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The von Neumann Facet and Optimal Cycles with with Small Discounting

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The von Neumann Facet and Optimal Cycles with with Small Discounting

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Discussion Paper #920

**The von Neumann Facet and Optimal
Cycles with Small Discounting**

by

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March 1995

The von Neumann Facet and Optimal Cycles with Small Discounting

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Abstract

This paper presents a standard two-sector optimal growth model with general neoclassical production functions: strictly quasi-concave, twice continuously differentiable homogeneous of degree one functions. The following two results will be established when the discount factor is sufficiently close to 1: a) under either capital intensity condition defined at an optimal steady state (OSS), a dynamical system displays simple dynamics: any optimal path converges either to OSS or to a cycle of period two. b) given the capital intensity condition such that the consumption sector is more capital intensive, and the discount factor is sufficiently close to one, then under some combination of

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parameters of technologies, a depreciation rate and a discount factor, any optimal path converges to a cycle of period two. In a subsidiary argument, it will also be established that if the discount factor is sufficiently close to one and some other conditions of parameter values hold for such a discount factor, the Turnpike Property holds.

1 Introduction

McKenzie [12] and [13] and Scheinkman [15] have proved that any optimal path converges asymptotically to a corresponding unique optimal steady state path (OSS) when a representative consumer does not discount the future heavily, in other words, the discount factor is sufficiently close to 1. This property is often referred to as the Turnpike Property. Furthermore, in Takahashi [17], [18] and [19] the Turnpike property is also established for a general neoclassical optimal growth model with many capital goods. On the other hand, Benhabib and Nishimura [1] have demonstrated that by applying the Hopf Bifurcation in a continuous-time optimal growth model, there exist optimal growth paths consisting of persistent cycles. After demonstrating this, Benhabib and Nishimura [2] gave sufficient conditions for the existence of optimal cycles of period two for a discrete-time optimal growth model. Furthermore, Boldrin and Montrucchio [6] and Deneckere and Pelikan [8] have demonstrated that an optimal path could show any behavior including chaotic and oscillating behaviors, especially when the discount factor is very small. The results obtained by the latter groups may give the impression that there exists an upper bound on the set of discount factors that produce oscillating or chaotic behaviors. Indeed, the recent work by Soger [16] demonstrated that there exists the upper bound of the discount factor especially with respect to the emergence of chaotic behaviors of optimal paths.

Other recent studies, however, have yielded different and somewhat contradictory results. Boldrin and Deneckere [7] set up a discrete-time two-sector optimal growth model where the consumption good is produced by the Cobb-Douglas technology, while the capital good is produced by the Leontief-type technology; their study demonstrated that at every level of discounting, optimal cycles of two can be found. They exploit the fact that factor intensity reversal is taking place. For a continuous-time case, Benhabib and Rustichini [3] have shown similar results by generalizing the method used in Benhabib and Nishimura [1] for constructing examples of persistent cycles. It seems

that we still cannot have full confidence that persistent cycles can generally occur at any rate of a discount factor, even at the rate sufficiently close to 1. In particular, the results of Boldrin and Deneckere [7] crucially depend on their model setting where the capital production sector uses the Leontief-type technology, which is a somewhat different model setting from a standard neo-classical optimal growth model where twice continuously differentiable, strictly quasi-concave, and homogeneous degree one production functions are usually assumed for all sectors. Nishimura and Yano [14] have constructed the two-sector model with Cobb-Douglas production functions i.e. no factor intensity reversal, and have demonstrated that for any discount value of the discount factor of future utility, cyclical optimal paths of period two appear. An optimal boundary path is fully used in their paper. Despite these results, it is still an open question whether optimal-interior two cycles can be found at every level of discounting, especially at a sufficiently small degree of discounting, when the standard neoclassical optimal growth model is considered.

To tackle this problem, a standard two-sector optimal growth model with general neoclassical production functions is presented in this paper. It may be characterized as a two-sector discrete-time version of the model used by Burmeister and Graham [5]. We will study the model under two types of capital intensity conditions. The first one is that the consumption sector is more capital intensive. The second one is that the capital good production sector is more capital intensive. In a two sector model, both conditions can be unambiguously defined. With the aid of this model, it will be shown that when the discount factor is sufficiently close to 1, a) under either of the capital intensity conditions defined at OSS, a dynamical system displays simple dynamics; any optimal path converges either to OSS or to a cycle of period two and b) given the first capital intensity condition, we can find some combination of values of parameters such as a technology, a depreciation rate and a discount factor so that any optimal path converges to a cycle of period two. In a subsidiary argument of this case, it will also be demonstrated that under some combinations of parameter values, the Turnpike Property holds. To establish these results, we fully use the Lyapunov Stability Theorem (the Neighborhood Turnpike Theorem) proved by McKenzie [13] and applied to a neoclassical optimal growth model with many capital goods studied by Takahashi [17], [18] and [19]. It also takes into consideration the result obtained by Benhabib and Nishimura [2]. The von Neumann Facet defined and exten-

sively studied by McKenzie[12] and [13] plays a crucial role in the application of both results to the model.

The organization of the paper is as follows: In Section 2, the model and key assumptions are presented. Section 3 is assigned to the explanation of the von Neumann Facet and provided the Lyapunov Stability. The main results are stated in Section 4 and some remarks are in the final section.

2 The Model and Assumptions

Our model is an exact discrete-time two-sector version of the one studied by Burmeister and Graham [5] except that the rate of population is fixed at zero:

$$\text{Maximize } \sum_{t=0}^{\infty} \rho^t y_0(t) \quad (1)$$

subject to

$$k(0) = \bar{k} \quad (2)$$

$$y_0(t) = f^0(k_0(t), \ell_0(t)) \quad (3)$$

$$y_1(t) = f^1(k_1(t), \ell_1(t)) \quad (4)$$

$$\sum_{i=0}^1 \ell_i(t) = 1 \quad (5)$$

$$\sum_{i=0}^1 k_i(t) = k(t) \quad (6)$$

$$y_1(t) + k(t) - \delta k(t) - k(t+1) = 0 \quad (7)$$

where $t=0,1,\dots$, and the notation is as follows:

- r = subjective rate of discount and $r \geq 0$,
 ρ = $1/(1+r)$
 $y_0(t) \in R_+$ = per-capita consumption good consumed at t ,
 $y_1(t) \in R_+$ = t^{th} period per-capita capital output,
 $k(0) \in R_+$ = initial per-capita capital stock,
 $f^j : R_+^2 \rightarrow R_+$ = per-capita production function of the j^{th} sector that is strictly quasi-concave, homogeneous of degree one and twice continuously differentiable on the interior of R_+^2 ,
 $k_j(t)$ = per-capita capital good used in the j^{th} sector in the t^{th} period,
 δ = depreciation rate of the capital good and given as $0 < \delta < 1$.

Let us posit that the good j ($j = 0, 1$) cannot be produced unless $k_j > 0$ or $\ell_j > 0$ for $j = 0, 1$. Then due to [1], the equations (3) through (6) are transformed into the social production function $y_0(t) = T(y(t), k(t))$ where T is concave and twice continuously differentiable on the interior of R_+^2 with $T_{22} = \partial^2 T / \partial k^2 < 0$ (see footnote 7 of [2]). If x and z stand for an initial and terminal capital stock respectively, the reduced form utility function $V(x, z)$ and the feasible set D can be defined as follows:

$$V(x, z) = T[z - (1 - \delta)x, x]$$

and

$$D = \{(x, z) \in R_+^n \times R_+^n : T[z - (1 - \delta)x, x] \geq 0\}.$$

Then, the above problem can be summarized as the following standard reduced-form problem:

$$\text{Max } \sum_{t=0}^{\infty} \rho^t V(k(t), k(t+1))$$

Let us denote the solution path of the above problem as $\{k(t)\}_0^{\infty}$. It must satisfy the following *Euler equation*:

$$V_2(k(t), k(t+1)) + \rho V_1(k(t+1), k(t+2)) = 0,$$

where $V_1 = \partial V / \partial k(t)$ and $V_2 = \partial V / \partial k(t + 1)$. Using the Euler equation, an optimal steady state path denoted by OSS from now on is defined below:

Definition 1 $k^\rho > 0$ is an optimal steady state (OSS) if

$$V_2(k^\rho, k^\rho) + \rho V_1(k^\rho, k^\rho) = 0.$$

Let us denote w_0 and w as the wage rate and the gross rental rate. Following Burmeister and Graham[5], some assumptions are then made in reference to a two-sector version:

Assumption 1. For all positive factor price vectors (w_0, w) , $a_{11} = k_1/y_1 > 0$, $a_{00} = \ell_0/y_0 > 0$ and $a_{01} = \ell_1/y_1 > 0$ hold.

Furthermore, from Benhabib and Nishimura [1](see pp.438 – 441), a_{ij} is a continuous function of (k, y_0, y_1) .

See Burmeister and Graham[5] or Burmeister and Dobell [4]for the definition of the viability.

Assumption 2. The matrix $\bar{a}^\rho(I - \Delta\bar{a}^\rho)^{-1}$ is the dominant characteristic root λ^* where

$$\bar{a}^\rho = \begin{bmatrix} 0 & 0 \\ a_{10}^\rho & a_{11}^\rho \end{bmatrix} \text{ and } \Delta = \begin{bmatrix} 0 & 0 \\ 0 & \delta \end{bmatrix}$$

and \bar{a}^ρ is uniquely chosen along a steady state path where ρ is given. The super fix ρ is used to express the fact that vectors and matrices depend on it.

Under these assumptions, we can prove:

Lemma 1 *There exists a unique optimal steady state path $k^\rho(>0)$ and a corresponding unique positive price vector p^ρ and positive factor price vector (w_0^ρ, w^ρ) .*

The problem presented by the reduced form has been extensively studied by Scheinkman [15] and McKenzie [12] and [13]. The Turnpike Property has been shown to demonstrate that any optimal path asymptotically converges to OSS, especially when the discount factor ρ is sufficiently close to

1. One crucial point is that under constant returns to scale assumption on production functions, the reduced form social production function V is not strictly concave but just concave. In other words, the function V has a flat segment in which OSS is contained. This flat segment is often referred to as the *von Neumann Facet*. As discussed by McKenzie [12] and [13], in this case the proof of the Turnpike Property will turn out to be more complicated. Indeed, as we will show later, there exists a one dimensional von Neumann Facet.

The Facet is formally defined in the reduced form model as follows:

Definition 2 *The von Neumann Facet $F(k^\rho, k^\rho)$ of the optimal steady state k^ρ is defined as:*

$$F(k^\rho, k^\rho) = \{(x, z) \in D : V(x, z) + \rho p^\rho z - p^\rho x = V(k^\rho, k^\rho) + \rho p^\rho z - p^\rho x\},$$

where p^ρ is a supporting price of k^ρ when the discount rate ρ is given and the price of the consumption good is normalized as 1.

As depicted in Figure 1, the von Neumann Facet is a projection onto the (x, z) space of the line segment of the function V , which contains OSS and is supported by the price vector $(-p^\rho, \rho p^\rho, 1)$. Some useful properties of the Facet in a multi-sector optimal growth model with many capital goods have been studied in Takahashi [17], [18], [19] and [20]. Reader is suggested to refer to the detailed formal arguments presented there, especially Takahashi [20]. Here, a more intuitive explanatory strategy focusing on diagrams will be pursued.

Since the Facet is supported by the price vector, due to the factor price equalization theorem, the unique production technology will be chosen at any point of the Facet. Let us denote this chosen technology matrix as:

$$A^\rho = \begin{bmatrix} a_{00}^\rho & a_{01}^\rho \\ a_{10}^\rho & a_{11}^\rho \end{bmatrix}$$

For this matrix we make the following capital intensity assumption such that, at OSS, the consumption sector is more capital intensive:

Assumption 4. $a_{11}^\rho/a_{01}^\rho < a_{10}^\rho/a_{00}^\rho$ with $\rho = 1$.

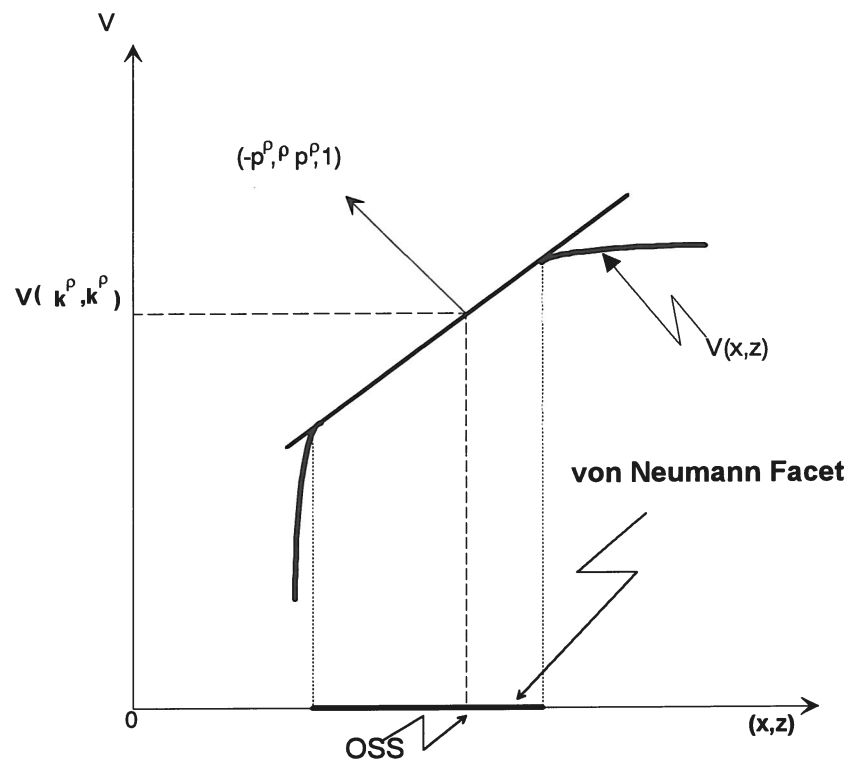


Figure 1:

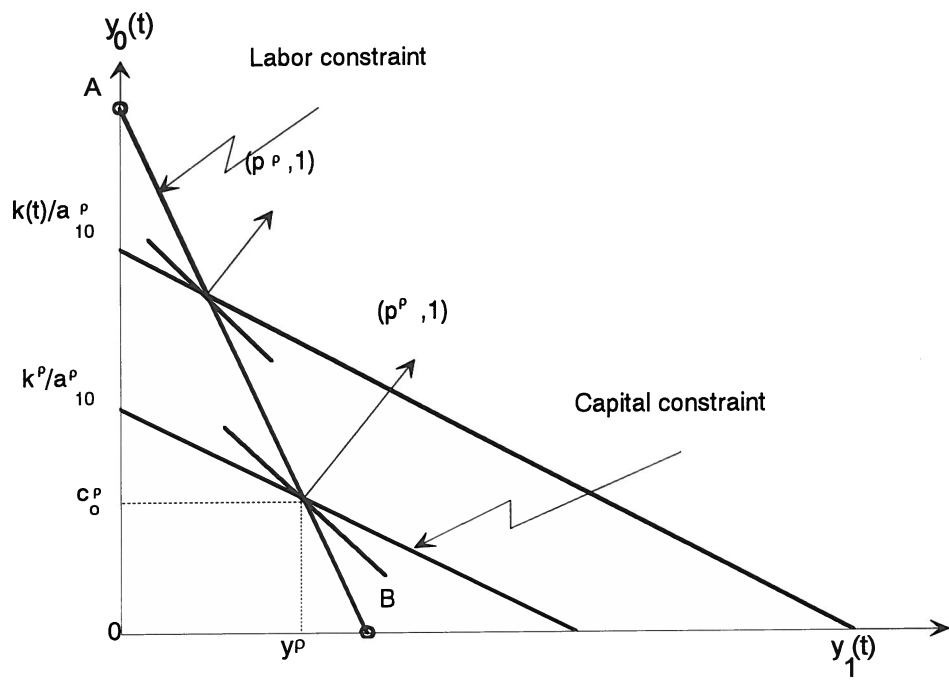


Figure 2:

It is straightforward that there exists a $\rho' > 0$ such that Assumption 4 holds for any $\rho \in [\rho', 1)$. As is often used in International Trade Theory, Figure 2 can be drawn on the output plane $(y_1(t), y_0(t))$, where the capital good constraint and the labor constraint are depicted. Note that (y_1^ρ, y_0^ρ) is the output pair corresponding to the optimal steady state k^ρ and is supported by the price vector $(p^\rho, 1)$. Also note that the diagram is drawn based on Assumption 4 where the consumption good production sector is more capital intensive than the capital good production sector.

At Points A and B, production specialization takes place. Suppose $\bar{k}(t) (< 1/a_{00}^\rho)$ is given. The capital constraint will then shift upward and the new equilibrium point E which is supported by the price vector $(p^\rho, 1)$ will be given as the intersection of the two constraints. At E, the new output pair $(y_1(t), y_0(t))$ is obtained. Using the accumulation equation (2), the capital stocks of the next period $\bar{k}(t+1)$ corresponding to $\bar{k}(t)$ will be calculated. Then the point $(\bar{k}(t), \bar{k}(t+1))$ obtained in this way can be plotted on the (x, z) space, i.e., the initial and terminal capital stock space. Changing $\bar{k}(t)$ and repeating this process for all the intersections of two constraints, the line segment AB in the (x, z) plane depicted as Figure 2, yields and corresponds to the labor constraint line AB of the output space. Line AB of (x, z) plane excluding the two end points A and B is the von Neumann Facet. The other case is also drawn as line A'B'.

From this argument, if there are n capital goods, the dimension of the Facet is n and will be redefined in the one capital good case as:

$F(k^\rho, k^\rho) = \{(k(t), k(t+1)) \in D: \text{there exists } y_0(t) \geq 0 \text{ and } y_1(t) \geq 0 \text{ such that}$

$$(i) \quad 1 = w_0^\rho a_{00}^\rho + w^\rho a_{10}^\rho,$$

$$(ii) \quad q^\rho = w_0^\rho a_{01}^\rho + w^\rho a_{11}^\rho,$$

$$(iii) \quad 1 = a_{00}^\rho y_0(t) + a_{01}^\rho y_1(t),$$

$$(iv) \quad k(t) = a_{10}^\rho y_0(t) + a_{11}^\rho y_1(t),$$

$$(v) \quad k(t+1) = [y_1(t) + (1 - \delta)k(t)],$$

where the price of the consumption good is normalized as one. (i) and (ii) are cost-minimization conditions. (iii) and (iv) are feasible conditions for labor

and capital goods, respectively. (v) is a capital accumulation equation.

From (iii) and (iv) of the definitions of the Facet,

$$y_1(t) = b^\rho k(t) + b_{10}^\rho \quad (8)$$

where b^ρ and b_{10}^ρ are the elements of the following inverse matrix of A^ρ :

$$(A^\rho)^{-1} = \begin{bmatrix} b_{00}^\rho & b_{01}^\rho \\ b_{10}^\rho & b^\rho \end{bmatrix}$$

Also from the accumulation equation (v) and (7),

$$k(t+1) = [b^\rho + (1 - \delta)]k(t) + b_{10}^\rho. \quad (9)$$

Defining $\eta(t) = k(t) - k^\rho$ yields

$$\eta(t+1) = [[b^\rho + (1 - \delta)]\eta(t)]. \quad (10)$$

In the next section the above difference equation is scrutinized.

One important result initially proved by McKenzie[12] and [13] for a reduced form model is Lyapunov Stability, often referred to as the "Neighborhood Turnpike Theorem". Takahashi [17], [19] and [20] has demonstrated that the same property will hold for a multi-sector neoclassical growth model with many capital goods. In the case of a neo-classical trade model the similar property was proved by Yano [21]. The Neighborhood Turnpike Theorem will be presented as Lemma 2 for later use and depicted in Figure 3:

Lemma 2 *Provided that there is no cyclic path on the von Neumann Facet $F(k^\rho, k^\rho)$ with $\rho = 1$, then for any $\epsilon > 0$, there exists a $\rho' > 0$ such that for any $\rho \in [\rho', 1)$ and a corresponding $\epsilon(\rho)$, any optimal path $k^\rho(t)$ is eventually attracted in the ϵ -neighborhood of k^ρ . Furthermore, as $\rho \rightarrow 1$, $\epsilon(\rho) \rightarrow 0$.*

Proof. Adopting the proof of Theorem 1 of Takahashi [20], the result follows.¹□

¹In Takahashi[17] and [18], a general objective function $\sum_0^\infty \rho^t u(y_0(t), c(t))$ where $c(t)$ is the vector of the consumable capital goods, is assumed and Lyapunov Stability is proved by showing the following three sufficient conditions obtained by McKenzie[13]: 1) there exists $\rho' > 0$ such that the facet $F(k^\rho, k^\rho)$ is a lower semi-continuous correspondence of $\rho \in [\rho', 1)$. 2) if (p, q) are support prices of some point (x, z) of $F(k^\rho, k^\rho)$ with $\rho = 1$, then $p = q = p^\rho$. 3) $F(k^\rho, k^\rho)$ with $\rho = 1$ has no cyclic path.

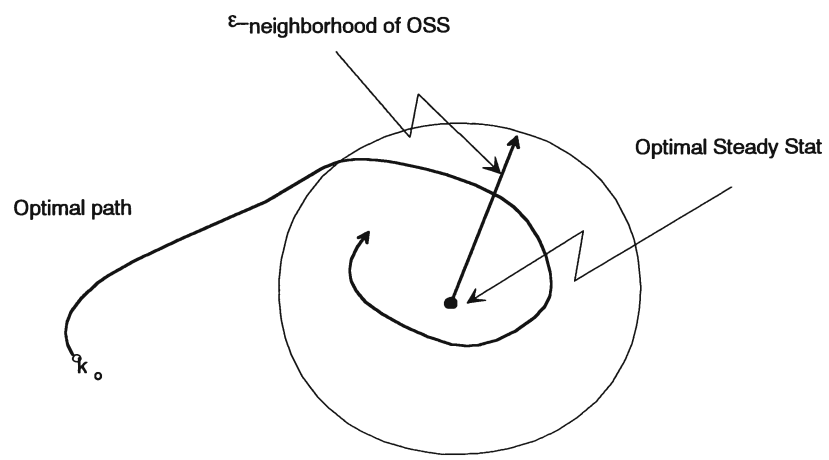


Figure 3:

3 Optimal Cyclic Paths

It follows from Theorem 1 that if the von Neumann Facet has no cyclic path, then we can restrict our argument to an ϵ -neighborhood of OSS. In other words, we can restrict our argument to the local analysis. The following important qualitative result on the optimal path trapped in the ϵ -neighborhood can be proved by adopting Theorem 3 of Benhabib and Nishimura[2]. Note that the important condition to establish the theorem is that $V_{22} < 0$ in N_ϵ (see Theorem 2 of Benhabib and Nishimura[2]). Since $V_{22}^\rho = T_{11}(k^\rho, k^\rho) = (b^\rho)^2 T_{22}(k^\rho, k^\rho) < 0$ (see P.299 in Benhabib and Nishimura[2]), taking ρ close enough to 1 satisfies the condition.

Lemma 3 *Suppose $\{k^\rho(t)\}_0^\infty$ is the optimal path attracted in the ϵ -neighborhood N_ϵ . Then if $V_{12} \neq 0$ is over all points of N_ϵ , an optimal path will either converge to an OSS or to a cyclic of period two.*

Proof. Replacing the region \bar{D} with N_ϵ and applying Theorem 3 of the argument provided by Benhabib and Nishimura [2], the result follows. \square

According to the term coined by Denekere and Pelikan [8], we may say that the dynamical system displays "simple dynamics".

Since an optimal path satisfies the Euler equation, a linear approximation of the Euler equation around OSS gives the following linear difference equation, if $V_{12}^\rho \neq 0$:

$$z_{t+1} = -(V_{12}^\rho)^{-1}(V_{11}^\rho + \rho^{-1}V_{22}^\rho)z_t - (\rho V_{12}^\rho)^{-1}V_{12}^\rho z_{t-1} \quad (11)$$

where $z_t = k(t) - k^\rho$, $V_{11}^\rho = \partial^2 V(k^\rho, k^\rho)/\partial x^2$, $V_{12}^\rho = \partial^2 V(k^\rho, k^\rho)/\partial x \partial z$, and $V_{22}^\rho = \partial^2 V(k^\rho, k^\rho)/\partial z^2$.

The next lemma was proved by Levhari and Liviatan [9] and will provide some important information concerning the local motion of the optimal path in conjunction with the von Neumann Facet:

Lemma 4 *If λ is a root of the characteristic equation related to Eq.(10), then $1/\rho\lambda$ is also a root.*

Proof. see Theorem 4 of Levhari and Liviatan [9]. \square

To establish the above, the condition that $V_{12}^\rho \neq 0$ plays an important role. Therefore we will demonstrate below when this condition holds.

Recall that Assumption 4 implies $b^\rho < 0$. Furthermore, from the argument in Benhabib and Nishimura [2] (see P.299), the following relation holds:

$$V_{12}^\rho = -(b^\rho)^2(\partial w/\partial k)[b^\rho + (1 - \delta)].$$

Since $T_{22} = \partial w/\partial k < 0$ by the concavity of T , we have $V_{12}^\rho = 0$ if $b^\rho = -(1 - \delta)$. With this in mind, consider the next four cases.

Case 1: $-(1 - \delta) - 1/\rho > b^\rho$

It follows from Eq.(9) that any path on the Facet is explosive. Thus there is no cyclic path on the Facet (see the line AB in Figure 4). Due to Lemma 2, the Liapunov stability theorem holds. This also implies that in the ϵ -neighborhood N_ϵ , the characteristic equation related to Eq.(10) must have $\lambda < -1$ as one of two eigenvalues. On the other hand, if ρ is close enough to unity due to Lemma 2, the other eigenvalue $1/\rho\lambda$ must satisfy $-1 < 1/\rho\lambda < 0$. Then the linear manifold spanned by the eigen vector associated with $1/\rho\lambda$ becomes a stable linear subspace and implies the existence of a stable manifold. Due to Lemma 16 of Scheinkman [15] (see also the sufficiency argument by Mangasarian[10]), any path on the stable manifold is optimal.

This local property is often referred to as the local stability of an optimal path². By Lemma 2, taking ρ close enough to unity, we can make the ϵ -neighborhood as small as possible. Then any optimal path must "jump on" the stable linear manifold and converge to an OSS.

Case 2: $-1 - (1 - \delta) < b^\rho < -(1 - \delta)$

²I have only proved here that OSS is saddle-point stable. Note that this, however, is not enough to prove the local stability of an optimal path. If the stable manifold does not stretch out in both directions on the $k(t)$ -axis, the trapped optimal path cannot "jump on" this stable manifold. This point was first mentioned in McKenzie [11] and later argued in detail by Scheinkman [15] for an optimal growth model. Adopting Scheinkman's argument, Takahashi [17] and [19] have proved that if V_{22}^ρ is negative definite, the saddle-point stability implies local stability. In our case, the condition $V_{22}^\rho < 0$ holds as we discussed before.

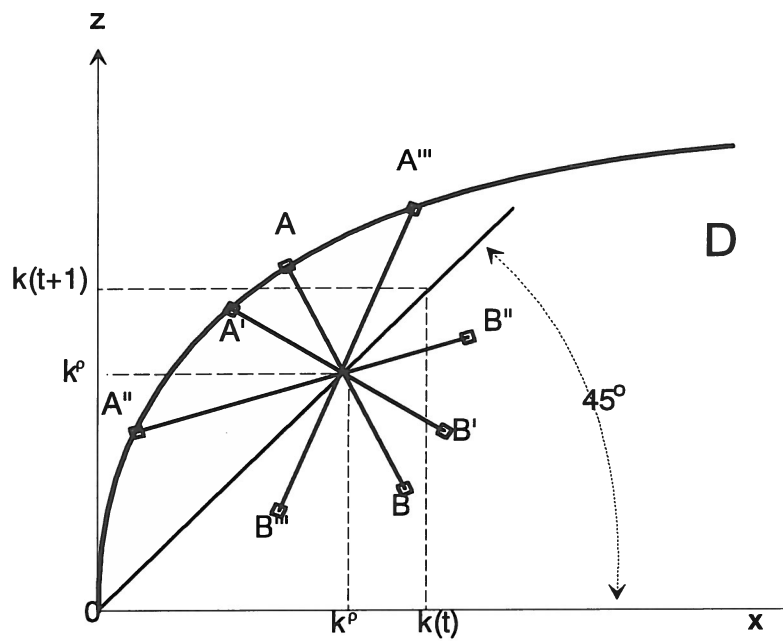


Figure 4:

From Eq.(9), any path on the Facet converges to OSS (see the line A'B' in Figure 4). This implies that the Facet itself is a stable manifold and there is no cyclic path. So the local stability and the Lyapunov Stability hold simultaneously. Thus by choosing ρ close enough to unity, the Turnpike Property is established; any optimal path must converge to an OSS.

Case 3: $-(1 - \delta) < b^\rho < 0$

Since it follows that $0 < b^\rho + (1 - \delta) < (1 - \delta) < 1$, the Facet is something like a line A''B'' depicted in Figure 4 and any path on the Facet is stable. So the one of two eigenvalue is less than one. From Lemma 4, the other eigenvalue is also less than one. So the local stability holds.

Note that in order to establish the first two results above, we never used Lemma 4. Let us summarize these two cases as a formal theorem:

Theorem 1. There exists a $\rho'' > 0$ such that for $\rho \in [\rho'', 1)$, any optimal path $k^\rho(t)$ converges to OSS if the conditions of Cases 1 through 3 are satisfied for such a ρ .

Case 4: $-(1 - \delta) - 1/\rho < b^\rho < -1 - (1 - \delta)$

In this case the Facet is something like A''B'' in Figure 4. From Eq.(9) again, any path on the Facet is explosive. This means that the Facet itself is an unstable manifold. Due to Lemma 4, $1/\rho\lambda$ is also the eigenvalue and is less than -1. So there exists an unstable manifold. Combining both of the above results implies that any path in an ϵ -neighborhood is explosive and never converges to OSS. I will apply Lemma 3 and show that an optimal path will eventually converge to a cyclic path of period two. The following theorem is a formal presentation.

Theorem 2. There exists a $\tilde{\rho} > 0$ such that for $\rho \in [\tilde{\rho}, 1)$, any optimal path $k^\rho(t)$ converges to a cyclic path of period two if the condition of Case 4 holds for such a ρ .

proof. Let us take a $\rho'' > 0$ such that for $\rho \in [\rho'', 1)$, there exists an ϵ -neighborhood so that Lemma 2 is established. Now define the ϵ -neighborhood $N_\epsilon(k^\rho)$ of (k^ρ, k^ρ) as follows:

$$N_\epsilon(k^\rho) = \{(x, z) : \|(x, z) - (k^\rho, k^\rho)\| < \epsilon\}$$

Suppose that $V_{12}(x, z) < 0$ for all $(x, z) \in N_\epsilon(k^\rho)$ does not hold for some $\rho \in [\rho'', 1)$. Since $V_{12}^\rho < 0$, we can find the following $\alpha' > 0$:

$$\alpha' = \limsup\{\alpha > 0 : V_{12}(x, z) < 0 \text{ for all } (x, z) \in N_\alpha(k^\rho)\}$$

where the *lim-sup* is taken over all k^ρ for $\rho \in [\rho'', 1)$.

Suppose $\epsilon \leq \alpha'$. Then there is an ϵ' -neighborhood where for $0 < \epsilon' < \epsilon$, the dynamics in $N_{\epsilon'}(k^\rho)$ can be described by Eq.(10). If ρ satisfies Case 4, from the argument of Case 4 any path in $N_{\epsilon'}(k^\rho)$ is explosive and never converges to OSS. Since $V_{12}(x, z) < 0$ holds for all $(x, z) \in N_\epsilon(k^\rho)$, Lemma 3 holds. This implies that an optimal path should converge to a cyclic path of period two.

If $\epsilon > \alpha'$, taking ρ closer to 1 we can find an $\epsilon'' > 0$ where $\epsilon'' \leq \alpha'$ and Lemma 2 holds. Thus, applying the same logic, the theorem will be proved. This completes the proof. \square

Benhabib and Nishimura [2] have reached similar conclusions under the much stronger condition that $V_{12} \neq 0$ over all interior points of the production possibility set D. This condition is unnecessary here. It is worth mentioning that when ρ is taken closer to 1, the interval of Case 4 shrinks and disappears at $\rho = 1$. This means that at $\rho = 1$, there are no optimal cycles and the Turnpike Property always emerges.

Case 5: $b^\rho = -(1 - \delta) - 1/\rho$ or $b^\rho = -1 - (1 - \delta)$

It is clear that there is a cyclic path on the Facet for both cases and Lemma 1 does not hold. We do not know, therefore, whether the cyclic path is optimal or not.

Finally, let us consider the case in which the following opposit capital intensity condition to Assumption 4 is assumed.

Assumption 5. $a_{11}^\rho/a_{01}^\rho > a_{10}^\rho/a_{00}^\rho$

This assumption implies $b^\rho > 0$. From the normalization of b^ρ , it follows that $b^\rho + (1 - \delta) > 1/\rho > 1$. Therefore any path on the Facet is explosive

(see line A''B'' in Figure 4). Applying the same argument as in Case 1, we have local stability. Then taking ρ close to 1, the Turnpike Property is obtained and there is no chance of cyclic optimal paths. This result may be summarized as the theorem:

Theorem 3. Under Assumption 5, there exists $\rho' > 0$ such that for $\rho \in [\rho', 1)$, any optimal path $k^\rho(t)$ converges to the OSS k^ρ .

This can also be proved by applying the theorem demonstrated by Benhabib and Nishimura[2] (see Theorem2), namely if $V_{12} \neq 0$ for all $(x, z) \in N_\epsilon$, then the solution paths are monotonic.

4 Concluding Remarks

One important unanswered question involves whether or not the results obtained above can be generalized into a multi-sector neoclassical optimal growth model. Under a generalized capital intensity condition such that the consumption sector is more capital-intensive, the results of Cases 1 through 3 have been extended to a multi-sector case by Takahashi [17], [19] and [20]. Especially, note that under the opposite generalized capital intensity condition: the capital good production sector uses more capital intensive technology, the Turnpike Property also has been established in a multi-sector case. It is not yet known, however, whether or not the result given in Case 4 could also be obtained in a multi-sector model.

A final note of caution: although these results suggest that there may be an upper bound of the discount factor, say, ρ' for emerging complicated dynamics, this never excludes the possibility of constructing a numerical example where the complicated dynamics could emerge at, say, $\rho = 0.99$. Since our conclusions are analytical rather than numerical, we cannot respond with numerical precision to the question of how close to 1 the lower bound ρ' should be.

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