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Article

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Future Business Journal

Provided in Cooperation with:

Faculty of Commerce and Business Administration, Future University

Suggested Citation: Ismail, Maha; El-Hefnawy, Ali; Saad, Abd El-Naser (2018) : New Deterministic Solution to a chance constrained linear programming model with Weibull Random Coefficients, Future Business Journal, ISSN 2314-7210, Elsevier, Amsterdam, Vol. 4, Iss. 1, pp. 109-120, <https://doi.org/10.1016/j.fbj.2018.02.001>

This Version is available at:

<https://hdl.handle.net/10419/187970>

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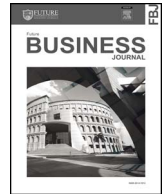
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Future Business Journal

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New Deterministic Solution to a chance constrained linear programming model with Weibull Random Coefficients

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ARTICLE INFO

Keywords:

Probabilistic Linear Programming
Chance Constrained Linear Programming
Sum of Weibull random variables
Linear Combination of Weibull random variables
Allocation of recruitment in manpower planning

ABSTRACT

Linear Programming model is an important tool used to solve constrained optimization problems. In fact, the real life problems are usually occurring in the presence of uncertainty. For instance, in managerial problems of assigning employees to different tasks with the aim of minimizing the total completion time, or maximizing the total productivity, which are better described as random variables. Therefore, the use of the Probabilistic Linear Programming model with random coefficients has drawn much attention in recent years. One of the most frequently used approaches to solve the Probabilistic Linear Programming model is the Chance Constrained Linear Programming approach. In this paper, a Chance Constrained Linear Programming model with Weibull random coefficients is proposed. The proposed model is introduced in the Bivariate form with two of the L.H.S technologic coefficients are random variables. Moreover, the performance of the proposed model is shown through an application of allocating recruitment in Manpower Planning so as to optimize the jobs' completion time. The obtained results are compared with the results of another model that depends on approximating the distribution of the sum of Weibull random variables to the Normal distribution. This comparison verified the good performance of the new proposed model.

1. Introduction

Linear Programming (LP) model is a mathematical model formulated to optimize linear objective function subject to some linear constraints (Taha, 2007). It is formulated generally as:

$$\text{Max (Min) } Z(x) = \sum_{j=1}^n c_j x_j \quad (1.1)$$

Subject to

$$\sum_{j=1}^n a_{ij} x_j \begin{cases} \leq & b_i \\ \geq & b_i \end{cases} \quad x_j \geq 0 \quad i = 1, 2, \dots, m, \quad j = 1, 2, \dots, n \quad (1.2)$$

where; $Z(\cdot)$: the objective function, c_j : the objective function coefficients, a_{ij} : the L.H.S technological coefficients, b_i : the R.H.S values,

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¹ This work is a part of the corresponding author master thesis.

representing the available resources, x_j : the decision variables, n : the number of decision variables, m : the number of constraints.

Many application areas, for instance, production planning, transportation, engineering, management, and finance, require optimization under uncertainty. Considering the data uncertainty is very crucial to get a complete model that represents the real system reasonably well and then the obtained optimal solution is optimal also for the real situation (Taha, 2007).

The data uncertainty is introduced to the model through random coefficients that follow a known probability distribution (Sengupta, Tintner & Millham, 1963; Atalay & Apaydin, 2011). The LP model (1.1)–(1.2) is then referred to as Probabilistic Linear Programming (PLP). In most cases, the PLP model is formulated by such a way that the researcher has a deterministic model, called the "base model". Then, observed that some or all of the coefficients in the model are random (Sengupta et al., 1963).

The technique for solving the PLP model works through converting it into an equivalent deterministic model whose feasible and optimal solution coincides with the feasible and optimal solution for the original PLP model (Symonds, 1967). The Chance Constrained Linear Programming (CCLP) is a very popular and commonly used approach to transform the PLP model into its deterministic equivalent model (Sengupta et al., 1963).

This approach is characterized by utilization of the information about the probability distribution of the random coefficient(s) to construct chance constraints. The chance constraints are defined to be the constraints that are satisfied with certain level of confidence rather than being always satisfied. This confidence level is usually not prescribed exactly, but a lower bound is given instead. Therefore, this approach permits for the random variations in the data that cannot be controlled, so that it provides reliable solutions with acceptable safety margins. Chance constraints may be imposed individually to each constraint "Individual Chance Constraints" (ICC) or jointly to some or all of the constraints that occur simultaneously with the same confidence level "Joint Chance Constraints" (JCCs) (Charnes, Cooper, Kirby & Raike, 1972; Sengupta, 1972b).

The CCLP model with ICC is defined as follows:

$$\text{Max. (Min.) } z = \sum_{j=1}^n c_j x_j \tag{1.3}$$

Subject to:

$$\Pr \left[\sum_{j=1}^n a_{ij} x_j \begin{matrix} \leq \\ \geq \end{matrix} b_i \right] \geq \alpha_i, \quad x_j \geq 0, \quad \alpha_i \in [0, 1] \\ i = 1, \dots, m, j = 1, \dots, n \tag{1.4}$$

α_i is the predetermined confidence measure (also known as the level of tolerance)². It indicates the extent to which the i th constraint is satisfied.

While, the k JCCs can be represented as follows:

$$\Pr \left[\bigcap_{i=1}^k \left(\sum_{j=1}^n a_{ij} x_j \begin{matrix} \leq \\ \geq \end{matrix} b_i \right) \right] \geq \alpha_i = 1, \dots, m, j = 1, \dots, n, \quad 2 \leq k \leq m \tag{1.5}$$

The literature of the CCLP is rich with many research works to introduce the deterministic equivalent to the CCLP model under different distributional assumptions. The Normal distribution has been frequently implemented in the framework of the CCLP. The usage of the Normal distribution is motivated by its properties (see Sengupta, 1972a, 1970b; Kataria, Elofsson & Hasler, 2009) such as:

- Under the conditions of the Central Limit Theorem (CLT), it is the limit distribution of many other probability distributions.
- The reproductive property (that is the sum of n independent r.v.'s from that distribution also belongs to the same distribution form).
- The availability of numerical tables which helps to get the value of $F(\cdot)$ and $F^{-1}(\cdot)$.

Examples for studies that assumed the Normal distribution for the random coefficient(s) in the CCLP model are; (Millar and Wagner, 1965; Symonds, 1967; Jagannathan, 1974; Acharya and Biswal, 2011; Ackooij, Henrion, Möller & Zorgati, 2011).

Although the advantages of the Normal distribution and the availability of much literature for the CCLP model under the normality assumption, it is not appropriate for all the optimization models. For example, the economic optimization models involves coefficient(s) that may represent price, cost, demand, profit, ...etc., which have to be nonnegative. Also, the reliability optimization model with the coefficients may represent time to failure which are nonnegative quantities.

Sengupta (1970b) argued that the optimal solution to the CCP model based on the Normality assumption may be sensitive to departures from normality. Kataria et al. (2009) formed a CCLP for a cost-minimization model and have shown that assuming the wrong distribution assumption leads to biased results. They proved that using the Normal distribution instead of the Truncated Normal distribution (as being the true distribution) leads to underestimation of the true cost.

Therefore, the researchers and scholars started to look for alternative Non-Normal distributions for the nonnegative r.v.'s. The Weibull distribution is a very good alternative for the Normal distribution for the nonnegative variables. It is widely used in a broad range of fields. The CCLP model with Weibull random coefficients has been handled by Jeeva, Rajagopal, Charles and Yadavalli (2004). They have introduced the deterministic model through approximating the sum of independent Weibull random variables to

² It worth noting that, the pre-assignment of α_i as high as unity is likely to result in a lack of feasible solution (Sengupta, 1970b).

the Normal distribution. In this paper, we introduce a new deterministic equivalent to the CCLP model, with Weibull random coefficients, that utilizes the exact information about the probability distribution.

Other distributions have been considered such as; the Exponential distribution (Sengupta, 1972b; Biswal, Biswal & Li, 1998) Gamma distribution (Lingaraj and Wolfe, 1974; Atalay and Apaydin, 2011), Chi-square distribution (Sengupta, 1970b; Sengupta, 1972a; Sengupta, 1972b), and General form of Distributions (Charles, V., Ansari, S.I., & Khalid, M.M. (2009)).

In this paper, we set our objectives to:

1. Introduce a new equivalent deterministic model to the CCLP model when the random coefficients in the L.H.S of the constraints follow the Weibull distribution.
2. Apply the obtained deterministic model to a problem of allocating recruitment in manpower planning to verify the model's performance and applicability.

This paper is organized as follows: Section 2 presents the definition and properties of the Weibull distribution. Section 3 introduces the proposed deterministic model to the CCLP when two of the L.H.S coefficients are independent non-negative continuous r.v.'s following the Weibull distribution. The application of the proposed model to allocation of recruitment problem is discussed in Section 4. Finally, Section 5 concludes and presents some suggested points for further research.

2. The Weibull distribution

The Weibull distribution is a generalization of the Exponential distribution. However, it has broader application (Lee & Wang, 2003). It has been used as a model in diverse disciplines such as reliability, survival analysis, human disease mortality studies, and financial context to conduct intangible asset valuation. It adequately describes observed failures of many different components and phenomena (Pham, 2006).

2.1. Forms of the Weibull distribution

According to Pham (2006), the Weibull distribution has appeared in five different forms. One of them can be defined as:

$$F(y) = 1 - \exp \left[- \left(\frac{y - \zeta}{\beta} \right)^\lambda \right]; \quad y \geq 0 \tag{2.1}$$

This distribution is characterized by 3 parameters, $\lambda > 0$ is the shape parameter, $\beta > 0$ is the scale parameter, and $\zeta \geq 0$ is the location parameter (Lee & Wang, 2003; Hager, 1963).

For $\zeta = 0$, the two-parameter Weibull distribution is obtained and defined by the following CDF form:

$$F(y) = 1 - \exp \left[- \left(\frac{y^\lambda}{\beta} \right) \right]; \quad y \geq 0 \tag{2.2}$$

Also, the standard Weibull is obtained by letting $\beta = 1$ (or $\lambda = 1$) in (2.2) (Pham, 2006).

2.2. Properties of the Weibull distribution

The Weibull distribution is characterized by many useful properties. Here, we refer to some of them and more can be found in Pham (2006), Lee and Wang (2003), Hager (1963):

1. The two-parameter Weibull distribution includes the exponential ($\beta = 1$) and the Rayleigh distribution ($\beta = 2$) as special cases (Pham, 2006).
2. The mean and variance of the two-parameter Weibull distribution are defined as (Pham, 2006).

$$E(Y) = \beta \Gamma (1/\lambda + 1)$$

$$Var(Y) = \beta^2 [\Gamma (2/\lambda + 1) - \Gamma^2 (1/\lambda + 1)]$$

3. Changing the value of the shape parameter affects the shape of the distribution as illustrated in the following figure.

As shown in Fig. (2-1), setting the scale parameter $\beta = 1$ and the shape parameter λ is greater than 1, the shape of the distribution curve becomes symmetric, while keeping λ less than or equal to 1, the shape gets skewed (Lee & Wang, 2003; Hager, 1963).

In the following section, we present the deterministic equivalent of the CCLP model with two random L.H.S coefficients from the two-parameter Weibull distribution.

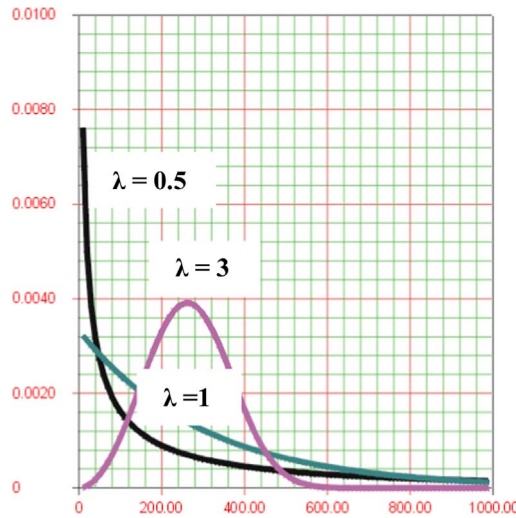


Fig. 2-1. The effect of the shape parameter, λ , on the Weibull p.d.f. with scale parameter, $\beta = 1$.

3. The proposed deterministic to the CCLP model with two L.H.S random coefficients

Actually, there is no general form for the equivalent deterministic model to the CCLP model. According to [Lingaraj and Wolfe \(1974\)](#), [Nishakova \(2010\)](#), it depends on the following factors:

- Whether the chance constraints are ICC or JCCs.
- The type of the constraint function (linear, convex or separable).
- The source of randomness; i.e., the position of the random coefficient(s) in the model.
- The nature of the random coefficients (continuous or discrete), and the independence or dependence relationships between them.
- The nature of the probability distribution of the random coefficients.

Here, we consider the CCLP model with linear ICC with the coefficients in the objective function and the L.H.S. technological coefficients are independent continuous random variables, and show the deterministic equivalent to this model.

3.1. The deterministic objective function

When the randomness occurs in the objective function coefficients, c_j 's, the equivalent deterministic model can be expressed by considering one of the following three forms (see; [Biswal et al., 1998](#); [Acharya & Biswal, 2011](#)):

- "The E-model"; the expected value of the objective function,
- "The V-model"; the generalized mean square of the objective function,
- "The P-model"; the probability of exceeding an aspiration level of the objective function.

3.2. The proposed deterministic constraints in case of random L.H.S coefficients

Assuming that the L.H.S coefficients a_{i1} and a_{i2} are two independent r.v.'s from the two-parameter Weibull distribution (defined above in [Section 2.1](#)). Then, the probabilistic constraint (1.4) represents the CDF of a linear combination of two Weibull r.v.'s evaluated at α_i (or $1-\alpha_i$, based on the direction of the constraint). First, we need to find the distribution of the linear combination; $W = a_{i1} x_1 + a_{i2} x_2$; $x_1, x_2 \in \mathcal{R}^+$. Actually, it can be shown that a scalar multiplied r.v., $T_j = a_{ij} x_j$; $i = 1, 2, \dots, m, j = 1, 2$, from the Weibull distribution again belongs to the same distribution.³ Thus, our concern now is to find the CDF of the sum of two independent Weibull r.v.'s, $W = T_1 + T_2$, as will be shown next.

It worth mentioning that it is workable to derive the deterministic equivalent constraints in the case when more than two random coefficients follow a probability distribution for which the distribution of the linear relation $\sum_{j=1}^n a_{ij} x_j$; $i = 1, \dots, m$ is defined, like the exponential and normal distributions. While, for the Weibull distribution, only the CDF of the sum of two Weibull random variables is defined in an approximate form, not even a closed form, as will be shown below. That is why we considered the case of two random L.H.S coefficients.

³ If a_{ij} follows the Weibull (λ_j, β_j) distribution, then $T_j = a_{ij} x_j$ follows the Weibull $(\lambda_j, \beta_j x_j^{\lambda_j})$ distribution.

3.3. The distribution of the sum of two independent Weibull r.v.'s

By reviewing the literature of the Weibull distribution, we found no exact form for the distribution of the sum of independent Weibull r.v.'s (Nadarajah, 2008). However, there are several attempts to derive approximations (see, Filho & Yacoub, 2006; Yilmaz & Alouini, 2009). Recently, El-Ayadi and Ismail (2012) proposed a highly accurate and simple approximation to the CDF of the sum and harmonic mean of two independent α - μ distributed r.v., T_j with p.d.f:

$$f_{T_j}(t_j) = \frac{\lambda_j \omega_j^{\omega_j}}{\eta_j^{\lambda_j \omega_j} \Gamma(\omega_j)} t_j^{\lambda_j \omega_j - 1} \exp \left[-\omega_j \left(\frac{t_j}{\eta_j} \right)^{\lambda_j} \right] ; t_j \geq 0 , j = 1, 2 \tag{3.1}$$

where λ_j is the shape parameter, $\omega_j = \frac{E^2(T_j^{\lambda_j})}{E(T_j^{2\lambda_j}) - E^2(T_j^{\lambda_j})}$, and $\eta_j = [E(T_j^{\lambda_j})]^{1/\lambda_j}$.

This p.d.f includes some important distributions such as the Weibull and Rayleigh distributions with appropriate ω_j and η_j .

Regarding the Sum $W = T_1 + T_2$, El-Ayadi and Ismail (2012) obtained an approximate form for its CDF. This approximate form is the difference between two terms, base and correction terms. The base term is defined as:

$$F_W^{Base}(w) = \hat{\gamma} \left(\omega_1, \omega_1 \left(\frac{w}{\eta_1} \right)^{\lambda_1} \right) \times \hat{\gamma} \left(\omega_2, \omega_2 \left(\frac{w}{\eta_2} \right)^{\lambda_2} \right) = \int_{u=0}^{\omega_1 \left(\frac{w}{\eta_1} \right)^{\lambda_1}} \frac{e^{-u}}{\Gamma(\omega_1)} u^{\omega_1 - 1} du \times \int_{v=0}^{\omega_2 \left(\frac{w}{\eta_2} \right)^{\lambda_2}} \frac{e^{-v}}{\Gamma(\omega_2)} v^{\omega_2 - 1} dv \tag{3.2}$$

where $\hat{\gamma}(\cdot, \cdot)$ is the normalized incomplete Gamma function.

First, we define the corresponding ω_j and η_j to the Weibull ($\lambda_j, \beta_j x_j^{\lambda_j}$) distribution as (see details on Appendix.A):

$$\omega_j = \frac{\beta_j^2 x_j^{2\lambda_j}}{2\beta_j^2 x_j^{2\lambda_j} - \beta_j^2 x_j^{\lambda_j}} = 1, \text{ and } \eta_j = \beta_j^{\frac{1}{\lambda_j}} x_j$$

Then, by substituting by ω_j and η_j in (3.2), we get the base term of the CDF of W:

$$\begin{aligned} F_W^{Base}(w) &= \hat{\gamma} \left(1, \left(\frac{w}{\beta_1^{\frac{1}{\lambda_1}} x_1} \right)^{\lambda_1} \right) \times \hat{\gamma} \left(1, \left(\frac{w}{\beta_2^{\frac{1}{\lambda_2}} x_2} \right)^{\lambda_2} \right) = \hat{\gamma} \left(1, \left(\frac{w^{\lambda_1}}{\beta_1 x_1^{\lambda_1}} \right) \right) \times \hat{\gamma} \left(1, \left(\frac{w^{\lambda_2}}{\beta_2 x_2^{\lambda_2}} \right) \right) \\ &= \int_{u=0}^{\frac{w^{\lambda_1}}{\beta_1 x_1^{\lambda_1}}} \frac{e^{-u}}{\Gamma(\omega_1)} u^{\omega_1 - 1} du \times \int_{v=0}^{\frac{w^{\lambda_2}}{\beta_2 x_2^{\lambda_2}}} \frac{e^{-v}}{\Gamma(\omega_2)} v^{\omega_2 - 1} dv \end{aligned}$$

Therefore,

$$F_W^{Base}(w) = 1 + \exp \left[-\frac{1}{\beta_1 x_1^{\lambda_1}} w^{\lambda_1} - \frac{1}{\beta_2 x_2^{\lambda_2}} w^{\lambda_2} \right] - \exp \left[-\frac{1}{\beta_1 x_1^{\lambda_1}} w^{\lambda_1} \right] - \exp \left[-\frac{1}{\beta_2 x_2^{\lambda_2}} w^{\lambda_2} \right] \tag{3.3}$$

While, the correction is defined by lower and upper bounds, $L_s(w; L)$ and $U_s(w; L)$, respectively:

$$L_s(w; L) = \sum_{l=0}^{L-1} [(\hat{\gamma}(\omega_1, u_{l+1}(w)) - \hat{\gamma}(\omega_1, u_l(w))) \times (\hat{\gamma}(\omega_2, v_0(w)) - \hat{\gamma}(\omega_2, v_l(w)))]$$

$$U_s(w; L) = \sum_{l=0}^{L-1} [(\hat{\gamma}(\omega_1, u_{l+1}(w)) - \hat{\gamma}(\omega_1, u_l(w))) \times (\hat{\gamma}(\omega_2, v_0(w)) - \hat{\gamma}(\omega_2, v_{l+1}(w)))]$$

Where,

$$(u_l(w), v_l(w)) = \left(\omega_1 \left(\frac{l w}{L \eta_1} \right)^{\lambda_1}, \omega_2 \left(\frac{(L-l) w}{L \eta_2} \right)^{\lambda_2} \right),$$

$$l = 0, 1, \dots, L, \omega_j = 1$$

$$\text{and } \eta_j = \beta_j^{\frac{1}{\lambda_j}} x_j$$

These bounds can be approximated as:

$$\begin{aligned} L_s(w; L) &= e^{-\frac{w^{\lambda_1}}{\beta_1 x_1^{\lambda_1}} \sum_{l=0}^{L-1} \frac{l^{\lambda_1}}{L^{\lambda_1}} - \frac{w^{\lambda_2}}{\beta_2 x_2^{\lambda_2}} \sum_{l=0}^{L-1} \frac{(L-l)^{\lambda_2}}{L^{\lambda_2}}} - e^{-\frac{w^{\lambda_1}}{\beta_1 x_1^{\lambda_1}} \sum_{l=0}^{L-1} \frac{l^{\lambda_1}}{L^{\lambda_1}} - \frac{(L-1) w^{\lambda_2}}{\beta_2 x_2^{\lambda_2}}} \\ &\quad - e^{-\frac{w^{\lambda_1}}{\beta_1 x_1^{\lambda_1}} \sum_{l=0}^{L-1} \frac{(l+1)^{\lambda_1}}{L^{\lambda_1}} - \frac{w^{\lambda_2}}{\beta_2 x_2^{\lambda_2}} \sum_{l=0}^{L-1} \frac{(L-l)^{\lambda_2}}{L^{\lambda_2}}} + e^{-\frac{w^{\lambda_1}}{\beta_1 x_1^{\lambda_1}} \sum_{l=0}^{L-1} \frac{(l+1)^{\lambda_1}}{L^{\lambda_1}} - \frac{(L-1) w^{\lambda_2}}{\beta_2 x_2^{\lambda_2}}} \end{aligned} \tag{3.4}$$

$$\begin{aligned}
 U_s(w; L) = & e^{-\frac{w^{\lambda_1}}{\beta_1 x_1^{\lambda_1}} \sum_{l=0}^{L-1} \frac{l^{\lambda_1}}{L^{\lambda_1}} - \frac{w^{\lambda_2}}{\beta_2 x_2^{\lambda_2}} \sum_{l=0}^{L-1} \frac{(L-l-1)^{\lambda_2}}{L^{\lambda_2}}} - e^{-\frac{w^{\lambda_1}}{\beta_1 x_1^{\lambda_1}} \sum_{l=0}^{L-1} \frac{l^{\lambda_1}}{L^{\lambda_1}} - \frac{(L-1)w^{\lambda_2}}{\beta_2 x_2^{\lambda_2}}} \\
 & - e^{-\frac{w^{\lambda_1}}{\beta_1 x_1^{\lambda_1}} \sum_{l=0}^{L-1} \frac{(l+1)^{\lambda_1}}{L^{\lambda_1}} - \frac{w^{\lambda_2}}{\beta_2 x_2^{\lambda_2}} \sum_{l=0}^{L-1} \frac{(L-l-1)^{\lambda_2}}{L^{\lambda_2}}} + e^{-\frac{w^{\lambda_1}}{\beta_1 x_1^{\lambda_1}} \sum_{l=0}^{L-1} \frac{(l+1)^{\lambda_1}}{L^{\lambda_1}} - \frac{(L-1)w^{\lambda_2}}{\beta_2 x_2^{\lambda_2}}}
 \end{aligned}
 \tag{3.5}$$

Therefore, the exact CDF of W is defined as:

$$F_W^{Base}(w) - U_s(w; L) \leq F_W(w) \leq F_W^{Base}(w) - L_s(w; L)
 \tag{3.6}$$

which can be approximated by taking the arithmetic mean of the lower and upper limits of (3.6) to get:

$$F_W(w) = F_W^{Base}(w) - \left(\frac{U_s(w; L) + L_s(w; L)}{2} \right)
 \tag{3.7}$$

Since this exact CDF $F_W(w)$, given in (3.7), cannot be given in a closed form, it will be embedded into the constraints as is. The next section presents a numerical example to show the applicability of the proposed deterministic model.

4. Numerical example: Allocation of recruitment in manpower planning

In this section, we show the applicability of the proposed model to a numerical example that is handled by in Jeeva et al. (2004), and the results will be compared with the results obtained when applying the deterministic model proposed by Jeeva et al. (2004). In this example, the L.H.S coefficients in the constraints and the objective function coefficients are assumed to be independent Weibull r.v.'s. The concern of this example is that a company has n types of jobs related to electronics for which N persons are recruited. Cluster-Analysis technique is used to divide the N eligible applicants into n homogeneous clusters (G_1, G_2, \dots, G_n) based on the similarity of specialization in one of the jobs. The problem is to allocate the persons from each cluster to each job so as to minimize the time required to complete all jobs.

The following CCLP model is formulated:

$$\text{Minimize } Z = \sum_{i=1}^n \sum_{j=1}^n t_{ij} x_{ij}
 \tag{4.1}$$

Subject to

$$\Pr \left[\sum_{j=1}^n t_{ij} x_{ij} \leq b_i \right] \geq \alpha_i \quad \text{for all } i = 1, 2, \dots, n
 \tag{4.2}$$

$$\Pr \left[\sum_{i=1}^n t_{ij} x_{ij} \leq b_j^* \right] \geq \alpha_j \quad \text{for all } j = 1, 2, \dots, n
 \tag{4.3}$$

$$\sum_{j=1}^n x_{ij} = x_i \quad \text{for all } i = 1, 2, \dots, n
 \tag{4.4}$$

$$\sum_{i=1}^n x_{ij} = y_j \quad \text{for all } j = 1, 2, \dots, n
 \tag{4.5}$$

$$\sum_{i=1}^n x_i = \sum_{j=1}^n y_j x_{ij} \geq 0 \quad \text{for all } (i, j = 1, 2, \dots, n)
 \tag{4.6}$$

where; t_{ij} : is the time taken by a person from the j th cluster to complete the i th job, x_{ij} : is the number of persons selected from the j th cluster for the i th job, b_i : is the upper bound for the completion time of the i th job, b_j^* : is the upper bound for the man-hours expected for each cluster, x_i : is the number of persons required for the i th job, y_j : is the number of persons in the j th cluster, and α_i & α_j : are the tolerance levels.

The time taken by persons from each cluster to complete each job, t_{ij} ; $i, j = 1, \dots, n$ is assumed to be independent r.v.'s following the Weibull distribution. The upper bounds for the time completion of the jobs, b_i , and the expected man-hours for the clusters, b_j^* , are fixed and predetermined.

To completely define the problem and to avoid infeasible solutions, Jeeva et al. (2004) made the following assumptions:

- The organization fixes a target on the number of employees for each job and for each cluster, x_i and y_j , respectively.
- The number of persons required for the jobs from any cluster should not exceed the number of persons available in that cluster, defined by constraint (4.5).
- The number of persons in the clusters put together equals the number of persons required to complete all the jobs, represented by constraint (4.6).

Here, we show the technique presented by Jeeva et al. (2004) to obtain the equivalent deterministic model to the CCLP, defined in

(4.1)–(4.6), assuming the case of two jobs and two clusters ($n = 2$). Also, without loss of generality, we assume that t_{ij} 's; $i, j = 1, 2$, have the 2-parameters Weibull distribution.

First, the MLE estimates of the parameters $\hat{\lambda}_{ij}, \hat{\beta}_{ij}$ are obtained. Then, the means and variances of the Weibull distribution; $\hat{\mu}_{ij}$ and $\hat{\sigma}_{ij}^2$ are calculated as:

$$\begin{aligned} \hat{\mu}_{ij} &= \hat{\beta}_{ij} \Gamma(1/\hat{\lambda}_{ij} + 1) \\ \hat{\sigma}_{ij}^2 &= \hat{\beta}_{ij}^2 [\Gamma(2/\hat{\lambda}_{ij} + 1) - \Gamma^2(1/\hat{\lambda}_{ij} + 1)] \end{aligned} \tag{4.8}$$

Then, the deterministic objective function is obtained by replacing the values of the r.v.'s; t_{ij} 's, with their expected values; $\hat{\mu}_{ij}$:

$$Z = \sum_{i=1}^2 \sum_{j=1}^2 \hat{\mu}_{ij} x_{ij} \tag{4.9}$$

Regarding the chance constraint (4.2), it is converted into deterministic as follows; the L.H.S linear combination; $r_i = \sum_{j=1}^2 t_{ij}x_{ij}$, is standardized using the transformation⁴ $Z_i = \frac{r_i - E(r_i)}{\sqrt{Var(r_i)}}$. Then, the Lyapunov CLT⁵ is applied to approximate Z_i to the Standard Normal distribution, $N(0, 1)$.

The nonlinear equivalent deterministic constraint is then obtained as:

$$\sum_{j=1}^2 \hat{\mu}_{ij} x_{ij} + Z_{\alpha i} \sqrt{\sum_{j=1}^2 \hat{\sigma}_{ij}^2 x_{ij}^2} \leq b_i \quad \text{for all } i = 1, 2 \tag{4.10}$$

Similarly, the equivalent deterministic to the probabilistic constraint (4.3) is of the form:

$$\sum_{i=1}^2 \hat{\mu}_{ij} x_{ij} + Z_{\alpha j} \sqrt{\sum_{i=1}^2 \hat{\sigma}_{ij}^2 x_{ij}^2} \leq b_j^* \quad \text{for all } j = 1, 2 \tag{4.11}$$

where Z_{α} is the inverse CDF of the Standard Normal variable at α .

Now, the Jeeva's deterministic nonlinear model can be defined by (4.9), (4.10), (4.11), (4.4), (4.6). To solve this model, the authors linearized it and applied the Simplex algorithm.

The Jeeva's equivalent deterministic model is criticized for the insufficient application of the Lyapunov CLT to approximate the linear combination of independent Weibull r.v.'s is by the Standard Normal distribution, since they ignored the *Lyapunov's condition* which is defined as:

For some $\delta > 0$,

$$\lim_{n \rightarrow \infty} \frac{1}{s_n^{2+\delta}} \sum_{i=1}^n [|X_i - \mu_i|^{2+\delta}] = 0$$

Also, the obtained deterministic model is linearized and this may lead to weak solutions in the sense of optimality. In addition, the decision variables are countable, so the integer programming is the suitable model over the linear programming model.

On the other hand, our proposed model corrects for the limitations of the Jeeva's model. It is characterized by using the exact information about the probability distribution of the r.v. Also, it will be solved without linearization.

Actually, our proposed model contains the same deterministic objective function in the Jeeva's proposed model, defined in (4.9). While, we obtain the deterministic constraints as follows:

First, we define, $W_i = \sum_{j=1}^2 t_{ij} x_{ij}$; $i = 1, 2$ and $W_j = \sum_{i=1}^2 t_{ij} x_{ij}$; $j = 1, 2$, then, applying the base term of the CDF, defined in (3.3),⁶ we can define the CDF of W_i

$$F_{W_i}(w_i) = 1 + \exp \left[-\frac{1}{\beta_{i1} x_{i1}} w_i^{\lambda_{i1}} - \frac{1}{\beta_{i2} x_{i2}} w_i^{\lambda_{i2}} \right] - \exp \left[-\frac{1}{\beta_{i1} x_{i1}} w_i^{\lambda_{i1}} \right] - \exp \left[-\frac{1}{\beta_{i2} x_{i2}} w_i^{\lambda_{i2}} \right] \tag{4.12}$$

Similarly, the CDF of W_j is defined as:

$$F_{W_j}(w_j) = 1 + \exp \left[-\frac{1}{\beta_{1j} x_{1j}} w_j^{\lambda_{1j}} - \frac{1}{\beta_{2j} x_{2j}} w_j^{\lambda_{2j}} \right] - \exp \left[-\frac{1}{\beta_{1j} x_{1j}} w_j^{\lambda_{1j}} \right] - \exp \left[-\frac{1}{\beta_{2j} x_{2j}} w_j^{\lambda_{2j}} \right] \tag{4.13}$$

Therefore, our proposed deterministic equivalents for the constraints (4.2) and (4.3) will take the forms:

⁴ $E(r_i) = \sum_{j=1}^n \mu_{ij} x_{ij}$ and $Var(r_i) = \sum_{j=1}^n \sigma_{ij}^2 x_{ij}^2$ are the mean and variance of r_i , respectively.
⁵ The Lyapunov CLT states that; If X_1, X_2, \dots, X_n is a sample of independent, but not necessarily identically distributed, r.v.'s, each with expected value μ_i and variance σ_i^2 , then $\frac{1}{s_n} \sum_{i=1}^n X_i - \sum_{i=1}^n \mu_i \xrightarrow{d} N(0, 1)$ where, $s_n^2 = \sum_{i=1}^n \sigma_i^2$ (Billingsley, 1979).
⁶ It was noted that the correction term is neglectable with large values of the r.v. W $w \geq 1.5$.

$$1 + \exp \left[-\frac{1}{\beta_{i1} x_{i1}^{\lambda_{i1}}} b_i^{\lambda_{i1}} - \frac{1}{\beta_{i2} x_{i2}^{\lambda_{i2}}} b_i^{\lambda_{i2}} \right] - \exp \left[-\frac{1}{\beta_{i1} x_{i1}^{\lambda_{i1}}} b_i^{\lambda_{i1}} \right] - \exp \left[-\frac{1}{\beta_{i2} x_{i2}^{\lambda_{i2}}} b_i^{\lambda_{i2}} \right] \geq \alpha_i ; \quad i = 1, 2 \tag{4.14}$$

$$1 + \exp \left[-\frac{1}{\beta_{1j} x_{1j}^{\lambda_{1j}}} (b_j^*)^{\lambda_{1j}} - \frac{1}{\beta_{2j} x_{2j}^{\lambda_{2j}}} (b_j^*)^{\lambda_{2j}} \right] - \exp \left[-\frac{1}{\beta_{1j} x_{1j}^{\lambda_{1j}}} (b_j^*)^{\lambda_{1j}} \right] - \exp \left[-\frac{1}{\beta_{2j} x_{2j}^{\lambda_{2j}}} (b_j^*)^{\lambda_{2j}} \right] \geq \alpha_j ; \quad j = 1, 2 \tag{4.15}$$

A comparison between the results of the two models is conducted based on the following criteria:

- **The optimal value of the objective function (Z^*)**; since the model aims at minimizing the Z value, then the model that gives smaller value of Z^* is the preferred model.
- **The solutions time**; the model that is solved faster is also the preferred one.

Actually, [Jeeva et al. \(2004\)](#) didn't refer to the data about the Weibull r.v.'s; t_{ij} , nor the values of the estimated parameters. So, we will assume known Weibull parameters λ_{ij}, β_{ij} . In order to keep the generality of the results and to avoid biasness, we will assume two cases with different values of the parameters to consider the behavior of the Weibull distribution w.r.t the Normal distribution.

Case I, when the values of the Weibull parameters are such that the distribution is symmetric. While, **Case II** when the values of the Weibull parameters are such that the distribution is skewed (see, [Lee et al., 2003](#)). Whereas, the values of the other coefficients of the model; the numbers of persons available in each cluster, y_j , the number of persons required for each job, x_i , the expected man-hours for persons in each cluster, b_j^* , and the completion time of each job, b_i , will be as they are mentioned in [Jeeva et al. \(2004\)](#).

The following table presents the expected values and variances (calculated applying the formulas (4.8)) as well as the other constraints' coefficients under the two cases.

Using the data presented in the above table, we applied, and the Mixed Integer Nonlinear Programming (MINLP) is used to solve the two obtained deterministic models using GAMS 24.9.1 software (assuming the confidence levels, $\alpha_i = \alpha_j = 95\%$).

The reported outputs of each model are; the optimal values of the objective function Z^* and the decision variables x_{ij}^* , as well as the time taken to solve each model. These outputs are presented in the following table.

Based on the results shown in [Table \(4.2\)](#), a comparison between the two models is conducted. For Case I, we can find that even if the values of the Weibull parameters make its shape symmetric and similar to that of the Normal distribution, the Normal approximation leads to inefficient results. On the other hand, the two models show the same performance and, approximately, in the same time for case II.

It worth noting that, small values of the Weibull parameters (in case II) yield small values of the expected values and the variances (see [Table 4.1](#)). This leads to a big gap between the two sides of the probabilistic constraints, which may affect the problem setting. Thus, we will assume the above example, while changing the R.H.S coefficients; the expected man-hours for persons in each cluster, b_j^* , and the completion time of each job, b_i , to be $b_1^* = 500$, $b_2^* = 250$ and $b_1 = 250$, $b_2 = 500$, respectively.

Using the data defined before in [Table \(4.1\)](#), with the new values of b_j^* and b_i , we can solve the two deterministic models under the two aforementioned cases, and the following results are obtained:

As shown in [Table \(4.3\)](#), our proposed model shows better performance (in terms of lower value of the objective function and faster solution time) than the Jeeva's proposed model for the two cases.

Table 4-1
The values of the distribution's means and variances and the model's deterministic coefficients under Cases I and II.

	Jobs	Clusters				Completion time (b_i)	Persons required (x_i)
		G_1		G_2			
		$\hat{\mu}_{ij}$	$\hat{\sigma}_{ij}^2$	$\hat{\mu}_{ij}$	$\hat{\sigma}_{ij}^2$		
Case I	1	13.7	11	11	9.34	500	20
	2	16.3	21	12.7	11	1000	30
	Expected man hours (b_j^*)		1000		500		
	Available persons (y_j)		30		20		
Case II	1	4.2	22.04		5	25	500
	2	5.65	51.0775		5.418	13.49	1000
	Expected man hours (b_j^*)		1000		500		
	Available persons (y_j)		30		20		

Table 4–2
The results of the obtained deterministic models.

		Objective Function	Decision Variables				Solution Time in seconds
		Z*	x ₁₁ *	x ₁₂ *	x ₂₁ *	x ₂₂ *	
Case I	Jeeva's Model	661.300	13	7	7	23	0.031
	The Proposed Model	590	0	19	0	30	0.015
Case II	Jeeva's Model	246.540	20	0	0	30	0.015
	The Proposed Model	246.540	20	0	0	30	0.016

Table 4–3
The results of the obtained deterministic models.

		Objective Function	Decision Variables				Solution Time in seconds
		Z*	x ₁₁ *	x ₁₂ *	x ₂₁ *	x ₂₂ *	
Case I	Jeeva's Model			Infeasible Solution			0.000
	The Proposed Model	608.2	0	20	2	28	0.125
Case II	Jeeva's Model	247.468	20	0	4	26	0.016
	The Proposed Model	245.54	15	4	0	30	0.000

5. Conclusion

In this paper we have proposed a new deterministic equivalent model to the bivariate CCLP model with two independent random L.H.S. technological coefficients from the Weibull distribution. The performance of the proposed model is shown through an application of allocating recruitment in Manpower Planning so as to optimize the jobs' completion time.

The obtained results verified that; although our proposed deterministic model are based on approximation of the CDF of the linear combination of independent Weibull random coefficients, it is still preferred compared with another model proposed by Jeeva et al. (2004), which depends on approximation to the Normal distribution.

At the end, we can suggest the following points for further research:

- Obtaining the deterministic equivalent model to the CCLP model when the random coefficients are dependent.
- Implementing the CCP approach when the random coefficients follow the Weibull distribution in the framework of Nonlinear programming and Goal Programming models.

Acknowledgement

The authors wish to deeply thank the anonymous reviewers for their valuable and helpful comments to the manuscript.

Appendix A. Finding the parameters, ω_j and η_j for the Weibull distribution

$$\omega_j = \frac{E^2(T_j^{\lambda_j})}{E(T_j^{\lambda_j}) - E^2(T_j^{\lambda_j})}$$

$$E(T_j^{\lambda_j}) = \int t_j^{\lambda_j} f(t_j) dt_j$$

$$E(T_j^{\lambda_j}) = \int_0^\infty t_j^{\lambda_j} \frac{\lambda_j}{\beta_j x_j^{\lambda_j}} t_j^{\lambda_j-1} e^{-\frac{1}{\beta_j x_j^{\lambda_j}} t_j^{\lambda_j}} dt_j$$

$$E(T_i^{\lambda_i}) = \int_0^\infty \frac{\lambda_i}{\beta_i x_i^{\lambda_i}} t_i^{2\lambda_i-1} e^{-\frac{1}{\beta_i x_i^{\lambda_i}} t_i^{\lambda_i}} dt_i$$

Applying integration by substitution method; Put, $y_i = \frac{1}{\beta_i x_i^{\lambda_i}} t_i^{\lambda_i}$, then $t_i = (\beta_i x_i^{\lambda_i} y_i)^{1/\lambda_i}$ and $dt_i = \frac{1}{\lambda_i} \beta_i x_i^{\lambda_i} (\beta_i x_i^{\lambda_i} y_i)^{\frac{1}{\lambda_i}-1} dy_i$

Then,

$$E(T_i^{\lambda_i}) = \int_0^\infty \frac{\lambda_i}{\beta_i x_i^{\lambda_i}} [(\beta_i x_i^{\lambda_i} y_i)^{1/\lambda_i}]^{2\lambda_i-1} e^{-y_i} \frac{1}{\lambda_i} \beta_i x_i^{\lambda_i} (\beta_i x_i^{\lambda_i} y_i)^{\frac{1}{\lambda_i}-1} dy_i$$

$$E (T_i^{\lambda_i}) = \int_0^\infty \beta_i x_i^{\lambda_i} y_i e^{-y_i} dy_i$$

$$E (T_i^{\lambda_i}) = \beta_i x_i^{\lambda_i} \int_0^\infty y_i^{2-1} e^{-y_i} dy_i$$

$$E (T_i^{\lambda_i}) = \beta_i x_i^{\lambda_i} \Gamma(2) = \beta_i x_i^{\lambda_i}$$

Similarly, $E (T_i^{2\lambda_i})$ is given by:

$$E (T_i^{2\lambda_i}) = \int t_i^{2\lambda_i} f(t_i) dt_i$$

$$E (T_i^{2\lambda_i}) = \int_0^\infty t_i^{2\lambda_i} \frac{\lambda_i}{\beta_i x_i^{\lambda_i}} t_i^{\lambda_i-1} e^{-\frac{1}{\beta_i x_i^{\lambda_i}} t_i^{\lambda_i}} dt_i$$

$$E (T_i^{2\lambda_i}) = \int_0^\infty \frac{\lambda_i}{\beta_i x_i^{\lambda_i}} t_i^{3\lambda_i-1} e^{-\frac{1}{\beta_i x_i^{\lambda_i}} t_i^{\lambda_i}} dt_i$$

$$E (T_i^{\lambda_i}) = \beta_i^2 x_i^{2\lambda_i} \Gamma(3) = 2\beta_i^2 x_i^{2\lambda_i}$$

Now, we can define ω_i as follows:

$$\therefore \omega_i = \frac{E^2 (T_i^{\lambda_i})}{E (T_i^{2\lambda_i}) - E^2 (T_i^{\lambda_i})} = \frac{\beta_i^2 x_i^{2\lambda_i}}{2\beta_i^2 x_i^{2\lambda_i} - \beta_i^2 x_i^{2\lambda_i}} = 1$$

- $\eta_j = [E (T_j^{\lambda_j})]^{1/\lambda_j} = \beta_j^{1/\lambda_j} x_j$

Appendix B. Derivation of the correction term

It's lower bound is given by:

$$L_s (w ; n) = \sum_{l=0}^{L-1} [(\hat{y} (\omega_1, u_{l+1}(w)) - \hat{y} (\omega_1, u_l(w))) \times (\hat{y} (\omega_2, v_0(w)) - \hat{y} (\omega_2, v_l(w)))]$$

Where,

$$(u_l(w), v_l(w)) = \left(\omega_1 \left(\frac{lw}{L\eta_1} \right)^{\lambda_1}, \omega_2 \left(\frac{(L-l)w}{L\eta_2} \right)^{\lambda_2} \right); \quad l = 0, 1, \dots, L, \quad \omega_j = 1 \quad \text{and} \quad \eta_j = \beta_j^{1/\lambda_j} x_j$$

Then

$$L_s (w ; L) = \sum_{l=0}^{L-1} [(\hat{y} (1, \left(\frac{(l+1)w}{L\eta_1} \right)^{\lambda_1}) - \hat{y} (1, \left(\frac{lw}{L\eta_1} \right)^{\lambda_1})) \times (\hat{y} (1, \left(\frac{Lw}{L\eta_2} \right)^{\lambda_2}) - \hat{y} (1, \left(\frac{(L-l)w}{L\eta_2} \right)^{\lambda_2}))]$$

$$L_s (w ; L) = \sum_{l=0}^{L-1} [(\hat{y} (1, \left(\frac{(l+1)w^{\lambda_1}}{L^{\lambda_1} \beta_1 x_1^{\lambda_1}} \right)) - \hat{y} (1, \left(\frac{l^{\lambda_1} w^{\lambda_1}}{L^{\lambda_1} \beta_1 x_1^{\lambda_1}} \right))) \times (\hat{y} (1, \left(\frac{w^{\lambda_2}}{\beta_2 x_2^{\lambda_2}} \right)) - \hat{y} (1, \left(\frac{(L-l)^{\lambda_2} w^{\lambda_2}}{L^{\lambda_2} \beta_2 x_2^{\lambda_2}} \right)))]$$

$$L_s (w ; L) = \sum_{l=0}^{L-1} [(\int_0^{(l+1)^{\lambda_1} w^{\lambda_1}} \frac{1}{L^{\lambda_1} \beta_1 x_1^{\lambda_1}} u^{l-1} e^{-u} du - \int_0^{l^{\lambda_1} w^{\lambda_1}} \frac{1}{L^{\lambda_1} \beta_1 x_1^{\lambda_1}} u^{l-1} e^{-u} du) \times (\int_0^{w^{\lambda_2}} \frac{1}{\beta_2 x_2^{\lambda_2}} v^{l-1} e^{-v} dv - \int_0^{(L-l)^{\lambda_2} w^{\lambda_2}} \frac{1}{L^{\lambda_2} \beta_2 x_2^{\lambda_2}} v^{l-1} e^{-v} dv)]$$

$$L_s (w ; L) = \sum_{l=0}^{L-1} [((-e^{-u})_0^{\frac{(l+1)^{\lambda_1} w^{\lambda_1}}{L^{\lambda_1} \beta_1 x_1^{\lambda_1}}} - (-e^{-u})_0^{\frac{l^{\lambda_1} w^{\lambda_1}}{L^{\lambda_1} \beta_1 x_1^{\lambda_1}}}) \times ((-e^{-v})_0^{\frac{w^{\lambda_2}}{\beta_2 x_2^{\lambda_2}}} - (-e^{-v})_0^{\frac{(L-l)^{\lambda_2} w^{\lambda_2}}{L^{\lambda_2} \beta_2 x_2^{\lambda_2}}})]$$

$$L_s (w ; L) = \sum_{l=0}^{L-1} [((1 - e^{-\frac{(l+1)^{\lambda_1} w^{\lambda_1}}{L^{\lambda_1} \beta_1 x_1^{\lambda_1}}}) - (1 - e^{-\frac{l^{\lambda_1} w^{\lambda_1}}{L^{\lambda_1} \beta_1 x_1^{\lambda_1}}})) \times ((1 - e^{-\frac{w^{\lambda_2}}{\beta_2 x_2^{\lambda_2}}}) - (1 - e^{-\frac{(L-l)^{\lambda_2} w^{\lambda_2}}{L^{\lambda_2} \beta_2 x_2^{\lambda_2}}}))]$$

$$L_s (w ; L) = \sum_{l=0}^{L-1} [(e^{-\frac{l^{\lambda_1} w^{\lambda_1}}{L^{\lambda_1} \beta_1 x_1^{\lambda_1}}} - e^{-\frac{(l+1)^{\lambda_1} w^{\lambda_1}}{L^{\lambda_1} \beta_1 x_1^{\lambda_1}}}) \times (e^{-\frac{(L-l)^{\lambda_2} w^{\lambda_2}}{L^{\lambda_2} \beta_2 x_2^{\lambda_2}}} - e^{-\frac{w^{\lambda_2}}{\beta_2 x_2^{\lambda_2}}})]$$

$$L_s (w ; L) = \sum_{l=0}^{L-1} [e^{-\frac{l^{\lambda_1} w^{\lambda_1}}{L^{\lambda_1} \beta_1 x_1^{\lambda_1}}} e^{-\frac{(L-l)^{\lambda_2} w^{\lambda_2}}{L^{\lambda_2} \beta_2 x_2^{\lambda_2}}} - e^{-\frac{l^{\lambda_1} w^{\lambda_1}}{L^{\lambda_1} \beta_1 x_1^{\lambda_1}}} e^{-\frac{w^{\lambda_2}}{\beta_2 x_2^{\lambda_2}}} - e^{-\frac{(l+1)^{\lambda_1} w^{\lambda_1}}{L^{\lambda_1} \beta_1 x_1^{\lambda_1}}} e^{-\frac{(L-l)^{\lambda_2} w^{\lambda_2}}{L^{\lambda_2} \beta_2 x_2^{\lambda_2}}} + e^{-\frac{(l+1)^{\lambda_1} w^{\lambda_1}}{L^{\lambda_1} \beta_1 x_1^{\lambda_1}}} e^{-\frac{w^{\lambda_2}}{\beta_2 x_2^{\lambda_2}}}]$$

$$L_s(w; L) = e^{-\frac{w^{\lambda_1}}{\beta_1 x_1^{\lambda_1}} \sum_{l=0}^{L-1} \frac{l^{\lambda_1}}{L^{\lambda_1}} - \frac{w^{\lambda_2}}{\beta_2 x_2^{\lambda_2}} \sum_{l=0}^{L-1} \frac{(L-l)^{\lambda_2}}{L^{\lambda_2}}} - e^{-\frac{w^{\lambda_1}}{\beta_1 x_1^{\lambda_1}} \sum_{l=0}^{L-1} \frac{l^{\lambda_1}}{L^{\lambda_1}} - \frac{(L-1)w^{\lambda_2}}{\beta_2 x_2^{\lambda_2}}} - e^{-\frac{w^{\lambda_1}}{\beta_1 x_1^{\lambda_1}} \sum_{l=0}^{L-1} \frac{(l+1)^{\lambda_1}}{L^{\lambda_1}} - \frac{w^{\lambda_2}}{\beta_2 x_2^{\lambda_2}} \sum_{l=0}^{L-1} \frac{(L-l)^{\lambda_2}}{L^{\lambda_2}}} + e^{-\frac{w^{\lambda_1}}{\beta_1 x_1^{\lambda_1}} \sum_{l=0}^{L-1} \frac{(l+1)^{\lambda_1}}{L^{\lambda_1}} - \frac{(L-1)w^{\lambda_2}}{\beta_2 x_2^{\lambda_2}}}$$

While, the upper bound is defined as:

$$U_s(w; L) = \sum_{l=0}^{L-1} [((\hat{\gamma}(w_1, u_{l+1}(w)) - (\hat{\gamma}(w_1, u_l(w))) \times ((\hat{\gamma}(w_2, v_0(w)) - (\hat{\gamma}(w_2, v_{l+1}(w))))]$$

$$U_s(w; L) = \sum_{l=0}^{L-1} [((\hat{\gamma}(1, \left(\frac{(l+1)w}{L\eta_1}\right)^{\lambda_1}) - \hat{\gamma}(1, \left(\frac{lw}{L\eta_1}\right)^{\lambda_1})) \times (\hat{\gamma}(1, \left(\frac{(L)w}{L\eta_2}\right)^{\lambda_2}) - \hat{\gamma}(1, \left(\frac{(L-l-1)w}{L\eta_2}\right)^{\lambda_2})))]$$

$$U_s(w; L) = \sum_{l=0}^{L-1} [((\hat{\gamma}(1, \left(\frac{(l+1)^{\lambda_1}w^{\lambda_1}}{L^{\lambda_1}\beta_1 x_1^{\lambda_1}}\right)) - \hat{\gamma}(1, \left(\frac{l^{\lambda_1}w^{\lambda_1}}{L^{\lambda_1}\beta_1 x_1^{\lambda_1}}\right))) \times (\hat{\gamma}(1, \left(\frac{w^{\lambda_2}}{\beta_2 x_2^{\lambda_2}}\right)) - \hat{\gamma}(1, \left(\frac{(L-l-1)^{\lambda_2}w^{\lambda_2}}{L^{\lambda_2}\beta_2 x_2^{\lambda_2}}\right)))]$$

$$U_s(w; L) = \sum_{l=0}^{L-1} [(\int_0^{\frac{(l+1)^{\lambda_1}w^{\lambda_1}}{L^{\lambda_1}\beta_1 x_1^{\lambda_1}}} u^{l-1} e^{-u} du - \int_0^{\frac{l^{\lambda_1}w^{\lambda_1}}{L^{\lambda_1}\beta_1 x_1^{\lambda_1}}} u^{l-1} e^{-u} du) \times (\int_0^{\frac{w^{\lambda_2}}{\beta_2 x_2^{\lambda_2}}} v^{l-1} e^{-v} dv - \int_0^{\frac{(L-l-1)^{\lambda_2}w^{\lambda_2}}{L^{\lambda_2}\beta_2 x_2^{\lambda_2}}} v^{l-1} e^{-v} dv)]$$

$$U_s(w; L) = \sum_{l=0}^{L-1} [((-e^{-u})_0^{\frac{(l+1)^{\lambda_1}w^{\lambda_1}}{L^{\lambda_1}\beta_1 x_1^{\lambda_1}}} - (-e^{-u})_0^{\frac{l^{\lambda_1}w^{\lambda_1}}{L^{\lambda_1}\beta_1 x_1^{\lambda_1}}}) \times ((-e^{-v})_0^{\frac{w^{\lambda_2}}{\beta_2 x_2^{\lambda_2}}} - (-e^{-v})_0^{\frac{(L-l-1)^{\lambda_2}w^{\lambda_2}}{L^{\lambda_2}\beta_2 x_2^{\lambda_2}}})]$$

$$U_s(w; L) = \sum_{l=0}^{L-1} [((1 - e^{-\frac{(l+1)^{\lambda_1}w^{\lambda_1}}{L^{\lambda_1}\beta_1 x_1^{\lambda_1}}}) - (1 - e^{-\frac{l^{\lambda_1}w^{\lambda_1}}{L^{\lambda_1}\beta_1 x_1^{\lambda_1}}})) \times ((1 - e^{-\frac{w^{\lambda_2}}{\beta_2 x_2^{\lambda_2}}}) - (1 - e^{-\frac{(L-l-1)^{\lambda_2}w^{\lambda_2}}{L^{\lambda_2}\beta_2 x_2^{\lambda_2}}}))]$$

$$U_s(w; L) = \sum_{l=0}^{L-1} [(e^{-\frac{l^{\lambda_1}w^{\lambda_1}}{L^{\lambda_1}\beta_1 x_1^{\lambda_1}}} - e^{-\frac{(l+1)^{\lambda_1}w^{\lambda_1}}{L^{\lambda_1}\beta_1 x_1^{\lambda_1}}}) \times (e^{-\frac{(L-l-1)^{\lambda_2}w^{\lambda_2}}{L^{\lambda_2}\beta_2 x_2^{\lambda_2}}} - e^{-\frac{w^{\lambda_2}}{\beta_2 x_2^{\lambda_2}}})]$$

$$U_s(w; L) = \sum_{l=0}^{L-1} [(e^{-\frac{l^{\lambda_1}w^{\lambda_1}}{L^{\lambda_1}\beta_1 x_1^{\lambda_1}}} e^{-\frac{(L-l-1)^{\lambda_2}w^{\lambda_2}}{L^{\lambda_2}\beta_2 x_2^{\lambda_2}}} - e^{-\frac{l^{\lambda_1}w^{\lambda_1}}{L^{\lambda_1}\beta_1 x_1^{\lambda_1}}} e^{-\frac{w^{\lambda_2}}{\beta_2 x_2^{\lambda_2}}} - e^{-\frac{(l+1)^{\lambda_1}w^{\lambda_1}}{L^{\lambda_1}\beta_1 x_1^{\lambda_1}}} e^{-\frac{(L-l-1)^{\lambda_2}w^{\lambda_2}}{L^{\lambda_2}\beta_2 x_2^{\lambda_2}}} + e^{-\frac{(l+1)^{\lambda_1}w^{\lambda_1}}{L^{\lambda_1}\beta_1 x_1^{\lambda_1}}} e^{-\frac{w^{\lambda_2}}{\beta_2 x_2^{\lambda_2}}})]$$

$$U_s(w; L) = e^{-\frac{w^{\lambda_1}}{\beta_1 x_1^{\lambda_1}} \sum_{l=0}^{L-1} \frac{l^{\lambda_1}}{L^{\lambda_1}} - \frac{w^{\lambda_2}}{\beta_2 x_2^{\lambda_2}} \sum_{l=0}^{L-1} \frac{(L-l-1)^{\lambda_2}}{L^{\lambda_2}}} - e^{-\frac{w^{\lambda_1}}{\beta_1 x_1^{\lambda_1}} \sum_{l=0}^{L-1} \frac{l^{\lambda_1}}{L^{\lambda_1}} - \frac{(L-1)w^{\lambda_2}}{\beta_2 x_2^{\lambda_2}}} - e^{-\frac{w^{\lambda_1}}{\beta_1 x_1^{\lambda_1}} \sum_{l=0}^{L-1} \frac{(l+1)^{\lambda_1}}{L^{\lambda_1}} - \frac{w^{\lambda_2}}{\beta_2 x_2^{\lambda_2}} \sum_{l=0}^{L-1} \frac{(L-l-1)^{\lambda_2}}{L^{\lambda_2}}} + e^{-\frac{w^{\lambda_1}}{\beta_1 x_1^{\lambda_1}} \sum_{l=0}^{L-1} \frac{(l+1)^{\lambda_1}}{L^{\lambda_1}} - \frac{(L-1)w^{\lambda_2}}{\beta_2 x_2^{\lambda_2}}}$$

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