c) = o(1)$. For a random variable X we define the space $L^q(X)$ as the equivalence class of all measurable functions of X with finite q th moment if $1 \leq q < \infty$; when $q = \infty$ with some abuse of notation we take $L^\infty(X)$ to mean the set of all bounded measurable $g : \mathcal{X} \rightarrow \mathbb{R}$ endowed with the sup norm $\|g\|_\infty = \sup_x |g(x)|$. Let $\langle \cdot, \cdot \rangle_X$ denote the inner product on $L^2(X)$. For matrix and vector norms, $\|\cdot\|_{\ell^q}$ denotes the vector ℓ^q norm when applied to vectors and the operator norm induced by the vector ℓ^q norm when applied to matrices. If a and b are scalars we let $a \vee b := \max\{a, b\}$ and $a \wedge b := \min\{a, b\}$. Minimum and maximum eigenvalues are denoted by λ_{\min} and λ_{\max} . If $\{a_n\}$ and $\{b_n\}$ are sequences of positive numbers, we say that $a_n \lesssim b_n$ if $\limsup_{n \rightarrow \infty} a_n/b_n < \infty$ and we say that $a_n \asymp b_n$ if $a_n \lesssim b_n$ and $b_n \lesssim a_n$.

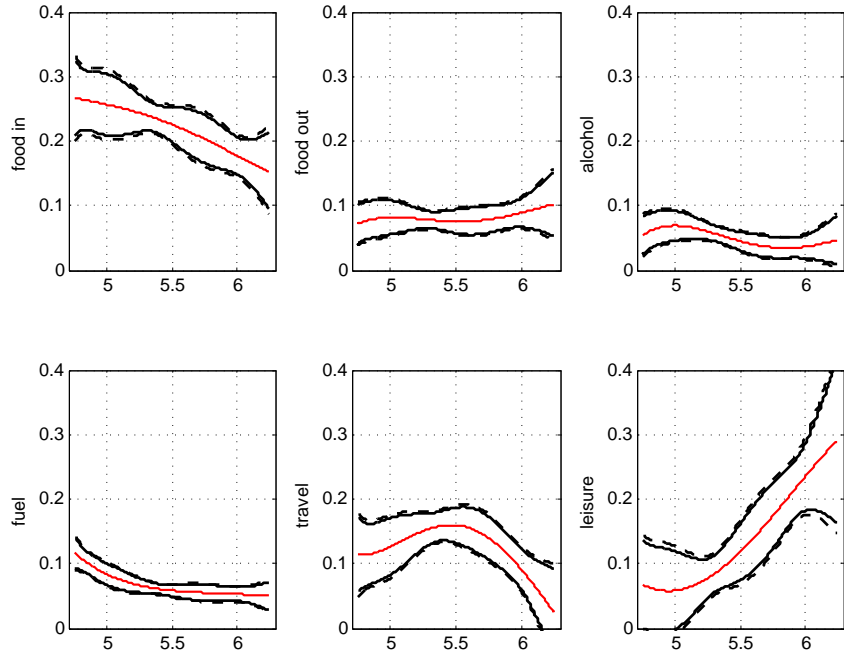


Figure 3: Estimated Engel curves (red line) with bootstrap uniform confidence bands (solid black lines are 90%, dashed black lines are 95%). The x -axis is log total household expenditure, the y -axis is household budget share.

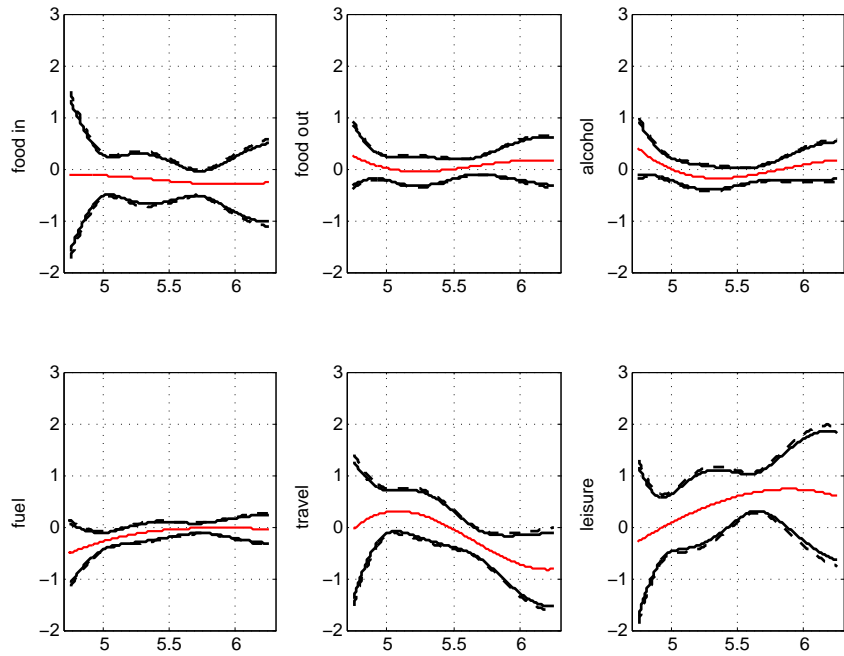


Figure 4: Estimated Engel curve derivatives (red line) with bootstrap uniform confidence bands (solid black lines are 90%, dashed black lines are 95%).

Sieve measure of ill-posedness. For a NPIV model (1), an important quantity is the *measure of ill-posedness* which, roughly speaking, measures how much the conditional expectation $h \mapsto E[h(X_i)|W_i = w]$ smoothes out h . To introduce a measure of ill-posedness, let $T : L^2(X) \rightarrow L^2(W)$ denote the conditional expectation operator given by

$$Th(w) = E[h(X_i)|W_i = w].$$

Let $\Psi_J = \text{clsp}\{\psi_{J1}, \dots, \psi_{JJ}\} \subset L^2(X)$ and $B_K = \text{clsp}\{b_{K1}, \dots, b_{KK}\} \subset L^2(W)$ denote the sieve spaces for the endogenous variables and instrumental variables, respectively. Let $\Psi_{J,1} = \{h \in \Psi_J : \|h\|_{L^2(X)} = 1\}$. The *sieve L^2 measure of ill-posedness* is

$$\tau_J = \sup_{h \in \Psi_J: h \neq 0} \frac{\|h\|_{L^2(X)}}{\|Th\|_{L^2(W)}} = \frac{1}{\inf_{h \in \Psi_{J,1}} \|Th\|_{L^2(W)}}.$$

Following [Blundell et al. \(2007\)](#), we call a NPIV model (1) with X_i being a d -dimensional random vector:

- (i) *mildly ill-posed* if $\tau_J = O(J^{\varsigma/d})$ for some $\varsigma > 0$; and
- (ii) *severely ill-posed* if $\tau_J = O(\exp(\frac{1}{2}J^{\varsigma/d}))$ for some $\varsigma > 0$.

See our second version ([Chen and Christensen, 2015a](#)) for simple consistent estimation of the sieve measure of ill-posedness τ_J .

3.1 Sup-norm convergence rates

We first introduce some basic conditions on the basic NPIV model (1) and the sieve spaces.

Assumption 1. (i) X_i has compact rectangular support $\mathcal{X} \subset \mathbb{R}^d$ with nonempty interior and the density of X_i is uniformly bounded away from 0 and ∞ on \mathcal{X} ; (ii) W_i has compact rectangular support $\mathcal{W} \subset \mathbb{R}^{d_w}$ and the density of W_i is uniformly bounded away from 0 and ∞ on \mathcal{W} ; (iii) $T : L^2(X) \rightarrow L^2(W)$ is injective; and (iv) $h_0 \in \mathcal{H} \subset L^\infty(X)$, and $\cup_J \Psi_J$ is dense in $(\mathcal{H}, \|\cdot\|_{L^\infty(X)})$.

Assumption 2. (i) $\sup_{w \in \mathcal{W}} E[u_i^2|W_i = w] \leq \bar{\sigma}^2 < \infty$; and (ii) $E[|u_i|^{2+\delta}] < \infty$ for some $\delta > 0$.

The following assumptions concern the basis functions. Define

$$\begin{aligned} G_\psi &= G_{\psi,J} &= E[\psi^J(X_i)\psi^J(X_i)'] &= E[\Psi'\Psi/n] \\ G_b &= G_{b,K} &= E[b^K(W_i)b^K(W_i)'] &= E[B'B/n] \\ S &= S_{KJ} &= E[b^K(W_i)\psi^J(X_i)'] &= E[B'\Psi/n]. \end{aligned}$$

We assume throughout that the basis functions are not linearly dependent, i.e. S has full column rank J and $G_{\psi,J}$ and $G_{b,K}$ are positive definite for each J and K , i.e. $e_J = \lambda_{\min}(G_{\psi,J}) > 0$ and

$e_{b,K} = \lambda_{\min}(G_{b,K}) > 0$, although e_J and $e_{b,K}$ could go to zero as $K \geq J$ goes to infinity. Let

$$\begin{aligned}\zeta_\psi &= \zeta_{\psi,J} = \sup_x \|G_\psi^{-1/2} \psi^J(x)\|_{\ell^2} & \zeta_b &= \zeta_{b,K} = \sup_w \|G_b^{-1/2} b^K(w)\|_{\ell^2} \\ \xi_\psi &= \xi_{\psi,J} = \sup_x \|\psi^J(x)\|_{\ell^1}\end{aligned}$$

for each J and K and define $\zeta = \zeta_J = \zeta_{b,K} \vee \zeta_{\psi,J}$. Note that $\zeta_{\psi,J}$ has some useful properties: $\|h\|_\infty \leq \zeta_{\psi,J} \|h\|_{L^2(X)}$ for all $h \in \Psi_J$, and $\sqrt{J} = (E[\|G_\psi^{-1/2} \psi^J(X)\|_{\ell^2}^2])^{1/2} \leq \zeta_{\psi,J} \leq \xi_{\psi,J}/\sqrt{e_J}$; clearly $\zeta_{b,K}$ has similar properties.

We say that the sieve basis for Ψ_J is Hölder continuous if there exist finite constants $\omega \geq 0, \omega' > 0$ such that $\|G_{\psi,J}^{-1/2} \{\psi^J(x) - \psi^J(x')\}\|_{\ell^2} \lesssim J^\omega \|x - x'\|_{\ell^2}^{\omega'}$ for all $x, x' \in \mathcal{X}$.

Assumption 3. (i) the basis spanning Ψ_J is Hölder continuous; (ii) $\tau_J \zeta^2 / \sqrt{n} = O(1)$; and (iii) $\zeta^{(2+\delta)/\delta} \sqrt{(\log n)/n} = o(1)$.

Let $\Pi_J : L^2(X) \rightarrow \Psi_J$ denote the $L^2(X)$ orthogonal (i.e. least squares) projection onto Ψ_J , namely $\Pi_J h_0 = \arg \min_{h \in \Psi_J} \|h_0 - h\|_{L^2(X)}$ and let $Q_J h_0 = \arg \min_{h \in \Psi_J} \|\Pi_K T(h_0 - h)\|_{L^2(W)}$ denote the sieve 2SLS projection of h_0 onto Ψ_J . We may write $Q_J h_0 = \psi^J(\cdot)' c_{0,J}$ where

$$c_{0,J} = [S' G_b^{-1} S]^{-1} S' G_b^{-1} E[b^K(W_i) h_0(X_i)].$$

Let $\Pi_K : L^2(W) \rightarrow B_K$ denote the $L^2(W)$ orthogonal (i.e. least-squares) projection onto B_K .

Assumption 4. (i) $\sup_{h \in \Psi_{J,1}} \|(\Pi_K T - T)h\|_{L^2(W)} = o(\tau_J^{-1})$; (ii) $\tau_J \times \|T(h_0 - \Pi_J h_0)\|_{L^2(W)} \leq \text{const} \times \|h_0 - \Pi_J h_0\|_{L^2(X)}$; and (iii) $\|Q_J(h_0 - \Pi_J h_0)\|_\infty \leq O(1) \times \|h_0 - \Pi_J h_0\|_\infty$.

Discussion of Assumptions. Assumption 1 is standard. Assumption 1(iii) is stronger than needed for convergence rates in sup-norm only. We impose it as a common sufficient condition for convergence rates in both sup-norm and L^2 -norm (Appendix B). For sup-norm convergence rate only, Assumption 1(iii) could be replaced by the following weaker identification condition:

Assumption 1 (iii-sup) $h_0 \in \mathcal{H} \subset L^\infty(X)$, and $T[h - h_0] = 0 \in L^2(W)$ for any $h \in \mathcal{H}$ implies that $\|h - h_0\|_\infty = 0$.

This in turn is implied by the injectivity of $T : L^\infty(X) \rightarrow L^2(W)$ (or the bounded completeness), which is weaker than the injectivity of $T : L^2(X) \rightarrow L^2(W)$ (i.e., the L^2 -completeness). Bounded completeness or L^2 -completeness condition is often assumed in models with endogeneity (e.g. Newey and Powell (2003); Carrasco et al. (2007); Blundell et al. (2007); Andrews (2011); Chen, Chernozhukov, Lee, and Newey (2014)) and is generically satisfied according to Andrews (2011). The parameter space \mathcal{H} for h_0 is typically taken to be a Hölder or Sobolev class of smooth functions. Assumption 1(i) could be relaxed to unbounded support, and the proofs need to be modified slightly using wavelet basis and weighted compact embedding results, see, e.g., Blundell et al. (2007); Chen and Pouzo (2012); Triebel (2006) and references therein. To present the sup-norm rate results in a clean way we stick to the simplest Assumption 1. Assumption 2 is also imposed for sup-norm convergence rates for series LS

regression under exogeneity (e.g., [Chen and Christensen \(2015b\)](#)). Assumption 3(i) is satisfied by many commonly used sieve bases, such as splines, wavelets, and cosine bases. Assumption 3(ii)(iii) restrict the rate at which J can grow with n . Upper bounds for $\zeta_{\psi,J}$ and $\zeta_{b,K}$ are known for commonly used bases. For instance, under Assumption 1(i)(ii), $\zeta_{b,K} = O(\sqrt{K})$ and $\zeta_{\psi,J} = O(\sqrt{J})$ for (tensor-product) polynomial spline, wavelet and cosine bases, and $\zeta_{b,K} = O(K)$ and $\zeta_{\psi,J} = O(J)$ for (tensor-product) orthogonal polynomial bases (see, e.g., [Huang \(1998\)](#) and online Appendix E). Assumption 4(i) is a mild condition on the approximation properties of the basis used for the instrument space and is similar to the first part of Assumption 5(iv) of [Horowitz \(2014\)](#). In fact, $\|(\Pi_K T - T)h\|_{L^2(W)} = 0$ for all $h \in \Psi_J$ when the basis functions for B_K and Ψ_J form either a Riesz basis or eigenfunction basis for the conditional expectation operator. Assumption 4(ii) is the usual L^2 “stability condition” imposed in the NPIV literature (cf. Assumption 6 in [Blundell et al. \(2007\)](#) and Assumption 5.2(ii) in [Chen and Pouzo \(2012\)](#)). Assumption 4(iii) is a new L^∞ “stability condition” to control the sup-norm bias. It turns out that Assumption 4(ii) and 4(iii) are also automatically satisfied by Riesz bases; see Appendix A for further discussions and sufficient conditions.

To derive the sup-norm (uniform) convergence rate we split $\|\hat{h} - h_0\|_\infty$ into so-called “bias” and “standard deviation” terms and derive sup-norm convergence rates for the two terms. Specifically, let

$$\tilde{h}(x) = \psi^J(x)' \tilde{c} \quad \text{with} \quad \tilde{c} = [\Psi' B(B'B)^{-1} B' \Psi]^{-1} \Psi' B(B'B)^{-1} B' H_0$$

where $H_0 = (h_0(X_1), \dots, h_0(X_n))'$. We refer loosely to $\|\tilde{h} - h_0\|_\infty$ as the “bias” term and $\|\hat{h} - \tilde{h}\|_\infty$ as the “standard deviation” (or sometimes “variance”) term. Both are random quantities. We first bound the sup-norm “standard deviation” term in the following lemma.

Lemma 3.1. *Let Assumptions 1(i)(iii), 2(i)(ii), 3(ii)(iii), and 4(i) hold. Then:*

(1) $\|\hat{h} - \tilde{h}\|_\infty = O_p(\tau_J \xi_{\psi,J} \sqrt{(\log J)/(ne_J)})$.

(2) *If Assumption 3(i) also holds, then:* $\|\hat{h} - \tilde{h}\|_\infty = O_p(\tau_J \zeta_{\psi,J} \sqrt{(\log n)/n})$.

Recall that $\sqrt{J} \leq \zeta_{\psi,J} \leq \xi_{\psi,J}/\sqrt{e_J}$. Result (2) of Lemma 3.1 provides a slightly tighter upper bound on the variance term than Result (1) does, while Result (1) allows for slightly more general basis to approximate h_0 . For splines and wavelets, we show in Appendix E that $\xi_{\psi,J}/\sqrt{e_J} \lesssim \sqrt{J}$, so Results (1) and (2) produce the same tight upper bound $\|\hat{h} - \tilde{h}\|_\infty = O_p(\tau_J \sqrt{(J \log n)/n})$ when $J \asymp n^r$ for some constant $r > 0$.

Before we present an upper bound on the “bias” term in Theorem 3.1 part (1) below, we mention one more property of the sieve space Ψ_J that is crucial for sharp bounds on the sup-norm bias term. Let $h_{0,J} \in \Psi_J$ denote the best approximation to h_0 in sup-norm, i.e. $h_{0,J}$ solves $\inf_{h \in \Psi_J} \|h_0 - h\|_\infty$. Then by Lebesgue’s Lemma ([DeVore and Lorentz, 1993](#), p. 30):

$$\|h_0 - \Pi_J h_0\|_\infty \leq (1 + \|\Pi_J\|_\infty) \times \|h_0 - h_{0,J}\|_\infty$$

where $\|\Pi_J\|_\infty$ is the Lebesgue constant for the sieve Ψ_J . Recently it has been established that $\|\Pi_J\|_\infty \lesssim 1$ when Ψ_J is spanned by a tensor product B-spline basis ([Huang \(2003\)](#)) or a tensor product Cohen-

Daubechies-Vial (CDV) wavelet basis (Chen and Christensen (2015b)).¹⁶ Boundedness of the Lebesgue constant is crucial for attaining optimal sup-norm rates.

Theorem 3.1. (1) Let Assumptions 1(iii), 3(ii) and 4 hold. Then:

$$\|\tilde{h} - h_0\|_\infty = O_p(\|h_0 - \Pi_J h_0\|_\infty).$$

(2) Let Assumptions 1(i)(iii)(iv), 2(i)(ii), 3(ii)(iii), and 4 hold. Then:

$$\|\hat{h} - h_0\|_\infty = O_p\left(\|h_0 - \Pi_J h_0\|_\infty + \tau_J \xi_{\psi, J} \sqrt{(\log J)/(ne_J)}\right).$$

(3) Further, if the linear sieve Ψ_J satisfies $\|\Pi_J\|_\infty \lesssim 1$ and $\xi_{\psi, J}/\sqrt{e_J} \lesssim \sqrt{J}$, then

$$\|\hat{h} - h_0\|_\infty = O_p\left(\|h_0 - h_{0, J}\|_\infty + \tau_J \sqrt{(J \log J)/n}\right).$$

Theorem 3.1(2)(3) follows directly from part (1) (for bias) and Lemma 3.1(1) (for standard deviation). See Appendix A for additional details about bound on sup-norm bias.

The following corollary provides concrete sup-norm convergence rates of \hat{h} and its derivatives. To introduce the result, let $B_{\infty, \infty}^p$ denote the Hölder space of smoothness $p > 0$ and $\|\cdot\|_{B_{\infty, \infty}^p}$ denote its norm (see Section 1.11.10 of Triebel (2006)). Let $B_\infty(p, L) = \{h \in B_{\infty, \infty}^p : \|h\|_{B_{\infty, \infty}^p} \leq L\}$ denote a Hölder ball of smoothness $p > 0$ and radius $L \in (0, \infty)$. Let $\alpha_1, \dots, \alpha_d$ be non-negative integers, let $|\alpha| = \alpha_1 + \dots + \alpha_d$, and define

$$\partial^\alpha h(x) := \frac{\partial^{|\alpha|} h}{\partial^{\alpha_1} x_1 \dots \partial^{\alpha_d} x_d} h(x).$$

Of course, if $|\alpha| = 0$ then $\partial^\alpha h = h$.¹⁷

Corollary 3.1. Let Assumptions 1(i)(ii)(iii) and 4 hold. Let $h_0 \in B_\infty(p, L)$, Ψ_J be spanned by a B-spline basis of order $\gamma > p$ or a CDV wavelet basis of regularity $\gamma > p$, B_K be spanned by a cosine, spline or wavelet basis.

(1) If Assumption 3(ii) holds, then

$$\|\partial^\alpha \tilde{h} - \partial^\alpha h_0\|_\infty = O_p\left(J^{-(p-|\alpha|)/d}\right) \quad \text{for all } 0 \leq |\alpha| < p.$$

(2) If Assumptions 2(i)(ii) and 3(ii)(iii) hold, then

$$\|\partial^\alpha \hat{h} - \partial^\alpha h_0\|_\infty = O_p\left(J^{-(p-|\alpha|)/d} + \tau_J J^{|\alpha|/d} \sqrt{(J \log J)/n}\right) \quad \text{for all } 0 \leq |\alpha| < p.$$

(2.a) Mildly ill-posed case: with $p \geq d/2$ and $\delta \geq d/(p + \varsigma)$, choosing $J \asymp (n/\log n)^{d/(2(p+\varsigma)+d)}$ implies

¹⁶See DeVore and Lorentz (1993) and Belloni et al. (2015) for examples of other bases with bounded Lebesgue constant or with Lebesgue constant diverging slowly with the sieve dimension.

¹⁷If $|\alpha| > 0$ then we assume h and its derivatives can be continuously extended to an open set containing \mathcal{X} .

that Assumption 3(ii)(iii) holds and

$$\|\partial^\alpha \widehat{h} - \partial^\alpha h_0\|_\infty = O_p((n/\log n)^{-(p-|\alpha|)/(2(p+\varsigma)+d)}).$$

(2.b) Severely ill-posed case: choosing $J = (c_0 \log n)^{d/\varsigma}$ with $c_0 \in (0, 1)$ implies that Assumption 3(ii)(iii) holds and

$$\|\partial^\alpha \widehat{h} - \partial^\alpha h_0\|_\infty = O_p((\log n)^{-(p-|\alpha|/\varsigma)}).$$

Corollary 3.1 shows that, for sieve NPIV estimators, taking derivatives has the same impact on the bias and standard deviation terms in terms of the order of convergence, and that the same choice of sieve dimension J can lead to optimal sup-norm convergence rates for estimating h_0 and its derivatives simultaneously (since they match the lower bounds in Theorem 3.2 below). When specializing to series LS regression (without endogeneity, i.e., $\tau_J = 1$), Corollary 3.1(2.a) with $\varsigma = 0$ automatically implies that spline and wavelet series LS estimators will also achieve the optimal sup-norm rates of Stone (1982) for estimating the derivatives of a nonparametric LS regression function. This strengthens the recent results in Belloni et al. (2015) and Chen and Christensen (2015b) for sup-norm rate optimality of spline and wavelet LS estimators of the regression function h_0 itself. This is in contrast to kernel based LS regression estimators where different choices of bandwidth are needed for the optimal rates of estimating h_0 and its derivatives.

Corollary 3.1 is useful for estimating functions with certain shape properties. For instance, if $h_0 : [a, b] \rightarrow \mathbb{R}$ is strictly monotone and/or strictly concave/convex, then knowing that $\partial \widehat{h}(x)$ and/or $\partial^2 \widehat{h}(x)$ converge uniformly to $\partial h_0(x)$ and/or $\partial^2 h_0(x)$ implies that \widehat{h} will also be strictly monotone and/or strictly concave/convex wpa1. In this paper, we shall illustrate the usefulness of Corollary 3.1 in controlling the nonlinear remainder terms for pointwise and uniform inferences on highly nonlinear (i.e., beyond quadratic) functionals of h_0 ; see Sections 4 and 5 for details.

3.2 Lower bounds

We now establish that the sup-norm rates obtained in Corollary 3.1 are the best possible (i.e. minimax) sup-norm convergence rates for estimating h_0 and its derivatives.

To establish a lower bound, we require a *link condition* that relates smoothness of T to the parameter space for h_0 . Let $\widetilde{\psi}_{j,k,G}$ denote a tensor-product CDV wavelet basis for $[0, 1]^d$ of regularity $\gamma > p$. Appendix E provides details on the construction and properties of this basis.

Condition LB (i) Assumption 1(i)–(iii) holds; (ii) $E[u_i^2 | W_i = w] \geq \underline{\sigma}^2 > 0$ uniformly for $w \in \mathcal{W}$; and (iii) there is a positive decreasing function ν s.t. $\|Th\|_{L^2(\mathcal{W})}^2 \lesssim \sum_{j,G,k} [\nu(2^j)]^2 \langle h, \widetilde{\psi}_{j,k,G} \rangle_X^2$ holds for all $h \in B_\infty(p, L)$.

Condition LB is standard in the optimal rate literature (see Hall and Horowitz (2005) and Chen and Reiss (2011)). The mildly ill-posed case corresponds to choosing $\nu(t) = t^{-\varsigma}$, and says roughly that the conditional expectation operator T makes p -smooth functions of X into $(\varsigma + p)$ -smooth functions of

W . The severely ill-posed case, which corresponds to choosing $\nu(t) = \exp(-\frac{1}{2}t^\varsigma)$ and says roughly that T maps smooth functions of X into “supersmooth” functions of W .

Theorem 3.2. *Let Condition LB hold for the NPIV model with a random sample $\{(X_i, Y_i, W_i)\}_{i=1}^n$. Then for any $0 \leq |\alpha| < p$:*

$$\liminf_{n \rightarrow \infty} \inf_{\hat{g}_n} \sup_{h \in B_\infty(p, L)} \mathbb{P}_h \left(\|\hat{g}_n - \partial^\alpha h\|_\infty \geq c(n/\log n)^{-(p-|\alpha|)/(2(p+\varsigma)+d)} \right) \geq c' > 0$$

in the mildly ill-posed case, and

$$\liminf_{n \rightarrow \infty} \inf_{\hat{g}_n} \sup_{h \in B_\infty(p, L)} \mathbb{P}_h \left(\|\hat{g}_n - \partial^\alpha h\|_\infty \geq c(\log n)^{-(p-|\alpha|)/\varsigma} \right) \geq c' > 0$$

in the severely ill-posed case, where $\inf_{\hat{g}_n}$ denotes the infimum over all estimators of $\partial^\alpha h$ based on the sample of size n , $\sup_{h \in B_\infty(p, L)} \mathbb{P}_h$ denotes the sup over $h \in B_\infty(p, L)$ and distributions of (X_i, W_i, u_i) that satisfy Condition LB with fixed ν , and the finite positive constants c, c' do not depend on n .

According to Theorem 3.2 and Theorem B.2 (in Appendix B), the minimax lower bounds in sup-norm for estimating h_0 and its derivatives coincide with those in L^2 for severely ill-posed NPIV problems, and are only a factor of $[\log(n)]^\epsilon$ (with $\epsilon = \frac{p-|\alpha|}{2(p+\varsigma)+d} < \frac{p}{2p+d} < \frac{1}{2}$) worse than those in L^2 for mildly ill-posed problems. Our proof of sup-norm lower bound for NPIV models is similar to that of [Chen and Reiss \(2011\)](#) for L^2 -norm lower bound. Similar sup-norm lower bounds for density deconvolution were recently obtained by [Lounici and Nickl \(2011\)](#).

3.3 Models with endogenous and exogenous regressors

In many empirical studies, some regressors might be endogenous while others are exogenous. Consider the model

$$Y_i = h_0(X_{1i}, Z_i) + u_i \tag{16}$$

where X_{1i} is a vector of endogenous regressors and Z_i is a vector of exogenous regressors. Let $X_i = (X'_{1i}, Z'_i)'$. Here the vector of instrumental variables W_i is of the form $W_i = (W'_{1i}, Z'_i)'$ where W_{1i} are instruments for X_{1i} . We refer to this as the “partially endogenous case”. The sieve NPIV estimator is implemented in exactly the same way as the “fully endogenous” setting in which X_i consists only of endogenous variables, just like 2SLS with endogeneous and exogenous regressors.¹⁸ Our convergence rates presented in Section 3.1 and Appendix B apply equally to the partially endogenous model (16) under the stated regularity conditions: all that differs between the two cases is the interpretation of the sieve measure of ill-posedness.

Consider first the fully endogenous case where $T : L^2(X) \rightarrow L^2(W)$ is compact under mild condi-

¹⁸ All that changes here is that J may grow more quickly as the degree of ill-posedness will be smaller. In contrast, other NPIV estimators based on estimating the conditional densities of the regressors and instrumental variables must be implemented separately for each value of z ([Hall and Horowitz, 2005](#); [Horowitz, 2011](#); [Gagliardini and Scaillet, 2012](#)).

tions on the conditional density of X given W (see, e.g., [Newey and Powell \(2003\)](#); [Blundell et al. \(2007\)](#); [Darolles et al. \(2011\)](#); [Andrews \(2011\)](#)). Then T admits a singular value decomposition (SVD) $\{\phi_{0j}, \phi_{1j}, \mu_j\}_{j=1}^{\infty}$ where $(T^*T)^{1/2}\phi_{0j} = \mu_j\phi_{0j}$, $\mu_j \geq \mu_{j+1}$ for each j and $\{\phi_{0j}\}_{j=1}^{\infty}$ and $\{\phi_{1j}\}_{j=1}^{\infty}$ are orthonormal bases for $L^2(X)$ and $L^2(W)$, respectively. Suppose that Ψ_J spans $\phi_{0j}, \dots, \phi_{0J}$. Then the sieve measure of ill-posedness is $\tau_J = \mu_J^{-1}$.

Now consider the partially endogenous case. Similar to [Horowitz \(2011\)](#), we suppose that for each value of z the conditional expectation operator $T_z : L^2(X_1|Z = z) \rightarrow L^2(W_1|Z = z)$ given by $(T_z h)(w_1) = E[h(X_1)|W_{1i} = w_1, Z_i = z]$ is compact. Then each T_z admits a SVD $\{\phi_{0j,z}, \phi_{1j,z}, \mu_{j,z}\}_{j=1}^{\infty}$ where $T_z\phi_{0j,z} = \mu_{j,z}\phi_{1j,z}$, $(T_z^*T_z)^{1/2}\phi_{0j,z} = \mu_{j,z}\phi_{0j,z}$, $(T_zT_z^*)^{1/2}\phi_{1j,z} = \mu_{j,z}\phi_{1j,z}$, $\mu_{j,z} \geq \mu_{j+1,z}$ for each j and z , and $\{\phi_{0j,z}\}_{j=1}^{\infty}$ and $\{\phi_{1j,z}\}_{j=1}^{\infty}$ are orthonormal bases for $L^2(X_1|Z = z)$ and $L^2(W_1|Z = z)$, respectively, for each z . The following result adapts Lemma 1 of [Blundell et al. \(2007\)](#) to the partially endogenous setting.

Lemma 3.2. *Let T_z be compact with SVD $\{\phi_{0j,z}, \phi_{1j,z}, \mu_{j,z}\}_{j=1}^{\infty}$ for each z . Let $\mu_j^2 = E[\mu_{j,Z_i}^2]$ and $\phi_{0j}(\cdot, z) = \phi_{0j,z}(\cdot)$ for each z and j . Then: (1) $\tau_J \geq \mu_J^{-1}$. (2) If, in addition, $\phi_{01}, \dots, \phi_{0J} \in \Psi_J$, then: $\tau_J \leq \mu_J^{-1}$.*

Consider the following partially-endogenous stylized example from [Hoderlein and Holzmann \(2011\)](#). Let X_{1i} , W_{1i} and Z_i be scalar random variables with

$$\begin{pmatrix} X_{1i} \\ W_{1i} \\ Z_i \end{pmatrix} \sim N \left(\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 & \rho_{xw} & \rho_{xz} \\ \rho_{xw} & 1 & \rho_{wz} \\ \rho_{xz} & \rho_{wz} & 1 \end{pmatrix} \right).$$

Then

$$\begin{pmatrix} \frac{X_{1i} - \rho_{xz}z}{\sqrt{1 - \rho_{xz}^2}} \\ \frac{W_{1i} - \rho_{wz}z}{\sqrt{1 - \rho_{wz}^2}} \end{pmatrix} \Bigg| Z_i = z \sim N \left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 & \rho_{xw|z} \\ \rho_{xw|z} & 1 \end{pmatrix} \right) \quad (17)$$

where

$$\rho_{xw|z} = \frac{\rho_{xw} - \rho_{xz}\rho_{wz}}{\sqrt{(1 - \rho_{xz}^2)(1 - \rho_{wz}^2)}}$$

is the partial correlation between X_{1i} and W_{1i} given Z_i . For each $j \geq 1$ let H_j denote the j th Hermite polynomial (the Hermite polynomials form an orthonormal basis with respect to Gaussian density). Since $T_z : L^2(X_1|Z = z) \rightarrow L^2(W_1|Z = z)$ is compact for each z , it follows from Mehler's formula that T_z has a SVD $\{\phi_{0j,z}, \phi_{1j,z}, \mu_{j,z}\}_{j=1}^{\infty}$ with

$$\phi_{0j,z}(x_1) = H_{j-1} \left(\frac{x_1 - \rho_{xz}z}{\sqrt{1 - \rho_{xz}^2}} \right), \quad \phi_{1j,z}(w_1) = H_{j-1} \left(\frac{w_1 - \rho_{wz}z}{\sqrt{1 - \rho_{wz}^2}} \right), \quad \mu_{j,z} = |\rho_{xw|z}|^{j-1}$$

for each z . Since $\mu_{J,z} = |\rho_{xw|z}|^{J-1}$ for each z , we have $\mu_J = |\rho_{xw|z}|^{J-1} \asymp |\rho_{xw|z}|^J$. If X_{1i} and W_{1i} are uncorrelated with Z_i , then $\mu_J = |\rho|^{J-1}$ where $\rho = \rho_{xw}$.

In contrast, consider the following fully-endogenous model in which X_i and W_i are bivariate with

$$\begin{pmatrix} X_{1i} \\ X_{2i} \\ W_{1i} \\ W_{2i} \end{pmatrix} \sim N \left(\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & \rho_1 & 0 \\ 0 & 1 & 0 & \rho_2 \\ \rho_1 & 0 & 1 & 0 \\ 0 & \rho_2 & 0 & 1 \end{pmatrix} \right)$$

where ρ_1 and ρ_2 are such that the covariance matrix is invertible. It is straightforward to verify that T has singular value decomposition with

$$\phi_{0j}(x) = H_{j-1}(x_1)H_{j-1}(x_2) \quad \phi_{1j}(w) = H_{j-1}(w_1)H_{j-2}(w_2), \quad \mu_j = |\rho_1\rho_2|^{j-1},$$

and $\mu_J = \rho^{2(J-1)}$ if $\rho_1 = \rho_2 = \rho$. Thus, the measure of ill-posedness diverges faster in the fully-endogenous case ($\mu_J = \rho^{2(J-1)}$) than that in the partially endogenous case ($\mu_J = |\rho|^{-(J-1)}$).

4 Uniform inference on collections of nonlinear functionals

In this section we apply our sup-norm rate results and tight bounds on random matrices (in online Appendix F) to show that the score bootstrap UCBs defined in (5) for collections of nonlinear (or linear) functionals $\{f_t(\cdot) : t \in \mathcal{T}\}$ of h_0 have asymptotically correct coverage.

We consider functionals $f_t : \mathcal{H} \subset L^\infty(X) \rightarrow \mathbb{R}$ for each $t \in \mathcal{T}$ for which $Df_t(h)[v] = \lim_{\delta \rightarrow 0^+} [\delta^{-1} f_t(h + \delta v)]$ exists for all $v \in \mathcal{H} - \{h_0\}$ for all h in a small neighborhood of h_0 (where the neighborhood is independent of t). This is trivially true for, say, $f_t(h) = h(t)$ with $\mathcal{T} \subseteq \mathcal{X}$ for UCBs for h_0 . Let $\Omega = E[u_i^2 b^K(W_i) b^K(W_i)']$. Then the ‘‘2SLS covariance matrix’’ for \hat{c} (given in (2)) is

$$\mathcal{U} = [S'G_b^{-1}S]^{-1}S'G_b^{-1}\Omega G_b^{-1}S[S'G_b^{-1}S]^{-1},$$

and the sieve variance for $f_t(\hat{h})$ is

$$[\sigma_n(f_t)]^2 = (Df_t(h_0)[\psi^J])' \mathcal{U} (Df_t(h_0)[\psi^J]).$$

Assumption 2 (continued). (iii) $E[u_i^2 | W_i = w] \geq \underline{\sigma}^2 > 0$ uniformly for all $w \in \mathcal{W}$; and (iv) $\sup_w E[|u_i|^3 | W_i = w] < \infty$.

Assumptions 2(iii)(iv) are reasonably mild conditions used to derive the uniform limit theory. Define

$$v_n(f_t)(x) = \psi^J(x)' [S'G_b^{-1}S]^{-1} Df_t(h_0)[\psi^J], \quad \hat{v}_n(f_t)(x) = \psi^J(x)' [S'G_b^{-1}S]^{-1} Df_t(\hat{h})[\psi^J],$$

where, for each fixed t , $v_n(f_t)$ could be viewed as a ‘‘sieve 2SLS Riesz representer’’. Note that $v_n(f_t) =$

$\widehat{v}_n(f_t)$ whenever f_t is linear. Under Assumption 2(i)(iii) we have that

$$[\sigma_n(f_t)]^2 \asymp Df_t(h_0)[\psi^J]'[S'G_b^{-1}S]^{-1}Df_t(h_0)[\psi^J] = \|\Pi_K T v_n(f_t)\|_{L^2(W)}^2 \quad \text{uniformly in } t.$$

Following [Chen and Pouzo \(2015\)](#), we call $f_t(\cdot)$ an irregular functional of h_0 (i.e., slower than \sqrt{n} -estimable) if $\sigma_n(f_t) \nearrow +\infty$ as $n \rightarrow \infty$. This includes the evaluation functionals $h_0(t)$ and $\partial^\alpha h_0(t)$ as well as $f_{CS,t}(h_0)$ and $f_{DL,t}(h_0)$. In this paper we shall focus on applications of sup-norm rates results to inferences on irregular functionals. See Section 6 for related applications to root- n estimable functionals.

Assumption 5. *Let η_n and η'_n be sequences of positive constants such that $\eta_n = o(1)$ and $\eta'_n = o(1)$. Let $\sigma_n(f_t) \nearrow +\infty$ as $n \rightarrow \infty$ for each $t \in \mathcal{T}$. Either (a) or (b) of the following holds:*

- (a) f_t is a linear functional for each $t \in \mathcal{T}$ and $\sup_{t \in \mathcal{T}} \sqrt{n}(\sigma_n(f_t))^{-1} |f_t(\widehat{h}) - f_t(h_0)| = O_p(\eta_n)$; or
- (b) (i) $v \mapsto Df_t(h_0)[v]$ is a linear functional for each $t \in \mathcal{T}$; (ii)

$$\sup_{t \in \mathcal{T}} \left| \sqrt{n} \frac{f_t(\widehat{h}) - f_t(h_0)}{\sigma_n(f_t)} - \sqrt{n} \frac{Df_t(h_0)[\widehat{h} - h_0]}{\sigma_n(f_t)} \right| = O_p(\eta_n);$$

and (iii) $\sup_{t \in \mathcal{T}} \frac{\|\Pi_K T(\widehat{v}_n(f_t) - v_n(f_t))\|_{L^2(W)}}{\sigma_n(f_t)} = O_p(\eta'_n)$.

Assumption 5(a)(b)(i)(ii) are similar to uniform-in- t versions of Assumption 3.5 of [Chen and Pouzo \(2015\)](#). Assumption 5(b)(iii) controls any additional error arising in the estimation of $\widehat{\sigma}_n(f_t)$ due to nonlinearity of $f_t(\cdot)$ and is not required when $f_t(\cdot)$ is a linear functional.

The next remark presents a set of sufficient conditions for Assumption 5 when $\{f_t : t \in \mathcal{T}\}$ are irregular functionals of h_0 . Since the functionals are irregular, the quantity $\underline{\sigma}_n := \inf_{t \in \mathcal{T}} \sigma_n(f_t)$ will typically satisfy $\underline{\sigma}_n \nearrow +\infty$ as $n \rightarrow \infty$. Our sup-norm rates for \widehat{h} and \widetilde{h} , together with divergence of $\underline{\sigma}_n$, helps to control the nonlinearity bias terms.

Remark 4.1. *Let $\mathcal{H}_n \subseteq \mathcal{H}$ be a sequence of neighborhoods of h_0 with $\widehat{h}, \widetilde{h} \in \mathcal{H}_n$ wpa1 and assume $\underline{\sigma}_n := \inf_{t \in \mathcal{T}} \sigma_n(f_t) > 0$ for each n . Then: Assumption 5(a) is implied by (a'), and Assumption 5(b) is implied by (b'), where*

- (a') (i) f_t is a linear functional for each $t \in \mathcal{T}$ and there exists α with $|\alpha| \geq 0$ s.t. $\sup_t |f_t(h - h_0)| \lesssim \|\partial^\alpha h - \partial^\alpha h_0\|_\infty$ for all $h \in \mathcal{H}_n$; and (ii) $\underline{\sigma}_n^{-1} \|\partial^\alpha \widetilde{h} - \partial^\alpha h_0\|_\infty = o_p(n^{-1/2})$; or
- (b') (i) $v \mapsto Df_t(h_0)[v]$ is a linear functional for each $t \in \mathcal{T}$ and there exists α with $|\alpha| \geq 0$ s.t. $\sup_t |Df_t(h_0)[h - h_0]| \lesssim \|\partial^\alpha h - \partial^\alpha h_0\|_\infty$ for all $h \in \mathcal{H}_n$;
- (ii) there are α_1, α_2 with $|\alpha_1|, |\alpha_2| \geq 0$ s.t.

$$(ii.1) \sup_t \left| f_t(\widehat{h}) - f_t(h_0) - Df_t(h_0)[\widehat{h} - h_0] \right| \lesssim \|\partial^{\alpha_1} \widehat{h} - \partial^{\alpha_1} h_0\|_\infty \|\partial^{\alpha_2} \widehat{h} - \partial^{\alpha_2} h_0\|_\infty \quad \text{and}$$

$$(ii.2) \underline{\sigma}_n^{-1} (\|\partial^{\alpha_1} \widehat{h} - \partial^{\alpha_1} h_0\|_\infty \|\partial^{\alpha_2} \widehat{h} - \partial^{\alpha_2} h_0\|_\infty + \|\partial^{\alpha_1} \widetilde{h} - \partial^{\alpha_1} h_0\|_\infty) = o_p(n^{-1/2});$$

and (iii) we have

$$\sup_{t \in \mathcal{T}} \frac{(\tau_J)^2 \left(\sum_{j=1}^J \left(Df_t(\hat{h})[(G_\psi^{-1/2} \psi^J)_j] - Df_t(h_0)[(G_\psi^{-1/2} \psi^J)_j] \right)^2 \right)}{[\sigma_n(f_t)]^2} = o_p(1).$$

Condition (a')(i) is automatically satisfied by functionals of the form $f_t(h) = \partial^\alpha h(t)$ with $\mathcal{T} \subseteq \mathcal{X}$ and $\mathcal{H}_n = \mathcal{H}$. Conditions (a')(i) and (b')(i)(ii) are sufficient conditions that are formulated to take advantage of the sup-norm rate results in Section 3. For example, condition (b')(i)(ii.1) is easily satisfied by exact CS and DL functionals (lemma A.1 of Hausman and Newey (1995)). Condition (b')(ii.2) can then be verified applying our sup-norm rate results. Condition (b')(iii) is a sufficient condition for Assumption 5(b)(iii), and is needed for uniform-in- t consistent estimation of $\sigma_n(f_t)$ by $\hat{\sigma}_n(f_t)$ only, and is automatically satisfied when $f_t(\cdot)$ is a linear functional.

The next assumption concerns the set of normalized sieve 2SLS Riesz representers, given by

$$u_n(f_t)(x) = v_n(f_t)(x)/\sigma_n(f_t).$$

Let d_n denote the semi-metric on \mathcal{T} given by $d_n(t_1, t_2)^2 = E[(u_n(f_{t_1})(X_i) - u_n(f_{t_2})(X_i))^2]$ and let $N(\mathcal{T}, d_n, \epsilon)$ denote the ϵ -entropy of \mathcal{T} with respect to d_n . Let $\delta_{h,n}$ be a sequence of positive constants such that $\|\hat{h} - h_0\|_\infty = O_p(\delta_{h,n}) = o_p(1)$ and $\delta_{V,n} = (\zeta_{b,K}^{(2+\delta)/\delta} \sqrt{(\log K)/n})^{\delta/(1+\delta)} + \tau_J \zeta \sqrt{(\log J)/n}$.

Assumption 6. (i) there is a sequence of finite constants $c_n > 0$ with $c_n^{-1} = o(1)$ such that

$$1 + \int_0^\infty \sqrt{\log N(\mathcal{T}, d_n, \epsilon)} \, d\epsilon = O(c_n);$$

and (ii) there is a sequence of constants $r_n > 0$ with $r_n = o(1)$ such that

$$\frac{\zeta_{b,K} J^2}{r_n^3 \sqrt{n}} = o(1) \quad \text{and} \quad \eta_n + \eta'_n \sqrt{J} + r_n + (\delta_{V,n} + \delta_{h,n} + \eta'_n) \times c_n = o(c_n^{-1})$$

where $\eta'_n \equiv 0$ if the f_t are linear.

Assumption 6(i) is a mild regularity condition requiring that the class $\{u_n(f_t) : t \in \mathcal{T}\}$ not be too complex; see Remark 4.2 below for a set of sufficient conditions to bound c_n . Assumption 6(ii) strengthens conditions on the growth rate of J . The first part $\frac{\zeta_{b,K} J^2}{r_n^3 \sqrt{n}} = o(1)$ is used to derive the Gaussian process strong approximation to the sieve t -statistic process, which could be improved if Assumption 2(iv) is replaced by a stronger conditional moment restriction on u_i . The remaining part ensures that nonlinear remainder terms and the error in estimating $\sigma_n(f_t)$ vanishes sufficiently quickly. While the consistency of $\hat{\sigma}_n(f_t)$ is enough for the pointwise asymptotic normality of the plug-in sieve t -statistic for $f(h_0)$ (see Theorem D.1 in the online Appendix D), we need the following rate of convergence for uniform inference

$$\sup_{t \in \mathcal{T}} \left| \frac{\sigma_n(f_t)}{\hat{\sigma}_n(f_t)} - 1 \right| = O_p(\delta_{V,n} + \delta_{h,n} + \eta'_n)$$

(see Lemma G.5 in the online Appendix G).

Remark 4.2. Let Assumptions 1(iii) and 4(i) hold. Let $\mathcal{T} \subset \mathbb{R}^{d_T}$ be compact and there exist positive sequences Γ_n and γ_n s.t.

$$\sup_{h \in \Psi_J: \|h\|_{L^2(X)}=1} |f_{t_1}(h) - f_{t_2}(h)| \leq \Gamma_n \|t_1 - t_2\|_{\ell^2}^{\gamma_n} \quad \text{if the } f_t \text{ are linear,}$$

or

$$\sup_{h \in \Psi_J: \|h\|_{L^2(X)}=1} |(Df_{t_1}(h_0)[h] - Df_{t_2}(h_0)[h])| \leq \Gamma_n \|t_1 - t_2\|_{\ell^2}^{\gamma_n} \quad \text{if the } f_t \text{ are nonlinear.}$$

Then: Assumption 6(i) holds with $c_n = 1 + \int_0^\infty \sqrt{d_T \{\log(\tau_J \Gamma_n \epsilon^{-\gamma_n} / \underline{\sigma}_n) \vee 0\}} d\epsilon$.

Our first result for this section is a Bahadur-type expansion for the sieve t -statistic process for (possibly) nonlinear functionals. Define

$$\widehat{\mathbb{Z}}_n(t) = \frac{(Df_t(h_0)[\psi^J])' [S' G_b^{-1} S]^{-1} S' G_b^{-1}}{\sigma_n(f_t)} \frac{1}{\sqrt{n}} \sum_{i=1}^n b^K(W_i) u_i.$$

Lemma 4.1. Let Assumptions 1(iii), 2(i)–(iii), 3(iii), 4(i), 5, and 6 hold with $\tau_J \zeta \sqrt{(J \log J)/n} = o(1)$. Then:

$$\sup_{t \in \mathcal{T}} \left| \frac{\sqrt{n}(f_t(\widehat{h}) - f_t(h_0))}{\widehat{\sigma}(f_t)} - \widehat{\mathbb{Z}}_n(t) \right| = o_p(c_n^{-1}).$$

Our second main result establishes consistency of the sieve score bootstrap for estimating the critical values of the uniform sieve t -statistic process $\sup_{t \in \mathcal{T}} |\widehat{\mathbb{Z}}_n(t)|$ for a NPIV model. Let \mathbb{P}^* denote the probability measure of the bootstrap innovations $\varpi_1, \dots, \varpi_n$ conditional on the data $Z^n := \{(X_1, Y_1, W_1), \dots, (X_n, Y_n, W_n)\}$. Recall that $\mathbb{Z}_n^*(t)$ is defined in (8).

Theorem 4.1. Let Assumptions 1(iii), 2, 3(iii), 4(i), 5, and 6 hold with $\tau_J \zeta \sqrt{(J \log J)/n} = o(1)$.

Then:

$$\sup_{s \in \mathbb{R}} \left| \mathbb{P} \left(\sup_{t \in \mathcal{T}} \left| \frac{\sqrt{n}(f_t(\widehat{h}) - f_t(h_0))}{\widehat{\sigma}(f_t)} \right| \leq s \right) - \mathbb{P}^* \left(\sup_{t \in \mathcal{T}} |\mathbb{Z}_n^*(t)| \leq s \right) \right| = o_p(1). \quad (18)$$

Theorem 4.1 appears to be the first to establish consistency of a sieve score bootstrap for uniform inference on general nonlinear functionals of NPIV under low-level conditions. When specializing to collections of linear functionals, Theorem 4.1 and Corollary 3.1 immediately imply the following result.

Corollary 4.1. Theorem 4.1 applies to UCBs for $\partial^\alpha h_0$ as a special case in which $f_t(h) = \partial^\alpha h(t)$ and $\mathcal{T} \subseteq \mathcal{X}$ provided $h_0 \in B_\infty(p, L)$, Assumptions 1(i)(ii)(iii), 2, and 4 hold, Ψ_J is formed from a B-spline basis of regularity $\gamma > (p \vee 2 + |\alpha|)$, B_K is spanned by a B-spline, wavelet, or cosine basis, $\sigma_n(f_t) \asymp \tau_J J^a$ uniformly in t with $a = \frac{1}{2} + \frac{|\alpha|}{d}$, and $J^{-p/d} = o(\tau_J \sqrt{(J \log n)/n})$, $\tau_J (J^a (\log J)^{3/2} \vee J) / \sqrt{n} = o(1)$, $J^{(2+\delta)/\delta} (\log J)^{(2\delta+1)/(1+\delta)} (\log n) = o(n^{\delta/(1+\delta)})$, and $J^5 (\log J)^6 / n = o(1)$.

Recently Horowitz and Lee (2012) developed a notion of UCBs for a NPIV function h_0 of a scalar endogenous regressor X_i based on interpolation over a growing number of grid points with h_0 estimated

via the modified orthogonal series NPIV estimator of [Horowitz \(2011\)](#).¹⁹ When specializing Corollary 4.1 to score bootstrap UCBs for a NPIV function of a scalar regressor, our sufficient conditions are comparable to theirs (see their theorem 4.1). Our score bootstrap UCBs would be computationally much simpler for a NPIV function of a multivariate endogenous regressor X_i , however.

When X_i is exogenous, Theorem 4.1 may be applied to establish the validity of sieve score bootstrap UCBs for collections of nonlinear functionals of a nonparametric regression function $E[Y_i|X_i = x]$.

Corollary 4.2. *When X_i is exogenous and $W_i = X_i$, $K = J$, and $b^K = \psi^J$, the sieve NPIV estimator \hat{h} reduces to the series LS estimator of the nonparametric regression $h_0(x) = E[Y_i|X_i = x]$. Let Assumptions 1(i), 2, 3(iii), 5, and 6 hold with $\zeta\sqrt{(J \log J)/n} = o(1)$, $\tau_J = 1$ and $S = G_b = G_\psi$. Then: Equation (18) holds with $\hat{\sigma}(f_t)$ and $\mathbb{Z}_n^*(t)$ reduced to*

$$[\hat{\sigma}(f_t)]^2 = (Df_t(\hat{h})[\psi^J])' \hat{G}_\psi^{-1} \hat{\Omega} \hat{G}_\psi^{-1} (Df_t(\hat{h})[\psi^J]) , \quad \hat{\Omega} = \frac{1}{n} \sum_{i=1}^n \hat{u}_i^2 \psi^J(X_i) \psi^J(X_i)' ,$$

$$\mathbb{Z}_n^*(t) = \frac{(Df_t(\hat{h})[\psi^J])' \hat{G}_\psi^{-1}}{\hat{\sigma}(f_t)} \left(\frac{1}{\sqrt{n}} \sum_{i=1}^n \psi^J(X_i) \hat{u}_i \varpi_i \right) \quad \text{for each } t \in \mathcal{T} .$$

In an important paper on series LS regression (without endogeneity), [Belloni et al. \(2015\)](#) derives UCBs results for collections of linear functionals of a regression function estimated via a series LS. Our Corollary 4.2 complements theirs by establishing the validity of score bootstrap UCBs for collections of *nonlinear* functionals of a nonparametric regression function estimated via a series LS.

4.1 Monte Carlo

We now evaluate the finite sample performance of our sieve score bootstrap UCBs for h_0 in NPIV model (1). We use the experimental design of [Newey and Powell \(2003\)](#), in which IID draws are generated from

$$\begin{pmatrix} u_i \\ V_i^* \\ W_i^* \end{pmatrix} \sim N \left(\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 & 0.5 & 0 \\ 0.5 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right)$$

from which we then set $X_i^* = W_i^* + V_i^*$. To ensure compact support of the regressor and instrument, we rescale X_i^* and W_i^* by defining $X_i = \Phi(X_i^*/\sqrt{2})$ and $W_i = \Phi(W_i^*)$ where Φ is the Gaussian cdf. We use $h_0(x) = 4x - 2$ for our *linear* design and $h_0(x) = \log(|16x - 8| + 1) \text{sgn}(x - \frac{1}{2})$ for our *nonlinear* design (our nonlinear h_0 is a re-scaled version of the h_0 used in [Newey and Powell \(2003\)](#)). Note that p for the nonlinear h_0 is between 1 and 2, so h_0 is not particularly smooth ($h_0'(x)$ has a kink at $x = \frac{1}{2}$).

We generate 1000 samples of length 1000 and implement our procedure using a B-spline basis for B_K and Ψ_J . For each simulation, we calculate the 90%, 95%, and 99% uniform confidence bands for

¹⁹Remark 4 in [Horowitz and Lee \(2012\)](#) mentioned that their notion of UCB is different from the standard UCBs. They also proved the consistency of their bootstrap confidence bands over fixed finite number of grid points.

h_0 over the support $[0.05, 0.95]$ with 1000 bootstrap replications for each simulation. We draw the bootstrap innovations ϖ_i from the two-point distribution of [Mammen \(1993\)](#). We then calculate the MC coverage probabilities of our uniform confidence bands.

Figure 5 displays the estimated structural function \hat{h} and confidence bands together with a scatterplot of the sample (X_i, Y_i) data for the nonlinear design. The true function h_0 is seen to lie inside the UCBs. The results of this MC experiment are presented in Table 2. Comparing the MC coverage probabilities with their nominal values, it is clear that the uniform confidence bands for the linear design are slightly too conservative. However, the uniform confidence bands for the nonlinear design using cubic B-splines to approximate h_0 have MC converge much closer to the nominal coverage probabilities.

5 Pointwise and uniform inference in nonparametric demand

We now illustrate how our sup-norm rates can be used to perform pointwise and uniform inferences on policy-relevant welfare functionals in nonparametric demand with endogeneity. In so doing, we make several contributions to the literature on nonparametric welfare analysis. First, we provide primitive sufficient conditions under which plug-in sieve t -statistics for exact CS and DL, and approximate CS functionals are asymptotically $N(0, 1)$, allowing for mildly and severely ill-posed inverse models (subsections 5.1 and 5.2). As a by-product, we also derive asymptotic normality of plug-in sieve t -statistics for these functionals under conditions weaker than those in the existing literature for models with exogenous prices and incomes (subsection 5.3). Finally, we establish validity of sieve score bootstrap UCBs for exact CS and DL functionals over a range of taxes and/or incomes (subsection 5.4).

Previously, [Hausman and Newey \(1995\)](#) provided sufficient conditions for asymptotic normality of plug-in estimators when prices and incomes are *exogenous* and demand is estimated via nonparametric LS regression. [Vanhems \(2010\)](#) studied consistency and convergence rates of kernel-based plug-in estimators of CS functionals allowing for endogeneity. [Blundell et al. \(2012\)](#) and [Hausman and Newey \(2016\)](#) estimated CS and DL functionals of nonparametric estimates of gasoline demand allowing for prices to be *endogenous*, but did not provide theoretical justification for their inference approach under endogeneity.

5.1 Exact CS and DL functionals with endogeneity

Here we present primitive regularity conditions for pointwise asymptotic normality of the sieve t -statistics for exact CS and DL. We suppress dependence of the functionals on $t = (\mathbf{p}^0, \mathbf{p}^1, y)$.

Let $\mathbf{X}_i = (\mathbf{P}_i, \mathbf{Y}_i)$. To rule out trivial cases, we assume in what follows that the support of both \mathbf{P}_i and \mathbf{Y}_i is bounded away from zero. If both \mathbf{P}_i and \mathbf{Y}_i are endogenous, let \mathbf{W}_i be a 2×1 vector of instruments. Let $T : L^2(\mathbf{X}) \rightarrow L^2(\mathbf{W})$ be compact and injective with singular value decomposition

Ψ_J	B_K	J	K	Design 1: Linear h_0			Design 2: nonlinear h_0		
				90% CI	95% CI	99% CI	90% CI	95% CI	99% CI
C	C	5	5	0.962	0.983	0.996	0.896	0.942	0.987
C	C	5	6	0.957	0.983	0.996	0.845	0.924	0.981
C	Q	5	5	0.961	0.982	0.996	0.884	0.939	0.985
C	Q	5	6	0.958	0.983	0.997	0.846	0.921	0.981
Q	Q	5	5	0.964	0.984	0.997	0.913	0.948	0.989
Q	Q	5	6	0.961	0.985	0.996	0.886	0.937	0.983

Table 2: MC coverage probabilities of uniform confidence bands for h_0 . Results are presented for cubic (C) and quartic (Q) B-spline bases for Ψ_J and B_K .

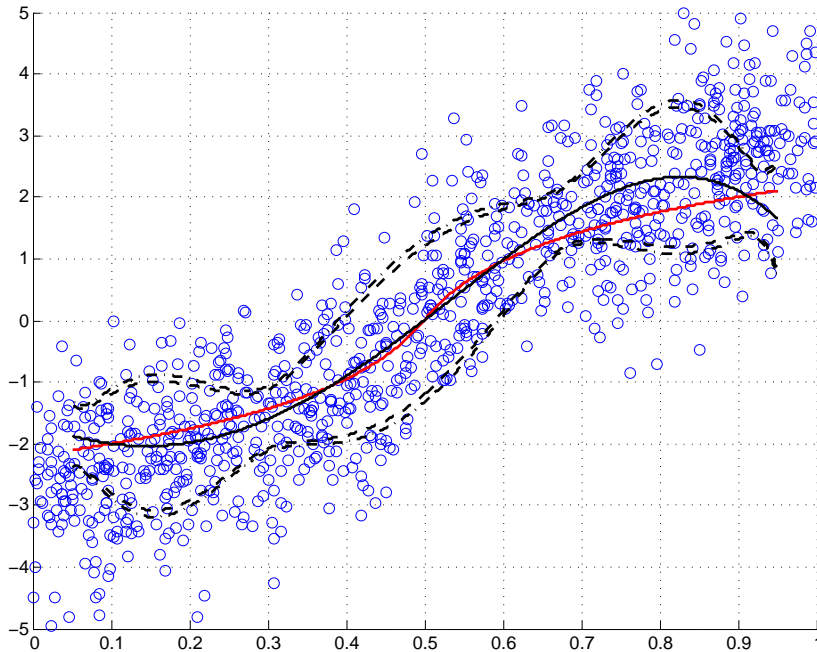


Figure 5: 90% and 95% uniform confidence bands for h_0 (dashed lines; innermost are 90%), NPIV estimate \hat{h} (solid black line), true structural function h_0 (solid red line) for the nonlinear design.

(SVD) $\{\phi_{0j}, \phi_{1j}, \mu_j\}_{j=1}^{\infty}$ where

$$T\phi_{0j} = \mu_j\phi_{1j}, \quad (T^*T)^{1/2}\phi_{0j} = \mu_j\phi_{0j}, \quad (TT^*)^{1/2}\phi_{1j} = \mu_j\phi_{1j}$$

and $\{\phi_{0j}\}_{j=1}^{\infty}$ and $\{\phi_{1j}\}_{j=1}^{\infty}$ are orthonormal bases for $L^2(\mathbf{X})$ and $L^2(\mathbf{W})$, respectively. If P_i is endogenous but Y_i is exogenous, we take $W_i = (W_{1i}, Y_i)'$ where W_{1i} an instrument for P_i . Let $T_y : L^2(\mathbf{P}|\mathbf{Y} = y) \rightarrow L^2(W_1|\mathbf{Y} = y)$ be compact and injective with SVD $\{\phi_{0j,y}, \phi_{1j,y}, \mu_{j,y}\}_{j=1}^{\infty}$ for each y where

$$T_y\phi_{0j,y} = \mu_{j,y}\phi_{1j,y}, \quad (T_y^*T_y)^{1/2}\phi_{0j,y} = \mu_{j,y}\phi_{0j,y}, \quad (T_yT_y^*)^{1/2}\phi_{1j,y} = \mu_{j,y}\phi_{1j,y}$$

and $\{\phi_{0j,y}\}_{j=1}^{\infty}$ and $\{\phi_{1j,y}\}_{j=1}^{\infty}$ are orthonormal bases for $L^2(\mathbf{P}|\mathbf{Y} = y)$ and $L^2(W_1|\mathbf{Y} = y)$, respectively. In this case, we define $\phi_{0j}(\mathbf{p}, y) = \phi_{0j,y}(\mathbf{p})$, $\phi_{1j}(\mathbf{w}_1, y) = \phi_{1j,y}(\mathbf{w}_1)$, and $\mu_j^2 = E[\mu_{j,Y_i}^2]$ (see Section 3.3 for further details).

In both cases, we follow [Chen and Pouzo \(2015\)](#) and assume that Ψ_J and B_K are Riesz bases in that they span $\phi_{01}, \dots, \phi_{0J}$ and $\phi_{11}, \dots, \phi_{1K}$, respectively. This implies that $\tau_J \asymp \mu_J^{-1}$. for fixed \mathbf{p}^0 , \mathbf{p}^1 , and y we define

$$a_j = a_j(\mathbf{p}^0, \mathbf{p}^1, y) = \int_0^1 \left(\phi_{0j}(\mathbf{p}(u), y - S_y(\mathbf{p}(u))) e^{-\int_0^u \partial_2 h_0(\mathbf{p}(v), y - S_y(\mathbf{p}(v))) \mathbf{p}'(v) dv} \mathbf{p}'(u) \right) du \quad (19)$$

for the exact CS functional.

Assumption CS (i) X_i and W_i both have compact rectangular support and densities bounded away from 0 and ∞ ; (ii) $h_0 \in B_{\infty}(p, L)$ with $p > 2$ and $0 < L < \infty$; (iii) $E[u_i^2|W_i = w]$ is uniformly bounded away from 0 and ∞ , $E[|u_i|^{2+\delta}]$ is finite, and $\sup_w E[u_i^2\{|u_i| > \ell(n)\}|W_i = w] = o(1)$ for any positive sequence with $\ell(n) \nearrow \infty$; (iv) Ψ_J is spanned by a (tensor-product) B-spline basis of order $\gamma > p$ or continuously differentiable wavelet basis of regularity $\gamma > p$ and B_K is spanned by a (tensor-product) B-spline, wavelet or cosine basis; and (v)

$$\frac{\sqrt{n}}{(\sum_{j=1}^J (a_j/\mu_j)^2)^{1/2}} \times \left(J^{-p/2} + \mu_J^{-2} \frac{J^2 \sqrt{\log J}}{n} \right) = o(1),$$

$$(J^{(2+\delta)/(2\delta)} \sqrt{\log n} \vee \mu_J^{-1} J^{3/2} \sqrt{\log J}) / \sqrt{n} = o(1).$$

Assumption CS(i)–(iv) is standard even for series LS regression without endogeneity. Let $[\sigma_n(f_{CS})]^2 = (Df_{CS}(h_0)[\psi^J])' \hat{U} (Df_{CS}(h_0)[\psi^J])$ be the sieve variance of the plug-in sieve NPIV estimator $f_{CS}(\hat{h}_0)$. Then these assumptions imply that $[\sigma_n(f_{CS})]^2 \asymp \sum_{j=1}^J (a_j/\mu_j)^2$. Assumption CS(v) is sufficient for Remark 4.1(b') for a fixed t ; see Corollary 5.1 below for more concrete sufficient conditions for Assumption CS(v).

Our first result is for CS functionals, established by applying Theorem D.1 in the online Appendix D. Let

$$\hat{\sigma}^2(f_{CS}) = Df_{CS}(\hat{h})[\psi^J]' \hat{U} Df_{CS}(\hat{h})[\psi^J]$$

with

$$Df_{CS}(\widehat{h})[\psi^J] = \int_0^1 \psi^J(\mathbf{p}(u), \mathbf{y} - \widehat{S}_y(\mathbf{p}(u))) e^{-\int_0^u \partial_2 \widehat{h}(\mathbf{p}(v), \mathbf{y} - \widehat{S}_y(\mathbf{p}(v))) \mathbf{p}'(v) dv} \mathbf{p}'(u) du.$$

Theorem 5.1. *Let Assumption CS hold. Then the sieve t -statistic for $f_{CS}(h_0)$ is asymptotically $N(0, 1)$, i.e.:*

$$\sqrt{n} \frac{f_{CS}(\widehat{h}) - f_{CS}(h_0)}{\widehat{\sigma}(f_{CS})} \rightarrow_d N(0, 1).$$

Corollary 5.1. *Let Assumption CS(i)–(iv) hold and let $a_j \asymp j^{a/2}$. Then:*

(1) *Mildly ill-posed case: let $\mu_j \asymp j^{-\varsigma/2}$ with $a + \varsigma > -1$ and $\delta \geq 2/(2 + \varsigma - (a \wedge 0))$ and let $nJ^{-(p+a+\varsigma+1)} = o(1)$ and $J^{3+\varsigma-(a \wedge 0)}(\log n)/n = o(1)$. Then: $[\sigma_n(f_{CS})]^2 \asymp J^{(a+\varsigma)+1}$, Assumption CS(v) holds, and the sieve t -statistic for $f_{CS}(h_0)$ is asymptotically $N(0, 1)$.*

(2) *Severely ill-posed case: let $\mu_j \asymp \exp(-\frac{1}{2}j^{\varsigma/2})$ and $a > 0$ and take $J = (\log(n/(\log n)^\varrho))^{2/\varsigma}$ where $\varrho > 0$ is chosen such that $2p > \varrho\varsigma - 2a$ and $\varrho\varsigma > 6 \vee (8 - 2a)$. Then:*

$$[\sigma_n(f_{CS})]^2 \gtrsim \frac{n}{(\log n)^\varrho} \times (\log(n/(\log n)^\varrho))^{2a/\varsigma},$$

Assumption CS(v) holds, and the sieve t -statistic for $f_{CS}(h_0)$ is asymptotically $N(0, 1)$.

Note that we may choose J satisfying the stated conditions for the mildly ill-posed case provided $p > 2 - a - (a \wedge 0)$. We may also choose such a ϱ for the severely ill-posed case whenever $2p > 6 \vee (8 - 2a) - 2a$.

Our next result is for DL functionals. Note that DL is the sum of CS and a tax receipts functional, namely $(\mathbf{p}^1 - \mathbf{p}^0)h_0(\mathbf{p}^1, \mathbf{y})$. Note that the tax receipts functional is typically less smooth and hence converges slower than that of CS functional. Therefore, $[\sigma_n(f_{DL})]^2 = (Df_{DL}(h_0)[\psi^J])' \widehat{U}(Df_{DL}(h_0)[\psi^J])$ will typically grow at the order of $(\tau_J \sqrt{J})^2$, which is the growth order of the sieve variance term for estimating the unknown NPIV function h_0 at a fixed point. For this reason we do not derive the joint asymptotic distribution of $f_{CS}(\widehat{h})$ and $f_{DL}(\widehat{h})$. The next result adapts Theorem 5.1 to derive asymptotic normality of plug-in sieve t -statistics for DL functionals. Let

$$\widehat{\sigma}^2(f_{DL}) = Df_{DL}(\widehat{h})[\psi^J]' \widehat{U} Df_{DL}(\widehat{h})[\psi^J]$$

with

$$Df_{DL}(\widehat{h})[\psi^J] = Df_{CS}(\widehat{h})[\psi^J] - (\mathbf{p}^1 - \mathbf{p}^0)\psi^J(\mathbf{p}^1, \mathbf{y}).$$

Theorem 5.2. *Let Assumption CS(i)–(iv) hold. Let $\sigma_n(f_{DL}) \asymp \mu_J^{-1}\sqrt{J}$, $\sqrt{n}\mu_J J^{-(p+1)/2} = o(1)$ and $(J^{(2+\delta)/(2\delta)}\sqrt{\log n} \vee \mu_J^{-1}J^{3/2}\sqrt{\log J})/\sqrt{n} = o(1)$. Then:*

$$\sqrt{n} \frac{f_{DL}(\widehat{h}) - f_{DL}(h_0)}{\widehat{\sigma}(f_{DL})} \rightarrow_d N(0, 1).$$

5.2 Approximate consumer surplus with endogeneity

Suppose instead that demand of consumer i for some good is estimated in logs, i.e.

$$\log Q_i = h_0(\log P_i, \log Y_i) + u_i. \quad (20)$$

As h_0 is the log-demand function, any linear functional of demand is a nonlinear functional of h_0 . One example are weighted average demand functionals of the form

$$f_A(h) = \int w(\mathbf{p}) e^{h(\log \mathbf{p}, \log y)} d\mathbf{p}$$

where $w(\mathbf{p})$ is a non-negative weighting function and y is fixed. With $w(\mathbf{p}) = \mathbb{1}\{\underline{\mathbf{p}} \leq \mathbf{p} \leq \bar{\mathbf{p}}\}$, the functional $f(h)$ may be interpreted as approximate CS. The functional is defined for fixed y , so it will typically be an irregular functional of h_0 .

The setup is similar to the previous subsection. Let $\mathbf{X}_i = (\log P_i, \log Y_i)$. If both P_i and Y_i are endogenous, we take W_i to be a 2×1 vector of instruments and let $T : L^2(\mathbf{X}) \rightarrow L^2(W)$ be compact with SVD $\{\phi_{0j}, \phi_{1j}, \mu_j\}_{j=1}^\infty$. If P_i is endogenous but Y_i is exogenous, we take $W_i = (W_{1i}, \log Y_i)'$ where W_{1i} an instrument for P_i . Let $T_y : L^2(\log P | \log Y = \log y) \rightarrow L^2(W_1 | \log Y = \log y)$ be compact with SVD $\{\phi_{0j,y}, \phi_{1j,y}, \mu_{j,y}\}_{j=1}^\infty$ for each y where $\{\phi_{0j,y}\}_{j=1}^\infty$ and $\{\phi_{1j,y}\}_{j=1}^\infty$ are orthonormal bases for $L^2(\log P | \log Y = \log y)$ and $L^2(W_1 | \log Y = \log y)$, respectively. In this case, we define $\phi_{0j}(\log \mathbf{p}, \log y) = \phi_{0j,y}(\log \mathbf{p})$, $\phi_{1j}(w_1, \log y) = \phi_{1j,y}(w_1)$, and $\mu_j^2 = E[\mu_{j,Y_i}^2]$ (see Section 3.3 for further details). We again assume that Ψ_J and B_K are Riesz bases in that they span $\phi_{01}, \dots, \phi_{0J}$ and $\phi_{11}, \dots, \phi_{1K}$, respectively. For each $j \geq 1$, define

$$a_j = a_j(y) = \int w(\mathbf{p}) e^{h_0(\log \mathbf{p}, \log y)} \phi_{0j}(\log \mathbf{p}, \log y) d\mathbf{p}.$$

The next result follows from Theorem D.1 (in the online Appendix D) and extends results in Newey (1997) for the case without endogeneity to models with endogeneity. Let

$$\hat{\sigma}^2(f_A) = Df_A(\hat{h})[\psi^J]' \hat{U} Df_A(\hat{h})[\psi^J]$$

with

$$Df_A(\hat{h})[\psi^J] = \int w(\mathbf{p}) e^{\hat{h}(\log \mathbf{p}, \log y)} \psi^J(\log \mathbf{p}, \log y) d\mathbf{p}.$$

Theorem 5.3. *Let Assumption CS(i)–(iv) hold for the log-demand model (20) with $p > 0$ and let*

$$\frac{\sqrt{n}}{(\sum_{j=1}^J (a_j/\mu_j)^2)^{1/2}} \times \left(J^{-p/2} + \mu_J^{-2} \frac{J^{3/2} \sqrt{\log J}}{n} \right) = o(1)$$

and $(J^{(2+\delta)/(2\delta)}\sqrt{\log n} \vee \mu_J^{-1}J)/\sqrt{n} = o(1)$. Then:

$$\frac{\sqrt{n}(f_A(\hat{h}) - f_A(h_0))}{\hat{\sigma}(f_A)} \rightarrow_d N(0, 1).$$

5.3 Inference on welfare functionals without endogeneity

In this section we present sufficient conditions for asymptotic normality of sieve t -statistics for exact CS and DL and approximate CS functionals under exogeneity. This is a special case of the preceding analysis with $X_i = W_i$, $J = K$, $b^K = \psi^J$, $\tau_J = 1$ and the sieve NPIV estimator reduces to the usual series LS estimator (Newey, 1997; Huang, 1998). For the following results, we use our sup-norm rates for h_0 and its derivatives from Corollary 3.1 to verify conditions in Chen and Christensen (2015b) for asymptotic normality of sieve t statistics for nonlinear functionals of a nonparametric LS regression h_0 estimated via spline and wavelet LS.

Corollary 5.2. *Let Assumption CS(i)–(iv) hold with $X_i = W_i$, $J = K$, $b^K = \psi^J$ and $\mu_J \asymp 1$ and let $\sum_{j=1}^J a_j^2 \asymp J^{a+1}$ with $0 \geq a > -1$.*

(1) *Let $nJ^{-(p+a+1)} = o(1)$, $J^{3-a}(\log J)/n = o(1)$, and $\delta \geq 2/(2-a)$. Then: the sieve t -statistic for $f_{CS}(h_0)$ is asymptotically $N(0, 1)$.*

(2) *Let $nJ^{-(p+1)} = o(1)$, $J^3(\log J)/n = o(1)$, and $\delta \geq 1$. Then: the sieve t -statistic for $f_{DL}(h_0)$ is asymptotically $N(0, 1)$.*

Hausman and Newey (1995) established the pointwise asymptotic normality of t -statistics for exact CS and DL based on plug-in kernel LS estimators of demand without endogeneity. They also established root- n asymptotic normality of t -statistics for averaged exact CS and DL (i.e. CS/DL averaged over a range of incomes) based on plug-in power series LS estimator of demand without endogeneity, under some regularity conditions including that $\sup_x E[|u_i|^4 | X_i = x] < \infty$ (which, in our notation, implies $\delta = 2$), $p = \infty$ (i.e., h_0 is infinitely times differentiable) and $J^{22}/n = o(1)$. Corollary 5.2 complements their work by providing conditions for the pointwise asymptotic normality of exact CS and DL functionals based on spline and wavelet LS estimators of demand.

Corollary 5.3. *Let Assumption CS(i)–(iv) hold for the log-demand model (20) with $X_i = W_i$, $J = K$, $b^K = \psi^J$, $\mu_J \asymp 1$ and $p > 0$, let $\sum_{j=1}^J a_j^2 \asymp J^{c+1}$ with $c > -1$, $nJ^{-(p+c+1)} = o(1)$, let $(J^2 \vee J^{2-c}(\log J))/n = o(1)$, and $\delta > 2/(1 - (c \wedge 0))$. Then: the sieve t statistic for $f_A(h_0)$ is asymptotically $N(0, 1)$.*

Previously, Newey (1997) established the pointwise asymptotic normality of t -statistics for approximate CS functionals based on plug-in series LS estimators of exogenous demand under some regularity conditions including that $\sup_x E[|u_i|^4 | X_i = x] < \infty$ (which implies $\delta = 2$), $nJ^{-p} = o(1)$ and either $J^6/n = o(1)$ for power series or $J^4/n = o(1)$ for splines.

5.4 UCBs for nonparametric exact CS and DL functionals

Here we apply our results from Theorem 4.1 to present regularity conditions for UCBs for exact CS and DL functionals to have asymptotically correct coverage. We maintain the setup described at the beginning of Subsection 5.1. We take $t = (\mathbf{p}^0, \mathbf{p}^1, \mathbf{y}) \in \mathcal{T} = [\underline{\mathbf{p}}^0, \overline{\mathbf{p}}^0] \times [\underline{\mathbf{p}}^1, \overline{\mathbf{p}}^1] \times [\underline{\mathbf{y}}, \overline{\mathbf{y}}]$ where the intervals $[\underline{\mathbf{p}}^0, \overline{\mathbf{p}}^0]$ and $[\underline{\mathbf{p}}^1, \overline{\mathbf{p}}^1]$ are in the interior of the support of \mathbf{P}_i and $[\underline{\mathbf{y}}, \overline{\mathbf{y}}]$ is in the interior of the support of \mathbf{Y}_i . For each $t \in \mathcal{T}$ we define

$$a_{j,t} = a_{j,t}(\mathbf{p}^0, \mathbf{p}^1, \mathbf{y}) = \int_0^1 \left(\phi_{0j}(\mathbf{p}(u), \mathbf{y} - \mathbf{S}_y(\mathbf{p}(u))) e^{-\int_0^u \partial_2 h_0(\mathbf{p}(v), \mathbf{y} - \mathbf{S}_y(\mathbf{p}(v))) \mathbf{p}'(v) dv} \mathbf{p}'(u) \right) du \quad (21)$$

for each $j \geq 1$ (where $\mathbf{p}(u)$ is a smooth price path from $\mathbf{p}^0 = \mathbf{p}(0)$ to $\mathbf{p}^1 = \mathbf{p}(1)$). Also define $\underline{\sigma}_n = \inf_{t \in \mathcal{T}} ((\sum_{j=1}^J (a_{j,t}/\mu_j)^2)^{1/2})$.

Assumption U-CS (i) $E[\mathbf{u}_i^2 | \mathbf{W}_i = w]$ is uniformly bounded away from 0, $E[|\mathbf{u}_i|^{2+\delta}]$ is finite, and $\sup_w E[|\mathbf{u}_i|^3 | \mathbf{W}_i = w]$ is finite; (ii) the Lipschitz condition in Remark 4.2 holds with $\gamma_n = \gamma$ and $\Gamma_n \leq L \underline{\sigma}_n$ for some finite positive constants γ and L ; (iii) $(J^{(2+\delta)/(2\delta)} \vee J^{3/2} \mu_J^{-1}) \sqrt{(\log n)/n} = o(1)$, $\log(\mu_J^{-1}) \times ((J^{(2+\delta)/2\delta} \sqrt{(\log n)/n})^{\delta/(1+\delta)} \vee \mu_J^{-1} \sqrt{J(\log J)/n} \vee J^{-p/2}) = o(1)$, and $J^{5/2}(\log(\mu_J^{-1}) \times \log \log n)^{3/2} / \sqrt{n} = o(1)$, and

$$\frac{\sqrt{n \log(\mu_J^{-1})}}{\underline{\sigma}_n} \times \left(J^{-p/2} + \mu_J^{-2} \frac{J^2 (\sqrt{J} \vee \sqrt{\log \mu_J^{-1}}) \sqrt{\log J}}{n} \right) = o(1).$$

Assumption U-CS (i) and (iii) are mild strengthenings of Assumptions CS(iii) and (v) respectively. Assumption U-CS(ii) (the Lipschitz condition) is made for simplicity to verify Assumption 6. This is but one sufficient condition for Assumption 6; Remark 4.2 can be used to derive others.

The next results are about the validity of score bootstrap UCBs for exact CS and DL functionals.

Theorem 5.4. *Let Assumptions CS(i)(ii)(iv) and U-CS hold. Then:*

$$\sup_{s \in \mathbb{R}} \left| \mathbb{P} \left(\sup_{t \in \mathcal{T}} \left| \frac{\sqrt{n}(f_{CS,t}(\hat{h}) - f_{CS,t}(h_0))}{\hat{\sigma}(f_{CS,t})} \right| \leq s \right) - \mathbb{P}^* \left(\sup_{t \in \mathcal{T}} |Z_n^*(t)| \leq s \right) \right| = o_p(1).$$

In the next theorem the condition $\underline{\sigma}_n \asymp \mu_J^{-1} \sqrt{J}$ is implied by the assumption that $\sigma_n(f_{DL,t}) \asymp \mu_J^{-1} \sqrt{J}$ uniformly for $t \in \mathcal{T}$, which is reasonable for the DL functional.

Theorem 5.5. *Let Assumptions CS(i)(ii)(iv) and U-CS hold with $\log J \lesssim \log(\mu_J^{-1})$ and $\underline{\sigma}_n \asymp \mu_J^{-1} \sqrt{J}$. Then:*

$$\sup_{s \in \mathbb{R}} \left| \mathbb{P} \left(\sup_{t \in \mathcal{T}} \left| \frac{\sqrt{n}(f_{DL,t}(\hat{h}) - f_{DL,t}(h_0))}{\hat{\sigma}(f_{DL,t})} \right| \leq s \right) - \mathbb{P}^* \left(\sup_{t \in \mathcal{T}} |Z_n^*(t)| \leq s \right) \right| = o_p(1).$$

6 Conclusion

This paper makes several important contributions to inference on nonparametric models with endogeneity. We derive the minimax sup-norm convergence rates for estimating the structural NPIV function h_0 and its derivatives. We also provide upper bounds for sup-norm convergence rates of computationally simple sieve NPIV (series 2SLS) estimators using any sieve basis to approximate unknown h_0 , and show that the sieve NPIV estimator using spline or wavelet basis can attain the minimax sup-norm rates. These rate results are particularly useful for establishing validity of pointwise and uniform inference procedures for nonlinear functionals of h_0 . In particular, we use our sup-norm rates to establish the uniform Gaussian process strong approximation and the validity of score bootstrap-based UCBs for collections of nonlinear functionals of h_0 under primitive conditions, allowing for mildly and severely ill-posed problems. We illustrate the usefulness of our UCBs procedure with two real data applications to nonparametric demand analysis with endogeneity. We establish the pointwise and uniform limit theories for sieve t -statistics for exact (and approximate) CS and DL nonlinear functionals under low-level conditions when the demand function is estimated via sieve NPIV. Our theoretical and empirical results for CS and DL are new additions to the literature on nonparametric welfare analysis.

We conclude the paper by mentioning some further extensions and applications of sup-norm convergence rates of sieve NPIV estimators.

Extensions to semiparametric IV models. Although our rate results are presented for purely nonparametric IV models, the results may be adapted easily to some semiparametric models with nonparametric endogeneity, such as partially linear IV regression (Ai and Chen, 2003; Florens, Johannes, and Van Bellegem, 2012), shape-invariant Engel curve IV regression (Blundell et al., 2007), and single index IV regression (Chen et al., 2014), to list a few. For example, consider the partially linear NPIV model

$$Y_i = X'_{1i}\beta_0 + h_0(X_{2i}) + u_i \quad E[u_i|W_{1i}, W_{2i}] = 0$$

where X_{1i} and X_{2i} are of dimensions d_1 and d_2 and do not contain elements in common, and $W_i = (W_{1i}, W_{2i})$ is the (conditional) IV. See Florens et al. (2012); Chen et al. (2014) for identification of (β_0, h_0) in this model. We can still estimate (β_0, h_0) via sieve NPIV or series 2SLS as before, replacing Ψ and B in equations (3)–(4) by:

$$\begin{aligned} \psi^J(x) &= (x'_1, \psi^J_2(x_2)')' & \psi^J_2(x) &= (\psi_{J1}(x_2), \dots, \psi_{JJ}(x_2))' \\ b^K(w) &= (w'_1, b^K_2(w_2)')' & b^K_2(w) &= (b_{K1}(w_2), \dots, b_{KK}(w_2))' \end{aligned}$$

where $x = (x'_1, x'_2)'$, $w = (w'_1, w'_2)'$, $\psi_{J1}, \dots, \psi_{JJ}$ denotes a sieve of dimension J for approximating $h_0(x_2)$ and b_{K1}, \dots, b_{KK} denotes a sieve of dimension K for the instrument space for W_2 . We then partition \hat{c} in (2) into $\hat{c} = (\hat{\beta}', \hat{c}'_2)'$ and set $\hat{h}(x) = \psi^J_2(x)'\hat{c}_2$. Note that $\hat{\beta}$ is root- n consistent and asymptotically normal for β_0 under mild conditions (see Ai and Chen (2003); Chen and Pouzo (2009)), and hence would not affect the optimal convergence rate of \hat{h} to h_0 . Our rate results may be slightly altered to derive sup-norm convergence rates for \hat{h} and its derivatives.

Nonparametric specification testing in NPIV models. Structural models may specify a parametric form $m_{\theta_0}(x)$ where $\theta_0 \in \Theta \subseteq \mathbb{R}^{d_\theta}$ for the unknown structural function $h_0(x)$ in NPIV model (1). We may be interested in testing the parametric model $\{m_\theta : \theta \in \Theta\}$ against a nonparametric alternative that only assumes some smoothness on h_0 . Specification tests for nonparametric regression without endogeneity have typically been performed via either a quadratic-form-based statistic or a Kolmogorov-Smirnov (KS) type sup statistic.²⁰ However, specification tests for NPIV models have so far only been performed via quadratic-form-based statistics; see, e.g., Horowitz (2006, 2011, 2012); Blundell and Horowitz (2007); Breunig (2015). Equipped with our sup-norm rate and UCBs results for NPIV function and its derivatives, one could also perform specification tests in NPIV models using KS type statistics of the form

$$T_n = \sup_x \frac{|\widehat{h}(x) - \widehat{m}(x, \widehat{\theta})|}{s_n(x)}$$

where $\widehat{\theta}$ is a first-stage estimator of θ_0 , and $\widehat{m}(x, \widehat{\theta})$ is obtained from series 2SLS regression of $m(X_1, \widehat{\theta}), \dots, m(X_n, \widehat{\theta})$ on the same basis functions as in \widehat{h} , and $s_n(x)$ is a normalization factor. Alternatively, one could consider a KS statistic formed in terms of the projection of $[\widehat{h}(x) - \widehat{m}(x, \widehat{\theta})]$ onto the instrument space. Sup-norm convergence rates and uniform limit theory derived in this paper would be useful in deriving the large-sample distribution of these KS type statistics. Further, based on our rate results (in sup- and L^2 -norm) for estimating derivatives of h_0 in a NPIV model, one could also perform nonparametric tests of significance by testing whether partial derivatives of NPIV function h_0 are identically zero, via KS or quadratic-form-based test statistics.

If one is interested in specifications or inferences on functionals directly, then one might consider KS type sup statistics for (possibly nonlinear) functionals directly. For example, if one is interested in exact CS functional of a demand and concerns about the potential endogeneity of price. Then one could estimate exact CS functional using a series LS estimated demand (under exogeneity) and series 2SLS estimated demand (under endogeneity), and then compare the two estimated exact CS functionals via KS type or quadratic-form-based test. In fact, the score bootstrap-based UCBs reported in Figure 2 indicates that such a test based on exact CS functional directly could be quite informative.

Semiparametric 2-step procedures with NPIV first stage. Many semiparametric two-step or multi-step estimation and inference procedures involve a nonparametric first stage. There are many theoretical results when the first stage is a purely nonparametric LS regression (without endogeneity) and its sup-norm convergence rate is used to assist subsequent analysis. For structural estimation and inference, it is natural to allow for the presence of nonparametric endogeneity in the first stage as well. For instance, if there is endogeneity present in the conditional moment inequality application of the famous intersection bound paper of Chernozhukov et al. (2013), one could simply use our sup-norm rate and UCBs results for sieve NPIV instead of their series LS regression in the first stage. As another

²⁰See, e.g., Bierens (1982), Hardle and Mammen (1993), Hong and White (1995), Fan and Li (1996), Lavergne and Vuong (1996), Stinchcombe and White (1998) and Horowitz and Spokoiny (2001) to list a few.

example, consider semiparametric two-step GMM models

$$E[g(Z_i, \theta_0, h_0(X_i))] = 0$$

where the finite-dimensional parameter of interest $\theta_0 \in \Theta \subseteq \mathbb{R}^{d_\theta}$, g is a \mathbb{R}^{d_g} -valued vector of moment functions with $d_g \geq d_\theta$, h_0 is the NPIV structural function and Z_i is an additional vector of random variables, possibly including W_i , X_i and Y_i . The estimator $\hat{\theta}$ of θ_0 is obtained as a solution to the semiparametric two-step GMM problem

$$\min_{\theta \in \Theta} \left(\frac{1}{n} \sum_{i=1}^n g(Z_i, \theta, \hat{h}(X_i)) \right)' \widehat{W} \left(\frac{1}{n} \sum_{i=1}^n g(Z_i, \theta, \hat{h}(X_i)) \right)$$

where \hat{h} is a sieve NPIV estimator of h_0 in model (1) and \widehat{W} is a weighting matrix. As an economic example related to Sections 2 and 5, one might be interested in estimating the average exact CS parameter θ_0 of a nonparametric demand function with endogeneity:

$$\theta_0 = \mathbb{E}[w(Y_i)S_{Y_i}(\mathbf{p}^0)], \quad (22)$$

which is the average CS functional associated with a price change from \mathbf{p}^0 to \mathbf{p}^1 over a range of incomes $\underline{y} \leq y \leq \bar{y}$. Here $w(y) > 0$ is a known weight function, and the expectation is taken wrt the income Y_i . The moment function is

$$g(Z_i, \theta, h(X_i)) = \theta - w(Y_i)S_{Y_i}(\mathbf{p}^0; h)$$

where $S_{Y_i}(\mathbf{p}^0; h)$ denotes the solution to (10) with h in place of h_0 and $\mathbf{y} = Y_i$. When h_0 enters the moment function $g(\cdot)$ nonlinearly, sup-norm convergence rates of \hat{h} to h_0 are useful in verifying the regularity conditions controlling the nonlinear remainder term (e.g., Assumption A.2(iii) in Chen and Liao (2015)), in order to derive an asymptotic distribution of $\hat{\theta}$ to θ_0 .

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A Bounds on sup-norm bias term

Before we provide a bound on the sup-norm “bias” term, we present some sufficient conditions for Assumption 4(iii). This involves three projections of h_0 onto the sieve approximating space Ψ_J . These projections imply different, but closely related, approximation biases for h_0 . Recall that $\Pi_J : L^2(X) \rightarrow \Psi_J$ is the $L^2(X)$ orthogonal (i.e. least squares) projection onto Ψ_J , namely $\Pi_J h_0 = \arg \min_{h \in \Psi_J} \|h_0 - h\|_{L^2(X)}$, and $Q_J h_0 = \arg \min_{h \in \Psi_J} \|\Pi_K T(h_0 - h)\|_{L^2(W)}$ is the sieve 2SLS projection of h_0 onto Ψ_J . Let $\pi_J h_0 = \arg \min_{h \in \Psi_J} \|T(h_0 - h)\|_{L^2(W)}$ denote the IV projection of h_0 onto Ψ_J . Note that each of these projections are non-random.

Instead of Assumption 4(iii), we could impose:

Assumption 4 (iii') $(\zeta_{\psi, J\tau_J}) \times \|(\Pi_K T - T)(Q_J h_0 - \pi_J h_0)\|_{L^2(W)} \leq \text{const} \times \|Q_J h_0 - \pi_J h_0\|_{L^2(X)}$.

Assumption 4(iii') seems mild and is automatically satisfied by Riesz basis. This is because $\|(\Pi_K T - T)h\|_{L^2(W)} = 0$ for all $h \in \Psi_J$ when the basis functions for B_K and Ψ_J form either a Riesz basis or eigenfunction basis for the conditional expectation operator. The following lemma collects some useful facts about the approximation properties of $\pi_J h_0$.

Lemma A.1. *Let Assumptions 1(iii) and 4(ii) hold. Then:*

- (1) $\|h_0 - \pi_J h_0\|_{L^2(X)} \asymp \|h_0 - \Pi_J h_0\|_{L^2(X)}$;
- (2) *If Assumption 4(i) also holds, then:* $\|Q_J h_0 - \pi_J h_0\|_{L^2(X)} \leq o(1) \times \|h_0 - \pi_J h_0\|_{L^2(X)}$.
- (3) *Further, if Assumption 4(iii') and*

$$\|\Pi_J h_0 - \pi_J h_0\|_\infty \leq \text{const} \times \|h_0 - \Pi_J h_0\|_\infty \quad (23)$$

hold then Assumption 4(iii) is satisfied.

In light of Lemma A.1 parts (1) and (2), Condition (23) seems mild. In fact, Condition (23) is trivially satisfied when the basis for Ψ_J is a Riesz basis because then $\pi_J h_0 = \Pi_J h_0$ (see section 6 in [Chen and Pouzo \(2015\)](#)). See Lemma G.2 in the online Appendix G for more detailed relations among $\Pi_J h_0$, $\pi_J h_0$ and $Q_J h_0$.

The next lemma provides a bound on the sup-norm ‘‘bias’’ term.

Lemma A.2. *Let Assumptions 1(iii), 3(ii) and 4 hold. Then:*

- (1) $\|\tilde{h} - \Pi_J h_0\|_\infty \leq O_p(1) \times \|h_0 - \Pi_J h_0\|_\infty$.
- (2) $\|\tilde{h} - h_0\|_\infty \leq O_p(1 + \|\Pi_J\|_\infty) \times \|h_0 - h_{0,J}\|_\infty$.

B Optimal L^2 -norm rates for derivatives

Here we show that the sieve NPIV estimator can attain the optimal L^2 -norm convergence rates for estimating h_0 and its derivatives under much weaker conditions. The optimal L^2 -norm rates for sieve NPIV derivative estimation presented in this section are new, and should be very useful for inference on some nonlinear functionals involving derivatives such as $f(h) = \|\partial^\alpha h\|_{L^2(X)}^2$.

Instead of Assumption 1(iii), we impose the following condition for identification in $(\mathcal{H}, \|\cdot\|_{L^2(X)})$:

Assumption 1 (iii') $h_0 \in \mathcal{H} \subset L^2(X)$, and $T[h - h_0] = 0 \in L^2(W)$ for any $h \in \mathcal{H}$ implies that $\|h - h_0\|_{L^2(X)} = 0$.

Theorem B.1. *Let Assumptions 1(iii') and 4(i)(ii) hold and let $\tau_J \zeta \sqrt{(\log J)/n} = o(1)$. Then:*

- (1) $\|\tilde{h} - h_0\|_{L^2(X)} \leq O_p(1) \times \|h_0 - \Pi_J h_0\|_{L^2(X)}$.
- (2) *Further, if Assumption 2(i) holds then*

$$\|\hat{h} - h_0\|_{L^2(X)} = O_p\left(\|h_0 - \Pi_J h_0\|_{L^2(X)} + \tau_J \sqrt{J/n}\right).$$

The following corollary provides concrete L^2 norm convergence rates of \hat{h} and its derivatives. Let $B_{2,2}^p$ denote the Sobolev space of smoothness $p > 0$, $\|\cdot\|_{B_{2,2}^p}$ denote a Sobolev norm of smoothness p , and $B_2(p, L) = \{h \in B_{2,2}^p : \|h\|_{B_{2,2}^p} \leq L\}$ where radius $0 < L < \infty$ ([Triebel, 2006](#), Section 1.11).

Corollary B.1. *Let Assumptions 1(i)(ii)(iii') and 4(i)(ii) hold. Let $h_0 \in B_2(p, L)$, Ψ_J be spanned by a cosine basis, B-spline basis of order $\gamma > p$, or CDV wavelet basis of regularity $\gamma > p$, B_K be spanned by a cosine, spline, or wavelet basis. Let $\tau_J \sqrt{(J \log J)/n} = o(1)$ hold. Then:*

(1) $\|\partial^\alpha \tilde{h} - \partial^\alpha h_0\|_{L^2(X)} = O_p(J^{-(p-|\alpha|)/d})$ for all $0 \leq |\alpha| < p$.

(2) Further if Assumption 2(i) holds, then

$$\|\partial^\alpha \hat{h} - \partial^\alpha h_0\|_{L^2(X)} = O_p\left(J^{-(p-|\alpha|)/d} + \tau_J J^{|\alpha|/d} \sqrt{J/n}\right) \quad \text{for all } 0 \leq |\alpha| < p.$$

(2.a) *Mildly ill-posed case: choosing $J \asymp n^{d/(2(p+\varsigma)+d)}$ yields $\tau_J \sqrt{(J \log J)/n} = o(1)$ and*

$$\|\partial^\alpha \hat{h} - \partial^\alpha h_0\|_{L^2(X)} = O_p(n^{-(p-|\alpha|)/(2(p+\varsigma)+d)}).$$

(2.b) *Severely ill-posed case: choosing $J = (c_0 \log n)^{d/\varsigma}$ for any $c_0 \in (0, 1)$ yields $\tau_J \sqrt{(J \log J)/n} = o(1)$ and*

$$\|\partial^\alpha \hat{h} - \partial^\alpha h_0\|_{L^2(X)} = O_p((\log n)^{-(p-|\alpha|/\varsigma)}).$$

The conclusions of Corollary B.1 remain true for an arbitrary basis B_K under the condition $\tau_J \zeta_b \sqrt{(\log J)/n} = o(1)$. Previously, assuming some rates on estimating the unknown operator T , Johannes, van Bellegem, and Vanhems (2011) obtained similar L^2 -norm rates for derivatives of iteratively Tikhonov-regularized estimators in a NPIV model with scalar regressor X_i and scalar instrument W_i .

Our next theorem shows that the rates obtained in Corollary B.1 are optimal. It extends the earlier work by Chen and Reiss (2011) on L^2 -norm lower-bounds for h_0 to lower bounds for derivative estimation.

Theorem B.2. *Let Condition LB hold with $B_2(p, L)$ in place of $B_\infty(p, L)$ for the NPIV model with a random sample $\{(X_i, Y_i, W_i)\}_{i=1}^n$. Then for any $0 \leq |\alpha| < p$:*

$$\liminf_{n \rightarrow \infty} \inf_{\hat{g}_n} \sup_{h \in B_2(p, L)} \mathbb{P}_h \left(\|\hat{g}_n - \partial^\alpha h\|_{L^2(X)} \geq cn^{-(p-|\alpha|)/(2(p+\varsigma)+d)} \right) \geq c' > 0$$

in the mildly ill-posed case, and

$$\liminf_{n \rightarrow \infty} \inf_{\hat{g}_n} \sup_{h \in B_2(p, L)} \mathbb{P}_h \left(\|\hat{g}_n - \partial^\alpha h\|_{L^2(X)} \geq c(\log n)^{-(p-|\alpha|/\varsigma)} \right) \geq c' > 0$$

in the severely ill-posed case, where $\inf_{\hat{g}_n}$ denotes the infimum over all estimators of $\partial^\alpha h$ based on the sample of size n , $\sup_{h \in B_2(p, L)} \mathbb{P}_h$ denotes the sup over $h \in B_2(p, L)$ and distributions of (X_i, W_i, u_i) that satisfy Condition LB with ν fixed, and the finite positive constants c, c' do not depend on n .

C Lower bounds for quadratic functionals

In this section we study quadratic functionals of the form

$$f(h) = \int (\partial^\alpha h(x))^2 \mu(x) dx$$

where $\mu(x) \geq \underline{\mu} > 0$ is a positive weighting function. These functionals are very important for non-parametric specification and goodness-of-fit testing, as outlined in the conclusion section. We derive lower bounds on convergence rates of estimators of the functional $f(h_0)$.

Theorem C.1. *Let Condition LB hold with $B_2(p, L)$ in place of $B_\infty(p, L)$ for the NPIV model with a random sample $\{(X_i, Y_i, W_i)\}_{i=1}^n$. Then for any $0 \leq |\alpha| < p$:*

$$\liminf_{n \rightarrow \infty} \inf_{\hat{g}_n} \sup_{h \in B_2(p, L)} \mathbb{P}_h (|\hat{g}_n - f(h)| > cr_n) \geq c' > 0$$

where

$$r_n = \begin{cases} n^{-1/2} & \text{in the mildly ill-posed case when } p \geq \varsigma + 2|\alpha| + d/4 \\ n^{-4(p-|\alpha|)/(4(p+\varsigma)+d)} & \text{in the mildly ill-posed case when } \varsigma < p < \varsigma + 2|\alpha| + d/4 \\ (\log n)^{-2(p-|\alpha|)/\varsigma} & \text{in the severely ill-posed case,} \end{cases}$$

$\inf_{\hat{g}_n}$ denotes the infimum over all estimators of $f(h_0)$ based on the sample of size n , $\sup_{h \in B_2(p, L)} \mathbb{P}_h$ denotes the sup over $h \in B_2(p, L)$ and distributions (X_i, W_i, u_i) which satisfy Condition LB with ν fixed, and the finite positive constants c, c' do not depend on n .

In the mildly ill-posed case, Theorem C.1 shows that the rate exhibits a so-called elbow phenomenon, in which $f(h_0)$ is \sqrt{n} -estimable when $p \geq \varsigma + 2|\alpha| + d/4$ and irregular otherwise. Moreover, $f(h_0)$ is always irregular in the severely ill-posed case.

Consider estimation using the plug-in estimator $f(\hat{h})$. Expanding the quadratic, we see that

$$f(\hat{h}) - f(h_0) = \int \partial^\alpha h_0(x) (\partial^\alpha \hat{h}(x) - \partial^\alpha h_0(x)) \mu(x) dx + \|\partial^\alpha \hat{h} - \partial^\alpha h_0\|_{L^2(\mu)}^2.$$

Under appropriate normalization, the first term on the right-hand side will be the ‘‘CLT term’’. Consider the quadratic remainder term. Since μ is bounded away from zero and the density of X_i is bounded away from zero and infinity, the quadratic remainder term behaves like $\|\partial^\alpha \hat{h} - \partial^\alpha h_0\|_{L^2(X)}^2$. In the mildly ill-posed case, the optimal convergence rate of this term has been shown to be $O_p(n^{-2(p-|\alpha|)/(2(p+\varsigma)+d)})$ (see Appendix B). This term vanishes faster than $n^{-1/2}$ provided that $p > \varsigma + 2|\alpha| + d/2$, which is a stronger condition than required for $f(h_0)$ to be \sqrt{n} -estimable. Therefore, when $\varsigma + 2|\alpha| + d/4 < p < \varsigma + 2|\alpha| + d/2$, then functional $f(h_0)$ is \sqrt{n} -estimable but the quadratic remainder term of plug-in estimators will dominate.

Online Appendix to

Optimal Sup-norm Rates and Uniform Inference on Nonlinear Functionals of Nonparametric IV Regression

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This online supplementary appendix contains material to support our paper “Optimal Sup-norm Rates and Uniform Inference on Nonlinear Functionals of Nonparametric IV Regression”. Appendix D presents pointwise normality of sieve t statistics for nonlinear functionals of NPIV under low-level sufficient conditions. Appendix E contains background material on B-spline and wavelet bases and the equivalence between Besov and wavelet sequence norms. Appendix F contains material on useful matrix inequalities and convergence results for random matrices. Appendix G contains additional technical lemmas and all of the proofs.

D Pointwise asymptotic normality of sieve t -statistics

In this section we derive the pointwise asymptotic normality of sieve t -statistics for nonlinear functionals of a NPIV function under low-level sufficient conditions. Previously under some high-level conditions, [Chen and Pouzo \(2015\)](#) established the pointwise asymptotic normality of sieve t statistics for (possibly) nonlinear functionals of h_0 satisfying general semi/nonparametric conditional moment restrictions including NPIV and nonparametric quantile IV models as special cases. As the sieve NPIV estimator \hat{h} has a closed-form expression and for the sake of easy reference, we derive the limit theory directly rather than appealing to the general theory in [Chen and Pouzo \(2015\)](#). Our low-level sufficient conditions are tailored to the case in which the functional $f(\cdot)$ is *irregular* in h_0 (i.e. slower than root- n estimable), so that they are directly comparable to the sufficient conditions for the uniform inference theory in Section 4.

We consider a functional $f : \mathcal{H} \subset L^\infty(X) \rightarrow \mathbb{R}$ for which $Df(h)[v] = \lim_{\delta \rightarrow 0^+} [\delta^{-1} f(h + \delta v)]$ exists for all $v \in \mathcal{H} - \{h_0\}$ for all h in a small neighborhood of h_0 . Recall that the sieve 2SLS Riesz representer of $Df(h_0)$ is

$$v_n(f)(x) = \psi^J(x)' [S' G_b^{-1} S]^{-1} Df(h_0) [\psi^J] ,$$

and let

$$[s_n(f)]^2 = \|\Pi_K T v_n(f)\|_{L^2(W)}^2 = (Df(h_0) [\psi^J])' [S' G_b^{-1} S]^{-1} Df(h_0) [\psi^J]$$

denote its weak norm. [Chen and Pouzo \(2015\)](#) called that the functional $f(\cdot)$ is an irregular (i.e. slower than \sqrt{n} -estimable) functional of h_0 if $s_n(f) \nearrow \infty$ and a regular (i.e. \sqrt{n} -estimable) functional of h_0 if $\lim_n s_n(f) < \infty$. Denote

$$\hat{v}_n(f)(x) = \psi^J(x)' [S' G_b^{-1} S]^{-1} Df(\hat{h}) [\psi^J] .$$

It is clear that $v_n(f) = \hat{v}_n(f)$ whenever $f(\cdot)$ is linear.

Recall that $\Omega = E[u_i^2 b^K(W_i) b^K(W_i)']$, and the ‘‘2SLS covariance matrix’’ for \hat{c} (given in equation (2)) is

$$\mathbb{U} = [S' G_b^{-1} S]^{-1} S' G_b^{-1} \Omega G_b^{-1} S [S' G_b^{-1} S]^{-1} ,$$

and the sieve variance for $f(\hat{h})$ is

$$[\sigma_n(f)]^2 = (Df(h_0)[\psi^J])' \mathbb{U} (Df(h_0)[\psi^J]) .$$

Under Assumption 2(i)(iii) we have that $[\sigma_n(f)]^2 \asymp [s_n(f)]^2$. Therefore $f(\cdot)$ is an irregular functional of h_0 iff $\sigma_n(f) \nearrow +\infty$ as $n \rightarrow \infty$. Recall the sieve variance estimator is

$$[\hat{\sigma}(f)]^2 = (Df(\hat{h})[\psi^J])' \hat{\mathbb{U}} (Df(\hat{h})[\psi^J]) .$$

where $\hat{\mathbb{U}}$ is defined in equation (6).

Assumption 2 (continued). (iv') $\sup_w E[u_i^2 \{ |u_i| > \ell(n) \} | W_i = w] = o(1)$ for any positive sequence with $\ell(n) \nearrow \infty$.

Assumption 2(iv') is a mild condition which is trivially satisfied if $E[|u_i|^{2+\epsilon} | W_i = w]$ is uniformly bounded for some $\epsilon > 0$.

Assumption 5' Assumption 5 holds with $f_t = f$ and \mathcal{T} a singleton.

Assumption 5'(a) and 5'(b)(i)(ii) is similar to Assumption 3.5 of Chen and Pouzo (2015). Assumption 5'(b)(iii) controls any additional error arising in the estimation of $\sigma_n(f)$ due to nonlinearity of $f(\cdot)$ and is automatically satisfied when $f(\cdot)$ is a linear functional.

Remark D.1. Remark 4.1 presents sufficient conditions for Assumption 5' as a special case, with $f_t = f$, $\underline{\sigma}_n = \sigma_n(f)$, and \mathcal{T} a singleton.

Again these sufficient conditions are formulated to take advantage of the sup-norm rate results in Section 3. Denote

$$\hat{\mathbb{Z}}_n = \frac{(Df(h_0)[\psi^J])' [S' G_b^{-1} S]^{-1} S' G_b^{-1}}{\sigma_n(f)} \frac{1}{\sqrt{n}} \sum_{i=1}^n b^K(W_i) u_i .$$

Theorem D.1. (1) Let Assumptions 1(iii), 2(i)(iii)(iv'), 4(i), and either 5'(a) or 5'(b)(i)(ii) hold, and let $\tau_J \zeta \sqrt{(J \log J)/n} = o(1)$. Then:

$$\sqrt{n} \frac{(f(\hat{h}) - f(h_0))}{\sigma_n(f)} = \hat{\mathbb{Z}}_n + o_p(1) \rightarrow_d N(0, 1) .$$

(2) If $\|\hat{h} - h_0\|_\infty = o_p(1)$ and Assumptions 2(ii) and 3(iii) hold (and 5'(b)(iii) also holds if f is nonlinear), then:

$$\left| \frac{\hat{\sigma}(f)}{\sigma_n(f)} - 1 \right| = o_p(1)$$

and

$$\sqrt{n} \frac{(f(\hat{h}) - f(h_0))}{\hat{\sigma}(f)} = \hat{\mathbb{Z}}_n + o_p(1) \rightarrow_d N(0, 1) .$$

By exploiting the closed form expression of the sieve NPIV estimator and by applying exponential inequalities for random matrices, Theorem D.1 derives the pointwise limit theory under lower-level sufficient conditions than those in Chen and Pouzo (2015) for irregular nonlinear functionals. One could also derive the pointwise asymptotic normality of sieve t -statistics for regular nonlinear functionals of a NPIV function under lower-level sufficient conditions by using our sup-norm rates results to verify Assumption 3.5(ii) and Remark 3.1 in Chen and Pouzo (2015).

E Spline and wavelet bases

In this section we bound the terms $\xi_{\psi,J}$, $e_J = \lambda_{\min}(G_{\psi,J})$ and $\kappa_{\psi}(J)$ for B-spline and CDV wavelet bases. Although we state the results for the space Ψ_J , they may equally be applied to B_K when B_K is constructed using B-spline or CDV wavelet bases.

E.1 Spline bases

We construct a univariate B-spline basis of order $r \geq 1$ (or degree $r - 1 \geq 0$) with $m \geq 0$ interior knots and support $[0, 1]$ in the following way. Let $0 = t_{-(r-1)} = \dots = t_0 \leq t_1 \leq \dots \leq t_m \leq t_{m+1} = \dots = t_{m+r} = 1$ denote the extended knot sequence and let $I_1 = [t_0, t_1), \dots, I_m = [t_m, t_{m+1}]$. A basis of order 1 is constructed by setting

$$N_{j,1}(x) = \begin{cases} 1 & \text{if } x \in I_j \\ 0 & \text{otherwise} \end{cases}$$

for $j = 0, \dots, m$. Bases of order $r > 1$ are generated recursively according to

$$N_{j,r}(x) = \frac{x - t_j}{t_{j+r-1} - t_j} N_{j,r-1}(x) + \frac{t_{j+r} - x}{t_{j+r} - t_{j+1}} N_{j+1,r-1}(x)$$

for $j = -(r-1), \dots, m$ where we adopt the convention $\frac{1}{0} := 0$ (see Section 5 of DeVore and Lorentz (1993)). This results in a total of $m + r$ splines of order r , namely $N_{-(r-1),r}, \dots, N_{m,r}$. Each spline is a polynomial of degree $r - 1$ on each interior interval I_1, \dots, I_m and is $(r - 2)$ -times continuously differentiable on $[0, 1]$ whenever $r \geq 2$. The mesh ratio is defined as

$$\text{mesh}(m) = \frac{\max_{0 \leq j \leq m} (t_{j+1} - t_j)}{\min_{0 \leq j \leq m} (t_{j+1} - t_j)}.$$

Clearly $\text{mesh}(m) = 1$ whenever the knots are placed evenly (i.e. $t_i = \frac{i}{m+1}$ for $i = 1, \dots, m$ and $m \geq 1$) and we say that the mesh ratio is *uniformly bounded* if $\text{mesh}(m) \lesssim 1$ as $m \rightarrow \infty$. Each of has continuous derivatives of orders $\leq r - 2$ on $(0, 1)$. We let the space $\text{BSpl}(r, m, [0, 1])$ be the closed linear span of the $m + r$ splines $N_{-(r-1),r}, \dots, N_{m,r}$.

We construct B-spline bases for $[0, 1]^d$ by taking tensor products of univariate bases. First generate d univariate bases $N_{-(r-1),r,i}, \dots, N_{m,r,i}$ for each of the d components x_i of x as described above. Then form the vector of

basis functions ψ^J by taking the tensor product of the vectors of univariate basis functions, namely:

$$\psi^J(x_1, \dots, x_d) = \bigotimes_{i=1}^d \begin{pmatrix} N_{-(r-1),r,i}(x_i) \\ \vdots \\ N_{m,r,i}(x_i) \end{pmatrix}.$$

The resulting vector ψ^J has dimension $J = (r + m)^d$. Let $\psi_{J1}, \dots, \psi_{JJ}$ denote its J elements.

Stability properties: The following two Lemmas bound $\xi_{\psi,J}$, and the minimum eigenvalue and condition number of $G_\psi = G_{\psi,J} = E[\psi^J(X_i)\psi^J(X_i)']$ when $\psi_{J1}, \dots, \psi_{JJ}$ is constructed using univariate and tensor-products of B-spline bases with uniformly bounded mesh ratio.

Lemma E.1. *Let X have support $[0, 1]$ and let $\psi_{J1} = N_{-(r-1),r}, \dots, \psi_{JJ} = N_{m,r}$ be a univariate B-spline basis of order $r \geq 1$ with $m = J - r \geq 0$ interior knots and uniformly bounded mesh ratio. Then: (a) $\xi_{\psi,J} = 1$ for all $J \geq r$; (b) If the density of X is uniformly bounded away from 0 and ∞ on $[0, 1]$, then there exists finite positive constants c_ψ and C_ψ such that $c_\psi J \leq \lambda_{\max}(G_\psi)^{-1} \leq \lambda_{\min}(G_\psi)^{-1} \leq C_\psi J$ for all $J \geq r$; (c) $\lambda_{\max}(G_\psi)/\lambda_{\min}(G_\psi) \leq C_\psi/c_\psi$ for all $J \geq r$.*

Lemma E.2. *Let X have support $[0, 1]^d$ and let $\psi_{J1}, \dots, \psi_{JJ}$ be a B-spline basis formed as the tensor product of d univariate bases of order $r \geq 1$ with $m = J^{1/d} - r \geq 0$ interior knots and uniformly bounded mesh ratio. Then: (a) $\xi_{\psi,J} = 1$ for all $J \geq r^d$; (b) If the density of X is uniformly bounded away from 0 and ∞ on $[0, 1]^d$, then there exists finite positive constants c_ψ and C_ψ such that $c_\psi J \leq \lambda_{\max}(G_\psi)^{-1} \leq \lambda_{\min}(G_\psi)^{-1} \leq C_\psi J$ for all $J \geq r^d$; (c) $\lambda_{\max}(G_\psi)/\lambda_{\min}(G_\psi) \leq C_\psi/c_\psi$ for all $J \geq r^d$.*

E.2 Wavelet bases

We construct a univariate wavelet basis with support $[0, 1]$ following [Cohen, Daubechies, and Vial \(1993\)](#) (CDV hereafter). Let (φ, ψ) be a Daubechies pair such that φ has support $[-N + 1, N]$. Given j such that $2^j - 2N > 0$, the orthonormal (with respect to the $L^2([0, 1])$ inner product) basis for the space V_j includes $2^j - 2N$ interior scaling functions of the form $\varphi_{j,k}(x) = 2^{j/2}\varphi(2^j x - k)$, each of which has support $[2^{-j}(-N + 1 + k), 2^{-j}(N + k)]$ for $k = N, \dots, 2^j - N - 1$. These are augmented with N left scaling functions of the form $\varphi_{j,k}^0(x) = 2^{j/2}\varphi_k^l(2^j x)$ for $k = 0, \dots, N - 1$ (where $\varphi_0^l, \dots, \varphi_{N-1}^l$ are fixed independent of j), each of which has support $[0, 2^{-j}(N + k)]$, and N right scaling functions of the form $\varphi_{j,2^j-k}(x) = 2^{j/2}\varphi_{-k}^r(2^j(x - 1))$ for $k = 1, \dots, N$ (where $\varphi_{-1}^r, \dots, \varphi_{-N}^r$ are fixed independent of j), each of which has support $[1 - 2^{-j}(1 - N - k), 1]$. The resulting 2^j functions $\varphi_{j,0}^0, \dots, \varphi_{j,N-1}^0, \varphi_{j,N}, \dots, \varphi_{j,2^j-N-1}, \varphi_{j,2^j-N}^1, \dots, \varphi_{j,2^j-1}^1$ form an orthonormal basis (with respect to the $L^2([0, 1])$ inner product) for their closed linear span V_j .

An orthonormal wavelet basis for the space W_j , defined as the orthogonal complement of V_j in V_{j+1} , is similarly constructed from the mother wavelet. This results in an orthonormal basis of 2^j functions, denoted $\psi_{j,0}^0, \dots, \psi_{j,N-1}^0, \psi_{j,N}, \dots, \psi_{j,2^j-N-1}, \psi_{j,2^j-N}^1, \dots, \psi_{j,2^j-1}^1$ (we use this conventional notation without confusion with the ψ_{Jj} basis functions spanning Ψ_J) where the ‘‘interior’’ wavelets $\psi_{j,N}, \dots, \psi_{j,2^j-N-1}$ are of the form $\psi_{j,k}(x) = 2^{j/2}\psi(2^j x - k)$. To simplify notation we ignore the 0 and 1 superscripts on the left and right wavelets and scaling functions henceforth. Let L_0 and L be integers such that $2N < 2^{L_0} \leq 2^L$. A wavelet space at resolution level L is the 2^{L+1} -dimensional set of functions given by

$$\text{Wav}(L, [0, 1]) = \left\{ \sum_{k=0}^{2^{L_0}-1} a_{L_0,k} \varphi_{L_0,k} + \sum_{j=L_0}^L \sum_{k=0}^{2^j-1} b_{j,k} \psi_{j,k} : a_{L_0,k}, b_{j,k} \in \mathbb{R} \right\}.$$

We say that $\text{Wav}(L, [0, 1])$ has *regularity* γ if $\psi \in C^\gamma$ (which can be achieved by choosing N sufficiently large) and write $\text{Wav}(L, [0, 1], \gamma)$ for a wavelet space of regularity γ with continuously differentiable basis functions.

We construct wavelet bases for $[0, 1]^d$ by taking tensor products of univariate bases. We again take L_0 and L to be integers such that $2N < 2^{L_0} \leq 2^L$. Let $\tilde{\psi}_{j,k,G}(x)$ denote an orthonormal tensor-product wavelet for $L^2([0, 1]^d)$ at resolution level j where $k = (k_1, \dots, k_d) \in \{0, \dots, 2^j - 1\}^d$ and where $G \in G_{j,L} \subseteq \{w_\varphi, w_\psi\}^d$ denotes which elements of the tensor product are ψ_{j,k_i} (indices corresponding to w_ψ) and which are φ_{j,k_i} (indices corresponding to w_φ). For example, $\tilde{\psi}_{j,k,w_\psi^d} = \prod_{i=1}^d \psi_{j,k_i}(x_i)$. Note that each $G \in G_{j,L}$ with $j > L$ has an element that is w_ψ (see [Triebel \(2006\)](#) for details). We have $\#(G_{L_0,L_0}) = 2^d$, $\#(G_{j,L_0}) = 2^d - 1$ for $j > L_0$. Let $\text{Wav}(L, [0, 1]^d, \gamma)$ denote the space

$$\text{Wav}(L, [0, 1]^d, \gamma) = \left\{ \sum_{j=L_0}^L \sum_{G \in G_{j,L_0}} \sum_{k \in \{0, \dots, 2^j - 1\}^d} a_{j,k,G} \tilde{\psi}_{j,k,G} : a_{j,k,G} \in \mathbb{R} \right\} \quad (24)$$

where each univariate basis has regularity γ . This definition clearly reduces to the above definition for $\text{Wav}(L, [0, 1], \gamma)$ in the univariate case.

Stability properties: The following two Lemmas bound $\xi_{\psi,J}$, as well as the minimum eigenvalue and condition number of $G_\psi = G_{\psi,J} = E[\psi^J(X_i)\psi^J(X_i)']$ when $\psi_{J1}, \dots, \psi_{JJ}$ is constructed using univariate and tensor-products of CDV wavelet bases.

Lemma E.3. *Let X have support $[0, 1]$ and let be a univariate CDV wavelet basis of resolution level $L = \log_2(J) - 1$. Then: (a) $\xi_{\psi,J} = O(\sqrt{J})$ for each sieve dimension $J = 2^{L+1}$; (b) If the density of X is uniformly bounded away from 0 and ∞ on $[0, 1]$, then there exists finite positive constants c_ψ and C_ψ such that $c_\psi \leq \lambda_{\max}(G_\psi)^{-1} \leq \lambda_{\min}(G_\psi)^{-1} \leq C_\psi$ for each J ; (c) $\lambda_{\max}(G_\psi)/\lambda_{\min}(G_\psi) \leq C_\psi/c_\psi$ for each J .*

Lemma E.4. *Let X have support $[0, 1]^d$ and let $\psi_{J1}, \dots, \psi_{JJ}$ be a wavelet basis formed as the tensor product of d univariate bases of resolution level L . Then: (a) $\xi_{\psi,J} = O(\sqrt{J})$ each J ; (b) If the density of X is uniformly bounded away from 0 and ∞ on $[0, 1]^d$, then there exists finite positive constants c_ψ and C_ψ such that $c_\psi \leq \lambda_{\max}(G_\psi)^{-1} \leq \lambda_{\min}(G_\psi)^{-1} \leq C_\psi$ for each J ; (c) $\lambda_{\max}(G_\psi)/\lambda_{\min}(G_\psi) \leq C_\psi/c_\psi$ for each J .*

Wavelet characterization of Besov norms: When the wavelet basis just described is of regularity $\gamma > 0$, the norms $\|\cdot\|_{B_{\infty,\infty}^p}$ for $p < \gamma$ can be restated in terms of the wavelet coefficients. We briefly explain the multivariate case as it nests the univariate case. Any $f \in L^2([0, 1]^d)$ may be represented as

$$f = \sum_{j,G,k} a_{j,k,G}(f) \tilde{\psi}_{j,k,G}$$

with the sum is understood to be taken over the same indices as in display (24). If $f \in B_{\infty,\infty}^p([0, 1]^d)$ then

$$\|f\|_{B_{\infty,\infty}^p} \asymp \|f\|_{b_{\infty,\infty}^p} := \sup_{j,k,G} 2^{j(p+d/2)} |a_{j,k,G}(f)|.$$

and if $f \in B_{2,2}^p([0, 1]^d)$ then

$$\|f\|_{B_{2,2}^p}^2 \asymp \|f\|_{b_{2,2}^p}^2 := \sum_{j,k,G} 2^{jp} |a_{j,k,G}(f)|^2$$

See [Johnstone \(2013\)](#) and [Triebel \(2006\)](#) for a more thorough discussion.

F Useful results on random matrices

F.1 Some matrix inequalities

The following Lemmas are used throughout the proofs in this paper and are stated here for convenience.

Lemma F.1 (Weyl's inequality). *Let $A, B \in \mathbb{R}^{r \times c}$ and let $s_i(A), s_i(B)$ denote the i th (ordered) singular value of A and B respectively, for $1 \leq i \leq (r \wedge c)$. Then: $|s_i(A) - s_i(B)| \leq \|A - B\|_{\ell^2}$ for all $1 \leq i \leq (r \wedge c)$. In particular, $|s_{\min}(A) - s_{\min}(B)| \leq \|A - B\|_{\ell^2}$.*

Lemma F.2. *Let $A \in \mathbb{R}^{r \times r}$ be nonsingular. Then: $\|A^{-1} - I_r\|_{\ell^2} \leq \|A^{-1}\|_{\ell^2} \|A - I_r\|_{\ell^2}$.*

Lemma F.3 (Schmitt (1992)). *Let $A, B \in \mathbb{R}^{r \times r}$ be positive definite. Then:*

$$\|A^{1/2} - B^{1/2}\|_{\ell^2} \leq \frac{1}{\sqrt{\lambda_{\min}(B)} + \sqrt{\lambda_{\min}(A)}} \|A - B\|_{\ell^2}.$$

Lemma F.4. *Let $A, B \in \mathbb{R}^{r \times c}$ with $r \leq c$ and let A and B have full row rank r . Then:*

$$\|B_l^- - A_l^-\|_{\ell^2} \leq \frac{1 + \sqrt{5}}{2} (s_{\min}(A)^{-2} \vee s_{\min}(B)^{-2}) \|A - B\|_{\ell^2}.$$

If, in addition, $\|A - B\|_{\ell^2} \leq \frac{1}{2} s_{\min}(A)$ then

$$\|B_l^- - A_l^-\|_{\ell^2} \leq 2(1 + \sqrt{5}) s_{\min}(A)^{-2} \|A - B\|_{\ell^2}.$$

Lemma F.5. *Let $A \in \mathbb{R}^{r \times c}$ with $r \leq c$ have full row rank r . Then: $\|A_l^-\|_{\ell^2} \leq s_{\min}(A)^{-1}$.*

Lemma F.6. *Let $A, B \in \mathbb{R}^{r \times c}$ with $r \leq c$ and let A and B have full row rank r . Then:*

$$\|A'(AA')^{-1}A - B'(BB')^{-1}B\|_{\ell^2} \leq (s_{\min}(A)^{-1} \vee s_{\min}(B)^{-1}) \|A - B\|_{\ell^2}.$$

F.2 Convergence of the matrix estimators

Before presenting the following lemmas, we define the *orthonormalized* matrix estimators

$$\begin{aligned} \widehat{G}_b^o &= G_b^{-1/2} \widehat{G}_b G_b^{-1/2} \\ \widehat{G}_\psi^o &= G_\psi^{-1/2} \widehat{G}_\psi G_\psi^{-1/2} \\ \widehat{S}^o &= G_b^{-1/2} \widehat{S} G_\psi^{-1/2} \end{aligned}$$

and let $G_b^o = I_K$, $G_\psi^o = I_J$ and S^o denote their respective expected values.

Lemma F.7. *The orthonormalized matrix estimators satisfy the exponential inequalities:*

$$\begin{aligned} \mathbb{P}\left(\|\widehat{G}_\psi^o - G_\psi^o\|_{\ell^2} > t\right) &\leq 2 \exp\left\{\log J - \frac{t^2/2}{\zeta_{\psi,J}^2(1+2t/3)/n}\right\} \\ \mathbb{P}\left(\|\widehat{G}_b^o - G_b^o\|_{\ell^2} > t\right) &\leq 2 \exp\left\{\log K - \frac{t^2/2}{\zeta_{b,K}^2(1+2t/3)/n}\right\} \\ \mathbb{P}\left(\|\widehat{S}^o - S^o\|_{\ell^2} > t\right) &\leq 2 \exp\left\{\log K - \frac{t^2/2}{(\zeta_{b,K}^2 \vee \zeta_{\psi,J}^2)/n + 2\zeta_{b,K}\zeta_{\psi,J}t/(3n)}\right\} \end{aligned}$$

and therefore

$$\begin{aligned}\|\widehat{G}_\psi^o - G_\psi^o\|_{\ell^2} &= O_p(\zeta_{\psi,J} \sqrt{(\log J)/n}) \\ \|\widehat{G}_b^o - G_b^o\|_{\ell^2} &= O_p(\zeta_{b,K} \sqrt{(\log K)/n}) \\ \|\widehat{S}^o - S^o\|_{\ell^2} &= O_p((\zeta_{b,K} \vee \zeta_{\psi,J}) \sqrt{(\log K)/n}).\end{aligned}$$

as $n, J, K \rightarrow \infty$ provided $(\zeta_{b,K} \vee \zeta_{\psi,J}) \sqrt{(\log K)/n} = o(1)$.

Lemma F.8 (Newey (1997), p. 162). *Let Assumption 2(i) hold. Then: $\|G_b^{-1/2} B' u/n\|_{\ell^2} = O_p(\sqrt{K/n})$.*

Lemma F.9. *Let $h_J(x) = \psi^J(x)' c_J$ for any deterministic $c_J \in \mathbb{R}^J$ and $H_J = (h_J(X_1), \dots, h_J(X_n))' = \Psi c_J$. Then:*

$$\begin{aligned}\|G_b^{-1/2} (B'(H_0 - \Psi c_J)/n - E[b^K(W_i)(h_0(X_i) - h_J(X_i))])\|_{\ell^2} \\ = O_p\left(\left(\sqrt{K/n} \times \|h_0 - h_J\|_\infty\right) \wedge \left(\zeta_{b,K}/\sqrt{n} \times \|h_0 - h_J\|_{L^2(X)}\right)\right).\end{aligned}$$

Lemma F.10. *Let $s_{JK}^{-1} \zeta \sqrt{(\log J)/n} = o(1)$ and let $J \leq K = O(J)$. Then:*

$$\begin{aligned}(a) \quad & \|(\widehat{G}_b^{-1/2} \widehat{S})_l^- \widehat{G}_b^{-1/2} G_b^{1/2} - (G_b^{-1/2} S)_l^- \|_{\ell^2} = O_p\left(s_{JK}^{-2} \zeta \sqrt{(\log J)/(ne_J)}\right) \\ (b) \quad & \|G_\psi^{1/2} \{(\widehat{G}_b^{-1/2} \widehat{S})_l^- \widehat{G}_b^{-1/2} G_b^{1/2} - (G_b^{-1/2} S)_l^-\} \|_{\ell^2} = O_p\left(s_{JK}^{-2} \zeta \sqrt{(\log J)/n}\right) \\ (c) \quad & \|G_b^{-1/2} S \{(\widehat{G}_b^{-1/2} \widehat{S})_l^- \widehat{G}_b^{-1/2} G_b^{1/2} - (G_b^{-1/2} S)_l^-\} \|_{\ell^2} = O_p\left(s_{JK}^{-1} \zeta \sqrt{(\log J)/n}\right).\end{aligned}$$

G Proofs

Notation: For a $r \times c$ matrix A with $r \leq c$ and full row rank r we let A_l^- denote its left pseudoinverse, namely $(A'A)^- A'$ where $'$ denotes transpose and $-$ denotes generalized inverse. We let $s_{\min}(A)$ denote the minimum singular value of a rectangular matrix A . For a positive-definite symmetric matrix A we let $\lambda_{\min}(A)$, $\lambda_{\max}(A)$ and $A^{1/2}$ denote its minimum and maximum eigenvalue and positive definite square root, respectively.

We also define $s_{JK} = s_{\min}(G_b^{-1/2} S G_\psi^{-1/2})$, which satisfies

$$s_{JK}^{-1} = \sup_{h \in \Psi_J: h \neq 0} \frac{\|h\|_{L^2(X)}}{\|\Pi_K T h\|_{L^2(W)}} \geq \tau_J$$

for all $K \geq J > 0$. The following result is used throughout the paper.

Lemma G.1. *Let Assumptions 1(iii) and 4(i) hold. Then: $(1 - o(1))s_{JK}^{-1} \leq \tau_J \leq s_{JK}^{-1}$ as $J \rightarrow \infty$.*

Proof of Lemma G.1. First note that $\tau_J > 0$ for all J by compactness and injectivity of T . Then:

$$s_{JK} = \inf_{h \in \Psi_J: \|h\|_{L^2(X)}=1} \|\Pi_K T h\|_{L^2(W)} \leq \inf_{h \in \Psi_J: \|h\|_{L^2(X)}=1} \|T h\|_{L^2(W)} = \tau_J^{-1}$$

holds uniformly in J because Π_K is a contraction, whence $\tau_J \leq s_{JK}^{-1}$. To derive a lower bound on τ_J , the triangle

inequality and Assumption 4(i) yield:

$$\begin{aligned}
s_{JK} &= \inf_{h \in \Psi_{J,1}} \|\Pi_K T h\|_{L^2(W)} \\
&\geq \inf_{h \in \Psi_{J,1}} \|T h\|_{L^2(W)} - \sup_{h \in \Psi_{J,1}} \|(\Pi_K T - T)h\|_{L^2(W)} \\
&= (1 - o(1))\tau_J^{-1}.
\end{aligned}$$

Therefore, $s_{JK}^{-1} \leq (1 - o(1))^{-1}\tau_J$. \square

Throughout the proofs in the appendix we use the identity

$$\begin{aligned}
\psi^J(x)'(G_b^{-1/2}S)_l^- &= \psi^J(x)'(S'G_b^{-1}S)^{-1}S'G_b^{-1/2} \\
&= \psi^J(x)'G_\psi^{-1/2}(G_\psi^{-1/2}S'G_b^{-1}SG_\psi^{-1/2})^{-1}G_\psi^{-1/2}S'G_b^{-1/2} \\
&= \psi^J(x)'G_\psi^{-1/2}(G_b^{-1/2}SG_\psi^{-1/2})_l^-
\end{aligned}$$

which implies that

$$\begin{aligned}
\|\psi^J(x)'(G_b^{-1/2}S)_l^-\|_{\ell^2} &\leq \|\psi^J(x)'G_\psi^{-1/2}\|_{\ell^2} \|(G_b^{-1/2}SG_\psi^{-1/2})_l^-\|_{\ell^2} \\
&\leq \zeta_{\psi,J} \|(G_b^{-1/2}SG_\psi^{-1/2})_l^-\|_{\ell^2} \\
&\leq \zeta_{\psi,J} s_{JK}^{-1}
\end{aligned} \tag{25}$$

by definition of $\zeta_{\psi,J}$ and the fact that $\|A_l^-\|_{\ell^2} \leq s_{\min}(A)^{-1}$ (see Lemma F.5).

G.1 Proofs for Appendix A and Section 3.1

Since the proofs of results in Section 3.1 built upon those for results in Appendix A, we shall present the proofs for Appendix A first.

G.1.1 Proofs for Appendix A

Proof of Lemma 3.1. Let $u = (u_1, \dots, u_n)'$. Let M_n be a sequence of positive constants diverging to $+\infty$, and decompose $u_i = u_{1,i} + u_{2,i}$ where

$$\begin{aligned}
u_{1,i} &= u_i \{|u_i| \leq M_n\} - E[u_i \{|u_i| \leq M_n\} | W_i] \\
u_{2,i} &= u_i \{|u_i| > M_n\} - E[u_i \{|u_i| > M_n\} | W_i] \\
u_1 &= (u_{1,1}, \dots, u_{1,n})' \\
u_2 &= (u_{2,1}, \dots, u_{2,n})'.
\end{aligned}$$

For **Result (1)**, recall that $\xi_{\psi,J} = \sup_x \|\psi^J(x)\|_{\ell^1}$. By Hölder's inequality we have

$$\|\widehat{h} - \widetilde{h}\|_{\infty} = \sup_x |\psi^J(x)'(\widehat{c} - \widetilde{c})| \leq \xi_{\psi,J} \|(\widehat{c} - \widetilde{c})\|_{\ell^{\infty}}.$$

To derive the sup-norm convergence rate of the standard deviation term $\widehat{h} - \widetilde{h}$, it suffices to bound the ℓ^{∞}

norm of the $J \times 1$ random vector $(\widehat{c} - \widetilde{c})$. Although this appears like a crude bound, $\xi_{\psi, J}$ grows slowly in J for certain sieves whose basis functions have local support. For such bases the above bound, in conjunction with the following result

$$\|\widehat{c} - \widetilde{c}\|_{\ell^\infty} = O_p \left(s_{JK}^{-1} \sqrt{(\log J)/(ne_J)} \right) \quad (26)$$

leads to a tight bound on the convergence rate of $\|\widehat{h} - \widetilde{h}\|_\infty$.

To prove (26), we begin by writing

$$\begin{aligned} \widehat{c} - \widetilde{c} &= (\widehat{G}_b^{-1/2} \widehat{S})_l^- \widehat{G}_b^{-1/2} B' u/n \\ &= (G_b^{-1/2} S)_l^- G_b^{-1/2} B' u/n + \{(\widehat{G}_b^{-1/2} \widehat{S})_l^- \widehat{G}_b^{-1/2} - (G_b^{-1/2} S)_l^- G_b^{-1/2}\} B' u/n \\ &=: T_1 + T_2. \end{aligned}$$

We will show that $\|T_1\|_{\ell^\infty} = O_p \left(s_{JK}^{-1} \sqrt{(\log J)/(ne_J)} \right)$ and $\|T_2\|_{\ell^\infty} = O_p \left(s_{JK}^{-1} \sqrt{(\log J)/(ne_J)} \right)$.

Control of $\|T_1\|_{\ell^\infty}$. Note that $T_1 = (G_b^{-1/2} S)_l^- G_b^{-1/2} B' u_1/n + (G_b^{-1/2} S)_l^- G_b^{-1/2} B' u_2/n$.

Let $(a)_j$ denote the j th element of a vector a . By the definition of $\|\cdot\|_{\ell^\infty}$ and the union bound,

$$\begin{aligned} \mathbb{P} \left(\|(G_b^{-1/2} S)_l^- G_b^{-1/2} B' u_1/n\|_{\ell^\infty} > t \right) &\leq \mathbb{P} \left(\bigcup_{j=1}^J |((G_b^{-1/2} S)_l^- G_b^{-1/2} B' u_1/n)_j| > t \right) \\ &\leq \sum_{j=1}^J \mathbb{P} \left(|((G_b^{-1/2} S)_l^- G_b^{-1/2} B' u_1/n)_j| > t \right) \\ &= \sum_{j=1}^J \mathbb{P} \left(\left| \sum_{i=1}^n q_{j,JK}(W_i) u_{1,i}/n \right| > t \right) \end{aligned} \quad (27)$$

where $q_{j,JK}(W_i) = ((G_b^{-1/2} S)_l^- G_b^{-1/2} b^K(W_i))_j$. The summands may be bounded by noting that

$$\begin{aligned} |q_{j,JK}(W_i)| &\leq \|(G_b^{-1/2} S)_l^-\|_{\ell^2} \|G_b^{-1/2} b^K(W_i)\|_{\ell^2} \\ &= \|G_\psi^{-1/2} [G_\psi^{-1/2} S' G_b^{-1} S G_\psi^{-1/2}]^{-1} G_\psi^{-1/2} S' G_b^{-1/2}\|_{\ell^2} \|G_b^{-1/2} b^K(W_i)\|_{\ell^2} \\ &\leq \|G_\psi^{-1/2}\|_{\ell^2} \| [G_\psi^{-1/2} S' G_b^{-1} S G_\psi^{-1/2}]^{-1} G_\psi^{-1/2} S' G_b^{-1/2}\|_{\ell^2} \|G_b^{-1/2} b^K(W_i)\|_{\ell^2} \\ &\leq \frac{\zeta_{b,K}}{s_{JK} \sqrt{e_J}} \end{aligned} \quad (28)$$

uniformly in i and j . Therefore,

$$|q_{j,JK}(W_i) u_{1,i}/n| \leq \frac{2M_n \zeta_{b,K}}{ns_{JK} \sqrt{e_J}} \quad (29)$$

uniformly in i and j .

Let $(A)_j$ denote the j th row of the matrix A and let $(A)_{jj}$ denote its j th diagonal element. The second moments

of the summands may be bounded by observing that

$$\begin{aligned}
E[q_{j,JK}(W_i)^2] &= E[((G_b^{-1/2}S)_l^-)_j | G_b^{-1/2} b^K(W_i)]^2 \\
&= E[((G_b^{-1/2}S)_l^-)_j | G_b^{-1/2} b^K(W_i) b^K(W_i)' G_b^{-1/2} ((G_b^{-1/2}S)_l^-)'_{jj}] \\
&= ((G_b^{-1/2}S)_l^- ((G_b^{-1/2}S)_l^-)')_{jj} \\
&= ((S' G_b^{-1} S)^{-1})_{jj} \\
&\leq \| (S' G_b^{-1} S)^{-1} \|_{\ell^2} \\
&= \| G_\psi^{-1/2} [G_\psi^{-1/2} S' G_b^{-1} S G_\psi^{-1/2}]^{-1} G_\psi^{-1/2} \|_{\ell^2} \\
&\leq \frac{1}{s_{JK}^2 e_J}
\end{aligned} \tag{30}$$

and so

$$E[(q_{j,JK}(W_i) u_{1,i}/n)^2] \leq \frac{\bar{\sigma}^2}{n^2 s_{JK}^2 e_J} \tag{31}$$

by Assumption 2(i) and the law of iterated expectations. Bernstein's inequality and expressions (27), (29) and (31) yield

$$\begin{aligned}
&\mathbb{P} \left(\| (G_b^{-1/2}S)_l^- G^{-1/2} B' u_1/n \|_{\ell^\infty} > C s_{JK}^{-1} \sqrt{(\log J)/(ne_J)} \right) \\
&\leq 2 \exp \left\{ \log J - \frac{C^2 (\log J)/(n s_{JK}^2 e_J)}{c_1/(n s_{JK}^2 e_J) + c_2 C M_n \zeta_{b,K} \sqrt{\log J}/(n^3 s_{JK}^2 e_J)} \right\} \\
&= 2 \exp \left\{ \log J - \frac{C^2 (\log J)/(n s_{JK}^2 e_J)}{1/(n s_{JK}^2 e_J) [c_1 + c_2 C M_n \zeta_{b,K} \sqrt{(\log J)/n}]} \right\}
\end{aligned} \tag{32}$$

for finite positive constants c_1 and c_2 . Then (32) is $o(1)$ for all large C provided $M_n \zeta_{b,K} \sqrt{(\log J)/n} = o(1)$.

By the triangle and Markov inequalities and (28), we have

$$\begin{aligned}
\mathbb{P} \left(\| (G_b^{-1/2}S)_l^- G^{-1/2} B' u_2/n \|_{\ell^\infty} > t \right) &= \mathbb{P} \left(\max_{1 \leq j \leq J} \left| \sum_{i=1}^n q_{j,JK}(W_i) u_{2,i}/n \right| > t \right) \\
&\leq \mathbb{P} \left(\frac{\zeta_{b,K}}{s_{JK} \sqrt{e_J}} \sum_{i=1}^n |u_{2,i}/n| > t \right) \\
&\leq \frac{2\zeta_{b,K}}{t s_{JK} \sqrt{e_J}} E[|u_i| \{ |u_i| > M_n \}] \\
&\leq \frac{2\zeta_{b,K}}{t s_{JK} \sqrt{e_J} M_n^{1+\delta}} E[|u_i|^{2+\delta} \{ |u_i| > M_n \}]
\end{aligned}$$

which, by Assumption 2(ii), is $o(1)$ when $t = C s_{JK}^{-1} \sqrt{(\log J)/(ne_J)}$ provided $\zeta_{b,K} \sqrt{n/(\log J)} = O(M_n^{1+\delta})$.

Choosing $M_n^{1+\delta} \asymp \zeta_{b,K} \sqrt{n/(\log J)}$ satisfies the condition $\zeta_{b,K} \sqrt{n/(\log J)} = O(M_n^{1+\delta})$ trivially, and satisfies the condition $M_n \zeta_{b,K} \sqrt{(\log J)/n} = o(1)$ provided $\zeta_{b,K}^{(2+\delta)/\delta} \sqrt{(\log J)/n} = o(1)$, which holds by Assumption 3(iii).

Control of $\|T_2\|_{\ell^\infty}$: Using the fact that $\|\cdot\|_{\ell^\infty} \leq \|\cdot\|_{\ell^2}$ on \mathbb{R}^J and Lemmas F.10(a) and F.8, we have:

$$\begin{aligned}
\|T_2\|_{\ell^\infty} &= \|\{(\widehat{G}_b^{-1/2}\widehat{S})_l^- \widehat{G}_b^{-1/2} G_b^{1/2} - (G_b^{-1/2} S)_l^-\} G_b^{-1/2} B' u/n\|_{\ell^\infty} \\
&\leq \|(\widehat{G}_b^{-1/2}\widehat{S})_l^- \widehat{G}_b^{-1/2} G_b^{1/2} - (G_b^{-1/2} S)_l^-\|_{\ell^2} \|G_b^{-1/2} B' u/n\|_{\ell^2} \\
&= O_p\left(s_{JK}^{-2} \zeta \sqrt{(\log K)/(ne_J)}\right) \times O_p(\sqrt{K/n}) \\
&= O_p\left(s_{JK}^{-1} \sqrt{(\log J)/(ne_J)}\right) \times O_p(s_{JK}^{-1} \zeta \sqrt{K/n}) \\
&= O_p\left(s_{JK}^{-1} \sqrt{(\log J)/(ne_J)}\right)
\end{aligned}$$

where the last equality follows from Assumption 3(ii) and the facts that $\zeta \geq \sqrt{K}$ and $J \asymp K$.

For **Result (2)**, we begin by writing

$$\begin{aligned}
\widehat{h}(x) - \widetilde{h}(x) &= \psi^J(x)' (\widehat{G}_b^{-1/2}\widehat{S})_l^- \widehat{G}_b^{-1/2} B' u/n \\
&= \psi^J(x)' (G_b^{-1/2} S)_l^- G_b^{-1/2} B' u/n + \psi^J(x)' \{(\widehat{G}_b^{-1/2}\widehat{S})_l^- \widehat{G}_b^{-1/2} - (G_b^{-1/2} S)_l^-\} G_b^{-1/2} B' u/n \\
&=: T_1 + T_2.
\end{aligned}$$

We will show that $\|T_1\|_\infty = O_p\left(\tau_J \zeta_{\psi,J} \sqrt{(\log n)/n}\right)$ and $\|T_2\|_\infty = O_p\left(\tau_J \zeta_{\psi,J} \sqrt{(\log n)/n}\right)$.

Control of $\|T_1\|_\infty$. Note that $T_1 = \psi^J(x)' (G_b^{-1/2} S)_l^- G_b^{-1/2} B' u_1/n + \psi^J(x)' (G_b^{-1/2} S)_l^- G_b^{-1/2} B' u_2/n$.

Let $\mathcal{X}_n \subset \mathcal{X}$ be a grid of finitely many points such that for each $x \in \mathcal{X}$ there exists a $\bar{x}_n(x) \in \mathcal{X}_n$ such that $\|x - \bar{x}_n(x)\| \lesssim (\zeta_{\psi,J} J^{-(\omega+\frac{1}{2})})^{1/\omega'}$, where ω, ω' are as in Assumption 3(i). By compactness and convexity of the support \mathcal{X} of X_i , we may choose \mathcal{X}_n to have cardinality $\#(\mathcal{X}_n) \lesssim n^\beta$ for some $0 < \beta < \infty$. Therefore,

$$\begin{aligned}
&\sup_x \|\psi^J(x)' (G_b^{-1/2} S)_l^- G_b^{-1/2} B' u_1/n\|_\infty \\
&\leq \max_{x_n \in \mathcal{X}_n} |\psi^J(x_n)' (G_b^{-1/2} S)_l^- G_b^{-1/2} B' u_1/n| + \sup_x |\{\psi^J(x) - \psi^J(\bar{x}_n(x))\}' (G_b^{-1/2} S)_l^- G_b^{-1/2} B' u_1/n| \\
&\leq \max_{x_n \in \mathcal{X}_n} |\psi^J(x_n)' (G_b^{-1/2} S)_l^- G_b^{-1/2} B' u_1/n| + C_\omega J^\omega \zeta_{\psi,J} J^{-(\omega+\frac{1}{2})} s_{JK}^{-1} \|G_b^{-1/2} B' u_1/n\|_{\ell^2} \\
&= \max_{x_n \in \mathcal{X}_n} |\psi^J(x_n)' (G_b^{-1/2} S)_l^- G_b^{-1/2} B' u_1/n| + C_\omega J^\omega \zeta_{\psi,J} J^{-(\omega+\frac{1}{2})} s_{JK}^{-1} \times O_p(\sqrt{J/n}) \\
&= \max_{x_n \in \mathcal{X}_n} |\psi^J(x_n)' (G_b^{-1/2} S)_l^- G_b^{-1/2} B' u_1/n| + o_p(s_{JK}^{-1} \zeta_{\psi,J} \sqrt{(\log J)/n})
\end{aligned}$$

for some finite positive constant C_ω , where the first inequality is by the triangle inequality, the second is by Hölder continuity of the basis for Ψ_J and similar reasoning to that used in equation (25), the first equality is by Lemma F.8 and the fact that $J \asymp K$, and the final equality is because $(\log J)^{-1/2} = o(1)$. For each $x_n \in \mathcal{X}_n$ we may write

$$\begin{aligned}
\psi^J(x_n)' (G_b^{-1/2} S)_l^- G_b^{-1/2} B' u_1/n &= \frac{1}{n} \sum_{i=1}^n g_{n,i}(x_n) u_{1,i}, \text{ where} \\
g_{n,i}(x_n) &= \psi^J(x_n)' (G_b^{-1/2} S)_l^- G_b^{-1/2} b^K(W_i).
\end{aligned}$$

It follows from equation (25) and the Cauchy-Schwarz inequality that the bounds

$$\begin{aligned}
|g_{n,i}(x_n)| &\leq s_{JK}^{-1} \zeta_{\psi,J} \zeta_{b,K} \\
E[g_{n,i}(x_n)^2] &= \psi^J(x_n)' (G_b^{-1/2} S)_l^- ((G_b^{-1/2} S)_l^-)' \psi^J(x_n) \leq s_{JK}^{-2} \zeta_{\psi,J}^2
\end{aligned}$$

hold uniformly for $x_n \in \mathcal{X}_n$. Therefore, by Assumption 2(i) and iterated expectations, the bounds

$$\begin{aligned} |g_{n,i}(x_n)u_{1,i}| &\leq 2s_{JK}^{-1}\zeta_{\psi,J}\zeta_{b,K}M_n \\ E[g_{n,i}(x_n)^2u_{1,i}^2] &\leq \bar{\sigma}^2\psi^J(x_n)'(G_b^{-1/2}S)_l^-((G_b^{-1/2}S)_l^-)' \psi^J(x_n) \leq \bar{\sigma}^2s_{JK}^{-2}\zeta_{\psi,J}^2 \end{aligned}$$

hold uniformly for $x_n \in \mathcal{X}_n$. It follows by the union bound and Bernstein's inequality that

$$\begin{aligned} &\mathbb{P}\left(\max_{x_n \in \mathcal{X}_n} |\psi^J(x_n)'(G_b^{-1/2}S)_l^-G_b^{-1/2}B'u_1/n| > Cs_{JK}^{-1}\zeta_{\psi,J}\sqrt{(\log n)/n}\right) \\ &\leq \#\mathcal{X}_n \max_{x_n \in \mathcal{X}_n} \mathbb{P}\left(\left|\frac{1}{n}\sum_{i=1}^n g_{n,i}(x_n)u_{1,i}\right| > Cs_{JK}^{-1}\zeta_{\psi,J}\sqrt{(\log n)/n}\right) \\ &\lesssim \exp\left\{\beta \log n - \frac{C^2\zeta_{\psi,J}^2(\log n)/(ns_{JK}^2)}{c_1\zeta_{\psi,J}^2/(ns_{JK}^2)[1 + (c_2/c_1)(CM_n\zeta_{b,K}\sqrt{(\log n)/n}]}\right\} \end{aligned} \quad (33)$$

for finite positive constants c_1 and c_2 . Then (33) is $o(1)$ for all large C provided $M_n\zeta_{b,K}\sqrt{(\log n)/n} = o(1)$.

By the triangle and Markov inequalities and equation (25), we have

$$\begin{aligned} \mathbb{P}\left(\|\psi^J(x)(G_b^{-1/2}S)_l^-G_b^{-1/2}B'u_2/n\|_\infty > t\right) &\leq \mathbb{P}\left(s_{JK}^{-1}\zeta_{\psi,J}\|G_b^{-1/2}B'u_2/n\|_{\ell^2} > t\right) \\ &\leq \mathbb{P}\left(s_{JK}^{-1}\zeta_{\psi,J}\zeta_{b,K}\sum_{i=1}^n |u_{2,i}/n| > t\right) \\ &\leq \frac{2\zeta_{\psi,J}\zeta_{b,K}}{ts_{JK}}E[|u_i|\{ |u_i| > M_n \}] \\ &\leq \frac{2\zeta_{\psi,J}\zeta_{b,K}}{ts_{JK}M_n^{1+\delta}}E[|u_i|^{2+\delta}\{ |u_i| > M_n \}] \end{aligned}$$

which, by Assumption 2(ii), is $o(1)$ when $t = Cs_{JK}^{-1}\zeta_{\psi,J}\sqrt{(\log n)/n}$ provided $\zeta_{b,K}\sqrt{n/(\log n)} = O(M_n^{1+\delta})$.

Choosing $M_n^{1+\delta} \asymp \zeta_{b,K}\sqrt{n/\log n}$ satisfies the condition $\zeta_{b,K}\sqrt{n/(\log n)} = O(M_n^{1+\delta})$ trivially, and satisfies the condition $M_n\zeta_{b,K}\sqrt{(\log n)/n} = o(1)$ provided $\zeta_{b,K}^{(2+\delta)/\delta}\sqrt{(\log n)/n} = o(1)$, which holds by Assumption 3(iii). We have therefore proved that $\|T_1\|_\infty = O_p(s_{JK}^{-1}\zeta_{\psi,J}\sqrt{(\log n)/n})$. It follows by the relation $\tau_J \asymp s_{JK}^{-1}$ (Lemma G.1) that $\|T_1\|_\infty = O_p(\tau_J\zeta_{\psi,J}\sqrt{(\log n)/n})$.

Control of $\|T_2\|_\infty$: Using the fact that $\|h\|_\infty \leq \zeta_{\psi,J}\|h\|_{L^2(X)}$ on Ψ_J and Lemmas F.10(b) and F.8 and the relation $\tau_J \asymp s_{JK}^{-1}$, we have:

$$\begin{aligned} \|T_2\|_\infty &\leq \zeta_{\psi,J}\|G_\psi^{1/2}\{(\widehat{G}_b^{-1/2}\widehat{S})_l^-\widehat{G}_b^{-1/2}G_b^{1/2} - (G_b^{-1/2}S)_l^-\}G_b^{-1/2}B'u/n\|_{\ell^2} \\ &\leq \zeta_{\psi,J}\|G_\psi^{1/2}\{(\widehat{G}_b^{-1/2}\widehat{S})_l^-\widehat{G}_b^{-1/2}G_b^{1/2} - (G_b^{-1/2}S)_l^-\}\|_{\ell^2}\|G_b^{-1/2}B'u/n\|_{\ell^2} \\ &= \zeta_{\psi,J}O_p\left(\tau_J^2\zeta\sqrt{(\log J)/n}\right) \times O_p(\sqrt{K/n}) \\ &= O_p\left(\tau_J\zeta_{\psi,J}\sqrt{(\log J)/n}\right) \times O_p(\tau_J\zeta\sqrt{K/n}) \\ &= O_p\left(\tau_J\zeta_{\psi,J}\sqrt{(\log J)/n}\right) \times O_p(1) \end{aligned}$$

where the last equality follows from Assumption 3(ii) and the fact that $\zeta \geq \sqrt{J} \asymp \sqrt{K}$. \square

It is clear that Lemma A.1 is implied by the following lemma.

Lemma G.2. *Let Assumptions 1(iii) and 4(ii) hold. Then:*

- (1) (a) $\|h_0 - \pi_J h_0\|_{L^2(X)} \asymp \|h_0 - \Pi_J h_0\|_{L^2(X)}$; and
 (b) $\tau_J \times \|T(h_0 - \pi_J h_0)\|_{L^2(W)} \leq \text{const} \times \|h_0 - \pi_J h_0\|_{L^2(X)}$.
- (2) If Assumption 4(i) also holds, then: (a) $\|Q_J h_0 - \pi_J h_0\|_{L^2(X)} \leq o(1) \times \|h_0 - \pi_J h_0\|_{L^2(X)}$; and
 (b) $\|h_0 - \Pi_J h_0\|_{L^2(X)} \asymp \|h_0 - Q_J h_0\|_{L^2(X)}$.
- (3) If Assumption 4(iii') also holds, then: $\|Q_J h_0 - \pi_J h_0\|_\infty \leq O(1) \times \|h_0 - \pi_J h_0\|_{L^2(X)}$.
- (4) Further, if Condition (23) also holds, then Assumption 4(iii) is satisfied.

Proof of Lemma G.2. In what follows, “const” denotes a generic positive constant that may be different from line to line. Assumption 1(iii) guarantees τ_J and $\pi_J h_0$ are well defined. For part (1.a), we have:

$$\begin{aligned}
\|h_0 - \Pi_J h_0\|_{L^2(X)} &\leq \|h_0 - \pi_J h_0\|_{L^2(X)} \\
&\leq \|h_0 - \Pi_J h_0\|_{L^2(X)} + \|\Pi_J h_0 - \pi_J h_0\|_{L^2(X)} \\
&\leq \|h_0 - \Pi_J h_0\|_{L^2(X)} + \tau_J \|T(\pi_J h_0 - \Pi_J h_0)\|_{L^2(W)} \\
&= \|h_0 - \Pi_J h_0\|_{L^2(X)} + \tau_J \|T\pi_J(h_0 - \Pi_J h_0)\|_{L^2(W)} \\
&\leq \|h_0 - \Pi_J h_0\|_{L^2(X)} + \tau_J \|T(h_0 - \Pi_J h_0)\|_{L^2(W)} \\
&= (1 + \text{const}) \times \|h_0 - \Pi_J h_0\|_{L^2(X)}
\end{aligned}$$

where the third line is by definition of τ_J , the fourth is because $\pi_J h = h$ for all $h \in \Psi_J$, the final line is by Assumption 4(ii), and the fifth is because π_J is a weak contraction under the norm $h \mapsto \|Th\|_{L^2(W)}$. More precisely by the definition of $\pi_J h_0$ we have:

$$\langle Th, T(h_0 - \pi_J h_0) \rangle_W = 0 \tag{34}$$

for all $h \in \Psi_J$, where $\langle \cdot, \cdot \rangle_W$ denotes the $L^2(W)$ inner product. With $h = \pi_J h_0 - \Pi_J h_0 \in \Psi_J$ this implies

$$\langle T(\pi_J h_0 - \Pi_J h_0), T(h_0 - \pi_J h_0) \rangle_W = 0.$$

$$\langle T(\pi_J h_0 - \Pi_J h_0), T(h_0 - \Pi_J h_0) \rangle_W = \langle T(\pi_J h_0 - \Pi_J h_0), T(\pi_J h_0 - \Pi_J h_0) \rangle_W.$$

Thus $\|T(\pi_J h_0 - \Pi_J h_0)\|_{L^2(W)} \leq \|T(h_0 - \Pi_J h_0)\|_{L^2(W)}$.

For part (1.b):

$$\begin{aligned}
\tau_J \|T(h_0 - \pi_J h_0)\|_{L^2(W)} &\leq \tau_J \|T(h_0 - \Pi_J h_0)\|_{L^2(W)} \\
&\leq \text{const} \times \|h_0 - \Pi_J h_0\|_{L^2(X)} \\
&\leq \text{const} \times \|h_0 - \pi_J h_0\|_{L^2(X)}
\end{aligned}$$

where the first and final inequalities are by definition of $\pi_J h_0$ and $\Pi_J h_0$ and the second inequality is by Assumption 4(ii).

For part (2.a), Lemma G.1 guarantees that $Q_J h_0$ is well defined and that $s_{JK}^{-1} \leq 2\tau_J$ for all J sufficiently large. By definition of $Q_J h_0$ we have:

$$\langle \Pi_K Th, T(h_0 - Q_J h_0) \rangle_W = 0 \tag{35}$$

for all $h \in \Psi_J$, where we use the fact that $\langle \Pi_K f, g \rangle_W = \langle \Pi_K f, \Pi_K g \rangle_W$ holds for any $f, g \in L^2(W)$ since Π_K is

a projection). Substituting $h = Q_J h_0 - \pi_J h_0 \in \Psi_J$ into the two equations (34) and (35) yields:

$$\langle (T - \Pi_K T)(Q_J h_0 - \pi_J h_0), T(h_0 - \pi_J h_0) \rangle_W + \langle \Pi_K T(Q_J h_0 - \pi_J h_0), T(h_0 - \pi_J h_0) \rangle_W = 0 \quad (36)$$

$$\langle \Pi_K T(Q_J h_0 - \pi_J h_0), T(h_0 - Q_J h_0) \rangle_W = 0. \quad (37)$$

By subtracting (37) from (36) we obtain

$$\langle (T - \Pi_K T)(Q_J h_0 - \pi_J h_0), T(h_0 - \pi_J h_0) \rangle_W + \|\Pi_K T(Q_J h_0 - \pi_J h_0)\|_{L^2(W)}^2 = 0$$

We have therefore proved

$$\|\Pi_K T(Q_J h_0 - \pi_J h_0)\|_{L^2(W)}^2 = |\langle (T - \Pi_K T)(Q_J h_0 - \pi_J h_0), T(h_0 - \pi_J h_0) \rangle_W|. \quad (38)$$

It follows from (38), the Cauchy-Schwarz inequality, and Assumption 4(i) that:

$$\begin{aligned} s_{JK}^2 \|Q_J h_0 - \pi_J h_0\|_{L^2(X)}^2 &\leq \|\Pi_K T(Q_J h_0 - \pi_J h_0)\|_{L^2(W)}^2 \\ &\leq \|(T - \Pi_K T)(Q_J h_0 - \pi_J h_0)\|_{L^2(W)} \|T(h_0 - \pi_J h_0)\|_{L^2(W)} \\ &\leq o(\tau_J^{-1}) \|Q_J h_0 - \pi_J h_0\|_{L^2(X)} \|T(h_0 - \pi_J h_0)\|_{L^2(W)}. \end{aligned} \quad (39)$$

$$\leq o(\tau_J^{-1}) \|Q_J h_0 - \pi_J h_0\|_{L^2(X)} \|T(h_0 - \pi_J h_0)\|_{L^2(W)}. \quad (40)$$

It follows by (40) and the relation $s_{JK}^{-1} \leq 2\tau_J$ for all J large that:

$$\begin{aligned} \|Q_J h_0 - \pi_J h_0\|_{L^2(X)} &\leq o(1) \times \tau_J \|T(h_0 - \pi_J h_0)\|_{L^2(W)} \\ &\leq o(1) \times \text{const} \times \|h_0 - \pi_J h_0\|_{L^2(X)} \end{aligned}$$

where the final line is by part (1.b). For part (2.b), by definition of Q_J , Π_J and results in part (1.a) and part (2.a), we have:

$$\begin{aligned} \|h_0 - \Pi_J h_0\|_{L^2(X)} &\leq \|h_0 - Q_J h_0\|_{L^2(X)} \\ &\leq \|h_0 - \pi_J h_0\|_{L^2(X)} + \|\pi_J h_0 - Q_J h_0\|_{L^2(X)} \\ &\leq \|h_0 - \pi_J h_0\|_{L^2(X)} + o(1) \times \|h_0 - \pi_J h_0\|_{L^2(X)} \\ &= (1 + \text{const}) \times \|h_0 - \Pi_J h_0\|_{L^2(X)}. \end{aligned}$$

This proves part (2.b).

For part (3), it follows from (39) and Assumption 4(iii') that

$$s_{JK}^2 \|Q_J h_0 - \pi_J h_0\|_{L^2(X)} \leq \text{const} \times (\zeta_{\psi, J\tau_J})^{-1} \|T(h_0 - \pi_J h_0)\|_{L^2(W)}.$$

and hence

$$\begin{aligned} \|Q_J h_0 - \pi_J h_0\|_{L^2(X)} &\leq \text{const} \times \zeta_{\psi, J}^{-1} \times \tau_J \|T(h_0 - \pi_J h_0)\|_{L^2(W)} \\ &\leq \zeta_{\psi, J}^{-1} \times \text{const} \times \|h_0 - \pi_J h_0\|_{L^2(X)} \end{aligned} \quad (41)$$

by Part (1.b) and the fact that $s_{JK}^{-1} \leq 2\tau_J$ for all J large. Therefore,

$$\begin{aligned} \|Q_J h_0 - \pi_J h_0\|_{\infty} &\leq \zeta_{\psi} \|Q_J h_0 - \pi_J h_0\|_{L^2(X)} \\ &\leq \text{const} \times \|h_0 - \pi_J h_0\|_{L^2(X)} \end{aligned}$$

where the last inequality is due to (41).

For part (4), by the triangle inequality, the results in part (1.a) and (3) and Condition (23) we have:

$$\begin{aligned}\|Q_J(h_0 - \Pi_J h_0)\|_\infty &\leq \|Q_J h_0 - \pi_J h_0\|_\infty + \|\pi_J h_0 - \Pi_J h_0\|_\infty \\ &\leq \text{const} \times \|h_0 - \Pi_J h_0\|_{L^2(X)} + \|h_0 - \Pi_J h_0\|_\infty \\ &\leq O(1) \times \|h_0 - \Pi_J h_0\|_\infty\end{aligned}$$

which completes the proof. \square

Note that we may write $\Pi_J h_0(x) = \psi^J(x)' c_J$ for some c_J in \mathbb{R}^J . We use this notation hereafter.

Proof of Lemma A.2. We first prove **Result (1)**. We begin by writing

$$\begin{aligned}\tilde{h}(x) - \Pi_J h_0(x) &= Q_J(h_0 - \Pi_J h_0)(x) \\ &\quad + \psi^J(x)' (G_b^{-1/2} S)_l^- \{G_b^{-1/2} (B'(H_0 - \Psi_{c_J})/n - E[b^K(W_i)(h_0(X_i) - \Pi_J h_0(X_i))])\} \\ &\quad + \psi^J(x)' \{(\widehat{G}_b^{-1/2} \widehat{S})_l^- \widehat{G}_b^{-1/2} G_b^{1/2} - (G_b^{-1/2} S)_l^- \} G_b^{-1/2} B'(H_0 - \Psi_{c_J})/n \\ &=: T_1 + T_2 + T_3\end{aligned}$$

where $Q_J : L^2(X) \rightarrow \Psi_J$ is the sieve 2SLS projection operator given by

$$Q_J h(x) = \psi^J(x)' [S' G_b^{-1} S]^{-1} S' G_b^{-1} E[b^K(W_i)h(X_i)].$$

Note that $Q_J h = h$ for all $h \in \Psi_J$.

Control of $\|T_1\|_\infty$: $\|T_1\|_\infty = O(1) \times \|h_0 - \Pi_J h_0\|_\infty$ by Assumption 4(iii).

Control of $\|T_2\|_\infty$: Using equation (25), the Cauchy-Schwarz inequality, and Lemma F.9, we obtain:

$$\begin{aligned}\|T_2\|_\infty &\leq \sup_x \|\psi^J(x) (G_b^{-1/2} S)_l^- \|_{\ell^2} \|G_b^{-1/2} (B'(H_0 - \Psi_{c_J})/n - E[b^K(W_i)(h_0(X_i) - \Pi_J h_0(X_i))])\|_{\ell^2} \\ &\leq \zeta_{\psi, J} s_{JK}^{-1} \|G_b^{-1/2} (B'(H_0 - \Psi_{c_J})/n - E[b^K(W_i)(h_0(X_i) - \Pi_J h_0(X_i))])\|_{\ell^2} \\ &= \zeta_{\psi, J} s_{JK}^{-1} \times O_p(\sqrt{K/n}) \times \|h_0 - \Pi_J h_0\|_\infty.\end{aligned}$$

It then follows by the relations $s_{JK}^{-1} \asymp \tau_J$ (Lemma G.1) and $\zeta_{\psi, J} \geq \sqrt{J} \asymp \sqrt{K}$ and Assumption 3(ii) that:

$$\|T_2\|_\infty = O_p(\tau_J \zeta_{\psi, J} \sqrt{J/n}) \times \|h_0 - \Pi_J h_0\|_\infty = O_p(1) \times \|h_0 - \Pi_J h_0\|_\infty.$$

Control of $\|T_3\|_\infty$: Similar to T_2 in the proof of Lemma 3.1, we may use Lemmas F.10(b) and G.1 to obtain:

$$\begin{aligned}\|T_3\|_\infty &\leq \zeta_{\psi, J} \|G_\psi^{1/2} \{(\widehat{G}_b^{-1/2} \widehat{S})_l^- \widehat{G}_b^{-1/2} G_b^{1/2} - (G_b^{-1/2} S)_l^- \} \|_{\ell^2} \|G_b^{-1/2} B'(H_0 - \Psi_{c_J})/n\|_{\ell^2} \\ &= \zeta_{\psi, J} \times O_p(\tau_J^2 \zeta \sqrt{(\log J)/n}) \times \|G_b^{-1/2} B'(H_0 - \Psi_{c_J})/n\|_{\ell^2}.\end{aligned}\tag{42}$$

Then by Lemma F.9 and the triangle inequality, we have:

$$\begin{aligned}\|G_b^{-1/2} B'(H_0 - \Psi_{c_J})/n\|_{\ell^2} &\leq O_p(\sqrt{K/n}) \times \|h_0 - \Pi_J h_0\|_\infty + \|\Pi_K T(h_0 - \Pi_J h_0)\|_{L^2(W)} \\ &\leq O_p(\sqrt{K/n}) \times \|h_0 - \Pi_J h_0\|_\infty + \|T(h_0 - \Pi_J h_0)\|_{L^2(W)}.\end{aligned}\tag{43}$$

Substituting (43) into (42) and using Assumptions 3(ii) and 4(ii):

$$\begin{aligned}
\|T_3\|_\infty &\leq O_p(\tau_J \zeta^2 / \sqrt{n}) \times \left(O_p(\tau_J \sqrt{K(\log J)/n}) \times \|h_0 - \Pi_J h_0\|_\infty + \tau_J \|T(h_0 - \Pi_J h_0)\|_{L^2(W)} \right) \\
&= O_p(1) \times (o_p(1) \times \|h_0 - \Pi_J h_0\|_\infty + O_p(1) \times \|h_0 - \Pi_J h_0\|_{L^2(X)}) \\
&\leq O_p(1) \times \|h_0 - \Pi_J h_0\|_\infty
\end{aligned}$$

where the final line is by the relation between the $L^2(X)$ and sup norms.

Result (2) then follows because

$$\begin{aligned}
\|\tilde{h} - h_0\|_\infty &\leq \|\tilde{h} - \Pi_J h_0\|_\infty + \|\Pi_J h_0 - h_0\|_\infty \\
&\leq (1 + O_p(1)) \|\Pi_J h_0 - h_0\|_\infty \\
&\leq (1 + O_p(1))(1 + \|\Pi_J\|_\infty) \|h_0 - h_{0,J}\|_\infty.
\end{aligned}$$

where the second inequality is by Result (1) and the final line is by Lebesgue's lemma. \square

G.1.2 Proofs for Section 3.1

Proof of Theorem 3.1. We decompose $\|\hat{h} - h_0\|_\infty$ into three parts:

$$\|\hat{h} - h_0\|_\infty \leq \|\hat{h} - \tilde{h}\|_\infty + \|\tilde{h} - \Pi_J h_0\|_\infty + \|\Pi_J h_0 - h_0\|_\infty.$$

where $\|\hat{h} - \tilde{h}\|_\infty = O_p(\tau_J \xi_{\psi,J} \sqrt{(\log J)/(ne_J)})$ by Lemma 3.1(1) and $\|\tilde{h} - \Pi_J h_0\|_\infty = O_p(1) \times \|\Pi_J h_0 - h_0\|_\infty$ by Lemma A.2. \square

Proof of Corollary 3.1. For **Result (1)**, note that Assumption 3(ii) is satisfied with $\zeta = O(J^{1/2})$ for Ψ_J and B_K being spline, or wavelet or cosine sieves. Next, by the lemmas in Appendix E, $\xi_{\psi,J}/\sqrt{e_J} = O(J^{1/2})$ for Ψ_J being spline or wavelet sieves. Also, $\|\Pi_J\|_\infty \lesssim 1$ for Ψ_J being a spline sieve (Huang (2003)) or a tensor product CDV wavelet sieve (Chen and Christensen (2015)). For $h_0 \in B_\infty(p, L)$ and Ψ_J being spline or wavelet sieves, Lemma A.2 implies that

$$\|\tilde{h} - h_0\|_\infty = O_p(J^{-p/d}).$$

Note that Bernstein inequalities (or inverse estimates) from approximation theory imply that

$$\|\partial^\alpha h\|_\infty = O(J^{|\alpha|/d}) \|h\|_\infty$$

for all $h \in \Psi_J$ (see Schumaker (2007) for splines and Cohen (2003) for wavelets on domains). Therefore,

$$\begin{aligned}
\|\partial^\alpha \tilde{h} - \partial^\alpha h_0\|_\infty &\leq \|\partial^\alpha \tilde{h} - \partial^\alpha (\Pi_J h_0)\|_\infty + \|\partial^\alpha (\Pi_J h_0) - \partial^\alpha h_0\|_\infty \\
&\leq O(J^{|\alpha|/d}) \|\tilde{h} - \Pi_J h_0\|_\infty + \|\partial^\alpha (\Pi_J h_0) - \partial^\alpha h_0\|_\infty \\
&\leq O_p(J^{-(p-|\alpha|)/d}) + \|\partial^\alpha (\Pi_J h_0) - \partial^\alpha h_0\|_\infty
\end{aligned}$$

Let h_J be any element of Ψ_J . Since $\Pi_J h_J = h_J$, we have:

$$\begin{aligned} \|\partial^\alpha(\Pi_J h_0) - \partial^\alpha h_0\|_\infty &= \|\partial^\alpha(\Pi_J(h_0 - h_J)) + \partial^\alpha h_J - \partial^\alpha h_0\|_\infty \\ &\leq O(J^{|\alpha|/d})\|\Pi_J(h_0 - h_J)\|_\infty + \|\partial^\alpha h_J - \partial^\alpha h_0\|_\infty \\ &\leq O(J^{|\alpha|/d}) \times \text{const} \times \|h_0 - h_J\|_\infty + \|\partial^\alpha h_J - \partial^\alpha h_0\|_\infty. \end{aligned}$$

The above inequality holds uniformly in $h_J \in \Psi_J$. Choosing h_J such that $\|h_0 - h_J\|_\infty = O(J^{-p/d})$ and $\|\partial^\alpha h_J - \partial^\alpha h_0\|_\infty = O(J^{-(p-|\alpha|)/d})$ yields the desired result.

For **Result (2)**, Theorem 3.1 implies that

$$\|\widehat{h} - h_0\|_\infty = O_p(J^{-p/d} + \tau_J \sqrt{(J \log J)/n}).$$

By similar arguments to the above, we have:

$$\begin{aligned} \|\partial^\alpha \widehat{h} - \partial^\alpha h_0\|_\infty &\leq \|\partial^\alpha \widehat{h} - \partial^\alpha \widetilde{h}\|_\infty + \|\partial^\alpha \widetilde{h} - \partial^\alpha h_0\|_\infty \\ &\leq O(J^{|\alpha|/d})\|\widehat{h} - \widetilde{h}\|_\infty + \|\partial^\alpha \widetilde{h} - \partial^\alpha h_0\|_\infty \\ &\leq O_p\left(J^{|\alpha|/d} \left(\tau_J \sqrt{(J \log J)/n}\right)\right) + \|\partial^\alpha \widetilde{h} - \partial^\alpha h_0\|_\infty \end{aligned}$$

and the result follows by Result (1).

For **Results (2.a) and (2.b)**, Assumption 3(ii)(iii) is satisfied if $\tau_J \times J/\sqrt{n} = O(1)$ and $J^{(2+\delta)/\delta}(\log n)/n = o(1)$. This is satisfied given the stated conditions with the optimal choice of J for mildly ill-posed case and severely ill-posed case respectively. \square

G.2 Proofs for Section 3.2

Proof of Theorem 3.2. Consider the Gaussian reduced-form NPIR model with known operator T :

$$\begin{aligned} Y_i &= Th_0(W_i) + u_i \\ u_i | W_i &\sim N(0, \sigma^2(W_i)) \end{aligned} \tag{44}$$

for $1 \leq i \leq n$, where W_i is continuously distributed over \mathcal{W} with density uniformly bounded away from 0 and ∞ . As in [Chen and Reiss \(2011\)](#), Theorem 3.2 is proved by (i) noting that the risk (in sup-norm loss) for the NPIV model is at least as large as the risk (in sup-norm loss) for the NPIR model, and (ii) calculating a lower bound (in sup-norm loss) for the NPIR model. Theorem 3.2 therefore follows from a sup-norm analogue of Lemma 1 of [Chen and Reiss \(2011\)](#) and Theorem G.1, which establishes a lower bound on minimax risk over Hölder classes under sup-norm loss for the NPIR model. \square

Theorem G.1. *Let Condition LB hold for the NPIR model (44) with a random sample $\{(W_i, Y_i)\}_{i=1}^n$. Then for any $0 \leq |\alpha| < p$:*

$$\liminf_{n \rightarrow \infty} \inf_{\widehat{g}_n} \sup_{h \in B_\infty(p, L)} \mathbb{P}_h \left(\|\widehat{g}_n - \partial^\alpha h\|_\infty \geq c(n/\log n)^{-(p-|\alpha|)/(2(p+\varsigma)+d)} \right) \geq c' > 0$$

in the mildly ill-posed case, and

$$\liminf_{n \rightarrow \infty} \inf_{\widehat{g}_n} \sup_{h \in B_\infty(p, L)} \mathbb{P}_h \left(\|\widehat{g}_n - \partial^\alpha h\|_\infty \geq c(\log n)^{-(p-|\alpha|)/\varsigma} \right) \geq c' > 0$$

in the severely ill-posed case, where $\inf_{\hat{g}_n}$ denotes the infimum over all estimators of $\partial^\alpha h$ based on the sample of size n , $\sup_{h \in B_\infty(p,L)} \mathbb{P}_h$ denotes the sup over $h \in B_\infty(p,L)$ and distributions (W_i, u_i) which satisfy Condition LB with ν fixed, and the finite positive constants c, c' depend only on p, L, d, ς and σ_0 .

Proof of Theorem G.1. We establish the lower bound by applying Theorem 2.5 of [Tsybakov \(2009\)](#) (see Theorem G.2 below). We first explain the scalar ($d = 1$) case in detail. Let $\{\phi_{j,k}, \psi_{j,k}\}_{j,k}$ be a wavelet basis of regularity $\gamma > p$ for $L^2([0,1])$ as described in Appendix E. Recall that this basis is generated by a Daubechies pair (φ, ψ) where φ has support $[-N+1, N]$. We will define a family of submodels in which we perturb h_0 by elements of the wavelet space W_j , where we choose j deterministically with n . For given j , recall that the wavelet space W_j consists of 2^j functions $\{\psi_{j,k}\}_{0 \leq k \leq 2^j - 1}$, such that $\{\psi_{j,k}\}_{r \leq k \leq 2^j - N - 1}$ are interior wavelets for which $\psi_{j,k}(\cdot) = 2^{j/2} \psi(2^j(\cdot) - k)$.

By construction, the support of each interior wavelet is an interval of length $2^{-j}(2r-1)$. Thus for all j sufficiently large (hence the \liminf in our statement of the Lemma) we may choose a set $M \subset \{r, \dots, 2^j - N - 1\}$ of interior wavelets with $\#(M) \gtrsim 2^j$ such that $\text{support}(\psi_{j,m}) \cap \text{support}(\psi_{j,m'}) = \emptyset$ for all $m, m' \in M$ with $m \neq m'$. Note also that by construction we have $\#(M) \leq 2^j$ (since there are $2^j - 2N$ interior wavelets).

Recall the norms $\|\cdot\|_{b_{\infty,\infty}^p}$ defined in Appendix E. Let $h_0 \in B_\infty(p,L)$ be such that $\|h_0\|_{B_{\infty,\infty}^p} \leq L/2$, and for each $m \in M$ let

$$h_m = h_0 + c_0 2^{-j(p+1/2)} \psi_{j,m}$$

where c_0 is a positive constant to be defined subsequently. Noting that

$$\begin{aligned} c_0 2^{-j(p+1/2)} \|\psi_{j,m}\|_{B_{\infty,\infty}^p} &\lesssim c_0 2^{-j(p+1/2)} \|\psi_{j,m}\|_{b_{\infty,\infty}^p} \\ &\leq c_0 \end{aligned}$$

it follows by the triangle inequality that $\|h_m\|_{B_{\infty,\infty}^p} \leq L$ uniformly in m for all sufficiently small c_0 . By Condition LB, let W_i be distributed such that X_i has uniform marginal distribution on $[0,1]$. For $m \in \{0\} \cup M$ let P_m be the joint distribution of $\{(W_i, Y_i)\}_{i=1}^n$ with $Y_i = Th_m(W_i) + u_i$ for the Gaussian NPIR model (44).

For any $m \in M$

$$\begin{aligned} \|\partial^\alpha h_0 - \partial^\alpha h_m\|_\infty &= c_0 2^{-j(p+1/2)} \|\partial^\alpha \psi_{j,m}\|_\infty \\ &= c_0 2^{-j(p-|\alpha|)} \|\psi^{(|\alpha|)}\|_\infty \end{aligned}$$

where $\psi^{(|\alpha|)}$ denotes the $|\alpha|$ th derivative of ψ . Moreover, for any $m, m' \in M$ with $m \neq m'$

$$\begin{aligned} \|\partial^\alpha h_m - \partial^\alpha h_{m'}\|_\infty &= c_0 2^{-j(p+1/2)} \|\partial^\alpha \psi_{j,m} - \partial^\alpha \psi_{j,m'}\|_\infty \\ &= 2c_0 2^{-j(p-|\alpha|)} \|\psi^{(|\alpha|)}\|_\infty \end{aligned}$$

by virtue of the disjoint support of $\{\psi_{j,m}\}_{m \in M}$.

By Condition LB(iii),

$$\|T\psi_{j,m}(W_i)\|_{L^2(W)} \lesssim \nu(2^j)^2 \langle \psi_{j,m}, \psi_{j,m} \rangle_X^2 = \nu(2^j)^2$$

(because $c_0 2^{-j(p+1/2)} \psi_{j,m} \in \mathcal{H}_2(p,L)$ for sufficiently small c_0) where $\nu(2^j) = 2^{-j\varsigma}$ in the mildly ill-posed case

and $\nu(2^j) = \exp(-2^{j\varsigma})$ in the severely ill-posed case. The KL distance $K(P_m, P_0)$ is

$$\begin{aligned} K(P_m, P_0) &\leq \frac{1}{2} \sum_{i=1}^n (c_0 2^{-j(p+1/2)})^2 E \left[\frac{(T\psi_{j,m}(W_i))^2}{\sigma^2(W_i)} \right] \\ &\leq \frac{1}{2} \sum_{i=1}^n (c_0 2^{-j(p+1/2)})^2 \frac{E[(T\psi_{j,m}(W_i))^2]}{\underline{\sigma}^2} \\ &\lesssim n(c_0 2^{-j(p+1/2)})^2 \nu(2^j)^2. \end{aligned}$$

In the mildly ill-posed case ($\nu(2^j) = 2^{-j\varsigma}$) we choose $2^j \asymp (n/(\log n))^{1/(2(p+\varsigma)+1)}$. This yields:

$$\begin{aligned} K(P_m, P_0) &\lesssim c_0^2 \log n \quad \text{uniformly in } m \\ \log(\#(M)) &\lesssim \log n + \log \log n. \end{aligned}$$

since $\#(M) \asymp 2^j$.

In the severely ill-posed case ($\nu(2^j) = \exp(-\frac{1}{2}2^{j\varsigma})$) we choose $2^j = (c_1 \log n)^{1/\varsigma}$ with $c_1 > 1$. This yields:

$$\begin{aligned} K(P_m, P_0) &\lesssim n^{-(c_1-1)} \quad \text{uniformly in } m \\ \log(\#(M)) &\lesssim \log \log n. \end{aligned}$$

In both the mildly and severely ill-posed cases, we may choose c_0 sufficiently small that both $\|h_m\|_{B_{\infty,\infty}^p} \leq L$ and $K(P_m, P_0) \leq \frac{1}{8} \log(\#(M))$ hold uniformly in m for all n sufficiently large. All conditions of Theorem 2.5 of [Tsybakov \(2009\)](#) are satisfied and hence we obtain the lower bound result.

In the multivariate case ($d > 1$) we let $\tilde{\psi}_{j,k,G}(x)$ denote an orthonormal tensor-product wavelet for $L^2([0, 1]^d)$ at resolution level j (see Appendix E). We construct a family of submodels analogously to the univariate case, setting $h_m = h_0 + c_0 2^{-j(p+d/2)} \tilde{\psi}_{j,m,G}$ where $\tilde{\psi}_{j,m,G}$ is now the tensor product of d interior univariate wavelets at resolution level j with $G = (w_\psi)^d$ and where $\#(M) \asymp 2^{jd}$. By condition LB we obtain

$$\|\partial^\alpha h_m - \partial^\alpha h_{m'}\|_\infty \gtrsim c_0 2^{-j(p-|\alpha|)}$$

for each $m, m' \in \{0\} \cup M$ with $m \neq m'$, and

$$K(P_m, P_0) \lesssim n(c_0 2^{-j(p+d/2)})^2 \nu(2^j)^2$$

for each $m \in M$, where $\nu(2^j) = 2^{-j\varsigma}$ in the mildly ill-posed case and $\nu(2^j) \asymp \exp(-2^{j\varsigma})$ in the severely ill-posed case. We choose $2^j \asymp (n/\log n)^{1/(2(p+\varsigma)+d)}$ in the mildly ill-posed case and $2^j = (c_1 \log n)^{1/\varsigma}$ in the severely ill-posed case. The result follows as in the univariate case. \square

The following theorem is a special case of Theorem 2.5 on p. 99 of [Tsybakov \(2009\)](#) which we use to prove the minimax lower bounds in sup- and L^2 -norm loss for h_0 and its derivatives. We state the result here for convenience.

Theorem G.2 ([Tsybakov \(2009\)](#)). *Assume that $\#(M) \geq 2$ and suppose that $(\mathcal{H}, \|\cdot\|_{\mathcal{H}})$ contains elements $\{h_m : m \in \{0\} \cup M\}$ such that:*

(i) $\|\partial^\alpha h_m - \partial^\alpha h_{m'}\|_{\mathcal{H}} \geq 2s > 0$ for each $m, m' \in M \cup \{0\}$ with $m \neq m'$;

(ii) $P_m \ll P_0$ for each $m \in M$ and

$$\frac{1}{\#(M)} \sum_{m \in M} K(P_m, P_0) \leq \mathbf{a} \log(\#(M))$$

with $0 < \mathbf{a} < \frac{1}{8}$ and where P_m denotes the distribution of the data when $h = h_m$ for each $m \in \{0\} \cup M$. Then:

$$\inf_{\hat{g}} \sup_{h \in \mathcal{H}} \mathbb{P}_h(\|\hat{g} - \partial^\alpha h\|_{\mathcal{H}} \geq s) \geq \frac{\sqrt{\#(M)}}{1 + \sqrt{\#(M)}} \left(1 - 2\mathbf{a} - \sqrt{\frac{2\mathbf{a}}{\log(\#(M))}} \right) > 0.$$

G.3 Proofs for Section 3.3

Proof of Lemma 3.2. We first prove **Result (1)**. Let $P_{J-1,z} = \text{clsp}\{\phi_{01,z}, \dots, \phi_{0J-1,z}\}$ and let $P_{J-1,z}^\perp$ denote its orthogonal complement in $L^2(X_1|Z=z)$. Observe that by definition of the singular values, for each z we have:

$$\begin{aligned} \sup_{h_z \in P_{J-1,z}^\perp: \|h_z\|_{L^2(X_1|Z=z)}=1} \|T_z h_z\|_{L^2(W_1|Z=z)}^2 &= \sup_{h_z \in P_{J-1,z}^\perp: \|h_z\|_{L^2(X_1|Z=z)}=1} \langle (T_z^* T_z) h_z, h_z \rangle_{X_1|Z=z} \\ &= \mu_{J,z}^2. \end{aligned} \quad (45)$$

Then let $P_{J-1}^\perp = \{h(x_1, z) \in L^2(X) : h(\cdot, z) \in P_{J-1,z}^\perp \text{ for each } z\}$. Note that $\phi_{0j} \in \{h \in P_{J-1}^\perp : \|h(\cdot, z)\|_{L^2(X_1|Z=z)} = 1 \forall z\}$ for each $j \geq J$. Then:

$$\begin{aligned} \tau_J^{-2} &= \inf_{h \in \Psi_J: \|h\|_{L^2(X)}=1} \|Th\|_{L^2(W)}^2 \\ &\leq \inf_{h \in \Psi_J \cap P_{J-1}^\perp: \|h\|_{L^2(X_1|Z=z)}=1 \forall z} \|Th\|_{L^2(W)}^2 \\ &\leq \sup_{h \in \Psi_J \cap P_{J-1}^\perp: \|h\|_{L^2(X_1|Z=z)}=1 \forall z} \|Th\|_{L^2(W)}^2 \\ &\leq \sup_{h \in P_{J-1}^\perp: \|h\|_{L^2(X_1|Z=z)}=1 \forall z} \|Th\|_{L^2(W)}^2. \end{aligned} \quad (46)$$

Let F_Z denote the distribution of Z . For any $h \in P_{J-1}^\perp$ let $h_z(x_1) = h(x_1, z)$ and observe that $h_z \in P_{J-1,z}^\perp$. By iterated expectations and (45), for any $h \in P_{J-1}^\perp$ with $\|h_z\|_{L^2(X_1|Z=z)} = 1$ for each z , we have:

$$\begin{aligned} \|Th\|_{L^2(W)}^2 &= \int \|E[h(X_{1i}, z)|W_{1i}, Z_i = z]\|_{L^2(W_1|Z=z)}^2 dF_Z(z) \\ &= \int \|T_z h_z\|_{L^2(W_1|Z=z)}^2 dF_Z(z) \\ &\leq \int \mu_{J,z}^2 \|h_z\|_{L^2(X_1|Z=z)}^2 dF_Z(z) \\ &= \int \mu_{J,z}^2 dF_Z(z) = E[\mu_{J,Z_i}^2]. \end{aligned} \quad (47)$$

It follows by substituting (47) into (46) that $\tau_J \geq E[\mu_{J,Z_i}^2]^{-1/2}$.

To prove **Result (2)**, note that any $h \in \Psi_J$ with $h \neq 0$ can be written as $\sum_{j=1}^J a_j \phi_{0j}$ for constants $a_j = a_j(h)$

where

$$\|h\|_{L^2(X)}^2 = E \left[E \left[\left(\sum_{j=1}^J a_j \phi_{0j}(X_{1i}, Z_i) \right)^2 \middle| Z_i \right] \right] = \sum_{j=1}^J a_j^2$$

since $E[\phi_{0j,z}(X_i)\phi_{0k,z}(X_i)|Z_i = z] = \delta_{jk}$ where δ_{jk} denotes the Kronecker delta. Moreover:

$$\begin{aligned} \|Th\|_{L^2(W)}^2 &= E \left[\left(E \left[\sum_{j=1}^J a_j \phi_{0j}(X_{1i}, Z_i) \middle| W_{1i}, Z_i \right] \right)^2 \right] \\ &= E \left[\left(E \left[\sum_{j=1}^J a_j \phi_{0j,Z_i}(X_{1i}) \middle| W_{1i}, Z_i \right] \right)^2 \right] \\ &= E \left[\left(\sum_{j=1}^J a_j \mu_{j,Z_i} \phi_{1j,Z_i}(W_{1i}) \right)^2 \right] \\ &= E \left[E \left[\left(\sum_{j=1}^J a_j \mu_{j,Z_i} \phi_{1j,Z_i}(W_{1i}) \right)^2 \middle| Z_i \right] \right] = \sum_{j=1}^J a_j^2 E[\mu_{j,Z_i}^2] \geq \|h\|_{L^2(X)}^2 E[\mu_{J,Z_i}^2] \end{aligned}$$

since $E[\phi_{1j,z}(W_{1i})\phi_{1k,z}(W_{1i})|Z_i = z] = \delta_{jk}$. Therefore,

$$\tau_J = \sup_{h \in \Psi_J} \frac{\|h\|_{L^2(X)}}{\|Th\|_{L^2(W)}} \leq \frac{1}{E[\mu_{J,Z_i}^2]^{1/2}}$$

as required. □

G.4 Proofs for Appendix D and Section 4

Since the proofs for uniform inference theories (in Section 4) built upon that for the pointwise normality Theorem D.1 (in Appendix D), we shall present the proof of Theorem D.1 first.

G.4.1 Proofs for Appendix D

Proof of Theorem D.1. We first prove **Result (1)**. By Assumption 5'(a) or 5'(b)(i)(ii) we have:

$$\sqrt{n} \frac{f(\hat{h}) - f(h_0)}{\sigma_n(f)} = \sqrt{n} \frac{Df(h_0)[\hat{h} - \tilde{h}]}{\sigma_n(f)} + o_p(1).$$

Define

$$Z_n(W_i) = \frac{(Df(h_0)[\psi^J])' [S' G_b^{-1} S]^{-1} S' G_b^{-1} b^K(W_i)}{\sigma_n(f)} = \Pi_K T u_n(f)(W_i)$$

where $u_n(f) = v_n(f)/\sigma_n(f)$ is the scaled sieve 2SLS Riesz representer. Note that $E[(Z_n(W_i)u_i)^2] = 1$. Then

$$\begin{aligned} \sqrt{n} \frac{Df(h_0)[\hat{h} - \tilde{h}]}{\sigma_n(f)} &= \frac{1}{\sqrt{n}} \sum_{i=1}^n Z_n(W_i)u_i \\ &\quad + \frac{(Df(h_0)[\psi^J])'((\hat{S}'\hat{G}_b^-\hat{S}) - \hat{S}'\hat{G}_b^- - (S'G_b^{-1}S)^{-1}S'G_b^{-1})(B'u/\sqrt{n})}{\sigma_n(f)} \\ &=: T_1 + T_2. \end{aligned}$$

We first show $T_1 \rightarrow_d N(0, 1)$ by the Lindeberg-Feller theorem. To verify the Lindeberg condition, note that

$$|Z_n(W_i)| \leq \frac{\left\| (Df(h_0)[\psi^J])'(S'G_b^{-1}S)^{-1}S'G_b^{-1/2} \right\| \left\| G_b^{-1/2}b^K(W_i) \right\|}{(\inf_w E[u_i^2|W_i = w])^{1/2} \left\| (Df(h_0)[\psi^J])'(S'G_b^{-1}S)^{-1}S'G_b^{-1/2} \right\|} \leq \underline{\sigma}^{-1}\zeta_b(K)$$

by the Cauchy-Schwarz inequality and Assumption 2(iii). Therefore,

$$E[u_i^2 Z_n(W_i)^2 \{ |Z_n(W_i)u_i| > \eta\sqrt{n} \}] \leq \sup_w E[u_i^2 \{ |u_i| \gtrsim \eta(\sqrt{n}/\zeta_b(K)) \} | W_i = w] = o(1)$$

by Assumption 2(iv') and the condition on J . Therefore, $T_1 \rightarrow_d N(0, 1)$.

For T_2 , observe that

$$\begin{aligned} |T_2| &= \left| \frac{(Df(h_0)[\psi^J])'((\hat{G}_b^{-1/2}\hat{S})_l^- \hat{G}_b^{-1/2}G_b^{1/2} - (G_b^{-1/2}S)_l^-)(G_b^{-1/2}B'u/\sqrt{n})}{\sigma_n(f)} \right| \\ &= \left| \frac{[(Df(h_0)[\psi^J])'(G_b^{-1/2}S)_l^-]G_b^{-1/2}S\{(\hat{G}_b^{-1/2}\hat{S})_l^- \hat{G}_b^{-1/2}G_b^{1/2} - (G_b^{-1/2}S)_l^-\}(G_b^{-1/2}B'u/\sqrt{n})}{\sigma_n(f)} \right| \\ &\leq \frac{\left\| (Df(h_0)[\psi^J])'(G_b^{-1/2}S)_l^- \right\| \left\| G_b^{-1/2}S\{(\hat{G}_b^{-1/2}\hat{S})_l^- \hat{G}_b^{-1/2}G_b^{1/2} - (G_b^{-1/2}S)_l^-\} \right\| \left\| G_b^{-1/2}B'u/\sqrt{n} \right\|}{(\inf_w E[u_i^2|W_i = w])^{1/2} \left\| (Df(h_0)[\psi^J])'(G_b^{-1/2}S)_l^- \right\|} \\ &\leq \underline{\sigma}^{-1} \left\| G_b^{-1/2}S\{(\hat{G}_b^{-1/2}\hat{S})_l^- \hat{G}_b^{-1/2}G_b^{1/2} - (G_b^{-1/2}S)_l^-\} \right\| \left\| G_b^{-1/2}B'u/\sqrt{n} \right\| \\ &= O_p(s_{JK}^{-1}\zeta\sqrt{(J\log J)/n}) \end{aligned}$$

where the first inequality is by the Cauchy-Schwarz inequality, the second is by Assumption 2(iii), and the final line is by Lemmas F.10(c) and F.8. The result follows by the equivalence $\tau_J \asymp s_{JK}^{-1}$ (see Lemma G.1) and the condition $\tau_J\zeta\sqrt{(J\log n)/n} = o(1)$.

Result (2) follows directly from Result (1) and Lemma G.3. \square

Lemma G.3. *Let Assumption 2(i)–(iii), 3(iii) and 4(i) hold, let $\tau_J\zeta\sqrt{(\log n)/n} = o(1)$, let $\|\hat{h} - h_0\|_\infty = o_p(1)$, and let Assumption 5'(b)(iii) hold if f is nonlinear. Then:*

$$\left| \frac{\hat{\sigma}_n(f)}{\sigma_n(f)} - 1 \right| = o_p(1).$$

Proof of Lemma G.3. First write

$$\begin{aligned} \frac{\widehat{\sigma}_n(f)^2}{\sigma_n(f)^2} - 1 &= \left(\frac{\widehat{\gamma}'_n \Omega^\circ \widehat{\gamma}_n}{\sigma_n(f)^2} - 1 \right) + \frac{\widehat{\gamma}'_n (\widehat{\Omega}^\circ - \Omega^\circ) \widehat{\gamma}_n}{\sigma_n(f)^2} \\ &= \left(\frac{(\widehat{\gamma}_n - \gamma_n)' \Omega^\circ (\widehat{\gamma}_n + \gamma_n)}{\sigma_n(f)^2} \right) + \frac{\widehat{\gamma}'_n (\widehat{\Omega}^\circ - \Omega^\circ) \widehat{\gamma}_n}{\sigma_n(f)^2} =: T_1 + T_2 \end{aligned}$$

where

$$\begin{aligned} \widehat{\Omega}^\circ &= G_b^{-1/2} \widehat{\Omega} G_b^{-1/2} & \widehat{\gamma}_n &= G_b^{1/2} \widehat{G}_b^{-1} \widehat{S} [\widehat{S}' \widehat{G}_b^{-1} \widehat{S}]^{-1} Df(\widehat{h})[\psi^J] \\ \Omega^\circ &= G_b^{-1/2} \Omega G_b^{-1/2} & \gamma_n &= G_b^{-1/2} S [S' G_b^{-1} S]^{-1} Df(h_0)[\psi^J] \end{aligned}$$

and observe that $\gamma'_n \Omega^\circ \gamma_n = \sigma_n(f)^2$ and $\widehat{\gamma}'_n \widehat{\Omega}^\circ \widehat{\gamma}_n = \widehat{\sigma}_n(f)^2$.

Control of T_1 : We first show that

$$\frac{\|\widehat{\gamma}_n - \gamma_n\|_{\ell^2}}{\sigma_n(f)} = o_p(1). \quad (48)$$

To simplify notation, let

$$\partial = \frac{Df(h_0)[\psi^J]}{s_n(f)} \quad \text{and} \quad \widehat{\partial} = \frac{Df(\widehat{h})[\psi^J]}{s_n(f)}$$

and note that $\|\partial'(G_b^{-1/2} S)_l^-\|_{\ell^2} = s_n(f)/\sigma_n(f) \asymp 1$ under Assumptions 2(i)(iii) and that $\widehat{\partial} = \partial$ if $f(\cdot)$ is linear. Then we have:

$$\begin{aligned} \frac{\|\widehat{\gamma}_n - \gamma_n\|_{\ell^2}}{\sigma_n(f)} &= \|\widehat{\partial}'(\widehat{G}_b^{-1/2} \widehat{S})_l^- \widehat{G}_b^{-1/2} G_b^{1/2} - \partial'(G_b^{-1/2} S)_l^-\|_{\ell^2} \\ &\leq \|\widehat{\partial}'(G_b^{-1/2} S)_l^-\|_{\ell^2} \|G_b^{-1/2} S \{(\widehat{G}_b^{-1/2} \widehat{S})_l^- \widehat{G}_b^{-1/2} G_b^{1/2} - (G_b^{-1/2} S)_l^-\}\|_{\ell^2} + \underline{\sigma}^{-1} \|(\widehat{\partial}' - \partial')(G_b^{-1/2} S)_l^-\|_{\ell^2} \\ &= O_p(1) \times O_p(s_{JK}^{-1} \zeta \sqrt{(\log J)/n}) + \underline{\sigma}^{-1} \frac{\|\Pi_K T(\widehat{v}_n(f) - v_n(f))\|_{L^2(W)}}{s_n(f)} \\ &= O_p(1) \times O_p(s_{JK}^{-1} \zeta \sqrt{(\log J)/n}) + o_p(1) \end{aligned}$$

where the third line is Lemma F.10(c) and the final line is by Assumption 5'(b)(iii). Therefore, (48) holds by the equivalence $s_{JK}^{-1} \asymp \tau_J$ (Lemma G.1) and the condition $\tau_J \zeta \sqrt{(\log n)/n}$.

Finally, since all eigenvalues of Ω° are bounded between $\underline{\sigma}^2$ and $\overline{\sigma}^2$ under Assumption 2(i)(iii), it follows from (48) and Cauchy-Schwarz that $|T_1| = o_p(1)$.

Control of T_2 : Equation (48) implies that $\|\widehat{\gamma}_n\|/\sigma_n(f) = O_p(1)$. Therefore, $|T_2| \leq O_p(1) \times \|\widehat{\Omega}^\circ - \Omega^\circ\|_{\ell^2} = o_p(1)$ by Lemma G.4. \square

Lemma G.4. *Let Assumptions 2(i)(ii) hold, let $\zeta_{b,K} \sqrt{(\log K)/n} = o(1)$, and let $\|\widehat{h} - h_0\|_\infty = O_p(\delta_{h,n})$ with $\delta_{h,n} = o(1)$. Then:*

$$\|\widehat{\Omega}^\circ - \Omega^\circ\|_{\ell^2} = O_p\left((\zeta_{b,K}^{(2+\delta)/\delta} \sqrt{(\log K)/n})^{\delta/(1+\delta)} + \delta_{h,n} \right)$$

Proof of Lemma G.4. By the triangle inequality:

$$\begin{aligned}
\|\widehat{\Omega}^\circ - \Omega^\circ\|_{\ell^2} &\leq \left\| G_b^{-1/2} \left(\frac{1}{n} \sum_{i=1}^n u_i^2 b^K(W_i) b^K(W_i)' \right) G_b^{-1/2} \right\|_{\ell^2} \\
&\quad + \left\| G_b^{-1/2} \left(\frac{1}{n} \sum_{i=1}^n 2u_i(\widehat{u}_i - u_i) b^K(W_i) b^K(W_i)' \right) G_b^{-1/2} \right\|_{\ell^2} \\
&\quad + \left\| G_b^{-1/2} \left(\frac{1}{n} \sum_{i=1}^n (\widehat{u}_i - u_i)^2 b^K(W_i) b^K(W_i)' \right) G_b^{-1/2} \right\|_{\ell^2} \\
&\leq O_p(\left(\zeta_{b,K}^{(2+\delta)/\delta} \sqrt{(\log K)/n}\right)^{\delta/(1+\delta)}) + \|\widehat{h} - h_0\|_\infty \times O_p(1) + \|\widehat{h} - h_0\|_\infty^2 \times O_p(1)
\end{aligned}$$

where the first term may easily be deduced from the proof of Lemma 3.1 of [Chen and Christensen \(2015\)](#), the second then follows because $2u_i(\widehat{u}_i - u_i) \leq 2(1 + u_i^2)\|\widehat{h} - h_0\|_\infty$, and the third follows similarly because $\|\widehat{G}_b^\circ\|_{\ell^2} = O_p(1)$ by Lemma F.7. \square

G.4.2 Proofs for Section 4

Proof of Lemma 4.1. By Assumption 5(a) or (b)(i)(ii), we have

$$\begin{aligned}
\sup_{t \in \mathcal{T}} \left| \sqrt{n} \frac{f_t(\widehat{h}) - f_t(h_0)}{\widehat{\sigma}_n(f_t)} - \widehat{Z}_n(t) \right| &\leq \sup_{t \in \mathcal{T}} \left| \sqrt{n} \frac{Df_t(h_0)[\widehat{h} - \widetilde{h}]}{\sigma_n(f_t)} - \widehat{Z}_n(t) \right| + O_p(\eta_n) \times \sup_{t \in \mathcal{T}} \left| \frac{\sigma_n(f_t)}{\widehat{\sigma}_n(f_t)} \right| \\
&\quad + \sup_{t \in \mathcal{T}} \left| \frac{\sigma_n(f_t)}{\widehat{\sigma}_n(f_t)} - 1 \right| \times \sup_{t \in \mathcal{T}} \left| \sqrt{n} \frac{Df_t(h_0)[\widehat{h} - \widetilde{h}]}{\sigma_n(f_t)} \right| \\
&=: T_1 + T_2 + T_3.
\end{aligned}$$

Control of T_1 : As in the proof of Theorem D.1,

$$\begin{aligned}
T_1 &= \sup_{t \in \mathcal{T}} \left| \frac{(Df_t(h_0)[\psi^J])'((\widehat{G}_b^{-1/2} \widehat{S})_l^- \widehat{G}_b^{-1/2} G_b^{1/2} - (G_b^{-1/2} S)_l^-) (G_b^{-1/2} B' u / \sqrt{n})}{\sigma_n(f_t)} \right| \\
&= \sup_{t \in \mathcal{T}} \left| \frac{[(Df_t(h_0)[\psi^J])' (G_b^{-1/2} S)_l^-] G_b^{-1/2} S \{ (\widehat{G}_b^{-1/2} \widehat{S})_l^- \widehat{G}_b^{-1/2} G_b^{1/2} - (G_b^{-1/2} S)_l^- \} (G_b^{-1/2} B' u / \sqrt{n})}{\sigma_n(f_t)} \right| \\
&\leq \sup_{t \in \mathcal{T}} \frac{\left\| (Df_t(h_0)[\psi^J])' (G_b^{-1/2} S)_l^- \right\| \left\| G_b^{-1/2} S \{ (\widehat{G}_b^{-1/2} \widehat{S})_l^- \widehat{G}_b^{-1/2} G_b^{1/2} - (G_b^{-1/2} S)_l^- \} \right\| \left\| G_b^{-1/2} B' u / \sqrt{n} \right\|}{(\inf_w E[u_i^2 | W_i = w])^{1/2} \left\| (Df_t(h_0)[\psi^J])' (G_b^{-1/2} S)_l^- \right\|} \\
&\leq \underline{\sigma}^{-1} \left\| G_b^{-1/2} S \{ (\widehat{G}_b^{-1/2} \widehat{S})_l^- \widehat{G}_b^{-1/2} G_b^{1/2} - (G_b^{-1/2} S)_l^- \} \right\| \left\| G_b^{-1/2} B' u / \sqrt{n} \right\| \\
&= O_p(\tau_J \zeta \sqrt{(J \log J)/n})
\end{aligned}$$

where the first inequality is by the Cauchy-Schwarz inequality, the second is by Assumption 2(iii), and the final line is by Lemmas F.10(c) and F.8 and the equivalence $\tau_J \asymp s_{JK}^{-1}$ (see Lemma G.1).

Control of T_2 : Lemma G.5 shows

$$\sup_{t \in \mathcal{T}} \left| \frac{\widehat{\sigma}_n(f_t)}{\sigma_n(f_t)} - 1 \right| = O_p(\delta_{V,n} + \delta_{h,n} + \eta'_n) = o_p(1)$$

from which it follows that $T_2 = O_p(\eta_n) \times O_p(1) = O_p(\eta_n)$.

Control of T_3 : By Lemma G.5 and the bound for T_1 , we have:

$$\begin{aligned}
T_3 &= O_p(\delta_{V,n} + \delta_{h,n} + \eta'_n) \times \sup_{t \in \mathcal{T}} \left| \sqrt{n} \frac{Df_t(h_0)[\widehat{h} - \widetilde{h}]}{\sigma_n(f_t)} \right| \\
&= O_p(\delta_{V,n} + \delta_{h,n} + \eta'_n) \times \sup_{t \in \mathcal{T}} |\widehat{\mathbb{Z}}_n(t)| \\
&= O_p(\delta_{V,n} + \delta_{h,n} + \eta'_n) \times (o_p(r_n) + \sup_{t \in \mathcal{T}} |\mathbb{Z}_n(t)|) \\
&= O_p(\delta_{V,n} + \delta_{h,n} + \eta'_n) \times (o_p(r_n) + O_p(c_n))
\end{aligned}$$

where the second-last line is by Part 2 and the final line is by Lemma G.6. Therefore we have proved:

$$\begin{aligned}
\sup_{t \in \mathcal{T}} \left| \sqrt{n} \frac{f_t(\widehat{h}) - f_t(h_0)}{\widehat{\sigma}_n(f_t)} - \widehat{\mathbb{Z}}_n(t) \right| &= O_p(\delta_{V,n} + \delta_{h,n} + \eta'_n) \times (o_p(r_n) + O(c_n)) + O_p(\eta_n) \\
&= o_p(c_n^{-1})
\end{aligned} \tag{49}$$

with the final line by Assumption 6(ii). \square

Lemma G.5. *Let Assumptions 1(iii), 2(i)–(iii), 3(iii) and 4(i) hold, let $\tau_J \zeta \sqrt{(\log n)/n} = o(1)$, let $\|\widehat{h} - h_0\|_\infty = O_p(\delta_{h,n})$ with $\delta_{h,n} = o(1)$, and let Assumption 5(b)(iii) hold if f is nonlinear. Then:*

$$\sup_{t \in \mathcal{T}} \left| \frac{\sigma_n(f_t)}{\widehat{\sigma}_n(f_t)} - 1 \right| = O_p(\delta_{V,n} + \delta_{h,n} + \eta'_n)$$

Proof of Lemma G.5. The proof follows by identical arguments to the proof of Lemma G.3. \square

Lemma G.6. *Let Assumption 6(i) hold. Then: both $E[\sup_{t \in \mathcal{T}} |\mathbb{Z}_n(t)|] \lesssim c_n$ and $\sup_{t \in \mathcal{T}} |\mathbb{Z}_n(t)| = O_p(c_n)$ hold.*

Proof of Lemma G.6. Observe that $d_n(t_1, t_2) := E[(\mathbb{Z}_n(t_1) - \mathbb{Z}_n(t_2))^2]^{1/2}$. By Corollary 2.2.8 of van der Vaart and Wellner (1996) and Assumption 6(i), there exists a universal constant C such that

$$E[\sup_t |\mathbb{Z}_n(t)|] \leq E[|\mathbb{Z}_n(\bar{t})|] + C \int_0^\infty \sqrt{\log N(\mathcal{T}, d_n, \epsilon)} \, d\epsilon$$

for any $\bar{t} \in \mathcal{T}$, where $E[|\mathbb{Z}_n(\bar{t})|] = \sqrt{2/\pi}$ because $\mathbb{Z}_n(\bar{t}) \sim N(0, 1)$. Therefore, $E[\sup_t |\mathbb{Z}_n(t)|] \lesssim c_n$. The second result follows by Markov's inequality. \square

Proof of Theorem 4.1. Step 1: Approximating $\widehat{\mathbb{Z}}_n(t)$ by a Gaussian process. Recall $\Omega^\circ = G_b^{-1/2} \Omega G_b^{-1/2}$. We use Yurinskii's coupling (Pollard, 2002, Theorem 10, p. 244) to show that there exists a sequence of $N(0, \Omega^\circ)$ random vectors \mathcal{Z}_n such that

$$\left\| \frac{1}{\sqrt{n}} \sum_{i=1}^n G_b^{-1/2} b^K(W_i) u_i - \mathcal{Z}_n \right\|_{\ell^2} = o_p(r_n). \tag{50}$$

By Assumption 2(iv) we have

$$\sum_{i=1}^n E[\|n^{-1/2} G_b^{-1/2} b^K(W_i) u_i\|_{\ell^2}^3] \lesssim n^{-1/2} \zeta_{b,K} E[\|G_b^{-1/2} b^K(W_i)\|_{\ell^2}^2] = \frac{\zeta_{b,K} K}{\sqrt{n}} = O\left(\frac{\zeta_{b,K} J}{\sqrt{n}}\right).$$

Existence of \mathbb{Z}_n follows under the condition

$$\frac{\zeta_{b,K} J^2}{r_n^3 \sqrt{n}} = o(1).$$

Then the process $\mathbb{Z}_n(t)$ given by

$$\mathbb{Z}_n(t) = \frac{(Df_t(h_0)[\psi^J])' [S' G_b^{-1} S]^{-1} S' G_b^{-1/2}}{\sigma_n(f_t)} \mathcal{Z}_n$$

is a centered Gaussian process with the covariance function

$$E[\mathbb{Z}_n(t_1)\mathbb{Z}_n(t_2)] = \frac{(Df_{t_1}(h_0)[\psi^J])' [S' G_b^{-1} S]^{-1} S' G_b^{-1} \Omega G_b^{-1} S [S' G_b^{-1} S]^{-1} Df_{t_2}(h_0)[\psi^J]}{\sigma_n(f_{t_1})\sigma_n(f_{t_2})}. \quad (51)$$

Now observe that

$$\sup_{t \in \mathcal{T}} \left\| \frac{(Df_t(h_0)[\psi^J])' [S' G_b^{-1} S]^{-1} S' G_b^{-1/2}}{\sigma_n(f_t)} \right\|_{\ell^2} = \sup_{t \in \mathcal{T}} \frac{s_n(f_t)}{\sigma_n(f_t)} \asymp 1 \quad (52)$$

by Assumption 2(i)(iii). Therefore,

$$\sup_{t \in \mathcal{T}} \left| \widehat{\mathbb{Z}}_n(t) - \mathbb{Z}_n(t) \right| = o_p(r_n) \quad (53)$$

by equations (50) and (52) and Cauchy-Schwarz.

Step 2: Approximating $\mathbb{Z}_n^*(t)$ by a Gaussian process. Each of the terms $n^{-1/2} G_b^{-1/2} b^K(W_i) \widehat{u}_i \varpi_i$ is centered under \mathbb{P}^* because $E[\varpi_i | Z^n] = 0$ for $i = 1, \dots, n$. Moreover,

$$\sum_{i=1}^n E[(n^{-1/2} G_b^{-1/2} b^K(W_i) \widehat{u}_i \varpi_i)(n^{-1/2} G_b^{-1/2} b^K(W_i) \widehat{u}_i \varpi_i)' | Z^n] = \widehat{\Omega}^o$$

where $G_b^{-1/2} \widehat{\Omega} G_b^{-1/2} = \widehat{\Omega}^o$, and

$$\sum_{i=1}^n E[\|n^{-1/2} G_b^{-1/2} b^K(W_i) \widehat{u}_i \varpi_i\|_{\ell^2}^3 | Z^n] \lesssim n^{-3/2} \sum_{i=1}^n E[\|G_b^{-1/2} b^K(W_i)\|_{\ell^2}^2 |\widehat{u}_i|^3]$$

because $E[|\varpi_i|^3 | Z^n] < \infty$ uniformly in i , and where

$$n^{-3/2} \sum_{i=1}^n E[\|G_b^{-1/2} b^K(W_i)\|_{\ell^2}^2 |\widehat{u}_i|^3] \lesssim \frac{\zeta_{b,K} K}{\sqrt{n}}$$

holds wpa1 (by Markov's inequality using $|\widehat{u}_i|^3 \lesssim |u_i|^3 + \|\widehat{h} - h_0\|_\infty^3$ and Assumption 2(iv)). A second application of Yurinskii's coupling conditional on the data Z^n then yields existence of a sequence of $N(0, \widehat{\Omega}^o)$ random vectors \mathcal{Z}_n^* such that

$$\left\| \frac{1}{\sqrt{n}} \sum_{i=1}^n G_b^{-1/2} b^K(W_i) \widehat{u}_i \varpi_i - \mathcal{Z}_n^* \right\|_{\ell^2} = o_{p^*}(r_n)$$

wpa1. Therefore:

$$\sup_{t \in \mathcal{T}} \left| \mathbb{Z}_n^*(t) - \frac{(Df_t(\widehat{h})[\psi^J])' [\widehat{S}' \widehat{G}_b^{-1} \widehat{S}]^{-1} \widehat{S}' \widehat{G}_b^{-1/2}}{\widehat{\sigma}_n(f_t)} \mathcal{Z}_n^* \right| = o_{p^*}(r_n) \quad (54)$$

wpa1. Now observe that we can define a centered Gaussian process \tilde{Z}_n^* under \mathbb{P}^* by

$$\tilde{Z}_n^*(t) = \frac{(Df_t(h_0)[\psi^J])'[S'G_b^{-1}S]^{-1}S'G_b^{-1/2}}{\sigma_n(f_t)}(\Omega^o)^{1/2}(\hat{\Omega}^o)^{-1/2}\mathcal{Z}_n^*$$

which has the same covariance function as Z_n (see equation (51)) whenever $\hat{\Omega}^o$ is invertible (which it is wpa1). Therefore, by Lemma G.7 we have:

$$\sup_{t \in \mathcal{T}} \left| \frac{(Df_t(\hat{h})[\psi^J])'[\hat{S}'\hat{G}_b^{-1}\hat{S}]^{-1}\hat{S}'\hat{G}_b^{-1/2}}{\hat{\sigma}_n(f_t)}\mathcal{Z}_n^* - \tilde{Z}_n^*(t) \right| = o_p^*(c_n^{-1}) \quad (55)$$

wpa1. It follows from equations (54) and (55) and Assumption 6(ii) that

$$\sup_{t \in \mathcal{T}} \left| Z_n^*(t) - \tilde{Z}_n^*(t) \right| = o_{p^*}(r_n) + o_{p^*}(c_n^{-1}) = o_p^*(c_n^{-1}) \quad (56)$$

wpa1.

Step 3: Consistency. By Lemma 4.1 and display (53), we have:

$$\sup_{t \in \mathcal{T}} \left| \frac{\sqrt{n}(f_t(\hat{h}) - f_t(h_0))}{\hat{\sigma}_n(f_t)} - Z_n(t) \right| = o_p(r_n) + o_p(c_n^{-1}) = o_p(c_n^{-1}).$$

Therefore, we may choose a sequence of positive constants ϵ_n with $\epsilon_n = o(1)$ such that

$$\sup_{t \in \mathcal{T}} \left| \frac{\sqrt{n}(f_t(\hat{h}) - f_t(h_0))}{\hat{\sigma}_n(f_t)} - Z_n(t) \right| \leq \epsilon_n c_n^{-1} \quad (57)$$

holds wpa1. By an anti-concentration inequality (Chernozhukov, Chetverikov, and Kato, 2014, Theorem 2.1) and Lemma G.6, we have:

$$\sup_{s \in \mathbb{R}} \mathbb{P} \left(\sup_{t \in \mathcal{T}} |Z_n(t) - s| \leq \epsilon_n c_n^{-1} \right) \lesssim \epsilon_n c_n^{-1} E[\sup_{t \in \mathcal{T}} |Z_n(t)|] \lesssim \epsilon_n = o(1)$$

which, together with (57), yields:

$$\sup_{s \in \mathbb{R}} \left| \mathbb{P} \left(\sup_{t \in \mathcal{T}} \left| \frac{\sqrt{n}(f_t(\hat{h}) - f_t(h_0))}{\hat{\sigma}_n(f_t)} \right| \leq s \right) - \mathbb{P} \left(\sup_{t \in \mathcal{T}} |Z_n(t)| \leq s \right) \right| = o(1). \quad (58)$$

Moreover, by (56) we may choose a sequence of positive constants ϵ'_n with $\epsilon'_n = o(1)$ such that

$$\sup_{t \in \mathcal{T}} \left| Z_n^*(t) - \tilde{Z}_n^*(t) \right| \leq \epsilon'_n c_n^{-1}$$

holds wpa1. Similar arguments then yield:

$$\sup_{s \in \mathbb{R}} \mathbb{P}^* \left(\sup_{t \in \mathcal{T}} |\tilde{Z}_n^*(t) - s| \leq \epsilon'_n c_n^{-1} \right) \lesssim \epsilon'_n = o(1)$$

wpa1. This, together with equation (56), yields:

$$\sup_{s \in \mathbb{R}} \left| \mathbb{P}^* \left(\sup_{t \in \mathcal{T}} |Z_n^*(t)| \leq s \right) - \mathbb{P}^* \left(\sup_{t \in \mathcal{T}} |\tilde{Z}_n^*(t)| \leq s \right) \right| = o_p(1). \quad (59)$$

The result is immediate from equations (58) and (59) and the fact that

$$\mathbb{P} \left(\sup_{t \in \mathcal{T}} |Z_n(t)| \leq s \right) = \mathbb{P}^* \left(\sup_{t \in \mathcal{T}} |\tilde{Z}_n^*(t)| \leq s \right)$$

holds uniformly in s wpa1. \square

Lemma G.7. *Let Assumptions 1(iii), 2, 3(iii), 4(i), and 6 hold, let $\tau_J \zeta \sqrt{(\log n)/n} = o(1)$, let $\|\hat{h} - h_0\|_\infty = O_p(\delta_{h,n})$ with $\delta_{h,n} = o(1)$, and let Assumption 5(b)(iii) hold if f is nonlinear. Let Z_n^* and \tilde{Z}_n be as in the proof of Theorem 4.1. Then:*

$$\sup_{t \in \mathcal{T}} \left| \frac{(Df_t(\hat{h})[\psi^J])' [\hat{S}' \hat{G}_b^{-1} \hat{S}]^{-1} \hat{S}' \hat{G}_b^{-1/2}}{\hat{\sigma}_n(f_t)} Z_n^* - \tilde{Z}_n(t) \right| = o_p^*(c_n^{-1})$$

with probability approaching one.

Proof of Lemma G.7. First note that because $Z_n^* \sim N(0, \hat{\Omega}^\circ)$ and the minimum and maximum eigenvalues of $\hat{\Omega}^\circ$ are uniformly bounded away from 0 and ∞ wpa1 (by Lemma G.4 and Assumptions 2(i)(iii)), we have $\|Z_n^*\| = O_{p^*}(\sqrt{K})$ wpa1 by Chebyshev's inequality.

Now, writing out term by term we have:

$$\begin{aligned} & \sup_{t \in \mathcal{T}} \left| \left(\frac{(Df_t(\hat{h})[\psi^J])' [\hat{S}' \hat{G}_b^{-1} \hat{S}]^{-1} \hat{S}' \hat{G}_b^{-1/2}}{\hat{\sigma}_n(f_t)} - \frac{(Df_t(h_0)[\psi^J])' [S' G_b^{-1} S]^{-1} S' G_b^{-1/2} (\Omega^\circ)^{1/2} (\hat{\Omega}^\circ)^{-1/2}}{\sigma_n(f_t)} \right) Z_n^* \right| \\ & \leq \sup_{t \in \mathcal{T}} \left| \frac{(Df_t(\hat{h})[\psi^J] - Df_t(h_0)[\psi^J])' (G_b^{-1/2} S)_l^{-1} G_b^{-1/2} S [\hat{S}' \hat{G}_b^{-1} \hat{S}]^{-1} \hat{S}' \hat{G}_b^{-1/2}}{\hat{\sigma}_n(f_t)} Z_n^* \right| \\ & + \sup_{t \in \mathcal{T}} \left| \frac{(Df_t(h_0)[\psi^J])' \left([\hat{S}' \hat{G}_b^{-1} \hat{S}]^{-1} \hat{S}' \hat{G}_b^{-1/2} - [S' G_b^{-1} S]^{-1} S' G_b^{-1/2} (\Omega^\circ)^{1/2} (\hat{\Omega}^\circ)^{-1/2} \right)}{\sigma_n(f_t)} Z_n^* \right| \times \sup_{t \in \mathcal{T}} \frac{\sigma_n(f_t)}{\hat{\sigma}_n(f_t)} \\ & + \sup_{t \in \mathcal{T}} \left| \frac{\sigma_n(f_t)}{\hat{\sigma}_n(f_t)} - 1 \right| \times \sup_{t \in \mathcal{T}} \left| \frac{(Df_t(h_0)[\psi^J])' [S' G_b^{-1} S]^{-1} S' G_b^{-1/2} (\Omega^\circ)^{1/2} (\hat{\Omega}^\circ)^{-1/2}}{\sigma_n(f_t)} Z_n^* \right| =: T_1 + T_2 + T_3. \end{aligned}$$

Control of T_1 : By Cauchy-Schwarz, we have:

$$\begin{aligned} T_1 & \leq \sup_{t \in \mathcal{T}} \frac{\|\Pi_K T(\hat{v}_n(f_t) - v_n(f_t))\|_{L^2(W)}}{\sigma_n(f_t)} \times \sup_{t \in \mathcal{T}} \frac{\sigma_n(f_t)}{\hat{\sigma}_n(f_t)} \times \left\| G_b^{-1/2} S [\hat{S}' \hat{G}_b^{-1} \hat{S}]^{-1} \hat{S}' \hat{G}_b^{-1/2} \right\|_{\ell^2} \times \|(\Omega^\circ)^{1/2} (\hat{\Omega}^\circ)^{-1/2}\|_{\ell^2} \times \|Z_n^*\|_{\ell^2} \\ & = o_p(\eta'_n) \times O_p(1) \times O_p(1) \times O_p(1) \times O_{p^*}(\sqrt{K}) \end{aligned}$$

where the first term is by Assumption 5(b)(iii) (or zero if the f_t are linear functionals), the second term is by Lemma G.5, the third is by Lemma F.10(c) (using the fact that $s_{JK} \asymp \tau_J$, see Lemma G.1) and the fact that $\|G_b^{-1/2} S [S' G_b^{-1} S]^{-1} S' G_b^{-1/2}\|_{\ell^2} = 1$, and the fourth term is by Lemma G.4. Therefore, $T_1 = O_{p^*}(\eta'_n \sqrt{J})$ wpa1 (since $K \asymp J$), and is therefore $o_p^*(c_n^{-1})$ wpa1 under Assumption 6(ii).

Control of T_2 : Let $\Delta\mathbb{Z}_n(t)$ denote the Gaussian process (under \mathbb{P}^*) defined by

$$\Delta\mathbb{Z}_n(t) = \frac{(Df_t(h_0)[\psi^J])' \left([\widehat{S}'\widehat{G}_b^{-1}\widehat{S}]^{-1}\widehat{S}'\widehat{G}_b^{-1/2} - [S'G_b^{-1}S]^{-1}S'G_b^{-1/2}(\Omega^o)^{1/2}(\widehat{\Omega}^o)^{-1/2} \right)}{\widehat{\sigma}_n(f_t)} \mathcal{Z}_n^*$$

for each $t \in \mathcal{T}$. The intrinsic semi-metric $\Delta d_n(t_1, t_2)$ of $\Delta\mathbb{Z}_n(t)$ is $\Delta d_n(t_1, t_2)^2 = E^*[(\Delta\mathbb{Z}_n(t_1) - \Delta\mathbb{Z}_n(t_2))^2]$ for each $t_1, t_2 \in \mathcal{T}$, where E^* denotes expectation under the measure \mathbb{P}^* . Observe that:

$$\begin{aligned} \Delta d_n(t_1, t_2) &= \left\| \left(\frac{Df_{t_1}(h_0)[\psi^J]}{\sigma_n(f_{t_1})} - \frac{Df_{t_2}(h_0)[\psi^J]}{\sigma_n(f_{t_2})} \right)' [S'G_b^{-1}S]^{-1}S'G_b^{-1/2}(\Omega^o)^{1/2} \right. \\ &\quad \times (\Omega^o)^{-1/2}G_b^{-1/2}S \left\{ [\widehat{S}'\widehat{G}_b^{-1}\widehat{S}]^{-1}\widehat{S}'\widehat{G}_b^{-1/2} - [S'G_b^{-1}S]^{-1}S'G_b^{-1/2}(\Omega^o)^{1/2}(\widehat{\Omega}^o)^{-1/2} \right\} (\widehat{\Omega}^o)^{1/2} \left. \right\|_{\ell^2} \\ &\lesssim d_n(t_1, t_2) \times \left\| G_b^{-1/2}S' \left\{ [\widehat{S}'\widehat{G}_b^{-1}\widehat{S}]^{-1}\widehat{S}'\widehat{G}_b^{-1/2} - [S'G_b^{-1}S]^{-1}S'G_b^{-1/2}(\Omega^o)^{1/2}(\widehat{\Omega}^o)^{-1/2} \right\} \right\|_{\ell^2} \end{aligned}$$

wpa1, where the first line uses the fact that $\widehat{\Omega}^o$ is invertible wpa1 and the second line uses the fact that Ω^o and $\widehat{\Omega}^o$ have eigenvalue uniformly bounded away from 0 and ∞ wpa1. It follows by Lemma F.10(c) and Lemmas G.4 and F.3 that

$$\begin{aligned} &\left\| G_b^{-1/2}S' \left\{ [\widehat{S}'\widehat{G}_b^{-1}\widehat{S}]^{-1}\widehat{S}'\widehat{G}_b^{-1/2} - [S'G_b^{-1}S]^{-1}S'G_b^{-1/2}(\Omega^o)^{1/2}(\widehat{\Omega}^o)^{-1/2} \right\} \right\|_{\ell^2} \\ &\leq \left\| G_b^{-1/2}S' \left\{ [\widehat{S}'\widehat{G}_b^{-1}\widehat{S}]^{-1}\widehat{S}'\widehat{G}_b^{-1/2} - [S'G_b^{-1}S]^{-1}S'G_b^{-1/2} \right\} \right\|_{\ell^2} + \left\| I - (\Omega^o)^{1/2}(\widehat{\Omega}^o)^{-1/2} \right\|_{\ell^2} \\ &= O_p(\tau_J \zeta \sqrt{(\log n)/n}) + O_p\left((\zeta_{b,K}^{(2+\delta)/\delta} \sqrt{(\log K)/n})^{\delta/(1+\delta)} + \delta_{h,n} \right) \\ &= O_p(\delta_{V,n} + \delta_{h,n}). \end{aligned}$$

Therefore,

$$\Delta d_n(t_1, t_2) \leq O_p(\delta_{V,n} + \delta_{h,n}) \times d_n(t_1, t_2)$$

wpa1. Moreover, by similar arguments we have

$$\sup_{t \in \mathcal{T}} E^*[(\Delta\mathbb{Z}_n(t))^2]^{1/2} \lesssim O_p(\delta_{V,n} + \delta_{h,n})$$

wpa1. Therefore, we can scale $\Delta\mathbb{Z}_n(t)$ by dividing through by a sequence of positive constants of order $(\delta_{V,n} + \delta_{h,n})$ to obtain

$$E^*[\sup_{t \in \mathcal{T}} |\Delta\mathbb{Z}_n(t)|] \lesssim O_p((\delta_{V,n} + \delta_{h,n})) \times c_n$$

wpa1 by identical arguments to the proof of Lemma G.6. Therefore,

$$T_2 \leq O_p((\delta_{V,n} + \delta_{h,n}) \times c_n) \times \sup_{t \in \mathcal{T}} \frac{\sigma_n(f_t)}{\widehat{\sigma}_n(f_t)} = O_p((\delta_{V,n} + \delta_{h,n}) \times c_n) \times O_p(1)$$

wpa1 by Lemma G.5 and so $T_2 = o_p^*(c_n^{-1})$ under Assumption 6(ii).

Control of T_3 : The second term in T_3 is the supremum of a Gaussian process with the same distribution (under \mathbb{P}^*) as $\mathbb{Z}_n(t)$ (under \mathbb{P}). Therefore, by Lemmas G.5 and G.6 we have:

$$T_3 = O_p(\delta_{V,n} + \delta_{h,n} + \eta'_n) \times O_p^*(c_n)$$

and so $T_3 = o_p^*(c_n^{-1})$ wpa1 under Assumption 6(ii). \square

Proof of Remark 4.2. For each $t_1, t_2 \in \mathcal{T}$ we have:

$$\begin{aligned}
d_n(t_1, t_2) &\leq \frac{2}{\sigma_n(f_{t_1}) \vee \sigma_n(f_{t_2})} \left\| \Omega^{1/2} G_b^{-1} S [S' G_b^{-1} S]^{-1} (Df_{t_1}(h_0)[\psi^J] - Df_{t_2}(h_0)[\psi^J]) \right\|_{\ell^2} \\
&\leq \frac{2\bar{\sigma}_{JK}^{-1}}{\underline{\sigma}_n} \left\| G_\psi^{-1/2} (Df_{t_1}(h_0)[\psi^J] - Df_{t_2}(h_0)[\psi^J]) \right\|_{\ell^2} \\
&\lesssim \frac{\tau_J \Gamma_n}{\underline{\sigma}_n} \|t_1 - t_2\|_{\ell^2}^{\gamma_n}
\end{aligned} \tag{60}$$

where the first inequality is because $\|x/\|x\| - y/\|y\|\| \leq 2\|x - y\|/(\|x\| \vee \|y\|)$ whenever $\|x\|, \|y\| \neq 0$ and the third is by the equivalence $s_{JK}^{-1} \asymp \tau_J$ (see Lemma G.1). By (60) and compactness of \mathcal{T} , we have $N(\mathcal{T}, d_n, \epsilon) \leq C(\tau_J \Gamma_n / \underline{\sigma}_n \epsilon^{\gamma_n})^{d_T} \vee 1$ for some finite constant C . \square

Proof of Corollary 4.1. By assumption we may take $\underline{\sigma}_n \asymp \tau_J J^a$ with $a = \frac{1}{2} + \frac{|\alpha|}{d}$. Assumption 5 is therefore satisfied with $\eta_n = \sqrt{n} \tau_J^{-1} J^{-(p/d+1/2)}$ by Remark 4.1(a') and Lemma A.2.

The continuity condition in Remark 4.2 holds with $\Gamma_n = O(J^{a'})$ for some $a' > 0$ and $\gamma_n = 1$ since Ψ_J is spanned by a B-spline basis of order $\gamma > (p \vee 2 + |\alpha|)$ (DeVore and Lorentz, 1993, Section 5.3). Assumption 6(i) therefore holds with $c_n = O(\sqrt{\log J})$ by Remark 4.2 because $\tau_J \Gamma_n \epsilon^{-\gamma_n} / \underline{\sigma}_n \lesssim J^{a'-a} \epsilon^{-1}$. We can therefore take $r_n = (\log J)^{-1}$ in Assumption 6(ii). The first condition in Assumption 6(ii) then holds provided $J^5 (\log J)^6 / n = o(1)$ and the second holds provided

$$\sqrt{n} \tau_J^{-1} J^{-(p/d+1/2)} + \left((J^{\frac{2+\delta}{2\delta}} \sqrt{(\log J)/n}^{\frac{\delta}{1+\delta}} + J^{-(p-|\alpha|)/d} + \tau_J J^a \sqrt{(\log J)/n}) \sqrt{\log J} \right) = o((\log J)^{-1/2})$$

(using Corollary 3.1 for $\delta_{h,n}$). In applying Corollary 3.1, we also require that the conditions $\tau_J J / \sqrt{n} = O(1)$ and $J^{(2+\delta)/2\delta} \sqrt{(\log n)/n} = o(1)$ hold. Sufficient conditions for these restrictions on J are $J^{-p/d} = o(\tau_J \sqrt{(J \log J)/n})$, $\tau_J (J^a (\log J)^{3/2} \vee J) / \sqrt{n} = o(1)$, and $J^{(2+\delta)/\delta} (\log J)^{(2\delta+1)/(1+\delta)} (\log n) = o(n^{\delta/(1+\delta)})$. \square

G.5 Proofs for Section 5

Proof of Theorem 5.1. The result will follow from Theorem D.1. Assumption 2(i)–(iii)(iv') is satisfied under Assumption CS(iii). Assumption 3(i)(ii)(iii) is satisfied by Assumption CS (iv) and the second part of Assumption CS(v), noting that $\zeta_{\psi,J} = O(\sqrt{J})$ and $\zeta_{b,K} = O(\sqrt{K}) = O(\sqrt{J})$. Since the basis spanning Ψ_J is a Riesz basis for T Assumption 4 is satisfied with $\tau_J \asymp \mu_J^{-1}$.

It remains to verify Assumption 5'(b). By the Riesz basis property and Assumption 2(i)–(iii) we have $[\sigma_n(f_{CS})]^2 \asymp \sum_{j=1}^J (a_j / \mu_j)^2$ (see Section 6 of Chen and Pouzo (2015)). For f_{CS} we have

$$Df_{CS}(h_0)[h - h_0] = \int_0^1 \left(\{h(\mathbf{p}(t), \mathbf{y} - S_y(t)) - h_0(t, \mathbf{y} - S_y(t))\} e^{-\int_0^t \partial_2 h_0(\mathbf{p}(v), \mathbf{y} - S_y(v)) \mathbf{p}'(v) dv} \mathbf{p}'(t) \right) dt$$

(Hausman and Newey, 1995, p. 1471) which is clearly a linear functional (Assumption 5'(b)(i)). Since $p > 2$ and $\mu_J^{-1} J^{3/2} \sqrt{(\log J)/n} = o(1)$, by Corollary 3.1 we have $\|\widehat{h} - h_0\|_{B_{\infty,\infty}^2} = o_p(1)$ and $\|\widehat{h} - h_0\|_{B_{\infty,\infty}^1} = o_p(1)$, and

$$\|\widehat{h} - h_0\|_{\infty} = O_p \left(J^{-p/2} + \mu_J^{-1} \sqrt{(J \log J)/n} \right), \quad \|\widehat{h} - h_0\|_{B_{\infty,\infty}^1} = O_p \left(\sqrt{J} \left(J^{-p/2} + \mu_J^{-1} \sqrt{(J \log J)/n} \right) \right),$$

and $\|\tilde{h} - h_0\|_\infty = O_p(J^{-p/2})$. Applying Lemma A1 of Hausman and Newey (1995), we obtain

$$\begin{aligned} \left| f_{CS}(\hat{h}) - f_{CS}(h_0) - Df_{CS}(h_0)[\hat{h} - h_0] \right| &= O_p \left(\sqrt{J} \left(J^{-p} + \mu_J^{-2} \frac{J \log J}{n} \right) \right) \\ \left| Df_{CS}(h_0)[\tilde{h} - h_0] \right| &= O_p(J^{-p/2}). \end{aligned}$$

Therefore, Assumption CS(v) guarantees that Assumption 5'(b)(ii) holds and $\sigma_n(f)^{-1} Df(h_0)[\tilde{h} - h_0] = o_p(n^{-1/2})$. Finally, for Assumption 5'(b)(iii), we have

$$\frac{\|\Pi_K T(\hat{v}_n(f_{CS}) - v_n(f_{CS}))\|_{L^2(W)}^2}{[\sigma_n(f_{CS})]^2} \lesssim \tau_J^2 \frac{\sum_{j=1}^J \left(Df_{CS}(\hat{h})[(G_\psi^{-1/2} \psi^J)_j] - Df_{CS}(h_0)[(G_\psi^{-1/2} \psi^J)_j] \right)^2}{[\sigma_n(f_{CS})]^2}$$

where $\tau_J \asymp \mu_J^{-1}$. Moreover,

$$\left| Df_{CS}(\hat{h})[(G_\psi^{-1/2} \psi^J)_j] - Df_{CS}(h_0)[(G_\psi^{-1/2} \psi^J)_j] \right| \lesssim \sqrt{J} \times O_p \left(\sqrt{J} \left(J^{-p/2} + \mu_J^{-1} \sqrt{(J \log J)/n} \right) \right)$$

(uniformly in $j = 1, \dots, J$) by Lemma A1 of Hausman and Newey (1995). Therefore

$$\frac{\|\Pi_K T(\hat{v}_n(f_{CS}) - v_n(f_{CS}))\|_{L^2(W)}}{\sigma_n(f_{CS})} \lesssim \frac{J^{3/2} \mu_J^{-1}}{\left(\sum_{j=1}^J (a_j / \mu_j)^2 \right)^{1/2}} \times O_p \left(J^{-p/2} + \mu_J^{-1} \sqrt{(J \log J)/n} \right).$$

which is $o_p(1)$ by Assumption CS(v). This proves the result. \square

Proof of Corollary 5.1. For **Result (1)**, since $\sigma_n(f_{CS}) \asymp J^{(a+\varsigma+1)/2}$, the first part of Assumption CS(v) is satisfied provided

$$\frac{\sqrt{n}}{J^{(a+\varsigma+1)/2}} \left(J^{-p/2} + J^{\varsigma+2} \sqrt{(\log J)/n} \right) = o(1)$$

for which a sufficient condition is $nJ^{-(p+a+\varsigma+1)} = o(1)$ and $J^{3+\varsigma-a}(\log n)/n = o(1)$. Moreover, the condition $\mu_J^{-1} J^{3/2} \sqrt{(\log J)/n} = o(1)$ is implied by $J^{3+\varsigma-(a \wedge 0)}(\log n)/n = o(1)$. The condition $J^{3+\varsigma-(a \wedge 0)}(\log n)/n = o(1)$ also implies that $J^{(2+\delta)/(2\delta)} \sqrt{(\log n)/n} = o(1)$ holds whenever $\delta \geq 2/(2 + \varsigma - (a \wedge 0))$.

For **Result (2)**, we have $\sigma_n(f_{CS})^2 \gtrsim a_J^2 / \mu_J^2 \asymp \exp(J^{\varsigma/2} + a \log J)$. Take $J = (\log(n/(\log n)^e))^{2/\varsigma}$. Then

$$\begin{aligned} \sigma_n(f_{CS})^2 &\gtrsim \exp \left(\log(n/(\log n)^e) + \log[(\log(n/(\log n)^e))^{2a/\varsigma}] \right) \\ &= \exp \left(\log[n/(\log n)^e] \times (\log(n/(\log n)^e))^{2a/\varsigma} \right) \\ &= n/(\log n)^e \times (\log(n/(\log n)^e))^{2a/\varsigma} \end{aligned}$$

and so

$$\sigma_n(f_{CS}) \gtrsim \frac{\sqrt{n}}{(\log n)^{e/2}} \times (\log(n/(\log n)^e))^{a/\varsigma}.$$

The first part of Assumption CS(v) is then satisfied provided

$$\frac{(\log n)^{e/2}}{(\log(n/(\log n)^e))^{a/\varsigma}} \left((\log(n/(\log n)^e))^{-p/\varsigma} + (\log n)^{-e} \times (\log(n/(\log n)^e))^{4/\varsigma} \log \log n \right) = o(1)$$

which holds provided $2p > \varrho_\varsigma - 2a$ and $\varrho_\varsigma > 8 - 2a$. The condition $J^{(2+\delta)/(2\delta)} \sqrt{(\log n)/n} = o(1)$ holds for any

$\delta > 0$. The remaining condition $\mu_J^{-1} J^{3/2} \sqrt{(\log J)/n} = o(1)$ is implied by

$$\frac{\sqrt{n}}{(\log n)^{\varrho/2}} (\log(n/(\log n)^\varrho))^{3/\varsigma} \sqrt{(\log \log n)/n} = o(1)$$

for which a sufficient condition is $\varrho\varsigma > 6$. Now, we may always choose $\varrho > 0$ so that $\varrho\varsigma > 6 \vee (8 - 2a)$. The remaining condition then holds provided $2p > 6 \vee (8 - 2a) - 2a$. \square

Proof of Theorem 5.2. The proof follows by identical arguments to Theorem 5.1, noting that

$$f_{DL}(h) - f_{CS}(h) = (\mathbf{p}^1 - \mathbf{p}^0)h(\mathbf{p}^1, \mathbf{y})$$

and so

$$\begin{aligned} f_{DL}(\widehat{h}) - f_{DL}(h_0) &= f_{CS}(\widehat{h}) - f_{CS}(h_0) + (\mathbf{p}^1 - \mathbf{p}^0) \left(\widehat{h}(\mathbf{p}^1, \mathbf{y}) - h_0(\mathbf{p}^1, \mathbf{y}) \right) \\ Df_{DL}(\widehat{h})[h - h_0] &= Df_{CS}(h_0)[h - h_0] + (\mathbf{p}^1 - \mathbf{p}^0) \left(\widehat{h}(\mathbf{p}^1, \mathbf{y}) - h_0(\mathbf{p}^1, \mathbf{y}) \right) \\ Df_{DL}(\widehat{h})[v] - Df_{DL}(h_0)[v] &= Df_{CS}(\widehat{h})[v] - Df_{CS}(h_0)[v] \end{aligned}$$

where clearly $(\mathbf{p}^1 - \mathbf{p}^0)(\widehat{h}(\mathbf{p}^1, \mathbf{y}) - h_0(\mathbf{p}^1, \mathbf{y})) \leq \text{const} \times \|\widehat{h} - h_0\|_\infty$. Since $\sigma_n(f_{DL}) \asymp \mu_J^{-1} \sqrt{J}$, to verify Assumption CS(v) we require

$$\frac{\sqrt{n}}{\mu_J^{-1} \sqrt{J}} \times \left(J^{-p/2} + \mu_J^{-2} \frac{J \log J}{n} \right) = o(1)$$

and $(J^{(2+\delta)/(2\delta)} \sqrt{(\log n)} \vee J^{3/2} \mu_J^{-1} \sqrt{(\log J)}) / \sqrt{n} = o(1)$. The conditions on J are satisfied by the conditions on J in the statement of the Theorem. \square

Proof of Theorem 5.3. Assumptions 2(i)–(iii)(iv'), 3(i)(ii)(iii), and 4 are verified as in the proof of Theorem 5.1. It remains to verify Assumption 5'(b). As in the proof of Theorem 5.1 we have $\tau_J \asymp \mu_J^{-1}$ and $[\sigma_n(f_A)]^2 \asymp \sum_{j=1}^J (a_j/\mu_j)^2$ (see Section 6 of Chen and Pouzo (2015)). Simple expansion of f_A yields

$$Df_A(h_0)[h - h_0] = \int w(\mathbf{p}) e^{h_0(\log \mathbf{p}, \log \mathbf{y})} (h(\log \mathbf{p}, \log \mathbf{y}) - h_0(\log \mathbf{p}, \log \mathbf{y})) d\mathbf{p}$$

which is clearly a linear functional (Assumption 5'(b)(i)), and

$$\begin{aligned} & \left| f_A(\widehat{h}) - f_A(h_0) - Df_A(h_0)[\widehat{h} - h_0] \right| \\ &= \int w(\mathbf{p}) \left(e^{\widehat{h}(\log \mathbf{p}, \log \mathbf{y}) - h_0(\log \mathbf{p}, \log \mathbf{y})} - 1 - (\widehat{h}(\log \mathbf{p}, \log \mathbf{y}) - h_0(\log \mathbf{p}, \log \mathbf{y})) \right) e^{h_0(\log \mathbf{p}, \log \mathbf{y})} d\mathbf{p}. \end{aligned}$$

Therefore, by Corollary 3.1 we have

$$\begin{aligned} \left| f_A(\widehat{h}) - f_A(h_0) - Df_A(h_0)[\widehat{h} - h_0] \right| &= O_p \left(J^{-p} + \mu_J^{-2} \frac{J \log J}{n} \right) \\ \left| Df_A(h_0)[\widehat{h} - h_0] \right| &= O_p(J^{-p/2}). \end{aligned}$$

Assumption ACS(v) therefore guarantees Assumption 5'(b)(ii) holds and $\sigma_n(f)^{-1} Df(h_0)[\widehat{h} - h_0] = o_p(n^{-1/2})$.

Finally, for Assumption 5'(b)(iii), we have

$$\frac{\|\Pi_{KT}(\widehat{v}_n(f_A) - v_n(f_A))\|_{L^2(W)}^2}{[\sigma_n(f_A)]^2} \lesssim \frac{\tau_J^2}{[\sigma_n(f_A)]^2} \left(\sum_{j=1}^J \left(Df_A(\widehat{h})[(G_\psi^{-1/2}\psi^J)_j] - Df_A(h_0)[(G_\psi^{-1/2}\psi^J)_j] \right)^2 \right)$$

where $\tau_J \asymp \mu_J^{-1}$ and where a first-order Taylor expansion of Df_A yields

$$\begin{aligned} \left| Df_A(\widehat{h})[(G_\psi^{-1/2}\psi^J)_j] - Df_A(h_0)[(G_\psi^{-1/2}\psi^J)_j] \right| &\lesssim \|(G_\psi^{-1/2}\psi^J)_j\|_\infty \times \|\widehat{h} - h_0\|_\infty \\ &= O_p \left(\sqrt{J} \left(J^{-p/2} + \mu_J^{-1} \sqrt{(J \log J)/n} \right) \right). \end{aligned}$$

It follows that

$$\frac{\|\Pi_{KT}(\widehat{v}_n(f_A) - v_n(f_A))\|_{L^2(W)}}{[\sigma_n(f_A)]} \lesssim \frac{\mu_J^{-1} J}{\left(\sum_{j=1}^J (a_j/\mu_j)^2 \right)^{1/2}} \times O_p \left(J^{-p/2} + \mu_J^{-1} \sqrt{(J \log J)/n} \right)$$

which is $o_p(1)$ by the condition on J in the statement of the result. \square

Proof of Corollary 5.2. For **Result (1)**, we apply Theorem 3.2 of [Chen and Christensen \(2015\)](#). Assumptions 1-4 of [Chen and Christensen \(2015\)](#) are satisfied by Assumption CS. We have

$$\|\widehat{h} - h_0\|_\infty = O_p \left(J^{-p/2} + \sqrt{(J \log J)/n} \right) \quad \|\widehat{h} - h_0\|_{B_{\infty,\infty}^1} = O_p \left(\sqrt{J} \left(J^{-p/2} + \sqrt{(J \log J)/n} \right) \right) \quad (61)$$

$$\|\widetilde{h} - h_0\|_\infty = O_p \left(J^{-p/2} \right) \quad \|\widetilde{h} - h_0\|_{B_{\infty,\infty}^1} = O_p \left(\sqrt{J} \left(J^{-p/2} \right) \right) \quad (62)$$

and $\|\widehat{h} - h_0\|_{B_{\infty,\infty}^2} = o_p(1)$ and $\|\widetilde{h} - h_0\|_{B_{\infty,\infty}^2} = o_p(1)$ by Corollary 3.1 and the conditions $p > 2$ and $J^{3/2} \sqrt{(\log J)/n} = o(1)$ and $J^{(2+\delta)/(2\delta)} \sqrt{(\log n)/n} = o(1)$. Let $\{C_n : n \geq 1\}$ be a positive sequence with $C_n \rightarrow \infty$ slowly enough that

$$C_n^2 \sqrt{n} J^{-a/2} \left(J^{-p} + \frac{J \log J}{n} \right) = o(1) \quad \text{and} \quad C_n J^{(2-a)/2} \left(J^{-p/2} + \sqrt{(J \log J)/n} \right) = o(1)$$

(note that we can always choose such a sequence by the conditions on J in the statement of the Corollary). Let $N_{J,n} = \{h \in B_\infty(2, \epsilon) \cap \Psi_J : \|h - h_0\|_\infty < C_n (J^{-p/2} + \sqrt{(J \log J)/n}) \text{ and } \|h - h_0\|_{B_{\infty,\infty}^1} < C_n \sqrt{J} (J^{-p/2} + \sqrt{(J \log J)/n})\}$ and note that $\widehat{h}, \widetilde{h} \in N_{J,n}$ wpa1. Moreover, by Lemma A1 of [Hausman and Newey \(1995\)](#) we have

$$\sup_{h \in N_{J,n}} \frac{\sqrt{n}}{\sigma_n(f_{CS})} |f_{CS}(h) - f_{CS}(h_0) - Df_{CS}(h_0)[h - h_0]| \lesssim C_n^2 \sqrt{n} J^{-(a+1)/2} \sqrt{J} \left(J^{-p} + \frac{J \log J}{n} \right) = o(1)$$

and

$$\frac{\sqrt{n}}{\sigma_n(f_{CS})} |Df_{CS}(h_0)[\widetilde{h} - h_0]| = O_p(\sqrt{n} J^{-(a+1+p)/2}) = o_p(1).$$

This verifies Assumption 9 of [Chen and Christensen \(2015\)](#). For their Assumption 10, by Lemma A1 of [Hausman and Newey \(1995\)](#) we have

$$\sup_{h \in N_{J,n}} \left| Df_{CS}(h)[(G_\psi^{-1/2}\psi^J)_j] - Df_{CS}(h_0)[(G_\psi^{-1/2}\psi^J)_j] \right| \lesssim C_n J (J^{-p/2} + \sqrt{(J \log J)/n})$$

uniformly in $j = 1, \dots, J$. Therefore,

$$\frac{1}{\sigma_n(f_{CS})} \sup_{h \in N_{J,n}} \left\| Df_{CS}(h)[G_\psi^{-1/2}\psi^J] - Df_{CS}(h_0)[G_\psi^{-1/2}\psi^J] \right\| \lesssim C_n J^{-(a+1)/2} J^{3/2} (J^{-p/2} + \sqrt{(J \log J)/n}) = o(1).$$

This verifies Assumption 10 of [Chen and Christensen \(2015\)](#).

Result (2) follows by similar arguments, choosing C_n such that

$$C_n^2 \sqrt{n} \left(J^{-p} + \frac{J \log J}{n} \right) = o(1) \quad \text{and} \quad C_n J (J^{-p/2} + \sqrt{(J \log J)/n}) = o(1)$$

and noting that

$$f_{DL}(h) - f_{DL}(h_0) - Df_{DL}(h_0)[h - h_0] = f_{CS}(h) - f_{CS}(h_0) - Df_{CS}(h_0)[h - h_0]$$

and hence

$$\sup_{h \in N_{J,n}} \frac{\sqrt{n}}{\sigma_n(f_{DL})} |f_{DL}(h) - f_{DL}(h_0) - Df_{DL}(h_0)[h - h_0]| \lesssim C_n^2 \sqrt{n} J^{-1/2} \sqrt{J} \left(J^{-p} + \frac{J \log J}{n} \right) = o(1)$$

and $Df_{DL}(h_0)[h - h_0] = Df_{CS}(h_0)[h - h_0] + (\mathbf{p}^1 - \mathbf{p}^0) \left(\widehat{h}(\mathbf{p}^1, \mathbf{y}) - h_0(\mathbf{p}^1, \mathbf{y}) \right)$, hence

$$\sqrt{n} J^{-1/2} |Df_{DL}(h_0)[\tilde{h} - h_0]| = O_p(\sqrt{n} J^{-(1+p)/2}) = o_p(1).$$

Finally, $Df_{DL}(h)[v] - Df_{DL}(h_0)[v] = Df_{CS}(h)[v] - Df_{CS}(h_0)[v]$, so we have

$$\frac{1}{\sigma_n(f_{DL})} \sup_{h \in N_{J,n}} \left\| Df_{DL}(h)[G_\psi^{-1/2}\psi^J] - Df_{DL}(h_0)[G_\psi^{-1/2}\psi^J] \right\| \lesssim C_n J^{-1/2} J^{3/2} (J^{-p/2} + \sqrt{(J \log J)/n}) = o(1).$$

□

Proof of Corollary 5.3. The result follows by similar arguments to the proof of Corollary 5.2. Note that

$$\|\widehat{h} - h_0\|_\infty = O_p \left(J^{-p/2} + \sqrt{(J \log J)/n} \right) = o_p(1) \quad \|\tilde{h} - h_0\|_\infty = O_p \left(J^{-p/2} \right) = o_p(1)$$

by Corollary 3.1 and the conditions $J/\sqrt{n} = o(1)$ and $J^{(2+\delta)/(2\delta)} \sqrt{(\log n)/n} = o(1)$. Let $\{C_n : n \geq 1\}$ be a positive sequence with $C_n \rightarrow \infty$ slowly enough that

$$C_n^2 \sqrt{n} J^{-(c+1)/2} \left(J^{-p} + \frac{J \log J}{n} \right) = o(1) \quad \text{and} \quad C_n J^{(1-c)/2} (J^{-p/2} + \sqrt{(J \log J)/n}) = o(1)$$

(note that we can always choose such a sequence by the conditions on J in the statement of the Corollary). Let $N_{J,n} = \{h \in \Psi_J : \|h - h_0\|_\infty < C_n (J^{-p/2} + \sqrt{(J \log J)/n})\}$ and note that $\widehat{h}, \tilde{h} \in N_{J,n}$ wpa1. By the proof of Theorem 5.3, we have

$$\sup_{h \in N_{J,n}} \frac{\sqrt{n}}{\sigma_n(f_A)} |f_A(h) - f_A(h_0) - Df_A(h_0)[h - h_0]| \lesssim C_n^2 \sqrt{n} J^{-(c+1)/2} \left(J^{-p} + \frac{J \log J}{n} \right) = o(1)$$

and

$$\frac{\sqrt{n}}{\sigma_n(f_A)} |Df_A(h_0)[\tilde{h} - h_0]| = O_p(\sqrt{n} J^{-(c+1+p)/2}) = o_p(1).$$

Finally,

$$\sup_{h \in N_{J,n}} \left| Df_A(h)[(G_\psi^{-1/2}\psi^J)_j] - Df_A(h_0)[(G_\psi^{-1/2}\psi^J)_j] \right| \lesssim C_n \sqrt{J}(J^{-p/2} + \sqrt{(J \log J)/n})$$

uniformly in $j = 1, \dots, J$. Therefore,

$$\frac{1}{\sigma_n(f_A)} \sup_{h \in N_{J,n}} \left\| Df_A(h)[G_\psi^{-1/2}\psi^J] - Df_A(h_0)[G_\psi^{-1/2}\psi^J] \right\| \lesssim C_n J^{-(c+1)/2} J(J^{-p/2} + \sqrt{(J \log J)/n}) = o(1).$$

□

Proof of Theorem 5.4. We verify the conditions of Theorem 4.1. Assumptions 1 and 2 are satisfied by Assumption CS(i)(ii)(iv) and U-CS(i). Assumption 3(iii) is satisfied by Assumption U-CS(iii). Assumption 4(i) is satisfied by the Riesz basis condition. For Assumption 5(b), we check the conditions of Remark 4.1(b'). It is clear that $Df_{CS,t}[h - h_0]$ (see display (14)) is a linear functional of $h - h_0$ for each $t \in \mathcal{T}$. Since $p > 2$, by Corollary 3.1 and the conditions $\mu_J^{-1} J^{3/2} \sqrt{(\log J)/n} = o(1)$ and $J^{(2+\delta)/(2\delta)} \sqrt{(\log n)/n} = o(1)$ we have $\|\hat{h} - h_0\|_{B_{\infty,\infty}^2} = o_p(1)$ and $\|\tilde{h} - h_0\|_{B_{\infty,\infty}^2} = o_p(1)$ and (61) and (62) hold by Corollary 3.1. Therefore, \hat{h} and \tilde{h} are within an ϵ neighborhood (in Hölder norm of smoothness 2) of h_0 wpa1. As $\mathcal{T} = [\underline{p}^0, \bar{p}^0] \times [\underline{p}^1, \bar{p}^1] \times [\underline{y}, \bar{y}]$ where the intervals $[\underline{p}^0, \bar{p}^0]$ and $[\underline{p}^1, \bar{p}^1]$ are in the interior of the support of P_i and $[\underline{y}, \bar{y}]$ is in the interior of the support of Y_i and $h_0 \in B_\infty(p, L)$ with $p > 2$ and $0 < L < \infty$, it is straightforward to extend Lemma A1 of Hausman and Newey (1995) to show

$$\begin{aligned} \sup_{t \in \mathcal{T}} \left| f_{CS,t}(\hat{h}) - f_{CS,t}(h_0) - Df_{CS,t}(h_0)[\hat{h} - h_0] \right| &= O_p \left(\sqrt{J} \left(J^{-p} + \mu_J^{-2} \frac{J \log J}{n} \right) \right) \\ \sup_{t \in \mathcal{T}} \left| Df_{CS,t}(h_0)[\tilde{h} - h_0] \right| &= O_p(J^{-p/2}). \end{aligned}$$

by Corollary 3.1. As in the proof of Theorem 5.1 we have $[\sigma_n(f_{CS,t})]^2 \asymp \sum_{j=1}^J (a_{j,t}/\mu_j)^2$ uniformly in t (see Section 6 of Chen and Pouzo (2015)). Therefore, Assumption U-CS(iii) guarantees that Assumption 5(b)(ii) holds with

$$\eta_n = \frac{\sqrt{n}}{\underline{\sigma}_n} \times \left(J^{-p/2} + \mu_J^{-2} \frac{J^{3/2} \log J}{n} \right).$$

For Assumption 5(b)(iii), we have

$$\sup_{t \in \mathcal{T}} \frac{\|\Pi_K T(\hat{v}_n(f_{CS,t}) - v_n(f_{CS,t}))\|_{L^2(W)}^2}{[\sigma_n(f_{CS,t})]^2} \lesssim \sup_{t \in \mathcal{T}} \frac{\tau_J^2 \sum_{j=1}^J \left(Df_{CS,t}(\hat{h})[(G_\psi^{-1/2}\psi^J)_j] - Df_{CS,t}(h_0)[(G_\psi^{-1/2}\psi^J)_j] \right)^2}{[\sigma_n(f_{CS,t})]^2}$$

where $\tau_J \asymp \mu_J^{-1}$. By straightforward extension of Lemma A1 of Hausman and Newey (1995) and (61) and (62):

$$\sup_{t \in \mathcal{T}} \left(Df_{CS,t}(\hat{h})[(G_\psi^{-1/2}\psi^J)_j] - Df_{CS,t}(h_0)[(G_\psi^{-1/2}\psi^J)_j] \right)^2 \lesssim \sqrt{J} \times O_p \left(\sqrt{J} \left(J^{-p/2} + \mu_J^{-1} \sqrt{(J \log J)/n} \right) \right)$$

whence Assumption 5(b)(iii) holds with

$$\eta'_n = \sup_{t \in \mathcal{T}} \frac{J^{3/2} \mu_J^{-1}}{\left(\sum_{j=1}^J (a_{j,t}/\mu_j)^2 \right)^{1/2}} \times \left(O_p \left(J^{-p/2} + \mu_J^{-1} \sqrt{J(\log J)/n} \right) \right).$$

which is $o(1)$ by Assumption U-CS(iii). This verifies Assumption 5(b).

Finally, Assumption 6(i) holds with $c_n = O(\sqrt{\log \tau_J})$ by Assumption U-CS(ii) and Remark 4.2. For Assumption

6(ii) take $r_n = 1/\sqrt{\log \tau_J \times \log \log n}$. The remaining conditions of Assumption 6(ii) are

$$\frac{\zeta_{b,K} J^2}{r_n^3 \sqrt{n}} = o(1)$$

which is implied by

$$J^{5/2} (\log \tau_J \times \log \log n)^{3/2} = o(\sqrt{n})$$

and

$$\eta_n + \eta'_n \sqrt{J} + (\delta_{V,n} + \delta_{h,n} + \eta'_n) \times c_n = o(c_n^{-1})$$

which are implied by the remaining conditions in Assumption U-CS(iii). \square

Proof of Theorem 5.5. Follows by similar arguments to the proofs of Theorems 5.2 and 5.4, noting that

$$|Df_{DL,t_1}(h_0)[h] - Df_{DL,t_2}(h_0)[h]| \leq |Df_{CS,t_1}(h_0)[h] - Df_{CS,t_2}(h_0)[h]| + |(\mathbf{p}_1^1 - \mathbf{p}_1^0)h(\mathbf{p}_1^1, y_1) - (\mathbf{p}_2^1 - \mathbf{p}_2^0)h(\mathbf{p}_2^1, y_2)|$$

and so $c_n = O(\sqrt{\log \tau_J} \vee \log J)$ by Assumption U-CS(ii) and Remark 4.2 (see the proof of Corollary 4.1). We also take $r_n = 1/\sqrt{\log(\tau_J \vee J) \times \log \log n}$. Also notice that $\log J \lesssim \log \tau_J$. \square

G.6 Proofs for Appendix B

Proof of Theorem B.1. As with the proof of Theorem 3.1, we first decompose the error into three parts:

$$\begin{aligned} \|\widehat{h} - h_0\|_{L^2(X)} &\leq \|\widehat{h} - \widetilde{h}\|_{L^2(X)} + \|\widetilde{h} - \Pi_J h_0\|_{L^2(X)} + \|\Pi_J h_0 - h_0\|_{L^2(X)} \\ &=: T_1 + T_2 + \|h_0 - \Pi_J h_0\|_{L^2(X)}. \end{aligned}$$

To prove **Result (1)** it is enough to show that $T_2 \leq O_p(1) \times \|h_0 - \Pi_J h_0\|_{L^2(X)}$. To do this, bound

$$\begin{aligned} T_2 &\leq \|G_\psi^{1/2} (S' G_b^{-1/2})_l^- G_b^{-1/2} B'(H_0 - \Psi c_J)/n\|_{\ell^2} \\ &\quad + \|G_\psi^{1/2} \{(\widehat{G}_b^{-1/2} \widehat{S})_l^- \widehat{G}_b^{-1/2} G_b^{1/2} - (G_b^{-1/2} S)_l^-\} G_b^{-1/2} B'(H_0 - \Psi c_J)/n\|_{\ell^2} =: T_{21} + T_{22}. \end{aligned}$$

For T_{21} ,

$$\begin{aligned} T_{21} &\leq s_{JK}^{-1} \|G_b^{-1/2} B'(H_0 - \Psi c_J)/n\|_{\ell^2} \\ &\leq O_p(\tau_J \zeta_{b,K} / \sqrt{n}) \times \|h_0 - \Pi_J h_0\|_{L^2(X)} + \tau_J \|\Pi_K T(h_0 - \Pi_J h_0)\|_{L^2(W)} \\ &\leq O_p(\tau_J \zeta_{b,K} / \sqrt{n}) \times \|h_0 - \Pi_J h_0\|_{L^2(X)} + \tau_J \|T(h_0 - \Pi_J h_0)\|_{L^2(W)} \\ &= O_p(1) \times \|h_0 - \Pi_J h_0\|_{L^2(X)} \end{aligned}$$

where the second line is by Lemma F.9 and the relations $J \asymp K$ and $\tau_J \asymp s_{JK}^{-1}$, and the final line is by Assumption 4(ii) and the condition $\tau_J \zeta \sqrt{(\log J)/n} = o(1)$. Similarly,

$$\begin{aligned} T_{22} &\leq \|G_\psi^{1/2} \{(\widehat{G}_b^{-1/2} \widehat{S})_l^- \widehat{G}_b^{-1/2} G_b^{1/2} - (G_b^{-1/2} S)_l^-\}\|_{\ell^2} \|G_b^{-1/2} B'(H_0 - \Psi c_J)/n\|_{\ell^2} \\ &\leq O_p(s_{JK}^{-2} \zeta \sqrt{(\log J)/n}) \times (O_p(\zeta_{b,K} / \sqrt{n}) \times \|h_0 - \Pi_J h_0\|_{L^2(X)} + \|T(h_0 - \Pi_J h_0)\|_{L^2(W)}) \\ &= o_p(\tau_J \zeta \sqrt{(\log J)/n}) \times \|h_0 - \Pi_J h_0\|_{L^2(X)} + O_p(1) \times \tau_J \|T(h_0 - \Pi_J h_0)\|_{L^2(W)} \\ &= O_p(1) \times \|h_0 - \Pi_J h_0\|_{L^2(X)} \end{aligned}$$

where the second line is by Lemmas F.9 and F.10(b) and the relations $J \asymp K$ and $\tau_J \asymp s_{JK}^{-1}$, and the final line is by the condition $\tau_J \zeta \sqrt{(\log J)/n} = o(1)$ and Assumption 4(ii). This proves Result (1).

To prove **Result (2)** it remains to control T_1 . To do this, bound

$$\begin{aligned} T_1 &\leq \|G_\psi^{1/2}(S'G_b^{-1/2})_l^- G_b^{-1/2} B'u/n\|_{\ell^2} + \|G_\psi^{1/2}\{(\widehat{G}_b^{-1/2}\widehat{S})_l^- \widehat{G}_b^{-1/2} G_b^{1/2} - (G_b^{-1/2}S)_l^-\} G_b^{-1/2} B'u/n\|_{\ell^2} \\ &=: T_{11} + T_{12}. \end{aligned}$$

For T_{11} , by definition of s_{JK} and Lemma F.8 we have:

$$T_{11} \leq s_{JK}^{-1} \|G_b^{-1/2} B'u/n\|_{\ell^2} = O_p(s_{JK}^{-1} \sqrt{K/n}) = O_p(\tau_J \sqrt{J/n})$$

where the final line is because $J \asymp K$ and $\tau_J \asymp s_{JK}^{-1}$ (Lemma G.1). Similarly,

$$\begin{aligned} T_{12} &\leq \|G_\psi^{1/2}\{(\widehat{G}_b^{-1/2}\widehat{S})_l^- \widehat{G}_b^{-1/2} G_b^{1/2} - (G_b^{-1/2}S)_l^-\}\|_{\ell^2} \|G_b^{-1/2} B'u/n\|_{\ell^2} \\ &= O_p(\tau_J^2 \zeta \sqrt{(\log J)/n}) \times O_p(\sqrt{J/n}) \\ &= o_p(\tau_J \sqrt{J/n}) \end{aligned}$$

where the second line is by Lemmas F.8 and F.10(b) and the relations $J \asymp K$ and $\tau_J \asymp s_{JK}^{-1}$, and the final line is by the condition $\tau_J \zeta \sqrt{(\log J)/n} = o(1)$. \square

Proof of Corollary B.1. Analogous to the proof of Corollary 3.1. \square

Proof of Theorem B.2. As in the proof of Theorem 3.2, it suffices to prove a lower bound for the Gaussian reduced-form NPIR model (44). Theorem G.3 below does just this. \square

Theorem G.3. *Let Condition LB hold with $B_2(p, L)$ in place of $B_\infty(p, L)$ hold for the NPIR model (44) with a random sample $\{(W_i, Y_i)\}_{i=1}^n$. Then for any $0 \leq |\alpha| < p$:*

$$\liminf_{n \rightarrow \infty} \inf_{\widehat{g}_n} \sup_{h \in B_2(p, L)} \mathbb{P}_h \left(\|\widehat{g}_n - \partial^\alpha h\|_{L^2(X)} \geq cn^{-(p-|\alpha|)/(2(p+\varsigma)+d)} \right) \geq c' > 0$$

in the mildly ill-posed case, and

$$\liminf_{n \rightarrow \infty} \inf_{\widehat{g}_n} \sup_{h \in B_2(p, L)} \mathbb{P}_h \left(\|\widehat{g}_n - \partial^\alpha h\|_{L^2(X)} \geq c(\log n)^{-(p-|\alpha|)/\varsigma} \right) \geq c' > 0$$

in the severely ill-posed case, in the severely ill-posed case, where $\inf_{\widehat{g}_n}$ denotes the infimum over all estimators of $\partial^\alpha h$ based on the sample of size n , $\sup_{h \in B_2(p, L)} \mathbb{P}_h$ denotes the sup over $h \in B_2(p, L)$ and distributions (W_i, u_i) which satisfy Condition LB with ν fixed, and the finite positive constants c, c' depend only on p, L, d, ς and σ_0 .

Proof of Theorem G.3. We use similar arguments to the proof of Theorem G.1, using Theorem 2.5 of Tsybakov (2009) (see Theorem G.2). Again, we first explain the scalar ($d = 1$) case in detail. Let $\{\phi_{j,k}, \psi_{j,k}\}_{j,k}$ be a wavelet basis of regularity $\gamma > p$ for $L^2([0, 1])$ as described in Appendix E.

By construction, the support of each interior wavelet is an interval of length $2^{-j}(2r-1)$. Thus for all j sufficiently large (hence the \liminf in our statement of the Lemma) we may choose a set $M \subset \{r, \dots, 2^j - r - 1\}$ of interior wavelets with cardinality $\#(M) \asymp 2^j$ such that $\text{support}(\psi_{j,m}) \cap \text{support}(\psi_{j,m'}) = \emptyset$ for all $m, m' \in M$ with $m \neq m'$.

Take $g_0 \in B(p, L/2)$ and for each $m \in M$ define $\theta = \{\theta_m\}_{m \in M}$ where each $\theta_m \in \{0, 1\}$ and define

$$h_\theta = g_0 + c_0 2^{-j(p+1/2)} \sum_{m \in M} \theta_m \psi_{j,m}$$

for each θ , where c_0 is a positive constant to be defined subsequently. Note that this gives $2^{\#(M)}$ such choices of h_θ . By the equivalence $\|\cdot\|_{B_{2,2}^p} \asymp \|\cdot\|_{b_{2,2}^p}$, for each θ we have:

$$\begin{aligned} \|h_\theta\|_{B_{2,2}^p} &\leq L/2 + \left\| c_0 2^{-j(p+1/2)} \sum_{m \in M} \theta_m \psi_{j,m} \right\|_{B_{2,2}^p} \\ &\leq L/2 + \text{const} \times \left\| c_0 2^{-j(p+1/2)} \sum_{m \in M} \theta_m \psi_{j,m} \right\|_{b_{2,2}^p} \\ &= L/2 + \text{const} \times c_0 2^{-j(p+1/2)} \left(\sum_{m \in M} \theta_m^2 2^{2jp} \right)^{1/2} \\ &\leq L/2 + \text{const} \times c_0. \end{aligned}$$

Therefore, we can choose c_0 sufficiently small that $h_\theta \in B_2(p, L)$ for each θ .

Since $\psi_{j,m} \in C^\gamma$ with $\gamma > |\alpha|$ is compactly supported and X_i has density bounded away from 0 and ∞ , we have $\|2^{j/2} \psi^{(|\alpha|)}(2^j x - m)\|_{L^2(X)} \asymp 1$ (uniformly in m). By this and the disjoint support of the $\psi_{j,m}$, for each θ, θ' we have:

$$\begin{aligned} \|\partial^\alpha h_\theta - \partial^\alpha h_{\theta'}\|_{L^2(X)} &= c_0 2^{-j(p-|\alpha|+1/2)} \left(\sum_{m \in M} (\theta_m - \theta'_m)^2 \|2^{j/2} \psi^{(|\alpha|)}(2^j \cdot - m)\|_{L^2(X)}^2 \right)^{1/2} \\ &\gtrsim c_0 2^{-j(p-|\alpha|+1/2)} \sqrt{\rho(\theta, \theta')} \end{aligned}$$

where $\rho(\theta, \theta')$ is the Hamming distance between θ and θ' . Take j large enough that $\#(M) \geq 8$. By the Varshamov-Gilbert bound (Tsybakov, 2009, Lemma 2.9) we may choose a subset $\theta^{(0)}, \theta^{(1)}, \dots, \theta^{(M^*)}$ such that $\theta^0 = (0, \dots, 0)$, $\rho(\theta^{(a)}, \theta^{(b)}) \geq \#(M)/8 \gtrsim 2^j$ for all $0 \leq a < b \leq M^*$ and $M^* \geq 2^{\#(M)/8}$ (recall that there were $2^{\#(M)}$ distinct vectors θ and $\#(M) \asymp 2^j$). For each $m \in \{0, 1, \dots, m^*\}$ let $h_m := h_{\theta^{(m)}}$. Then we have

$$\|\partial^\alpha h_m - \partial^\alpha h_{m'}\|_{L^2(X)} \gtrsim c_0 2^{-j(p-|\alpha|)}$$

for each $0 \leq m < m' \leq M^*$.

For each $0 \leq m \leq M^*$, let P_m denote the joint distribution of $\{(W_i, Y_i)\}_{i=1}^n$ with $Y_i = Th_m(W_i) + u_i$ for the Gaussian NPIR model (44). It follows from Condition LB(ii)(iii) that for each $1 \leq m \leq M^*$ the KL distance $K(P_m, P_0)$ is

$$\begin{aligned} K(P_m, P_0) &\leq \frac{1}{2} \sum_{i=1}^n (c_0 2^{-jp})^2 E \left[\frac{(T \sum_{k \in M} \theta_k^{(m)} (\psi_{j,k}(W_i)))^2}{\sigma^2(W_i)} \right] \\ &\leq \frac{n}{2\sigma^2} (c_0 2^{-jp})^2 \nu(2^j)^2 \sum_{k \in M} (\theta_k^{(m)})^2 \|\psi_{j,k}\|_{L^2(X)}^2 \\ &\lesssim n c_0^2 2^{-j(2p+1)} \nu(2^j)^2 \end{aligned}$$

where the final line is because X_i has density bounded away from 0 and ∞ and $\sum_{k \in M} (\theta_k^{(m)})^2 \leq \#(M) \asymp 2^j$

for each $1 \leq m \leq M^*$.

In the mildly ill-posed case ($\nu(2^j) = 2^{-j^\varsigma}$) we choose $2^j \asymp n^{1/(2(p+\varsigma)+1)}$. This yields:

$$\begin{aligned} K(P_m, P_0) &\lesssim c_0^2 \quad \text{uniformly in } m \\ \log(M^*) &\lesssim \log n. \end{aligned}$$

since $M^* \asymp 2^j$.

In the severely ill-posed case ($\nu(2^j) = \exp(-\frac{1}{2}2^{j^\varsigma})$) we choose $2^j = (c_1 \log n)^{1/\varsigma}$ with $c_1 > 1$. This yields:

$$\begin{aligned} K(P_m, P_0) &\lesssim n^{-(c_1-1)} \quad \text{uniformly in } m \\ \log(M^*) &\lesssim \log \log n. \end{aligned}$$

In both the mildly and severely ill-posed cases, the result follows by choosing c_0 sufficiently small that both $\|h_m\|_{B_{2,2}^p} \leq L$ and $K(P_m, P_0) < \frac{1}{8} \log(M^*)$ hold uniformly in m for all n sufficiently large. All conditions of Theorem 2.5 of [Tsybakov \(2009\)](#) are satisfied and hence we obtain the lower bound result.

In the multivariate case ($d > 1$) we let $\tilde{\psi}_{j,k,G}(x)$ denote an orthonormal tensor-product wavelet for $L^2([0,1]^d)$ at resolution level j . We construct a family of submodels analogously to the univariate case, setting $h_\theta = g_0 + c_0 2^{-j(p+d/2)} \sum_{m \in M} \theta_m \tilde{\psi}_{j,m,G}$ where $\psi_{j,m}$ is now the product of d interior univariate wavelets at resolution level j with $G = (w_\psi)^d$ (see [Appendix E](#)) and where $\#(M) \asymp 2^{jd}$. We then use the Varshamov-Gilbert bound to reduce this to a family of models h_m with $0 \leq m \leq M^*$ and $M^* \asymp 2^{jd}$. We then have:

$$\|\partial^\alpha h_m - \partial^\alpha h_{m'}\|_\infty \gtrsim c_0 2^{-j(p-|\alpha|)}$$

for each $0 \leq m < m' \leq M^*$, and

$$K(P_m, P_0) \lesssim n(c_0 2^{-j(p+d/2)})^2 \nu(2^j)^2$$

for each $1 \leq m \leq M^*$, where $\nu(2^j) = 2^{-j^\varsigma}$ in the mildly ill-posed case and $\nu(2^j) \asymp \exp(-2^{j^\varsigma})$ in the severely ill-posed case. We choose $2^j \asymp n^{1/(2(p+\varsigma)+d)}$ in the mildly ill-posed case and $2^j = (c_1 \log n)^{1/\varsigma}$ in the severely ill-posed case. The result follows as in the univariate case. \square

G.7 Proofs for Appendix C

Proof of Theorem C.1. As in the proof of [Theorem 3.2](#), this follows from the lower bound for NPIR in [Theorem G.4](#). \square

The following is a slightly stronger ‘‘in probability’’ version of [Lemma 1](#) in [Yu \(1997\)](#), which is used to prove [Theorem G.4](#). Let \mathcal{P} be a family of probability measures, let $\theta(P)$ be a parameter with values in a pseudo-metric space (\mathcal{D}, d) for some distribution $P \in \mathcal{P}$, and let $\hat{\theta}(P)$ be an estimator of $\theta(P)$ taking values in (\mathcal{D}, d) . If $\theta \in \mathcal{D}$ and $D \subset \mathcal{D}$, we let $d(\theta, D) = \inf_{\theta' \in D} d(\theta, \theta')$. Let $\text{co}(\mathcal{P})$ denote the convex hull of a set of measures \mathcal{P} . Finally, if $\mathbb{P}, \mathbb{Q} \in \mathcal{P}$ we let $\|\mathbb{P} - \mathbb{Q}\|_{TV}$ denote the total variation distance and $\text{aff}(\mathbb{P}, \mathbb{Q}) = 1 - \|\mathbb{P} - \mathbb{Q}\|_{TV}$ denote the affinity between \mathbb{P} and \mathbb{Q} .

Lemma G.8. *Suppose there are subsets $D_1, D_2 \subset \mathcal{D}$ that are 2δ separated for some $\delta > 0$ (i.e. $d(s_1, s_2) \geq 2\delta$ for all $s_1 \in D_1$ and $s_2 \in D_2$) and subsets $\mathbb{P}_1, \mathbb{P}_2 \subset \mathcal{P}$ for which $\theta(\mathbb{P}) \in D_1$ for all $\mathbb{P} \in \mathbb{P}_1$ and $\theta(\mathbb{P}) \in D_2$ for all*

$\mathbb{P} \in \mathcal{P}_2$. Then:

$$2 \sup_{\mathbb{P} \in \mathcal{P}} \mathbb{P}(d(\widehat{\theta}, \theta(\mathbb{P})) \geq \delta) \geq \sup_{\mathbb{P}_1 \in co(\mathcal{P}_1), \mathbb{P}_2 \in co(\mathcal{P}_2)} \text{aff}(\mathbb{P}_1, \mathbb{P}_2).$$

Proof of Lemma G.8. We proceed as in the proof of Lemma 1 in Yu (1997). Let $P_1 \in \mathcal{P}_1$ and $P_2 \in \mathcal{P}_2$. Then:

$$\begin{aligned} 2 \sup_{\mathbb{P} \in \mathcal{P}} \mathbb{P}(d(\widehat{\theta}, \theta(\mathbb{P})) \geq \delta) &\geq \mathbb{P}_1(d(\widehat{\theta}, \theta(\mathbb{P}_1)) \geq \delta) + \mathbb{P}_2(d(\widehat{\theta}, \theta(\mathbb{P}_2)) \geq \delta) \\ &\geq \mathbb{P}_1(d(\widehat{\theta}, D_1) \geq \delta) + \mathbb{P}_2(d(\widehat{\theta}, D_2) \geq \delta). \end{aligned}$$

Since the inequality $2 \sup_{\mathbb{P} \in \mathcal{P}} \mathbb{P}(d(\widehat{\theta}, \theta(\mathbb{P})) \geq \delta) \geq \mathbb{P}_1(d(\widehat{\theta}, D_1) \geq \delta) + \mathbb{P}_2(d(\widehat{\theta}, D_2) \geq \delta)$ holds for any fixed $\mathbb{P}_1 \in \mathcal{P}_1$ and $\mathbb{P}_2 \in \mathcal{P}_2$, it must also hold for any $\mathbb{P}_1 \in co(\mathcal{P}_1)$ and $\mathbb{P}_2 \in co(\mathcal{P}_2)$. Also note that

$$\begin{aligned} \mathbb{1}\{d(\widehat{\theta}, D_1) \geq \delta\} + \mathbb{1}\{d(\widehat{\theta}, D_2) \geq \delta\} &\geq \mathbb{1}\{d(\widehat{\theta}, D_1) + d(\widehat{\theta}, D_2) \geq 2\delta\} \\ &\geq \mathbb{1}\{d(D_1, D_2) \geq 2\delta\} = 1 \end{aligned}$$

because $d(D_1, D_2) \geq 2\delta$. Now by definition of $\alpha(\cdot, \cdot)$, for any $\mathbb{P}_1 \in co(\mathcal{P}_1)$ and $\mathbb{P}_2 \in co(\mathcal{P}_2)$ we have:

$$\begin{aligned} 2 \sup_{\mathbb{P} \in \mathcal{P}} \mathbb{P}(d(\widehat{\theta}, \theta(\mathbb{P})) \geq \delta) &\geq \mathbb{P}_1(d(\widehat{\theta}, D_1) \geq \delta) + \mathbb{P}_2(d(\widehat{\theta}, D_2) \geq \delta) \\ &\geq \inf\{\mathbb{P}_1 f + \mathbb{P}_2 g : f, g \text{ non negative and measurable with } f + g \geq 1\} \\ &= \text{aff}(\mathbb{P}_1, \mathbb{P}_2). \end{aligned}$$

The result follows by taking the supremum of the right-hand side over \mathbb{P}_1 and \mathbb{P}_2 . \square

Theorem G.4. Let Condition LB hold with $B_2(p, L)$ in place of $B_\infty(p, L)$ for the NPIR model (44) with a random sample $\{(W_i, Y_i)\}_{i=1}^n$. Then for any $0 \leq |\alpha| < p$:

$$\liminf_{n \rightarrow \infty} \inf_{\widehat{g}_n} \sup_{h \in B_2(p, L)} \mathbb{P}_h(|\widehat{g}_n - f(h)| > cr_n) \geq c' > 0$$

where

$$r_n = \begin{cases} n^{-1/2} & \text{in the mildly ill-posed case when } p \geq \varsigma + 2|\alpha| + d/4 \\ n^{-4(p-|\alpha|)/(4(p+\varsigma)+d)} & \text{in the mildly ill-posed case when } \varsigma < p < \varsigma + 2|\alpha| + d/4 \\ (\log n)^{-2(p-|\alpha|)/\varsigma} & \text{in the severely ill-posed case,} \end{cases}$$

$\inf_{\widehat{g}_n}$ denotes the infimum over all estimators of $f(h_0)$ based on the sample of size n , $\sup_{h \in B_2(p, L)} \mathbb{P}_h$ denotes the sup over $h \in B_2(p, L)$ and distributions (W_i, u_i) which satisfy Condition LB with ν fixed, and the finite positive constants c, c' do not depend on n .

Proof of Theorem G.4. We first prove the result for the scalar ($d = 1$) case, then describe the modifications required in the multivariate case.

Let $\{\phi_{j,k}, \psi_{j,k}\}_{j,k}$ be a CDV wavelet basis of regularity $\gamma > p$ for $L^2([0, 1])$, as described in Appendix E. As in the proof of Theorem G.3, we choose a set $M \subset \{r, \dots, 2^j - r - 1\}$ of interior wavelets with cardinality $\mathfrak{m} := \#(M) \asymp 2^j$ such that $\text{support}(\psi_{j,m}) \cap \text{support}(\psi_{j,m'}) = \emptyset$ for all $m, m' \in M$ with $m \neq m'$. Let $\theta = \{\theta_m\}_{m \in M}$ where each $\theta_m \in \{-1, 1\}$ and for each $\theta \in \{-1, 1\}^{\mathfrak{m}}$ define:

$$h_\theta = \sum_{m \in M} \frac{\theta_m c_0 2^{-jp}}{\sqrt{\mathfrak{m}}} \psi_{j,m}.$$

and let $h_0 = 0$. By the equivalence $\|\cdot\|_{b_{2,2}^p} \asymp \|\cdot\|_{B_{2,2}^p}$, we have:

$$\|h_\theta\|_{B_{2,2}^p} \lesssim \|h_\theta\|_{b_{2,2}^p} = \left(2^{2jp} \sum_{m \in M} \frac{\theta_m^2 c_0^2 2^{-2jp}}{\mathfrak{m}} \right)^{1/2} = c_0.$$

Therefore, we may choose c_0 sufficiently small that $h_\theta \in B_2(p, L)$ for all $\theta \in \{-1, 1\}^m$.

Let $\psi^{(|\alpha|)}$ denote the $|\alpha|$ th derivative of ψ . By disjoint support of the $\psi_{j,m}(x) = 2^{j/2}\psi(2^j x - m)$, $\mu(x) \geq \underline{\mu} > 0$, and a change of variables, we have:

$$\begin{aligned} |f(h_\theta) - f(h_0)| &= \mathfrak{m}^{-1} \sum_{m \in M} \int (c_0 \theta_m 2^{-jp} \psi_{j,m}^{(|\alpha|)}(x))^2 \mu(x) dx \\ &\gtrsim \mathfrak{m}^{-1} \sum_{m \in M} \int (c_0 \theta_m 2^{-jp} \psi_{j,m}^{(|\alpha|)}(x))^2 dx \\ &= c_0^2 2^{-2jp} \mathfrak{m}^{-1} \sum_{m \in M} \int 2^{(2|\alpha|+1)j} \psi^{(|\alpha|)}(2^j x - m)^2 dx \\ &= c_0^2 2^{-2j(p-|\alpha|)} \int \psi^{(|\alpha|)}(u)^2 du \gtrsim c_0^2 2^{-2j(p-|\alpha|)}. \end{aligned}$$

Therefore, there exists a constant $c_* > 0$ such that

$$|f(h_\theta) - f(h_0)| > 2c_* 2^{-2j(p-|\alpha|)} \quad (63)$$

holds for all for each $\theta \in \{-1, 1\}^m$ whenever j is sufficiently large.

Let P_0 (respectively P_θ) denote the joint distribution of $\{(W_i, Y_i)\}_{i=1}^n$ with $Y_i = Th_0(W_i) + u_i$ (respectively $Y_i = Th_\theta(W_i) + u_i$) for the Gaussian NPIR model (44) where, under Condition LB, we may assume that X_i and W_i have uniform marginals and that the joint density $f_{XW}(x, w)$ of (X_i, W_i) has wavelet expansion

$$f_{XW}(x, w) = \sum_{k=0}^{2^{r_0}-1} \lambda_{r_0} \varphi_{r_0,k}(x) \varphi_{r_0,k}(w) + \sum_{j=r_0}^{\infty} \sum_{k=0}^{2^j-1} \lambda_j \psi_{j,k}(x) \psi_{j,k}(w).$$

Observe that

$$T\psi_{j,k}(w) = \int \psi_{j,k}(x) f_{XW}(x, w) dx = \lambda_j \psi_{j,k}(w)$$

for each $0 \leq k \leq 2^j - 1$ and each $j \geq r_0$ (r_0 is fixed) and that $|\lambda_j| \asymp \nu(2^j)$ by Condition LB(iii). Let P^* denote the mixture distribution obtained by assigning weight 2^{-m} to P_θ for each of the 2^m realizations of θ . Lemma G.9 yields

$$\|P^* - P_0\|_{TV}^2 \lesssim \frac{n^2 2^{-4jp} \nu(2^j)^4}{\mathfrak{m}}. \quad (64)$$

In the mildly ill-posed case ($\nu(2^j) = 2^{j^\varsigma}$) we have

$$\|P^* - P_0\|_{TV}^2 \lesssim n^2 2^{-j(4(p+\varsigma)+1)}$$

because $\mathfrak{m} \asymp 2^j$. Choose $2^j \asymp cn^{2/(4(p+\varsigma)+1)}$ with c sufficiently small so $\|P^* - P_0\|_{TV} \leq 1 - \epsilon$ for some $1 > \epsilon > 0$ and all n large enough, whence:

$$\text{aff}(P^*, P_0) = 1 - \|P^* - P_0\|_{TV} \geq \epsilon \quad (65)$$

for all n sufficiently large. It now follows by Lemma G.8 and equations (63) and (65) that for all n sufficiently large, any estimator \widehat{g}_n of $f(h)$ obeys the bound

$$\sup_{h \in B_2(p, L)} \mathbb{P}_h \left(|\widehat{g}_n - f(h)| > c_* 2^{-2j(p-|\alpha|)} \right) \geq \epsilon/2 \quad (66)$$

where $2^{-2j(p-|\alpha|)} \asymp n^{-4(p-|\alpha|)/(4(p+\varsigma)+1)}$. This is slower than $n^{-1/2}$ whenever $p \leq \varsigma + 2|\alpha| + 1/4$.

In the severely ill-posed case ($\nu(2^j) = \exp(-\frac{1}{2}2^{\varsigma j})$) we choose $2^j = (c \log n)^{1/\varsigma}$ for some $c \in (0, 1)$. This yields $\|P^* - P_0\|_{TV} = o(1)$ by (64) and hence there exists $\epsilon > 0$ such that $\text{aff}(P^*, P_0) \geq \epsilon$ for all n sufficiently large. Then by Lemma G.8 and equation (63), for all n sufficiently large, any estimator \widetilde{f}_n of $f(h)$ obeys the same bound (66) with $2^{-2j(p-|\alpha|)} \asymp (\log n)^{-2(p-|\alpha|)/\varsigma}$.

In the multivariate case ($d > 1$) we let $\widetilde{\psi}_{j,k,G}(x)$ denote an orthonormal tensor-product wavelet for $L^2([0, 1]^d)$ at resolution level j , as described in Appendix E. We may choose a subset M of $\{0, \dots, 2^j - 1\}^d$ with $\#(M) \asymp 2^{dj}$ for which each $m \in M$ indexes a tensor-product of interior wavelets of the form $2^{j/2} \psi(2^j x_l - m_l)$, which we denote by $\widetilde{\psi}_{j,m}(x)$, such that $\widetilde{\psi}_{j,m}$ and $\widetilde{\psi}_{j,m'}$ have disjoint support for each $m, m' \in M$ with $m \neq m'$. For each $\theta \in \{-1, 1\}^m$ we define

$$h_\theta = \sum_{m \in M} \frac{\theta_m c_0 2^{-jp}}{\sqrt{\mathfrak{m}}} \widetilde{\psi}_{j,m}(x)$$

with c_0 sufficiently small such that $h_\theta \in B_2(p, L)$ for each θ . Let $h_0(x) = 0$ for all $x \in [0, 1]^d$. By disjoint support of the $\widetilde{\psi}_{j,m}$ and a change of variables, we have:

$$\begin{aligned} |f(h_\theta) - f(h_0)| &= c_0^2 2^{-2jp} \mathfrak{m}^{-1} \sum_{m \in M} \int \left(\prod_{i=1}^d 2^{(2\alpha_i+1)j} \psi^{(\alpha_i)}(2^j x_i - m_i)^2 \right) \mu(x) dx \\ &\gtrsim c_0^2 2^{-2jp} \mathfrak{m}^{-1} \sum_{m \in M} \int \left(\prod_{i=1}^d 2^{(2\alpha_i+1)j} \psi^{(\alpha_i)}(2^j x_i - m_i)^2 \right) dx \gtrsim c_0^2 2^{-2j(p-|\alpha|)}. \end{aligned}$$

Letting P_0, P_θ , and P^* be defined analogously to in the univariate case, we let X_i and W_i have uniform marginals on $[0, 1]^d$ and their joint density $f_{XW}(x, w)$ has wavelet expansion

$$f_{XW}(x, w) = \sum_{j=r_0}^{\infty} \sum_{G \in \mathcal{G}_{j,r_0}} \sum_k \lambda_j \widetilde{\psi}_{j,k,G}(x) \widetilde{\psi}_{j,k,G}(w) \quad (67)$$

with $|\lambda_j| \asymp \nu(2^j)$. Lemma G.9 again yields

$$\|P^* - P_0\|_{TV}^2 \lesssim \frac{n^2 2^{-4jp} \nu(2^j)^4}{\mathfrak{m}}.$$

The result follows by choosing $2^j \asymp cn^{2/(4(p+\varsigma)+d)}$ with sufficiently small c in the mildly ill-posed case and $2^j = (c \log n)^{1/\varsigma}$ for some $c \in (0, 1)$ in the severely ill-posed case. \square

Lemma G.9. *Let the Condition LB hold with $B_2(p, L)$ in place of $B_\infty(p, L)$ for the NPIR model (44), let P^* and P_0 be as described in the proof of Theorem G.4, and let $2^{-jp} \nu(2^j) = o(1)$. Then:*

$$\|P^* - P_0\|_{TV}^2 \lesssim \frac{n^2 2^{-4jp} \nu(2^j)^4}{\mathfrak{m}}.$$

Proof of Lemma G.9. We prove the result for the multivariate case. For each $\theta \in \{-1, 1\}^m$, the density of

P_θ with respect to P_0 is

$$\begin{aligned}\frac{dP_\theta}{dP_0} &= \prod_{i=1}^n \exp \left\{ \frac{-1}{2\sigma_0^2} ([T(h_\theta - h_0)(W_i)]^2 - 2u_i[T(h_\theta - h_0)(W_i)]) \right\} \\ &= \prod_{i=1}^n \exp \left\{ \frac{-1}{2\sigma_0^2} \left(\left[\sum_{m \in M} \frac{\theta_m c_0 2^{-jp} \lambda_j}{\sqrt{\mathfrak{m}}} \tilde{\psi}_{j,m}(W_i) \right]^2 - 2u_i \left[\sum_{m \in M} \frac{\theta_m c_0 2^{-jp} \lambda_j}{\sqrt{\mathfrak{m}}} \tilde{\psi}_{j,m}(W_i) \right] \right) \right\}.\end{aligned}$$

Since the $\tilde{\psi}_{j,m}$ have disjoint support, we have

$$\begin{aligned}\frac{dP_\theta}{dP_0} &= \prod_{i=1}^n \exp \left\{ -\frac{1}{2} \sum_{m \in M} \frac{c_0^2 2^{-2jp} \lambda_j^2}{\sigma_0^2 \mathfrak{m}} \tilde{\psi}_{j,m}(W_i)^2 + \frac{u_i}{\sigma_0} \sum_{m \in M} \theta_m \frac{c_0 2^{-jp} \lambda_j}{\sigma_0 \sqrt{\mathfrak{m}}} \tilde{\psi}_{j,m}(W_i) \right\} \\ &= \prod_{i=1}^n \exp \left\{ \sum_{m \in M} \left(-\frac{1}{2} \Delta_{i,j,m}^2 + \theta_m \frac{u_i}{\sigma_0} \Delta_{i,j,m} \right) \right\}\end{aligned}$$

where $u_i \sim N(0, \sigma_0^2)$ under P_0 and

$$\Delta_{i,j,m} = \frac{c_0 2^{-jp} \lambda_j}{\sigma_0 \sqrt{\mathfrak{m}}} \tilde{\psi}_{j,m}(W_i).$$

Therefore:

$$\frac{dP_\theta}{dP_0} = \prod_{i=1}^n (1 + A_{i,j}(\theta))$$

where

$$\begin{aligned}A_{i,j}(\theta) &= \exp \left\{ \sum_{m \in M} \left(-\frac{1}{2} \Delta_{i,j,m}^2 + \theta_m \frac{u_i}{\sigma_0} \Delta_{i,j,m} \right) \right\} - 1 \\ &= \left\{ \sum_{m \in M} \exp \left(-\frac{1}{2} \Delta_{i,j,m}^2 + \theta_m \frac{u_i}{\sigma_0} \Delta_{i,j,m} \right) \right\} - 1\end{aligned}$$

and the second line is again by disjoint support of the $\tilde{\psi}_{j,m}$ (which implies $\Delta_{i,j,m}$ is nonzero for at most one m for each i).

Let E_0 be expectation under the measure P_0 and observe that $E_0[A_{i,j}(\theta)] = 0$ for each $\theta \in \{-1, 1\}^{\mathfrak{m}}$. For each θ, θ' we define the vector $\kappa_{\theta, \theta'} \in \mathbb{R}^n$ whose i th element is:

$$\begin{aligned}\kappa_{\theta, \theta'}(i) &= E_0[A_{i,j}(\theta)A_{i,j}(\theta')] \\ &= \sum_{m \in M} \sum_{m' \in M} E_0 \left[\exp \left\{ -\frac{1}{2} \Delta_{i,j,m}^2 - \frac{1}{2} \Delta_{i,j,m'}^2 + \frac{u_i}{\sigma_0} (\theta_m \Delta_{i,j,m} + \theta'_{m'} \Delta_{i,j,m'}) \right\} - 1 \right] \\ &= \sum_{m \in M} \sum_{m' \in M} E_0 \left[e^{\theta_m \theta'_{m'} \Delta_{i,j,m} \Delta_{i,j,m'}} - 1 \right] \\ &= \sum_{m \in M} E_0 \left[e^{\theta_m \theta'_m \Delta_{i,j,m}^2} - 1 \right].\end{aligned}$$

where the final line is again by disjoint support of the $\tilde{\psi}_{j,m}$. Using $|\tilde{\psi}_{j,k}| \lesssim 2^{dj/2}$, $E_0[\tilde{\psi}_{j,k}(X_i)^2] = 1$, and $\mathfrak{m} \asymp 2^{dj}$,

it is straightforward to derive the bounds:

$$|\Delta_{i,j,m}| \lesssim \frac{c_0 2^{-jp} \lambda_j}{\sigma_0} \quad (68)$$

$$E_0[\Delta_{i,j,m}^2] = \frac{c_0^2 2^{-2jp} \lambda_j^2}{\sigma_0^2 m}. \quad (69)$$

By Taylor's theorem:

$$\begin{aligned} \kappa_{\theta, \theta'}(i) &= \sum_{m \in M} \theta_m \theta'_m E_0[\Delta_{i,j,m}^2] + \sum_{m \in M} \frac{1}{2} E_0[\Delta_{i,j,m}^4] + r_3(\theta, \theta') \\ &= \sum_{m \in M} \theta_m \theta'_m E_0[\Delta_{i,j,m}^2] + r_2(\theta, \theta') \end{aligned}$$

where r_2 and r_3 are remainder terms. Using the Lagrange remainder formula and (68) and (69), we may deduce that

$$r_2(\theta, \theta') \leq \frac{1}{2} \sum_m E_0 \left[e^{\Delta_{i,j,m}^2} \Delta_{i,j,m}^4 \right] \lesssim \exp \left\{ \frac{C c_0^2 2^{-2jp} \lambda_j^2}{\sigma_0^2} \right\} \frac{c_0^4 2^{-4jp} \lambda_j^4}{\sigma_0^4 m}.$$

and

$$r_3(\theta, \theta') \leq \frac{1}{6} \sum_m E_0 \left[e^{\Delta_{i,j,m}^2} \Delta_{i,j,m}^6 \right] \lesssim \exp \left\{ \frac{C c_0^2 2^{-2jp} \lambda_j^2}{\sigma_0^2} \right\} \frac{c_0^6 2^{-6jp} \lambda_j^6}{\sigma_0^6 m}.$$

where C is a finite positive constant and we again used the fact that $\Delta_{i,j,m}$ is nonzero for at most one m for each i .

Lemma 22 of [Pollard \(2000\)](#) provides the bound:

$$\|P^* - P_0\|_{TV}^2 \leq 2^{-2m} \sum_{\theta \in \{-1,1\}^m} \sum_{\theta' \in \{-1,1\}^m} \Upsilon(\kappa_{\theta, \theta'})$$

where for any vector $c = (c_1, \dots, c_n)' \in \mathbb{R}^n$ the function $\Upsilon(c)$ is defined as

$$\Upsilon(c) = -1 + \prod_{i=1}^n (1 + c_i) = \sum_{i=1}^n c_i + \sum_{i_1=1}^n \sum_{i_2=i_1+1}^n c_{i_1} c_{i_2} + \text{higher-order terms}$$

where the higher-order terms are sums over triples, quadruples, etc, with all distinct indices, up to $c_1 c_2 \dots c_n$. Therefore:

$$\begin{aligned} \|P^* - P_0\|_{TV}^2 &\leq 2^{-2m} \sum_{\theta, \theta'} \left[\sum_{i=1}^n \left(\sum_{m \in M} \theta_m \theta'_m E_0[\Delta_{i,j,m}^2] + \sum_{m \in M} \frac{1}{2} E_0[\Delta_{i,j,m}^4] + r_3(\theta, \theta') \right) \right. \\ &\quad \left. + \sum_{i_1=1}^n \sum_{i_2=i_1+1}^n \left\{ \left(\sum_{m \in M} \theta_m \theta'_m E_0[\Delta_{i_1,j,m}^2] + r_2(\theta, \theta') \right) \left(\sum_{m' \in M} \theta_{m'} \theta'_{m'} E_0[\Delta_{i_2,j,m'}^2] + r_2(\theta, \theta') \right) \right\} \right. \\ &\quad \left. + \text{higher order terms} \right]. \quad (70) \end{aligned}$$

Since $\sum_{\theta, \theta'} \theta_m \theta'_{m'} = 0$ for all $m, m' \in M$ and $\sum_{\theta, \theta'} 1 = 2^{2m}$, the first-order sum in (70) is:

$$\begin{aligned}
& 2^{-2m} \sum_{i=1}^n \sum_{\theta, \theta'} \left(\sum_m \theta_m \theta'_{m'} E_0[\Delta_{i,j,m}^2] + \frac{1}{2} \sum_m E_0[\Delta_{i,j,m}^4] + r_3(\theta, \theta') \right) \\
&= \frac{n}{2} \sum_m E_0[\Delta_{i,j,m}^4] + n 2^{-2m} \sum_{\theta, \theta'} r_3(\theta, \theta') \\
&\lesssim \frac{n c_0^4 2^{-4jp} \lambda_j^4}{\sigma_0^4 m} \left(1 + \exp \left\{ \frac{c_0^2 2^{-2jp} \lambda_j^2}{\sigma_0^2} \right\} \frac{c_0^2 2^{-2jp} \lambda_j^2}{\sigma_0^2 m} \right). \tag{71}
\end{aligned}$$

Also observe that

$$\sum_{\theta} \sum_{\theta'} (\theta_m \theta'_m) (\theta_{m'} \theta'_{m'}) = \begin{cases} 0 & \text{if } m \neq m' \\ 2^{2m} & \text{if } m = m'. \end{cases}$$

The second-order sum in (70) is therefore:

$$\begin{aligned}
& \frac{n(n-1)}{2} \left(m E_0[\Delta_{i,j,m}^2]^2 + O(m \max_{m \in M} E[\Delta_{i,j,m}^2] \times \max_{\theta, \theta'} r_2(\theta, \theta')) + O(\max_{\theta, \theta'} r_2(\theta, \theta')^2) \right) \\
&\asymp \frac{n^2 c_0^4 2^{-4jp} \lambda_j^4}{\sigma_0^4 m} \left(1 + O \left(\exp \left\{ \frac{2C c_0^2 2^{-2jp} \lambda_j^2}{\sigma_0^2} \right\} \left(2^{-2jp} \lambda_j^2 + \frac{2^{-4jp} \lambda_j^4}{m} \right) \right) \right). \tag{72}
\end{aligned}$$

The higher-order terms in (70) will be of asymptotically smaller order because $2^{-jp} \lambda_j \asymp 2^{-jp\nu} (2^j) = o(1)$. Substituting (71) and (72) into (70) yields:

$$\|P^* - P_0\|_{TV}^2 \lesssim \frac{n^2 2^{-4jp} \lambda_j^4}{m} (1 + o(1)) \asymp \frac{n^2 2^{-4jp\nu} (2^j)^4}{m}$$

as required. \square

G.8 Proofs for Appendix E

Proof of Lemma E.1. Part (a) is equation (3.4) on p. 141 of DeVore and Lorentz (1993). For part (b), let $v \in \mathbb{R}^J$, let $f_X(x)$ denote the density of X_i and let $\underline{f}_X = \inf_x f_X(x)$ and $\bar{f}_X = \sup_x f_X(x)$. Then for any $v \in \mathbb{R}^J$:

$$\begin{aligned}
v' E[\psi^J(X_i) \psi^J(X_i)'] v &\geq \underline{f}_X \int_0^1 (\psi^J(x) v)^2 dx \\
&\geq \underline{f}_X c_1^2 \frac{\min_{-r+1 \leq j \leq m} (t_{j+r} - t_r)}{r} \|v\|_{\ell^2}^2 \\
&\geq \underline{f}_X c_1^2 \frac{c_2 J^{-1}}{r} \|v\|_{\ell^2}^2
\end{aligned}$$

for some finite positive constant c_1 , where the first inequality is by Assumption 1(i), the second is by Theorem 4.2 (p. 145) of DeVore and Lorentz (1993) with $p = 2$, and the third is by uniform boundedness of the mesh ratio. By the variational characterization of eigenvalues of selfadjoint matrices, we have:

$$\lambda_{\min}(G_\psi) = \min_{v \in \mathbb{R}^J, v \neq 0} \frac{v' E[\psi^J(X_i) \psi^J(X_i)'] v}{\|v\|_{\ell^2}^2} \geq \underline{f}_X c_1^2 \frac{c_2 J^{-1}}{r}$$

This establishes the upper bound on $\lambda_{\min}(G_\psi)^{-1}$. The proof of the lower bound for $\lambda_{\max}(G_\psi)^{-1}$ follows analogously by Theorem 4.2 (p. 145) of DeVore and Lorentz (1993) with $p = 2$. Part (c) then follows directly from

part (b). \square

Proof of Lemma E.2. The ℓ^1 norm of the tensor product of vectors equals the product of the ℓ^1 norms of the factors, whence part (a) follows from Lemma E.1. As $\psi^J(x)$ is formed as the tensor-product of univariate B-splines, each element of $\psi^J(x)$ is of the form $\prod_{l=1}^d \psi_{Ji_l}(x_l)$ where $\psi_{Ji_l}(x_l)$ denotes the i_l th element of the vector of univariate B-splines. Let $v \in \mathbb{R}^J$. We may index the elements of v by the multi-indices $i_1, \dots, i_d \in \{1, \dots, m+r\}^d$. By boundedness of f_X away from zero and Fubini's theorem, we have:

$$\begin{aligned} v' E[\psi^J(X_i)\psi^J(X_i)']v &\geq \underline{f}_X \int_0^1 \cdots \int_0^1 \left(\sum_{i_1, \dots, i_d} v_{i_1, \dots, i_d} \prod_{l=1}^d \psi_{Ji_l}(x_l) \right)^2 dx_1 \cdots dx_d \\ &= \underline{f}_X \int_0^1 \cdots \int_0^1 \sum_{i_2, \dots, i_d} \sum_{j_2, \dots, j_d} \left(\prod_{l=2}^d \psi_{Ji_l}(x_l) \right) \left(\prod_{l=2}^d \psi_{Jj_l}(x_l) \right) \\ &\quad \left\{ \int_0^1 \sum_{i_1} \sum_{j_1} v_{i_1, \dots, i_d} v_{j_1, \dots, j_d} \psi_{Ji_1}(x_1) \psi_{Jj_1}(x_1) dx_1 \right\} dx_2 \cdots dx_d. \end{aligned}$$

Applying Theorem 4.2 (p. 145) of DeVore and Lorentz (1993) to the term in braces, and repeating for x_2, \dots, x_d , we have:

$$\begin{aligned} v' E[\psi^J(X_i)\psi^J(X_i)']v &\geq \underline{f}_X \left(c_1^2 \frac{\min_{-r+1 \leq j \leq m}(t_{j+r} - t_r)}{r} \right)^d \|v\|_{\ell^2}^2 \\ &\geq \underline{f}_X c_1^{2d} \frac{c_2^d J^{-1}}{r^d} \|v\|_{\ell^2}^2 \end{aligned}$$

where the second inequality is by uniform boundedness of the mesh ratio. The rest of the proof follows by identical arguments to Lemma E.1. \square

Proof of Lemma E.3. Each of the interior $\varphi_{j,k}$ and $\psi_{j,k}$ have support $[2^{-j}(-N+1+k), 2^{-j}(N+k)]$, therefore $\varphi_{j,k}(x) \neq 0$ (respectively $\psi_{j,k}(x) \neq 0$) for less than or equal to $2N$ interior $\varphi_{j,k}$ (resp. $\psi_{j,k}$) and for any $x \in [0, 1]$. Further, there are only N left and right $\varphi_{j,k}$ and $\psi_{j,k}$. Therefore, $\varphi_{j,k}(x) \neq 0$ (respectively $\psi_{j,k}(x) \neq 0$) for less than or equal to $3N$ of the $\varphi_{j,k}$ (resp. $\psi_{j,k}$) at resolution level j for each $x \in [0, 1]$. By construction of the basis, each of $\varphi, \varphi_{j,k}^l, \psi_{j,k}^l$ for $k = 0, \dots, N-1$ and $\varphi_{j,-k}^r, \psi_{j,-k}^r$ for $k = 1, \dots, N$ are continuous and therefore attain a finite maximum on $[0, 1]$. Therefore, each of the $\varphi_{j,k}$ and $\psi_{j,k}$ are uniformly bounded by some multiple of $2^{j/2}$ and so:

$$\xi_{\psi,J} \lesssim 3N \times \left(\underbrace{2^{L_0/2}}_{\text{for the } \varphi_{L_0,k}} + \underbrace{2^{L_0/2}}_{\text{for the } \psi_{L_0,k}} + \dots + \underbrace{2^{L/2}}_{\text{for the } \psi_{L,k}} \right) \lesssim 2^{L/2}$$

The result then follows because $J = 2^{L+1}$. For part (b), because f_X is uniformly bounded away from 0 and ∞ and the wavelet basis is orthonormal for $L^2[0, 1]$, we have

$$v' E[\psi^J(X_i)\psi^J(X_i)']v \asymp v' \left(\int_0^1 \psi^J(x)\psi^J(x)' dx \right) v = \|v\|_{\ell^2}^2$$

and so all eigenvalues of G_ψ are uniformly (in J) bounded away from 0 and ∞ . Part (c) follows directly. \square

Proof of Lemma E.4. Lemma E.3 implies that each of the factor vectors in the tensor product at level j has ℓ^1 norm of order $O(2^{dj/2})$ uniformly for $x = (x_1, \dots, x_d)' \in [0, 1]^d$ and in j . There are at most 2^d such tensor products at each resolution level. Therefore, $\xi_{\psi,J} = O(2^{dL/2}) = O(\sqrt{J})$ since $J = O(2^{dL})$. Parts (b) and

(c) follow by the same arguments of the proof of Lemma E.3 since the tensor-product basis is orthonormal for $L^2([0, 1]^d)$. \square

G.9 Proofs for Appendix F

Proof of Lemma F.2. $\|A^{-1} - I_r\|_{\ell^2} = \|A^{-1}(A - I_r)\|_{\ell^2} \leq \|A^{-1}\|_{\ell^2} \|A - I_r\|_{\ell^2}$. \square

Proof of Lemma F.4. The first assertion is immediate by Theorem 3.3 of Stewart (1977) and definition of A_l^- and B_l^- . For the second part, Weyl's inequality implies that $s_{\min}(B) \geq \frac{1}{2}s_{\min}(A)$ whenever $\|A - B\|_{\ell^2} \leq \frac{1}{2}s_{\min}(A)$. \square

Proof of Lemma F.5. $\|A_l^-\|_{\ell^2}^2 = \lambda_{\max}(A_l^-(A_l^-)') = \lambda_{\max}((A'A)^{-1}) = 1/\lambda_{\min}(A'A) = s_{\min}(A)^{-2}$. \square

Proof of Lemma F.6. The result follows from Li, Li, and Cui (2013) (see also Stewart (1977)). \square

Proof of Lemma F.7. We prove the results for \widehat{S}^o ; convergence of \widehat{G}_ψ^o and \widehat{G}_b^o is proved in Lemma 2.1 of Chen and Christensen (2015). Note that

$$\widehat{S}^o - S^o = \sum_{i=1}^n n^{-1} G_b^{-1/2} \{b^K(W_i) \psi^J(X_i)' - E[b^K(W_i) \psi^J(X_i)']\} G_\psi^{-1/2} =: \sum_{i=1}^n \Xi_i^o$$

where $\|\Xi_i\|_{\ell^2} \leq 2n^{-1} \zeta_{b,K} \zeta_{\psi,J}$. Also,

$$\begin{aligned} \left\| \sum_{i=1}^n E[\Xi_i^o \Xi_i^{o'}] \right\|_{\ell^2} &\leq n^{-1} \|E[G_b^{-1/2} b^K(W_i) \psi^J(X_i)' G_\psi^{-1} \psi^J(X_i) b^K(W_i)' G_b^{-1/2}]\|_{\ell^2} \\ &\leq n^{-1} \zeta_{\psi,J}^2 \|E[G_b^{-1/2} b^K(W_i) b^K(W_i)' G_b^{-1/2}]\|_{\ell^2} \\ &= n^{-1} \zeta_{\psi,J}^2 \|I\|_{\ell^2} \\ &= n^{-1} \zeta_{\psi,J}^2 \end{aligned}$$

by the fact that $\|I\|_{\ell^2} = 1$. An identical argument yields the bound $\|\sum_{i=1}^n E[\Xi_i^{o'} \Xi_i^o]\|_{\ell^2} \leq n^{-1} \zeta_{b,K}^2$. Applying a Bernstein inequality for random matrices (Tropp, 2012, Theorem 1.6) yields

$$\mathbb{P}\left(\|\widehat{S}^o - S^o\|_{\ell^2} > t\right) \leq 2 \exp\left\{\log K - \frac{-t^2/2}{(\zeta_{b,K}^2 \vee \zeta_{\psi,J}^2)/n + 2\zeta_{b,K} \zeta_{\psi,J} t/(3n)}\right\}.$$

The convergence rate $\|\widehat{S}^o - S^o\|_{\ell^2}$ from this inequality under appropriate choice of t . \square

Proof of Lemma F.9. Let $\tilde{b}^K(x) = G_b^{-1/2} b^K(x)$ and denote $\tilde{b}^K(x)' = (\tilde{b}_{K1}(x), \dots, \tilde{b}_{KK}(x))$. As the summands have expectation zero, we have

$$\begin{aligned} E\left[\|G_b^{-1/2} \{B'(H_0 - H_J)/n - E[b^K(W_i)(h_0(X_i) - h_J(X_i))]\}\|_{\ell^2}^2\right] &\leq \frac{1}{n} E\left[\sum_{k=1}^K (\tilde{b}_{Kk}(W_i))^2 (h_0(X_i) - h_J(X_i))^2\right] \\ &\leq \frac{K}{n} \|h_0 - h_J\|_{\infty}^2 \wedge \frac{\zeta_{b,K}^2}{n} \|h_0 - h_J\|_{L^2(X)}^2. \end{aligned}$$

The result follows by Chebyshev's inequality. \square

Proof of Lemma F.10. We begin by rewriting the target in terms of the orthonormalized matrices

$$\begin{aligned} (\widehat{G}_b^{-1/2}\widehat{S})_l^- \widehat{G}_b^{-1/2} G_b^{1/2} - (G_b^{-1/2}S)_l^- &= G_\psi^{-1/2} \{ (\widehat{S}' \widehat{G}_b^{o-1} \widehat{S}^o)^{-1} \widehat{S}' \widehat{G}_b^{o-1} - (S' S^o)^{-1} S'^o \} \\ &= G_\psi^{-1/2} \{ ((\widehat{G}_b^o)^{-1/2} \widehat{S}^o)_l^- (\widehat{G}_b^o)^{-1/2} - (S^o)_l^- \}. \end{aligned} \quad (73)$$

We first bound the term in braces. By the triangle inequality,

$$\begin{aligned} &\|((\widehat{G}_b^o)^{-1/2} \widehat{S}^o)_l^- (\widehat{G}_b^o)^{-1/2} - (S^o)_l^-\|_{\ell^2} \\ &\leq \|((\widehat{G}_b^o)^{-1/2} \widehat{S}^o)_l^- - (S^o)_l^-\|_{\ell^2} \|(\widehat{G}_b^o)^{-1/2}\|_{\ell^2} + \|(\widehat{G}_b^o)^{-1/2} - I\|_{\ell^2} \|(S^o)_l^-\|_{\ell^2}. \end{aligned} \quad (74)$$

Lemma F.7 provides that

$$\|\widehat{G}_b^o - I_K\|_{\ell^2} = O_p(\zeta_{b,K} \sqrt{(\log K)/n}) \quad (75)$$

$$\|\widehat{S}^o - S^o\|_{\ell^2} = O_p((\zeta_{b,K} \vee \zeta_{\psi,J}) \sqrt{(\log K)/n}). \quad (76)$$

Let \mathcal{A}_n denote the event upon which $\|\widehat{G}_b^o - I_K\|_{\ell^2} \leq \frac{1}{2}$ and note that $\mathbb{P}(\mathcal{A}_n^c) = o(1)$ because $\|\widehat{G}_b^o - I_K\|_{\ell^2} = o_p(1)$. Then by Lemmas F.2 and F.3 we have

$$\begin{aligned} \|(\widehat{G}_b^o)^{-1/2} - I_K\|_{\ell^2} &\leq \sqrt{2} \|(\widehat{G}_b^o)^{1/2} - I_K\|_{\ell^2} \\ &\leq \frac{2}{1 + \sqrt{2}} \|\widehat{G}_b^o - I_K\|_{\ell^2} \end{aligned}$$

on \mathcal{A}_n . It follows by expression (75) and the fact that $\mathbb{P}(\mathcal{A}_n^c) = o(1)$ that

$$\|(\widehat{G}_b^o)^{-1/2} - I\|_{\ell^2} = O_p(\zeta_{b,K} \sqrt{(\log K)/n}) \quad (77)$$

which in turn implies that $\|(\widehat{G}_b^o)^{-1/2}\| = 1 + o_p(1)$.

To bound $\|((\widehat{G}_b^o)^{-1/2} \widehat{S}^o)_l^- - (S^o)_l^-\|_{\ell^2}$, it follows by equations (76) and (77) and the fact that $\|S^o\|_{\ell^2} \leq 1$ that:

$$\begin{aligned} \|(\widehat{G}_b^o)^{-1/2} \widehat{S}^o - S^o\|_{\ell^2} &\leq \|(\widehat{G}_b^o)^{-1/2} - I_K\|_{\ell^2} \|\widehat{S}^o\|_{\ell^2} + \|\widehat{S}^o - S^o\|_{\ell^2} \\ &= O_p((\zeta_{b,K} \vee \zeta_{\psi,J}) \sqrt{(\log K)/n}). \end{aligned} \quad (78)$$

Let $\mathcal{A}_{n,1} \subseteq \mathcal{A}_n$ denote the event on which $\|(\widehat{G}_b^o)^{-1/2} \widehat{S}^o - S^o\|_{\ell^2} \leq \frac{1}{2} s_{JK}$ and note that $\mathbb{P}(\mathcal{A}_{n,1}^c) = o(1)$ by virtue of the condition $s_{JK}^{-1} (\zeta_{b,K} \vee \zeta_{\psi,J}) \sqrt{(\log K)/n} = o(1)$. Lemma F.4 provides that

$$\|((\widehat{G}_b^o)^{-1/2} \widehat{S}^o)_l^- - (S^o)_l^-\|_{\ell^2} \leq 2(1 + \sqrt{5}) s_{JK}^{-2} \|(\widehat{G}_b^o)^{-1/2} \widehat{S}^o - S^o\|_{\ell^2} \quad (79)$$

on $\mathcal{A}_{n,1}$, and so

$$\|((\widehat{G}_b^o)^{-1/2} \widehat{S}^o)_l^- - (S^o)_l^-\|_{\ell^2} = O_p(s_{JK}^{-2} (\zeta_{b,K} \vee \zeta_{\psi,J}) \sqrt{(\log K)/n}) \quad (80)$$

by (78) and (79). It follows from equations (80) and (73) that:

$$\|(\widehat{G}_b^{-1/2} \widehat{S})_l^- \widehat{G}_b^{-1/2} G_b^{1/2} - (G_b^{-1/2} S)_l^-\|_{\ell^2} = O_p\left(s_{JK}^{-2} (\zeta_{b,K} \vee \zeta_{\psi,J}) \sqrt{(\log K)/(ne_J)}\right)$$

which, together with the condition $J \leq K = O(J)$, proves part (a). Part (b) follows similarly.

For part (c), we pre and post multiply terms in the product by $G_b^{-1/2}$ and $G_\psi^{-1/2}$ to obtain:

$$\begin{aligned}
& \|G_b^{-1/2} S\{(\widehat{G}_b^{-1/2} \widehat{S})_l^- \widehat{G}_b^{-1/2} G_b^{1/2} - (G_b^{-1/2} S)_l^-\}\|_{\ell^2} \\
&= \|S^o[\widehat{S}^{o'}(\widehat{G}_b^o)^- \widehat{S}^o]^- \widehat{S}^{o'}(\widehat{G}_b^o)^- - S^o[S^{o'} S^o]^{-1} S^{o'}\|_{\ell^2} \\
&\leq \|S^o[\widehat{S}^{o'}(\widehat{G}_b^o)^- \widehat{S}^o]^- \widehat{S}^{o'}(\widehat{G}_b^o)^{-1/2} ((\widehat{G}_b^o)^{-1/2} - I_K)\|_{\ell^2} \\
&\quad + \|(S^o - (\widehat{G}_b^o)^{-1/2} \widehat{S}^o)[\widehat{S}^{o'}(\widehat{G}_b^o)^- \widehat{S}^o]^- \widehat{S}^{o'}(\widehat{G}_b^o)^{-1/2}\|_{\ell^2} \\
&\quad + \|(\widehat{G}_b^o)^{-1/2} \widehat{S}^o[\widehat{S}^{o'}(\widehat{G}_b^o)^- \widehat{S}^o]^- \widehat{S}^{o'}(\widehat{G}_b^o)^{-1/2} - S^o[S^{o'} S^o]^{-1} S^{o'}\|_{\ell^2}. \tag{81}
\end{aligned}$$

Note that $\|((\widehat{G}_b^o)^{-1/2} \widehat{S}^o)_l^-\|_{\ell^2} \leq 2s_{JK}^{-1}$ on $\mathcal{A}_{n,1}$ by Lemma F.5, so

$$\|((\widehat{G}_b^o)^{-1/2} \widehat{S}^o)_l^-\|_{\ell^2} = O_p(s_{JK}^{-1}). \tag{82}$$

It follows by substituting (77), (78), and (82) into (81) that

$$\begin{aligned}
& \|G_b^{-1/2} S\{(\widehat{G}_b^{-1/2} \widehat{S})_l^- \widehat{G}_b^{-1/2} G_b^{1/2} - (G_b^{-1/2} S)_l^-\}\|_{\ell^2} \\
&\leq O_p(s_{JK}^{-1}(\zeta_{b,K} \vee \zeta_{\psi,J})\sqrt{(\log K)/n}) \\
&\quad + \|(\widehat{G}_b^o)^{-1/2} \widehat{S}^o[\widehat{S}^{o'}(\widehat{G}_b^o)^- \widehat{S}^o]^- \widehat{S}^{o'}(\widehat{G}_b^o)^{-1/2} - S^o[S^{o'} S^o]^{-1} S^{o'}\|_{\ell^2}. \tag{83}
\end{aligned}$$

The remaining term on the right-hand side of (83) is the ℓ^2 norm of the difference between the orthogonal projection matrices associated with S^o and $(\widehat{G}_b^o)^{-1/2} \widehat{S}^o$. Applying Lemma F.6, we obtain:

$$\|(\widehat{G}_b^o)^{-1/2} \widehat{S}^o[\widehat{S}^{o'}(\widehat{G}_b^o)^- \widehat{S}^o]^- \widehat{S}^{o'}(\widehat{G}_b^o)^{-1/2} - S^o[S^{o'} S^o]^{-1} S^{o'}\|_{\ell^2} \leq 2s_{JK}^{-1} \|(\widehat{G}_b^o)^{-1/2} \widehat{S}^o - S^o\|_{\ell^2}$$

on $\mathcal{A}_{n,1}$. Result (c) then follows by (78) and (83). \square

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