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Queen's Economics Department Working Paper No. 28

THE IMPLICATIONS OF STEADY STATE GROWTH FOR ENDOGENOUS AND EMBODIED TECHNOLOGICAL CHANGE*

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THE IMPLICATIONS OF STEADY STATE GROWTH FOR ENDOGENOUS AND EMBODIED TECHNOLOGICAL CHANGE*

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Discussion Paper No. 28

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THE IMPLICATIONS OF STEADY STATE GROWTH FOR ENDOGENOUS AND EMBODIED TECHNOLOGICAL CHANGE.

1. Introduction.

The concern of this paper is with necessary, rather than sufficient, conditions for the existence of steady state growth paths. In a sense this is a negative approach to growth theory but it does have its positive aspects. Firstly, it enables us to appreciate the restrictive nature of our assumptions concerning technological change if a steady state growth path is to exist. Secondly, it emphasizes the fact that if one wishes to study the implications of technological change in any but its simplest forms, it will be fruitless to investigate the properties of steady state growth paths.

One may classify growth models according to the technological change assumptions that they embody. Technological change may be either exogenous or endogenous. Under either of these categories one may further classify the model according to whether technological change is embodied or disembodied. Lastly, if technological change is embodied, one then classifies the model according to whether there is a general or fixed coefficient production function.

The simplest class of models are those in which technological change is both exogenous and disembodied. Swan [9] has shown the well known result that for a steady state growth path to exist for all time, technological change must be labor augmenting and change at a constant exponential rate. Not so well known is the extension of this result to the case where technological change is embodied and there is a fixed coefficient production function. Recently Inada [3] has shown that Swan's result is applicable in this case also if we assume the existence of ths steady state growth path for any value of the savings ratio 's' in a non-degenerate closed interval and also that the economic lifetime of capital goods is constant on this path.

If the latter assumption is dropped the problem remains open. Also open is the same problem but with a general production function replacing the previous fixed coefficient production function.

Sufficient conditions for a steady state growth path to exist when technological change is endogenous and embodied and there is a fixed coefficient production function have been studied by Arrow [1] and the model extended to a general production function by Levhari [5]. Sufficient conditions have also been studied by Sheshinski [8] for the case where technological change is endogenous but disembodied. In all these learning by doing models technological change is purely labor augmenting and a power function of gross cumulative investment. From now on this will be referred to as Arrow Type technological change. Necessary conditions have recently been studied by Inada [4] for the disembodied case and also for the embodied but fixed coefficient production function case. Levhari and Sheshinski [7] have also studied necessary conditions for the disembodied case. Under the conditions of Theorem 1 below it was shown that for a balanced growth path to exist technological change must be Arrow Type.

The purpose of this paper is to enquire whether this result can be extended to the case of a general production function where technological change is endogenous and embodied. In fact, it is shown that the result does carry over to this case.

However, central to the proof to be illustrated below, is Inada's result [4] on the pseudo-production function (i.e., the function that relates total output to total cumulative gross investment and total labor supply in a vintage model). In fact, it is precisely this critical dependence that makes the procedure shown below inapplicable to the case where technological change is exogenous and embodied and there is a general

production function. In this case Fisher [2] has shown that a capital aggregate exists, and this is a necessary requirement for the existence of a pseudo-production function, if and only if technological change is purely capital augmenting. It is therefore deemed necessary to first explain Inada's result.

Let us now introduce the following notation:

G: Cumulative gross investment at time t.

L: Total labor employment at time t.

Q: Total output at time t from all vintages.

σ: Exponential rate of growth of the labor supply.

ω: Rate of growth of G on the balanced growth path.

Inada first observes that on a balanced growth path the following relation holds:

$$G^m/L = G_o^m/L_o = k(s) \text{ where } m = \sigma/\omega$$

and k is a constant for any given value of the savings ratio. Clearly G_o and ω may depend on s. Now suppose that the pseudo-production function, $Q = Q(G,L)$, exists. As the equation of motion is:

$$\dot{G} = sQ(G,L)$$

it is seen that on a steady state growth path:

$$\frac{\omega}{s} = Q(G,L)/G \equiv \Phi(G,L) \text{ say.}$$

Then $\Phi(G,L)$ takes the same value for all (G,L) which locate on the curve

$$G^{m(s)}/L = k(s) \text{ in Fig. 1 below.}$$

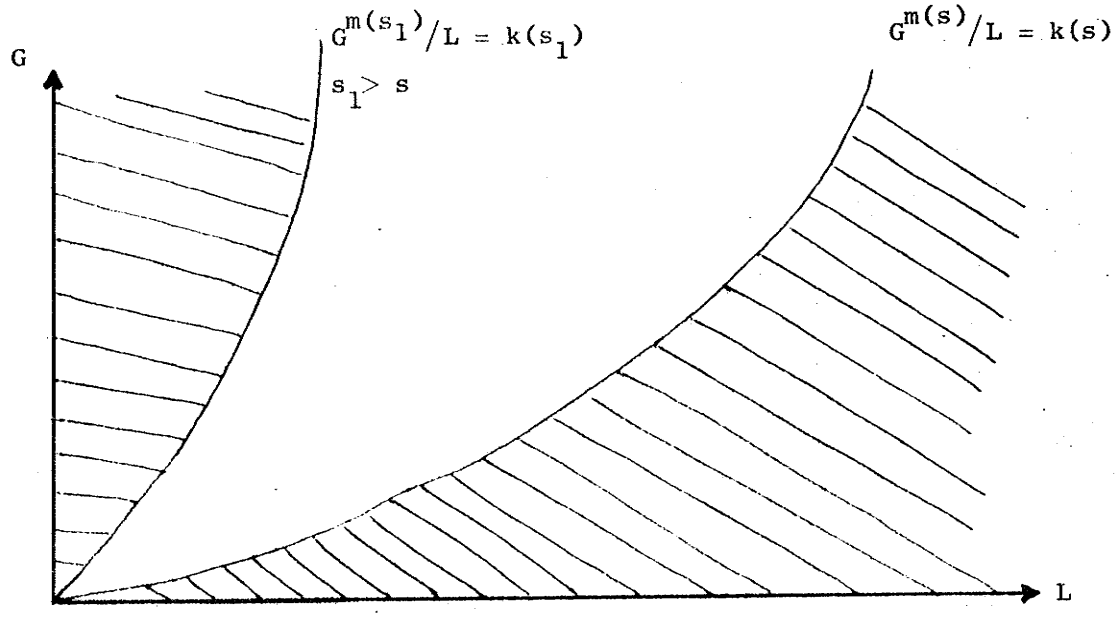


Fig. 1.

Inada then shows that as s increases this curve shifts to the left, no two curves intersecting, and Φ decreases. (The proof requires that there be full employment of labor at least for a while. In the model to be considered in the next section positive marginal products and continuous production functions are assumed and so this full employment assumption is trivially satisfied.) Thus the value of Φ is completely determined by $k(s)$ and so one may write:

$$\Phi(G,L) \equiv H(G^{-m}L)$$

for any value of (G,L) in the non-shaded area above. This area is delineated by the extreme values between which the savings ratio must be chosen.

It now becomes clear that $Q = GH(G^{-m}L)$. Levhari and Sheshinski [7] have also shown this result but under alternative conditions. This is summarized by:

Theorem 1. [Inada, Levhari and Sheshinski].

A necessary condition for the existence of a balanced growth path,

(a) for all initial conditions or

(b) for all time and for any value of the savings ratio in a non-degenerate closed interval,

is that the pseudo-production function exist and have the following representation:

$$Q = G.H(G^{-m}L) = G.H(G^{n}L/G) \quad \text{where} \quad n = 1 - m.$$

In particular, it is noticed that the pseudo-production function must be homogeneous of degree one in G and $G^{n}L$.

Proof.

For a complete proof see [4] and [7].‖

Theorem 1 will be used in the final section of this paper even though it is not possible to find the explicit form of the pseudo-production function. When this theorem is applied to the pseudo-production function obtained when the vintage production functions are of the fixed coefficient variety and technological change is endogenous and embodied it is quickly seen that it must be Arrow Type.

In the next section the model and the assumption will be explained and also the implicit representation of the pseudo-production function will be derived. In the final section Theorem 1 will be applied to this implicit representation in order to derive results concerning what type of technological change assumptions are compatible with the existence of steady state growth paths.

2. The Model and the Implicit Representation of the Pseudo-Production Function.

The following additional notation will be used in this section:

$Q_{v,t}$: Output at time t from machines of vintage v.

I_v : Investment in machines of vintage v.

$L_{v,t}$: Labor force employed at time t with machines of vintage v.

$\varphi(G_v)$: The capital augmenting technological progress function which is assumed to depend only on G_v. φ is a continuous function.

$\lambda(G_v)$: The labor augmenting technological progress function which is assumed to depend only on G_v. λ is a continuous function and so also is $\lambda'(G)$. It should be noted that technological progress is assumed to be factor augmenting.

G_v : Gross capital accumulation at time v.

$m(t)$: The age of the oldest capital good in use at time t.

w_t : The wage rate at time t.

The vintage production function is written as:

$$Q_{v,t} = F[\varphi(G_v)I_v, \lambda(G_v)L_{v,t}] \qquad (1)$$

It is assumed that:

1. $F[I,0] = 0$ i.e., it is impossible to produce without labor.
2. F is homogeneous of degree one in its arguments, i.e., constant returns to scale apply.
3. F is concave with positive first partial derivatives and negative second partial derivatives.

As was mentioned earlier, a constant savings ratio is assumed, and also:

$$L_t = L_o e^{\sigma t} = \text{labor supply at time } t. \qquad (2)$$

Clearly:

$$L_t = \int_{t-m(t)}^{t} L_{v,t}\, dv = \text{demand for labor at time } t. \tag{3}$$

$$I_v = \dot{G}_v \tag{4}$$

$$G_t = \int_{-\infty}^{t} I_v dv \tag{5}$$

Perfect competition is assumed so total output is maximized. This also implies that there is full employment of labor and that the marginal product of labor will be the same in all vintages and equal to the current wage rate. Therefore:

$$\partial F/\partial L_{v,t} = w_t \tag{6}$$

By Assumption 2 above:

$$\partial F/\partial(\lambda(G_v)L_{v,t}) \equiv h[\lambda(G_v)L_{v,t}/\varphi(G_v)I_v] \quad \text{say:} \tag{7}$$

where, by Assumption 3, $h' < 0$ and the prime denotes the derivative.

Define the function $f(f \equiv h^{-1})$ by:

$$\lambda(G_v)L_{v,t}/\varphi(G_v)I_v \equiv f[\partial F/\partial(\lambda(G_v)L_{v,t})] \tag{8}$$

where, clearly, $f' < 0$. Then, by (6) and (7),

$$w_t = \lambda(G_v)h[\lambda(G_v)L_{v,t}/\varphi(G_v)I_v] \tag{9}$$

Therefore:

$$L_{v,t} = \frac{\varphi(G_v)}{\lambda(G_v)} f[w_t/\lambda(G_v)]I_v \tag{10}$$

At this point it is necessary to stipulate our assumptions concerning $h(0)$ and also the technological progress functions. It is trivially assumed that $\varphi(G_v)$ and $\lambda(G_v)$ are always positive and normalized so that $\varphi(0) = \lambda(0) = 1$. Also, either one of the following two mutually exclusive assumptions will be made:

Assumption 1. $h(0)$ is finite, $\lambda'(G_v) \geq 0$ and $\varphi'(G_v) \geq 0$.

Assumption 2. $h(0)$ is infinite.

If Assumption 1 is applicable it is not necessarily efficient to employ all machines but the assumptions concerning technological change guarantee that the more recent vintages will always be employed in preference to older ones. If Assumption 2 is applicable it is clear that one will employ all vintages: there is no economic obsolescence. Therefore, in either case one may write:

$$Q_t \equiv \int_{t-m(t)}^{t} Q_{v,t}\,dv \tag{11}$$

where, in the latter case, $m(t)$ is infinite and so no additional technological change assumptions are required.

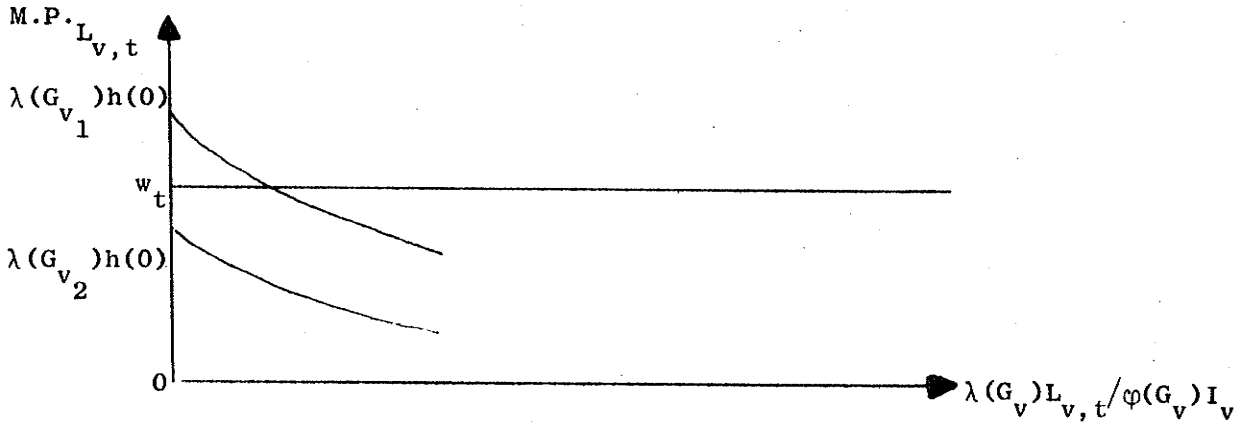
Remark.

It appears at first sight that the following possibility is not included by either Assumption 1 or 2 above: namely, $h(0)$ finite but $\lambda'(G_v) \equiv 0$. However, in general, the marginal product of labor employed at time t on vintage v (i.e., $MP_{L_{v,t}}$) is:

$$M.P._{L_{v,t}} = \lambda(G_v)h[\lambda(G_v)L_{v,t}/\varphi(G_v)I_v].$$

Therefore, as illustrated in the diagram below which is drawn for a particular time period t, the marginal productivity curve may shift downward as we consider older vintages. In the case illustrated one would

employ machines of vintage v_1, as $w_t < \lambda(G_{v_1})h(0)$, but not machines of vintage v_2 (where $v_2 < v_1$). But if $\lambda'(G_v) \equiv 0$ then the curve will not shift. In other words, if one employs machines of vintage v_1, one will employ all machines for the same reason as the employment of any of them. This means that this case is effectively included in the analysis under Assumption 2.



It is now possible to show:

Theorem 1.

If either Assumption 1 or 2 is applicable the pseudo-production function exists and has the following implicit representation:

$$\frac{\partial Q}{\partial G} + \frac{\partial Q}{\partial L}\frac{\varphi(G)}{\lambda(G)} f[\partial Q/\partial L /\!/ \lambda(G)] \equiv F[1,\ f[\partial Q/\partial L /\!/ \lambda(G)]]\varphi(G)$$

Proof.

As $F[.,.]$ is homogeneous of degree one in its arguments:

$$F[\varphi(G_v)I_v, \lambda(G_v)L_{v,t}] = F[1,\ \lambda(G_v)L_{v,t}/\varphi(G_v)I_v]\varphi(G_v)I_v$$

Relations (1), (10) and (11) imply:

$$Q_t = \int_{t-m(t)}^{t} F[1,\ f[w_t/\lambda(G_v)]]\varphi(G_v)I_v dv \qquad (12)$$

Also (3) and (10) imply:

$$L_t = \int_{t-m(t)}^{t} \frac{\varphi(G_v)}{\lambda(G_v)} f[w_t/\lambda(G_v)]I_v dv \qquad (13)$$

Substitute $z = G_v$, and therefore $I_v dv = dz$, into (12) and (13). Then:

$$Q_t = \int_{G_{t-m(t)}}^{G_t} F[1, f[w_t/\lambda(z)]]\varphi(z)dz \qquad (14)$$

$$L_t = \int_{G_{t-m(t)}}^{G_t} \frac{\varphi(z)}{\lambda(z)} f[w_t/\lambda(z)]dz \qquad (15)$$

The object now is to eliminate $G_{t-m(t)}$ and w_t from (14) and (15) in order to obtain the implicit representation of the pseudo-production function. In other words, it is necessary to eliminate two unknowns from two equations. It is at this point that one must distinguish carefully between Assumptions 1 and 2. In the latter case an infinite marginal product of labor at the origin implies that all machines are in use so:

$$G_{t-m(t)} = 0 \qquad (16)$$

but w_t is unknown. Under A.1, $G_{t-m(t)}$ is unknown, but it is now possible to determine the wage rate in terms of $G_{t-m(t)}$. Therefore in either case there is one less variable to be eliminated. In fact, under Assumption 1 the wage rate will be equal to the marginal product of labor on the oldest machine in use. Therefore by (9):

$$w_t = \lambda(G_{t-m(t)})\ h(0) \qquad (17)$$

Suppose Assumption 1 is applicable. Substitute (17) into (15) and one obtains:

$$L_t = \int_{G_{t-m(t)}}^{G_t} \frac{\varphi(z)}{\lambda(z)} f[\lambda(G_{t-m(t)})h(0)/\lambda(z)]dz \equiv T(G_t, G_{t-m(t)}) \quad \text{say.} \tag{18}$$

Now differentiate this partially with respect to $G_{t-m(t)}$. Therefore:

$$\partial T/\partial G_{t-m(t)} \equiv \int_{G_{t-m(t)}}^{G_t} h(0) \frac{\varphi(z)}{\lambda^2(z)} f'[\lambda(G_{t-m(t)})h(0)/\lambda(z)]\lambda'(G_{t-m(t)})dz \tag{19}$$

It is clear that (18) may be solved for $G_{t-m(t)}$ if and only if $\partial T/\partial G_{t-m(t)}$ is one signed. As $f' < 0$ this is certainly the case if $\lambda'(G_v) > 0$ which, in fact, was assumed. Therefore one may write:

$$G_{t-m(t)} \equiv g(G_t, L_t) \quad \text{say.} \tag{20}$$

Substituting (17) and (19) into (14) and also (19) into (18) the following <u>identities</u> are seen to hold (where the time subscript has been dropped as it is now redundant):

$$Q \equiv \int_{g(G,L)}^{G} F[1, f[\lambda(g(G,L))h(0)/\lambda(z)]] \varphi(z)dz \tag{21}$$

$$L \equiv \int_{g(G,L)}^{G} \frac{\varphi(z)}{\lambda(z)} f[\lambda(g(G,L))h(0)/\lambda(z)]dz \tag{22}$$

In order to obtain the required implicit representation of the pseudo-production function it remains only to eliminate $g(G,L)$ between (21) and (22). Relationships (21) and (22) imply the following:

$$\partial Q/\partial G \equiv F[1,\ f[\lambda(g)h(0)/\lambda(G)]]\ \varphi(G) + [h^2(0)\lambda(g)\int_g^G A(z)\partial g/\partial G.dz]\lambda'(g) \qquad (23)$$

$$\partial Q/\partial L \equiv h^2(0)\ \lambda(g)\ \lambda'(g) \int_g^G A(z)\ \partial g/\partial L.dz \qquad (24)$$

$$\partial L/\partial G = 0 \equiv \frac{\varphi(G)}{\lambda(G)}\ f[\lambda(g)h(0)/\lambda(G)] + \lambda'(g)\ h(0) \int_g^G A(z)\ \partial g/\partial G.dz \qquad (25)$$

$$\partial L/\partial L = 1 \equiv \lambda'(g)\ h(0) \int_g^G A(z)\ \partial g/\partial L.dz \qquad (26)$$

where $A(z) = \varphi(z)\ f'[\lambda(g)h(0)/\lambda(z)]/\lambda^2(z)$.

One now notes, as expected, that (24) and (26) imply:

$$\partial Q/\partial L \equiv h(0)\lambda(g) = w \quad \text{by (17)} \qquad (27)$$

From (23) and (27):

$$\partial Q/\partial G + \frac{\varphi(G)}{\lambda(G)}\ f[\partial Q/\partial L /\!\!/ \lambda(G)]\partial Q/\partial L = F[1, f[\partial Q/\partial L /\!\!/ \lambda(G)]]\varphi(G) \qquad (28)$$

$$+\ h(0)\lambda'(G)\partial Q/\partial L \int_g^G A(z)\ \frac{\partial g}{\partial G}\ dz$$

$$+\ \frac{\varphi(G)}{\lambda(G)}\ f[\partial Q/\partial L /\!\!/ \lambda(G)]\partial Q/\partial L$$

By (25) the last two terms of (28) sum to zero so the theorem follows.

Now suppose Assumption 2 is applicable. Substitute (16) into (15):

$$L = \int_0^G \frac{\varphi(z)}{\lambda(z)}\ f[w/\lambda(z)]dz$$

The partial derivative of the right hand side of this expression with respect to w is always a negative function of w, as $f' < 0$, irrespective

of any assumptions that might be made concerning the technological change functions. Therefore one may solve for the wage rate as a function of G and L:

$$w = w(G,L) \quad \text{say.}$$

It can quickly be verified, that by following the same procedure as before, one again obtains (27) and hence the Theorem. Therefore the same result is applicable under either assumption.‖

Remark.

As either Assumption 1 or 2 leads to the same result it will not be necessary, in the next section, to distinguish the two. In the above analysis they were essentially sufficiency conditions for the existence of the pseudo-production function.

3. The Derivation of the Necessary Conditions for Steady State Growth.

The existence of a steady state growth path for all time and for any value of the savings ratio in a non-degenerate closed interval, or for all initial conditions, implies, as was seen earlier, that the pseudo-production function takes the form:

$$Q \equiv G\, H(G^{-m}L) \tag{1}$$

Therefore:

$$\partial Q/\partial G \equiv H(y) - myH'(y) \tag{2}$$

$$\partial Q/\partial L \equiv H'(y)G^{n} \tag{3}$$

where $y = G^{-m}L$.

The partial derivatives, $\partial Q/\partial G$ and $\partial Q/\partial L$, must be positive (unless L is zero so, as $F[I,0] = 0$, $\partial Q/\partial G$ is zero also: but this is trivial) as $\partial F/\partial I_v$ and $\partial F/\partial L_{v,t}$ are positive by assumption.

Now using (1), (2), (3) and Theorem (2.1), one obtains:

$$H(y) - myH'(y) \equiv \varphi(G)\left\{F[1,f[\pi(G)H'(y)]] - \pi(G)f[\pi(G)H'(y)]H'(y)\right\} \qquad (4)$$

where $\pi(G) \equiv G^n/\lambda(G)$.

This is an identity in G and y. First, differentiate both sides of (4) partially with respect to G and re-arrange. Therefore:

$$\varphi'(G)[H(y) - myH'(y)] \equiv \varphi(G)\pi'(G)H'(y)f[\pi(G)H'(y)] \qquad (5)$$

Divide both sides of the above by $\varphi(G)\partial Q/\partial G$ to obtain:

$$\frac{\varphi'(G)}{\varphi(G)} \equiv \frac{\pi'(G)H'(y)f[\pi(G)H'(y)]}{H(y) - myH'(y)} \qquad (6)$$

Lastly, differentiate both sides of (6) partially with respect to y, multiply through by $[H(y) - myH'(y)]f[\pi(G)H'(y)]/H'(y)$ and re-arrange to get:

$$\pi'(G)\left\{\frac{f'[\pi(G)H'(y)]}{f[\pi(G)H'(y)]}H''(y)\pi(G) + \frac{H''(y)}{H'(y)} - \frac{H'(y)(1-m) - myH''(y)}{H(y) - myH'(y)}\right\} \equiv 0 \qquad (7)$$

Alternatively:

$$\pi'(G)R(G,y) \equiv 0 \quad \text{where} \qquad (8)$$

$$R(G,y) \equiv \frac{\partial \text{Log}.f[\pi(G)H'(y)]}{\partial y} - \frac{H'(y)(1-m) - myH''(y)}{H(y) - myH'(y)} + \frac{H''(y)}{H'(y)} \qquad (9)$$

We have therefore shown:

Lemma 1.

Under the conditions of Theorem 1.1., G and L must always satisfy:

$$\pi'(G)R(G,y) \equiv 0$$

where $R(G,y)$ is given by (9), $y = G^{-m}L$ and $\pi(G) = G^n/\lambda(G)$.

Proof.

As above.‖

Lemma 2.

If $\pi'(G) \equiv 0$ for all G belonging to some open interval then technological change must have an Arrow Type representation on this same interval.

Proof.

As $\pi'(G) \equiv 0$ and $\partial Q/\partial G \neq 0$ equation (5) implies $\varphi'(G) \equiv 0$. By the definition of $\pi(G)$, $\pi'(G) \equiv 0$ implies $\lambda(G) \equiv G^n$ and hence the lemma.‖

Lemma 3.

If G^* and y^* are such that $R(G^*,y^*) = 0$ then, assuming $\pi'(G)R(G,y) \equiv 0$, either:

(a) $\pi'(G^*) = 0$ or

(b) $R(G,y^*) = 0$ for all $G \in N_\varepsilon(G^*)$ and for all small ε less than some $\bar{\varepsilon}$ where $N_\varepsilon(G^*) = [G: \ 0 < |G-G^*| < \varepsilon]$.

Proof.

Suppose the lemma is false. Then:

(1) $\pi'(G^*) \neq 0$ and

(2) there exists G^o and $\varepsilon_1 < \bar{\varepsilon}$, $G^o \in N_{\varepsilon_1}(G^*)$ such that $R(G^o,y^*) \neq 0$. Now as $\lambda'(G)$ is a continuous function so also is $\pi'(G)$. Therefore (1) implies there exists ε_2, say, such that $\pi'(G) \neq 0$ for all $G \in N_{\varepsilon_2}(G^*)$. Now choose $\bar{\varepsilon} < \varepsilon_2$. As $\varepsilon_1 < \bar{\varepsilon}$ we have $G^o \in N_{\varepsilon_2}(G^*)$ so $\pi'(G^o) \neq 0$. But this, together with (2), violates the condition $\pi'(G^o)R(G^o,y^*) = 0$. Therefore (1) and (2) cannot hold simultaneously and so the lemma is proved.‖

Lemma 4.

Suppose $\bar{G}$ is such that $\pi'(\bar{G}) \neq 0$. Then $R(G,y) = 0$ for all y and all $G \in N_\varepsilon(\bar{G})$.

Proof.

As $\pi'(\bar{G}) \neq 0$ and $\pi'(G)R(G,y) \equiv 0$ one must have $R(\bar{G},y) = 0$ for all y. Then Lemma 4 follows from Lemma 3.‖

Remark.

The purpose of Lemma 4 is the following. If $\pi'(G) = 0$ for all $G \in N_\varepsilon(G)$ then Lemma 2 is applicable which is what we want. Therefore one may as well assume that there exists G^* such that $\pi'(G^*) \neq 0$ and so by Lemma 4, $R(G,y) = 0$ for all $G \in N_\varepsilon(G^*)$. Under these <u>two</u> conditions it is then possible to show Lemma 6 below. But Lemma 6 and Lemma 7 are contradictory so there does not exist G^* such that $\pi'(G^*) \neq 0$. This information is summarized in Theorem 1 below.

By the above argument we have circumvented the following difficulty. There may exist a G^*, say, such that $\pi'(G^*) = 0$ but $\pi'(G) \neq 0$ for all $G \neq G^*$ yet $G \in N_\varepsilon(G^*)$. In this case Lemma 2 would not be applicable.

Lemma 5.

If technological change is purely capital augmenting one can aggregate investments of different vintages in terms of efficiency units: specifically,

$$\lambda'(G_v) \equiv 0 \quad \text{implies} \quad Q \equiv F\Big[\int_{-\infty}^{t} \varphi(G_v) I_v dv,\ L\Big].$$

Proof.

As $\lambda'(G_v) \equiv 0$ all vintages will be employed regardless of $h(0)$. Therefore (2.10) implies:

$$L_{v,t} = \varphi(G_v) f(w_t) I_v \tag{10}$$

Also (2.14) together with (10) implies:

$$Q_t = \int_o^{G_t} F[1,f(w_t)]\ \varphi(z)dz = F[1,f(w_t)] \int_o^{G_t} \varphi(z)dz \qquad (11)$$

Similarly (2.15) implies:

$$L_t = f(w_t) \int_o^{G_t} \varphi(z)dz \qquad (12)$$

Equations (11) and (12) then show that:

$$Q = F[L/f(w),L] = F[\int_o^{G_t} \varphi(z)dz, L] = F[\int_{-\infty}^{t} \varphi(G_v)I_v dv, L]$$

as was required. ‖

Lemma 6.

Suppose there exists G^* such that $\pi'(G^*) \neq 0$. Then the vintage and pseudo-production functions have a Cobb-Douglas representation for all $G \in N_\epsilon(G^*)$.

Proof.

In the proof to follow all identities will be understood to refer to the interval $N_\epsilon(G^*)$.

As $\pi'(G^*) \neq 0$ we have $R(G,y) \equiv 0$ by Lemma 4.

Define $\theta(y)$ by:

$$\theta'(y) = [H'(y)(1-m)-myH''(y)]/[H(y)-myH'(y)] - H''(y)/H'(y) \qquad (13)$$

Then, by the definition of $R(G,y)$ in (9), and as $R(G,y) \equiv 0$, the following identity must hold:

$$\text{Log.}f[\pi(G)H'(y)] \equiv \theta(y) + \mathcal{K}(G) \quad \text{say,} \qquad (14)$$

or:

$$f[\pi(G)H'(y)] \equiv e^{\theta(y)}e^{\mathcal{K}(G)} \tag{15}$$

Differentiate (15) partially with respect to y to obtain (16) and with respect to G to obtain (17):

$$\pi(G)H''(y)f'[\pi(G)H'(y)] \equiv \theta'(y)e^{\theta(y)}e^{\mathcal{K}(G)} \equiv \theta'(y)f[\pi(G)H'(y)] \tag{16}$$

$$\pi'(G)H'(y)f'[\pi(G)H'(y)] \equiv \mathcal{K}'(G)e^{\theta(y)}e^{\mathcal{K}(G)} \equiv \mathcal{K}'(G)f[\pi(G)H'(y)] \tag{17}$$

Now multiply (17) by $\pi(G)H''(y)$, substitute into (16), and divide through by $f[\pi(G)H'(y)]$ (note that $L \neq 0$ so this is valid) to get:

$$\pi'(G)H'(y)\theta'(y) \equiv \mathcal{K}'(G)\pi(G)H''(y) \tag{18}$$

As $\pi'(G^*) \neq 0$ by assumption and $\pi'(G)$ is a continuous function, $\pi'(G) \neq 0$ for all $G \in N_{\epsilon}(G^*)$. Therefore (18) may be re-written:

$$H'(y)\theta'(y) \equiv a(G)H''(y) \quad \text{say.} \tag{19}$$

Then differentiating both sides of (19) partially with respect to G one sees that $a'(G) \equiv 0$ so $a(G) \equiv a$ say.

Alternatively, (18) may be written:

$$\mathcal{K}'(G) = b(y)\pi'(G)/\pi(G) \quad \text{say,} \tag{20}$$

as $H''(y) \neq 0$ by nature of our assumptions on the vintage production functions. Again, differentiate both sides of (20) with respect to y and one sees that $b'(y) = 0$ or $b(y) \equiv b$ say.

Substituting (19) and (20) with $a(G) \equiv a$ and $b(y) \equiv b$ into (18) yields $a = b$. Therefore (19) and (20) imply:

$$\theta'(y) \equiv aH''(y)/H'(y) \equiv a\text{dLog}.H'(y)/dy \tag{21}$$

$$\mathcal{H}'(G) \equiv a\pi'(G)/\pi(G) \equiv a\text{dLog}.\pi(G)/dG \tag{22}$$

From (21) and (22), respectively:

$$e^{\theta(y)} \equiv k_1[H'(y)]^a \quad \text{say,} \tag{23}$$

$$e^{\mathcal{H}(G)} \equiv k_2[\pi(G)]^a \quad \text{say.} \tag{24}$$

Now substitute (23) and (24) into (15) to get:

$$f[\pi(G)H'(y)] \equiv k_o[\pi(G)H'(y)]^a \quad \text{where} \quad k_o = k_1k_2 \tag{25}$$

Put $x = \pi(G)H'(y)$ into (25) to get:

$$f(x) \equiv k_o x^a \tag{26}$$

Then by the definition of f in terms of h:

$$h(x) \equiv k_o^{-1/a}\, x^{1/a} \tag{27}$$

Therefore one may write:

$$\partial Q_{v,t}/\partial[\lambda(G_v)L_{v,t}] \equiv h[\lambda(G_v)L_{v,t}/\varphi(G_v)I_v] \equiv k_o^{-1/a}[\lambda(G_v)L_{v,t}/\varphi(G_v)I_v]^{1/a} \tag{28}$$

by (27) where $Q_{v,t} = F[\varphi(G_v)I_v,\ \lambda(G_v)L_{v,t}]$.

Integrating (28) with respect to $\lambda(G_v)L_{v,t}$ while holding $\varphi(G_v)I_v$ constant yields:

$$Q_{v,t} = k_o^{-1/a}.a/(1+a).[\varphi(G_v)I_v]^{-1/a}.[\lambda(G_v)L_{v,t}]^{(1+a)/a}+Y[\varphi(G_v)I_v] \qquad (29)$$

where $Y[.]$ is the arbitrary constant of integration.

However, F is homogeneous of degree one in $\varphi(G_v)I_v$ and $\lambda(G_v)L_{v,t}$. Therefore:

$$Y[c\varphi(G_v)I_v] \equiv Y[\varphi(G_v)I_v].c.$$

where c is an arbitrary constant. Therefore Y is a linear homogeneous function. Using this fact, together with (29), and setting $L_{v,t} = 0$ so $Q_{v,t} = 0$ as it is impossible to produce without labor, it is seen that $Y[.] = 0$. (Earlier in this derivation $L = 0$ was excluded as meaningless. This is not at variance with setting $L_{v,t} = 0$ as clearly this is possible even with a positive labor force.) Therefore:

$$Q_{v,t} = k_o^{-1/a}.a/(1+a).[\varphi(G_v)I_v]^{-1/a}[\lambda(G_v)L_{v,t}]^{(1+a)/a} \qquad (30)$$

i.e., the vintage production function has a Cobb-Douglas representation which is the first part of the Lemma.

As the vintage production function is Cobb-Douglas $h(0) = \omega$ so there is no restriction on the technological change assumptions. Therefore assume technological change is purely capital augmenting: this involves no loss of generality in the Cobb-Douglas case. Putting $\lambda(G_v) \equiv 1$ in (30) and applying Lemma 5 one obtains:

$$Q_t \equiv [k_o^{-1/a}a/(1+a)][\int_o^{G_t} \varphi(z)dz]^{-1/a}L_t^{(1+a)/a} \qquad (31)$$

so the second part of Lemma 6 is also true.‖

Lemma 7.

If the vintage production function is given by (30) for $G \in N_\epsilon(G)$

then:

(a) $Q \equiv j^{n-m/a_G} G^{n-m/a_L} L^{1+1/a}$ where j is a constant and $G \in N_\epsilon(G)$.

and (b) $\pi'(G) \equiv 0$ for all $G \in N_\epsilon(G)$.

Proof.

Again, all identities refer to $G \in N_\epsilon(G)$. We first show part (a). From (26):

$$f[w(G,L)/\lambda(z)] \equiv k_o[w(G,L)/\lambda(z)]^a \tag{32}$$

Equations (2.14), (2.15) and (2.16) and the fact that (in the $h(0) = \omega$ case) one can solve for a solution $w_t = w(G,L)$, imply, after substitution from (30) and (32):

$$Q \equiv k_o^{-1/a} a/(1+a). \int_o^G \left\{ k_o[w(G,L)/\lambda(z)]^a \right\}^{(1+a)/a} \varphi(z)dz \tag{33}$$

$$L \equiv k_o \int_o^G \varphi(z)[w(G,L)/\lambda(z)]^a/\lambda(z).\, dz \tag{34}$$

Then (33) and (34) imply:

$$\partial \mathrm{Log} Q/\partial L \equiv 1/rL \quad \text{where} \quad r = a/(1+a) \quad \text{as} \quad w = \partial Q/\partial L.$$

Therefore:

$$Q \equiv [S(G)L]^{1/r} \equiv [G^n L]^{1/r}\ [G^{-n} S(G)]^{1/r} \tag{35}$$

where $S(G)$ is the arbitrary constant of integration: a function of G as Q is a function of G and L. Using (35) and the fact, by Theorem 1.1, that Q is homogeneous of degree one in G and $G^n L$ it can be verified that:

$$S(pG) = p^{r+n-1} S(G) \tag{36}$$

where p is an arbitrary positive constant.

As (36) is valid for any p it can be differentiated with respect to p, holding G constant, to obtain:

$$GS'(pG) \equiv (r+n-1)p^{r+n-2}S(G)$$

which, after substituting into (36) and setting $p = 1$, yields:

$$d\text{Log}.S(G)/dG \equiv S'(G)/S(G) \equiv (r+n-1)/G$$

Therefore:

$$S(G) \equiv (jG)^{r+n-1} \tag{37}$$

where j is the constant of integration. Now substitute (37) into (35) to get:

$$Q \equiv j^{(r+n-1)/r}\, G[G^{-m}L]^{1/r} \equiv j^{n-m/a}\, G^{n-m/a}\, L^{1+1/a}$$

which is part (a) of the Lemma.

Noting that (30) implies (31) and by using part (a) of the Lemma we have:

$$G^{n-m/a} \equiv \left[\int_o^G \varphi(z)dz\right]^{-1/a} \tag{38}$$

Differentiating both sides of (38) w.r.t. G yields:

$$\varphi(G) \equiv [1 - n(1+a)]\, G^{-n(1+a)} \tag{39}$$

But (30) shows that purely capital augmenting technological change is equivalent to purely labor augmenting technological change providing:

$$\varphi(G)^{-1/a} \equiv \lambda(G)^{1+1/a} \tag{40}$$

The (39) and (40) imply $\lambda(G) \equiv bG^n$ so $\pi'(G) \equiv 0$. As the pseudo-production function is Cobb-Douglas one can assume $\varphi(G) \equiv 1$ without loss of generality: so part (b) of the lemma is true.‖

Finally we obtain the desired result given by:

Theorem 1.

Under the conditions of Theorem 1.1. if technological change is factor augmenting and a function only of gross cumulative investment at the time of installation and also $\varphi(.)$, $\lambda(.)$ and $\lambda'(.)$ are continuous functions then:

If either Assumption 2.1. or 2.2. is applicable technological change must be Arrow Type.

Proof.

If there exists G^* such that $\pi'(G^*) \neq 0$ then by Lemma 6, (30) is applicable for all $G \in N_\varepsilon(G^*)$. But then part (b) of Lemma 7 requires $\pi'(G^*) = 0$ which is a contradiction. Therefore $\pi'(G) = 0$ for all G and the Theorem follows from Lemma 2.‖

Remark 1.

As the applicability of Theorem 1.1(b) pertains to points (G,L) in the non-shaded area of Figure 1 the same restriction must also be made with respect to Theorem 1 above.

Remark 2.

Theorem 2.1. implicitly defines the pseudo-production function $Q \equiv Q(G,L)$. It is clear that in general the sign of $\partial^2 Q/\partial G^2$ is indeterminate: in particular it may well be positive. However, if the conditions of Theorem 1.1. are applicable, then one must have:

$$Q(\rho G, \rho^m L) = \rho Q(G,L)$$

where ρ is an arbitrary positive constant. Differentiate both sides of the above with respect to ρ to get (after setting $\rho = 1$):

$$Q(G,L) \equiv G\partial Q/\partial G + (1-n)L\partial Q/\partial L$$

Differentiate both sides of the above with respect to G to get,

$$G\partial^2 Q/\partial G^2 + (1-n)L\partial^2 Q/\partial G\partial L \equiv 0$$

and with respect to L to get:

$$[(1-n)L\partial^2 Q/\partial L^2 - n\partial Q/\partial L] + G\partial^2 Q/\partial L\partial G \equiv 0.$$

As n is a positive fraction and $\partial^2 Q/\partial L^2 < 0$ the term in brackets is negative. Therefore $\partial^2 Q/\partial L\partial G > 0$. But equation (3.1) can be used to verify the equality of the cross partial derivatives. Therefore $\partial^2 Q/\partial G^2 < 0$. In other words, even though the production process operates under conditions of increasing returns to scale so that $\partial^2 Q/\partial G^2$ may be positive, if a balanced growth path is to exist these returns to scale must not be large enough to cause increasing returns to capital taken alone.

Remark 3.

Lemma 5 gives a simple aggregation result in the event that technological change is purely capital augmenting ($\lambda'(G) \equiv 0$, $\varphi'(G)$ arbitrary). If technological change is Arrow Type then a balanced growth path exists (see Levhari [5]) so that Theorem 1.1. gives a simple aggregation result in this case also.

Furthermore if technological change is Hick's neutral and:

$$\lambda(G) \equiv \varphi(G) \equiv G^n \tag{41}$$

then, as shown below, $Q = Q(G,L)$ is homogeneous of degree $1+n$ in its arguments providing $h(0)$ is finite. Substitute (41) into (2.12) and (2.13) and, as $h(0)$ is finite, substitute (2.17) also. Then:

$$Q_t = \int_{t-m(t)}^{t} F[1, f[h(0)G^n_{t-m(t)}/G^n_v]]\ G^n_v I_v dv \tag{42}$$

$$L_t = \int_{t-m(t)}^{t} f[h(0)G^n_{t-m(t)}/G^n_v]\ I_v dv \tag{43}$$

One observes from (2.14) and (2.15) that output does not depend on the time profile of investment. Therefore one can multiply G_t by ρ (a positive constant) by multiplying I_v (and hence G_v and $G_{t-m(t)}$) by ρ so the right hand sides of (42) and (43) become, respectively:

$$\int_{t-m(t)}^{t} F[1, f[h(0)\rho^n G^n_{t-m(t)}/\rho^n G^n_v]]\ \rho^n G^n_v\ I_v dv = \rho^{1+n} Q \quad \text{by (42)}$$

$$\int_{t-m(t)}^{t} f[h(0)\rho^n G^n_{t-m(t)}/\rho^n G^n_v]\ \rho I_v dv = \rho L \quad \text{by (43)}.$$

Therefore $\rho^{n+1} Q = Q(\rho G, \rho L)$ as required.

Remark 4.

Suppose that technological change depends also on gross cumulative output at the time of installation. In other words, the vintage production function is:

$$Q_{v,t} = F[\varphi(G_v, S_v)I_v,\ \lambda(G_v, S_v)L_{v,t}]$$

where:

$$S_v = \int_{v-m(v)}^{v} Q_{r,v}\,dr = \text{gross cumulative output at time } v.$$

If the conditions of Theorem 1.1(b) are applicable so there is a constant savings ratio:

$$G_v - G_o = s(S_v - S_o)$$

so, in particular:

$$S_v \equiv S_v(G_v) \quad \text{say.}$$

Therefore one may write:

$$\varphi(G_v, S_v) = \bar{\varphi}(G_v) \quad \text{and} \quad \lambda(G_v, S_v) = \bar{\lambda}(G_v).$$

It is now seen that under the conditions of Theorem 3.1. the same result must apply. That is, one must have $\bar{\varphi}'(G_v) \equiv 0$ and so $\varphi(G_v, S_v)$ is constant and also $\bar{\lambda}(G_v) \equiv G_v^n$. Thus technological change must be purely labor augmenting. Also:

$$\bar{\lambda}(G_v) \equiv \lambda(G_v, S_v(G_v)) \equiv G_v^n$$

But as the existence of a balanced growth path is assumed for all time $S_v(\rho G_v) = \rho S_v(G_v)$. Therefore the $\lambda(.)$ function must be homogeneous of degree n in its arguments. In fact, this was assumed by Levhari [6] in his study of sufficient conditions.

With this formulation, the same implicit representation of the pseudo-production function (Theorem 2.1.) would hold, with the required changes in the $\varphi(.)$ and $\lambda(.)$ functions.

This hypothesis has nothing to say about Lundberg's Horndal Effect

where output per man-hour rose over time even without any new investment. It may be considered better to replace $\varphi(G_v,S_v)$ by $\varphi(G_v,S_t)$ or $\varphi(G_t,S_v)$ and to make corresponding changes in the $\lambda(.)$ function. This would then explain the Horndal Effect. However, in this case Theorem 2.1. does not hold and I have been unable to obtain the necessary conditions for the existence of a balanced growth path.

Remark 5.

For completeness one might consider the technological change functions $\varphi(G_t,S_t)$ and $\lambda(G_t,S_t)$ which again would explain the Horndal Effect. The vintage production functions then aggregate to:

$$Q = F[\varphi(G,S)G,\ \lambda(G,S)L].$$

This may be shown rigorously from the preceding analysis but is intuitively clear: For in this case learning by doing has been separated from embodiment so the vintage concept plays no role and the capital stock and labor force may be aggregated in efficiency units. Note that even if $h(0)$ is finite there will not be any economic obsolescence. Now Inada [4] has shown that if the production function is of the form $Q = F[G,L,K(G)]$ then necessary conditions for balanced growth require that technological change, $K(G)$, be Arrow Type. Thus it is clear that in our case $\varphi(G,S)$ is constant and $\lambda(G,S)$ is homogeneous of degree n in its arguments.

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