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# Logics for Belief as Maximally Plausible Possibility

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## Abstract

We consider a basic logic with two primitive uni-modal operators: one for *certainty* and the other for *plausibility*. The former is assumed to be a normal operator (corresponding - semantically - to a binary Kripke relation), while the latter is merely a classical operator (corresponding - semantically - to a neighborhood structure). We then define belief, interpreted as “maximally plausible possibility”, in terms of these two notions: the agent believes  $\phi$  if (1) she cannot rule out  $\phi$  (that is, it is not the case that she is certain that  $\neg\phi$ ), (2) she judges  $\phi$  to be plausible and (3) she does not judge  $\neg\phi$  to be plausible. We consider several interaction properties between certainty and plausibility and study how these properties translate into properties of belief (positive and negative introspection, their converses, conjunction, etc.). We then prove that all the logics considered are minimal logics for the highlighted theorems. We also consider a number of possible interpretations of plausibility and identify the corresponding logics.

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## 1 Introduction

In both theoretical discussions and applications it is common to attribute to individuals two different epistemic levels, often referred to as *knowledge* and *belief*. While the first represents the entire range of possibilities that the individual has in mind (possibly reflecting the available evidence), the latter represents a less “cautious” judgment, reflecting an assessment of likelihood or plausibility. An example of such a dual epistemic/doxastic state is expressed in the following statement: “I believe, but am not certain, that Palestine is not a party to the statute of the International Court of Justice” (Beddor and Goldstein (2018)).

One strand in the literature<sup>1</sup> takes both knowledge and belief as primitives and focuses on the interaction properties between the two notions: for example, whether it is reasonable to postulate that if the individual believes  $\phi$  then she should believe that she knows  $\phi$  ( $B\phi \rightarrow BK\phi$ ), whether knowledge implies belief ( $K\phi \rightarrow B\phi$ ), etc.

In a second strand of the literature<sup>2</sup> belief is not a primitive but a derived notion: it is derived from knowledge and/or plausibility. This is done either by postulating a preference ordering over possible worlds – so that  $\phi$  is believed if it is known to be true in the most preferred (most plausible) worlds – or by postulating a plausibility measure over events – so that an individual is said to believe  $\phi$  if and only if she knows that  $\phi$  is more plausible than  $\neg\phi$ .

In both strands of the literature knowledge is typically assumed to satisfy the S5 logic and belief is either assumed to satisfy the KD45 logic or properties of the primitives are postulated that yield the KD45 logic for belief.

In this paper we follow the second strand in the literature, *by not taking belief as a primitive*. We define belief in terms of two notions: *certainty* and *plausibility*. We start with a basic logic, called  $\mathbb{L}$ , where the *certainty operator*  $C$  is assumed to be a normal operator,<sup>3</sup> with no restrictions imposed on the logic of certainty, and a *plausibility operator*  $P$ , which is assumed to be a classical operator,<sup>4</sup> with no restrictions imposed on the logic of plausibility. We then define belief as *maximally plausible possibility* in the sense that the agent believes that  $\phi$  if and only if:

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<sup>1</sup>See, for example, Hintikka (1962), van der Hoek and Meyer (1995), Kraus and Lehmann (1988), Lenzen (1978), Stalnaker (2006), Voorbraak (1992).

<sup>2</sup>See, for example, Friedman and Halpern (1997), Lamarin and Shoham (1994), Moses and Shoham (1993).

<sup>3</sup>Represented, semantically, by a binary relation on the set of possible worlds.

<sup>4</sup>Represented by a neighborhood function on possible worlds)

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1. she cannot rule out  $\phi$ , that is, it is not the case that she is certain that  $\neg\phi$ ,
2. she judges  $\phi$  to be plausible, and
3. she does not judge  $\neg\phi$  to be plausible.<sup>5</sup>

Formally:

$$B\phi \leftrightarrow (\neg C\neg\phi \wedge P\phi \wedge \neg P\neg\phi)$$

First we show that it is a theorem of this very basic logic that belief satisfies consistency, that is, it cannot be the case that the agent believes  $\phi$  and also  $\neg\phi$ :

$$B\phi \rightarrow \neg B\neg\phi \quad (\text{Proposition 1})$$

Then we consider minimal extensions of logic  $\mathbb{L}$  that yield as theorems various schemata of interest such as:

1. Interaction between certainty and belief:

- (a) Certainty implies belief:

$$C\phi \rightarrow B\phi \quad (\text{Proposition 5})$$

- (b) (Absence of) belief implies certainty of (absence of) belief:

$$B\phi \rightarrow CB\phi \quad (\text{Proposition 6})$$

$$\neg B\phi \rightarrow C\neg B\phi \quad (\text{Proposition 7})$$

- (c) Belief of certainty implies belief:

$$BC\phi \rightarrow B\phi \quad (\text{Proposition 8})$$

2. Introspection properties of belief:

- (a) Positive and negative introspection:

$$B\phi \rightarrow BB\phi \quad (\text{Proposition 9})$$

$$\neg B\phi \rightarrow B\neg B\phi \quad (\text{Proposition 10})$$

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<sup>5</sup>We use the expression '*maximally* plausible possibility' rather than 'plausible possibility' because of Point 3: if  $\phi$  is plausible, but so is  $\neg\phi$ , then – according to our definition – it is not the case that the agent believes  $\phi$ .

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(b) and their converses:

$$BB\phi \rightarrow B\phi \quad (\text{Proposition 11})$$

$$B\neg B\phi \rightarrow \neg B\phi \quad (\text{Proposition 12})$$

3. Conjunction properties of belief:

$$B(\phi \wedge \psi) \rightarrow B\phi \wedge B\psi \quad (\text{Proposition 17})$$

$$B\phi \wedge B\psi \wedge \neg C\neg(\phi \wedge \psi) \rightarrow B(\phi \wedge \psi) \quad (\text{Proposition 20})$$

For none of these results does one need to assume that certainty satisfies the S5 logic of knowledge, or indeed even the KD45 logic.<sup>6</sup> Furthermore, with the exception of the conjunction properties of belief (Propositions 17 and 20) no substantive properties of plausibility are required, except for some interaction properties between certainty and plausibility.

All the results are proved syntactically by providing derivations for each theorem (Sections 2-5). Section 6 discusses the semantics of certainty and plausibility and provides semantic characterizations of all the axioms considered. Furthermore, the semantics is used to show that the extensions of the basic logic  $\mathbb{L}$  considered in the previous sections are minimal extensions for the corresponding theorems. In Section 8 we consider possible interpretations of the notion of plausibility (e.g. an event  $E$  is plausible if it has positive probability) and identify the corresponding extension of logic  $\mathbb{L}$  that is implied by such an interpretation. Section 9 discusses related literature and Section 10 concludes.

## 2 The basic logic

We consider a modal logic with two modal operators:  $C$  interpreted as “absolute certainty” (or “certainty” for short) and  $P$  interpreted as “plausibility”. Thus  $C\phi$  means that the individual in question, from now on called “the agent”, is certain that  $\phi$ , and  $P\phi$  means that the agent judges  $\phi$  to be plausible.  $C$  will be taken to be a normal operator, while we will impose no restrictions on the operator  $P$  (besides assuming that is a classical operator).

The formal language is built in the usual way from a countable set of propositional variables (or atoms)  $\text{At}$ , the connectives  $\neg$  (for “not”) and  $\vee$  (for “or”)

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<sup>6</sup>Most of these results do not rely on both positive and negative introspection of certainty, or even on either of them

and the modal operators.<sup>7</sup> Thus the set  $\Phi$  of formulas is defined inductively as follows:  $q \in \Phi$  for every atomic proposition  $q \in \mathbf{At}$ , and if  $\phi, \psi \in \Phi$  then all of the following belong to  $\Phi$ :  $\neg\phi$ ,  $\phi \vee \psi$ ,  $C\phi$  and  $P\phi$ . We denote by  $\mathbb{L}_0$  the logic determined by the following axioms and rules of inference.<sup>8</sup>

AXIOMS:

1. All propositional tautologies.

2. Axiom *K* for *C*:

$$C(\phi \rightarrow \psi) \rightarrow (C\phi \rightarrow C\psi) \quad (\mathbf{K}_C)$$

RULES OF INFERENCE:

1. Modus Ponens:

$$\frac{\phi, \phi \rightarrow \psi}{\psi} \quad (\mathbf{MP})$$

2. Necessitation for *C*:

$$\frac{\phi}{C\phi} \quad (\mathbf{Nec}_C)$$

3. Rule *RE* for *P*:

$$\frac{\phi \leftrightarrow \psi}{P\phi \leftrightarrow P\psi} \quad (\mathbf{R}_E^P)$$

**Remark 1** (Derived rules of inference). *It is well-known that from  $\mathbf{K}_C$ ,  $(\mathbf{MP})$  and  $(\mathbf{Nec}_C)$  one can derive the following rules of inference:*

<sup>7</sup>See, for example, [Chellas \(1984\)](#), [Blackburn et al. \(2001\)](#). The connectives  $\wedge$  (for “and”),  $\rightarrow$  (for “if ... then ...”) and  $\leftrightarrow$  (for “if and only if”) are defined as usual:  $\phi \wedge \psi = \neg(\neg\phi \vee \neg\psi)$ ,  $\phi \rightarrow \psi = \neg\phi \vee \psi$  and  $\phi \leftrightarrow \psi = (\phi \rightarrow \psi) \wedge (\psi \rightarrow \phi)$ .

<sup>8</sup>Throughout the paper, the naming of axioms and rules of inference follows [Chellas \(1984\)](#).

$$\frac{\phi \rightarrow \psi}{C\phi \rightarrow C\psi} \quad (\mathbf{R}_K^C)$$

$$\frac{\phi \leftrightarrow \psi}{C\phi \leftrightarrow C\psi} \quad (\mathbf{R}_E^C)$$

$$\frac{\phi \rightarrow \psi}{\neg C\neg\phi \rightarrow \neg C\neg\psi} \quad (\mathbf{R}_K^{C\neg})$$

$$\frac{\phi \leftrightarrow \psi}{\neg C\neg\phi \leftrightarrow \neg C\neg\psi} \quad (\mathbf{R}_E^{C\neg})$$

**Remark 2** (Normality of C). *It is also well-known that the following are theorems of the basic logic  $\mathbb{L}_0$ :*

$$(M_C) \quad C(\phi \wedge \psi) \rightarrow (C\phi \wedge C\psi)$$

$$(C_C) \quad (C\phi \wedge C\psi) \rightarrow C(\phi \wedge \psi)$$

$$(M_C^{dual}) \quad \neg C\neg(\phi \vee \psi) \rightarrow (\neg C\neg\phi \vee \neg C\neg\psi)$$

$$(C_C^{dual}) \quad (\neg C\neg\phi \vee \neg C\neg\psi) \rightarrow \neg C\neg(\phi \vee \psi)$$

**Remark 3.** *In virtue of axiom  $R_K^C$ , the following is a theorem of logic  $\mathbb{L}_0$ :*<sup>9</sup>

$$\neg C\neg(\phi_1 \wedge \dots \wedge \phi_n) \rightarrow (\neg C\neg\phi_1 \wedge \dots \wedge \neg C\neg\phi_n)$$

We now extend the logic by adding a third operator, namely the *belief operator*  $B$ , which is derived from  $C$  and  $P$ . The interpretation of  $B\phi$  is “the agent believes that  $\phi$ ”. Belief is defined as “maximally plausible possibility” in the sense that the agent believes that  $\phi$  if and only if:

1. she cannot rule out  $\phi$ , that is, it is not the case that she is certain that  $\neg\phi$ ,
2. she judges  $\phi$  to be plausible, and

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<sup>9</sup>*Proof.* For every  $i = 1, \dots, n$ , since  $\neg\phi_i \rightarrow \neg(\phi_1 \wedge \dots \wedge \phi_n)$  is a tautology, by  $R_K^C$  we get that  $C\neg\phi_i \rightarrow C\neg(\phi_1 \wedge \dots \wedge \phi_n)$  is a theorem. Thus, by propositional logic (PL), we get that  $\neg C\neg(\phi_1 \wedge \dots \wedge \phi_n) \rightarrow \neg C\neg\phi_i$ , from which we obtain (again, by PL) that  $\neg C\neg(\phi_1 \wedge \dots \wedge \phi_n) \rightarrow (\neg C\neg\phi_1 \wedge \dots \wedge \neg C\neg\phi_n)$ .

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3. she does not judge  $\neg\phi$  to be plausible.

**Definition 2.1.** The operator  $B$  is defined as follows:

$$B\phi \leftrightarrow (\neg C\neg\phi \wedge P\phi \wedge \neg P\neg\phi) \quad (\text{DefB})$$

We denote by  $\mathbb{L}$  the logic obtained by extending the syntax to include formulas of the form  $B\phi$  and by adding axiom (DefB) to the basic logic  $\mathbb{L}_0$ . In later sections we will consider extensions of logic  $\mathbb{L}$ .

We now show that, without imposing any further axioms, we obtain consistency of beliefs: the property that the agent does not simultaneously believe a proposition and also its negation; that is, the following is a theorem of logic  $\mathbb{L}$ :

$$B\phi \rightarrow \neg B\neg\phi. \quad (\text{D}_B)$$

We write  $\mathbb{L} \vdash \phi$  to denote the fact that formula  $\phi$  is a theorem of logic  $\mathbb{L}$ .

**Proposition 1.**

$$\mathbb{L} \vdash B\phi \rightarrow \neg B\neg\phi$$

In all the proofs, ‘PL’ stands for ‘Propositional Logic’.<sup>10</sup>

*Proof.*

1.  $B\phi \leftrightarrow (\neg C\neg\phi \wedge P\phi \wedge \neg P\neg\phi)$  (DefB)
2.  $(\neg C\neg\phi \wedge P\phi \wedge \neg P\neg\phi) \rightarrow (P\phi \wedge \neg P\neg\phi)$  (tautology)
3.  $(P\phi \wedge \neg P\neg\phi) \rightarrow (P\phi \vee \neg P\neg\phi)$  (tautology)
4.  $B\phi \rightarrow (P\phi \vee \neg P\neg\phi)$  (1, 2, 3, PL)
5.  $B\neg\phi \leftrightarrow (\neg C\phi \wedge P\neg\phi \wedge \neg P\phi)$  (DefB)
6.  $(\neg C\phi \wedge P\neg\phi \wedge \neg P\phi) \leftrightarrow \neg(C\phi \vee \neg P\neg\phi \vee P\phi)$  (PL)
7.  $\neg B\neg\phi \leftrightarrow (C\phi \vee \neg P\neg\phi \vee P\phi)$  (5, 6, PL)
8.  $(P\phi \vee \neg P\neg\phi) \rightarrow (C\phi \vee \neg P\neg\phi \vee P\phi)$  (tautology)
9.  $B\phi \rightarrow (C\phi \vee \neg P\neg\phi \vee P\phi)$  (4, 8, PL)
10.  $B\phi \rightarrow \neg B\neg\phi.$  (7, 9, PL)

□

<sup>10</sup> Note that in Step 5 of the following proof we also implicitly use the theorem  $P\phi \leftrightarrow P\neg\neg\phi$  which is obtained from the fact that  $\phi \leftrightarrow \neg\neg\phi$  is a tautology, so that – by rule  $R_E^P$  – we get that  $P\phi \leftrightarrow P\neg\neg\phi$  is a theorem. The same applies to  $C\phi \leftrightarrow C\neg\neg\phi$ .



The next proposition shows that the rule of inference ( $R_E$ ) applies also to the belief operator.

**Proposition 2.** *The following is a derived rule of inference of logic  $\mathbb{L}$ :*

$$\frac{\phi \leftrightarrow \psi}{B\phi \leftrightarrow B\psi} \quad (R_E^B)$$

*Proof.*

1.  $\phi \leftrightarrow \psi$  (Hypothesis)
2.  $P\phi \leftrightarrow P\psi$  (1, rule  $R_E^P$ )
3.  $\neg\phi \leftrightarrow \neg\psi$  (1, PL)
4.  $P\neg\phi \leftrightarrow P\neg\psi$  (3, rule  $R_E^P$ )
5.  $\neg P\neg\phi \leftrightarrow \neg P\neg\psi$  (4, PL)
6.  $C\neg\phi \leftrightarrow C\neg\psi$  (3, rule  $R_E^C$ )
7.  $\neg C\neg\phi \leftrightarrow \neg C\neg\psi$  (6, PL)
8.  $(\neg C\neg\phi \wedge P\phi \wedge \neg P\neg\phi) \leftrightarrow (\neg C\neg\psi \wedge P\psi \wedge \neg P\neg\psi)$  (2, 5, 7, PL)
9.  $B\phi \leftrightarrow (\neg C\neg\phi \wedge P\phi \wedge \neg P\neg\phi)$  (DefB)
10.  $B\psi \leftrightarrow (\neg C\neg\psi \wedge P\psi \wedge \neg P\neg\psi)$  (DefB)
11.  $B\phi \leftrightarrow B\psi$ . (8, 9, 10, PL)

□

### 3 Extensions of logic $\mathbb{L}$

In this section we list a number of axioms that can be used to obtain extensions of logic  $\mathbb{L}$ .

#### 3.1 Candidate axioms for certainty

Natural axioms to consider for the certainty operator are the following:

- (D<sub>C</sub>) CONSISTENCY:  $C\phi \rightarrow \neg C\neg\phi$
- (4<sub>C</sub>) POSITIVE INTROSPECTION:  $C\phi \rightarrow CC\phi$
- (5<sub>C</sub>) NEGATIVE INTROSPECTION:  $\neg C\phi \rightarrow C\neg C\phi$

Consistency rules out the possibility that the agent may simultaneously be certain of a proposition and also of its negation.

Positive Introspection says that if the agent is certain of  $\phi$  then she is certain that she is certain of  $\phi$ .

Negative Introspection says that if the agent is *not* certain of  $\phi$  then she is certain that she is not certain of  $\phi$ .

### 3.2 Candidate axioms for the interaction of $C$ and $P$

We will consider the following axioms on the interaction between certainty and plausibility:

$$(CP_1) \quad C\phi \rightarrow P\phi$$

$$(CP_2) \quad C\phi \rightarrow \neg P\neg\phi$$

$$(CP_3) \quad \neg P\phi \rightarrow C\neg P\phi$$

$$(CP_4) \quad P\phi \rightarrow CP\phi.$$

Axiom  $(CP_1)$  says that if the agent is certain that  $\phi$ , then she must consider  $\phi$  plausible, while axiom  $(CP_2)$  says that if the agent is certain that  $\phi$  then she cannot judge the negation of  $\phi$  as plausible.

Axioms  $(CP_3)$  and  $(CP_4)$  are introspective properties: the former says that if the agent does not consider  $\phi$  plausible, then she is certain that she does not consider  $\phi$  plausible, while the latter says that if the agent considers  $\phi$  plausible then she is certain that she considers  $\phi$  plausible.

We denote the fact that formula  $\phi$  is a theorem of the extension of  $\mathbb{L}$  obtained by adding axioms  $\phi_1, \dots, \phi_n$  as follows:

$$\mathbb{L} + \left\{ \begin{array}{l} \phi_1 \\ \dots \\ \phi_n \end{array} \right\} \vdash \phi$$

The following proposition shows that Axiom  $(CP_2)$  is a rather strong axiom: it implies the reduction of  $B\phi$  to  $(P\phi \wedge \neg P\neg\phi)$ .

**Proposition 3.**

$$\mathbb{L} + \left\{ (CP_2) \quad C\phi \rightarrow \neg P\neg\phi \right\} \vdash \boxed{B\phi \leftrightarrow (P\phi \wedge \neg P\neg\phi)}$$

*Proof.*

1.  $B\phi \leftrightarrow (\neg C\neg\phi \wedge P\phi \wedge \neg P\neg\phi)$  (DefB)
2.  $B\phi \rightarrow (P\phi \wedge \neg P\neg\phi)$  (1, PL)
3.  $C\neg\phi \rightarrow \neg P\phi$  (Axiom CP<sub>2</sub>)
5.  $P\phi \rightarrow \neg C\neg\phi$  (3, PL)
6.  $(P\phi \wedge \neg P\neg\phi) \rightarrow (\neg C\neg\phi \wedge P\phi \wedge \neg P\neg\phi)$  (5, PL)
7.  $(P\phi \wedge \neg P\neg\phi) \rightarrow B\phi$  (6, 1, PL)
8.  $B\phi \leftrightarrow (P\phi \wedge \neg P\neg\phi)$ . (2, 7, PL)

□

The next proposition shows that consistency of certainty (D<sub>C</sub>) is provable from the conjunction of Axioms (CP<sub>1</sub>) and (CP<sub>2</sub>).

**Proposition 4.**

$$\mathbb{L} + \left\{ \begin{array}{l} (CP_1) \quad C\phi \rightarrow P\phi \\ (CP_2) \quad C\phi \rightarrow \neg P\neg\phi \end{array} \right\} \vdash \boxed{C\phi \rightarrow \neg C\neg\phi}$$

*Proof.*

1.  $C\neg\phi \rightarrow \neg P\phi$  (Axiom CP<sub>2</sub>)
2.  $P\phi \rightarrow \neg C\neg\phi$  (1, PL)
3.  $C\phi \rightarrow P\phi$  (Axiom CP<sub>1</sub>)
4.  $C\phi \rightarrow \neg C\neg\phi$ . (3, 2, PL)

□

In the following sections we consider theorems of extensions of  $\mathbb{L}$  concerning the interaction of certainty and belief, introspection properties of belief and conjunction properties of belief.

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## 4 Theorems on the interaction between $C$ and $B$

The first result is that, by adding Axioms  $(CP_1)$  and  $(CP_2)$  to  $\mathbb{L}$ , one obtains as a theorem that certainty implies belief.

**Proposition 5.**

$$\mathbb{L} + \left\{ \begin{array}{l} (CP_1) \quad C\phi \rightarrow P\phi \\ (CP_2) \quad C\phi \rightarrow \neg P\neg\phi \end{array} \right\} \vdash \boxed{C\phi \rightarrow B\phi}$$

The proof is obvious, since the conjunction of the consequents of  $(CP_1)$  and  $(CP_2)$  is  $(P\phi \wedge \neg P\neg\phi)$  which, by Proposition 3, is equivalent to  $B\phi$ .

The next result says that if Axioms  $(CP_1)$ ,  $(CP_3)$  and  $(CP_4)$  are assumed, then, whenever the agent believes something, she is certain that she believes it.

**Proposition 6.**

$$\mathbb{L} + \left\{ \begin{array}{l} (CP_1) \quad C\phi \rightarrow P\phi \\ (CP_3) \quad \neg P\phi \rightarrow C\neg P\phi \\ (CP_4) \quad P\phi \rightarrow CP\phi \end{array} \right\} \vdash \boxed{B\phi \rightarrow CB\phi}$$

1.  $B\phi \leftrightarrow (\neg C\neg\phi \wedge P\phi \wedge \neg P\neg\phi)$  (DefB)
2.  $P\phi \rightarrow CP\phi$  (Axiom  $CP_4$ )
3.  $\neg P\neg\phi \rightarrow C\neg P\neg\phi$  (Axiom  $CP_3$ )
4.  $B\phi \rightarrow (CP\phi \wedge C\neg P\neg\phi)$  (1, 2, 3, PL)
5.  $C\neg\phi \rightarrow P\neg\phi$  (Axiom  $CP_1$ )
6.  $\neg C\neg C\neg\phi \rightarrow \neg C\neg P\neg\phi$  (5,  $R_K^{C\neg}$ )
7.  $C\neg P\neg\phi \rightarrow C\neg C\neg\phi$  (6, PL)
8.  $B\phi \rightarrow (C\neg C\neg\phi \wedge CP\phi \wedge C\neg P\neg\phi)$  (4, 7, PL)
9.  $(C\neg C\neg\phi \wedge CP\phi \wedge C\neg P\neg\phi) \rightarrow C(\neg C\neg\phi \wedge P\phi \wedge \neg P\neg\phi)$  ( $C_C$ : Remark 2)
10.  $B\phi \rightarrow C(\neg C\neg\phi \wedge P\phi \wedge \neg P\neg\phi)$  (8, 9, PL)
11.  $CB\phi \leftrightarrow C(\neg C\neg\phi \wedge P\phi \wedge \neg P\neg\phi)$  (1,  $R_E^C$ )
12.  $B\phi \rightarrow CB\phi$  (10, 11, PL)

□

**Remark 4.** The schema  $B\phi \rightarrow CB\phi$  is also a theorem of the logic obtained by replacing Axiom  $(CP_1)$  with Axiom  $(5_C)$  in Proposition 6, that is,

$$\mathbb{L} + \left\{ \begin{array}{l} (5_C) \quad \neg C\phi \rightarrow C\neg C\phi \\ (CP_3) \quad \neg P\phi \rightarrow C\neg P\phi \\ (CP_4) \quad P\phi \rightarrow CP\phi \end{array} \right\} \vdash \boxed{B\phi \rightarrow CB\phi}$$

The proof consists of Steps 1-4 above while Steps 5-7 are replaced by the single step  $\neg C\neg\phi \rightarrow C\neg C\neg\phi$ , which is an instance of Axiom  $(5_C)$ , and then continues with Steps 8-12 above.

The next proposition says that if positive introspection of certainty (Axiom  $4_C$ ) and Axioms  $(CP_3)$  and  $(CP_4)$  are assumed then, if the agent does not believe  $\phi$ , then she is certain that she does not believe  $\phi$ .

**Proposition 7.**

$$\mathbb{L} + \left\{ \begin{array}{l} (4_C) \quad C\phi \rightarrow CC\phi \\ (CP_3) \quad \neg P\phi \rightarrow C\neg P\phi \\ (CP_4) \quad P\phi \rightarrow CP\phi \end{array} \right\} \vdash \boxed{\neg B\phi \rightarrow C\neg B\phi}$$

*Proof.*

1.  $C\neg\phi \rightarrow CC\neg\phi$  (Axiom  $4_C$ )
2.  $\neg CC\neg\phi \rightarrow \neg C\neg\phi$  (1, PL)
3.  $\neg P\phi \rightarrow C\neg P\phi$  (Axiom  $CP_3$ )
4.  $\neg C\neg P\phi \rightarrow P\phi$  (3, PL)
5.  $P\neg\phi \rightarrow CP\neg\phi$  (Axiom  $CP_4$ )
6.  $\neg CP\neg\phi \rightarrow \neg P\neg\phi$  (5, PL)
7.  $(\neg CC\neg\phi \wedge \neg C\neg P\phi \wedge \neg CP\neg\phi) \rightarrow (\neg C\neg\phi \wedge P\phi \wedge \neg P\neg\phi)$  (2, 4, 6, PL)
8.  $(\neg C\neg\phi \wedge P\phi \wedge \neg P\neg\phi) \leftrightarrow B\phi$  (DefB)
9.  $(\neg CC\neg\phi \wedge \neg C\neg P\phi \wedge \neg CP\neg\phi) \rightarrow B\phi$  (7, 8, PL)
10.  $\neg C\neg(\neg C\neg\phi \wedge P\phi \wedge \neg P\neg\phi) \leftrightarrow \neg C\neg B\phi$  (8,  $R_E^{C^-}$ )
11.  $\neg C\neg(\neg C\neg\phi \wedge P\phi \wedge \neg P\neg\phi) \rightarrow (\neg CC\neg\phi \wedge \neg C\neg P\phi \wedge \neg CP\neg\phi)$  (Remark 3)
12.  $\neg C\neg B\phi \rightarrow B\phi$  (10, 11, 9, PL)
13.  $\neg B\phi \rightarrow C\neg B\phi$  (12, PL)

□

**Remark 5.** The schema  $\neg B\phi \rightarrow C\neg B\phi$  is also a theorem of the logic obtained by replacing  $(4_C)$  with the conjunction of  $(CP_1)$  and  $(CP_2)$  in Proposition 7, that is,

$$\mathbb{L} + \left\{ \begin{array}{l} (CP_1) \quad C\phi \rightarrow P\phi \\ (CP_2) \quad C\phi \rightarrow \neg P\neg\phi \\ (CP_3) \quad \neg P\phi \rightarrow C\neg P\phi \\ (CP_4) \quad P\phi \rightarrow CP\phi \end{array} \right\} \vdash \boxed{\neg B\phi \rightarrow C\neg B\phi}$$

*Proof.*

1.  $B\phi \leftrightarrow (P\phi \wedge \neg P\neg\phi)$  (Proposition 3)
2.  $\neg B\phi \leftrightarrow (\neg P\phi \vee P\neg\phi)$  (1, PL)
3.  $C\neg B\phi \leftrightarrow C(\neg P\phi \vee P\neg\phi)$  (2,  $R_E^C$ )
4.  $\neg P\phi \rightarrow C\neg P\phi$  (Axiom  $CP_3$ )
5.  $P\neg\phi \rightarrow CP\neg\phi$  (Axiom  $CP_4$ )
6.  $\neg B\phi \rightarrow (C\neg P\phi \vee CP\neg\phi)$  (2, 4, 5, PL)
7.  $\neg P\phi \rightarrow (\neg P\phi \vee P\neg\phi)$  (tautology)
8.  $C\neg P\phi \rightarrow C(\neg P\phi \vee P\neg\phi)$  (7,  $R_p^C$ )
9.  $P\neg\phi \rightarrow (\neg P\phi \vee P\neg\phi)$  (tautology)
10.  $CP\neg\phi \rightarrow C(\neg P\phi \vee P\neg\phi)$  (9,  $R_p^C$ )
11.  $(C\neg P\phi \vee CP\neg\phi) \rightarrow C(\neg P\phi \vee P\neg\phi)$  (8, 10, PL)
12.  $\neg B\phi \rightarrow C\neg B\phi$  (6, 11, 3, PL)

□

The next proposition says that if we extend logic  $\mathbb{L}$  by adding negative introspection of certainty (Axiom  $5_C$ ) as well as Axioms  $(CP_1)$  and  $(CP_2)$  then we get as theorem that if the agent believes that she is certain that  $\phi$  then she believes that  $\phi$ .

**Proposition 8.**

$$\mathbb{L} + \left\{ \begin{array}{l} (5_C) \quad \neg C\phi \rightarrow C\neg C\phi \\ (CP_1) \quad C\phi \rightarrow P\phi \\ (CP_2) \quad C\phi \rightarrow \neg P\neg\phi \end{array} \right\} \vdash \boxed{BC\phi \rightarrow B\phi}$$

*Proof.*

1.  $C\phi \rightarrow B\phi$  (Proposition 5)
2.  $\neg B\phi \rightarrow \neg C\phi$  (1, PL)
3.  $\neg C\phi \rightarrow C\neg C\phi$  (Axiom 5<sub>C</sub>)
4.  $C\neg C\phi \rightarrow B\neg C\phi$  (Proposition 5)
5.  $B\neg C\phi \rightarrow \neg BC\phi$  (Proposition 1)
6.  $\neg B\phi \rightarrow \neg BC\phi$  (2, 3, 4, 5, PL)
7.  $BC\phi \rightarrow B\phi$ . (6, PL)

□

## 5 Theorems on introspection properties of belief

In this section we identify minimal extensions of  $\mathbb{L}$  that yield the following theorems:

- (4<sub>B</sub>)  $B\phi \rightarrow BB\phi$
- (5<sub>B</sub>)  $\neg B\phi \rightarrow B\neg B\phi$
- (4<sub>B</sub><sup>cnv</sup>)  $BB\phi \rightarrow B\phi$
- (5<sub>B</sub><sup>cnv</sup>)  $B\neg B\phi \rightarrow \neg B\phi$ .

(4<sub>B</sub>) is the property of positive introspection of belief, (5<sub>B</sub>) is the property of negative introspection, (4<sub>B</sub><sup>cnv</sup>) is the converse of (4<sub>B</sub>) and (5<sub>B</sub><sup>cnv</sup>) is the converse of (5<sub>B</sub>) (the latter two say that the agent has correct beliefs about what she believes and what she does not believe).

**Proposition 9.**

$$\mathbb{L} + \left\{ \begin{array}{l} (CP_1) \quad C\phi \rightarrow P\phi \\ (CP_2) \quad C\phi \rightarrow \neg P\neg\phi \\ (CP_3) \quad \neg P\phi \rightarrow C\neg P\phi \\ (CP_4) \quad P\phi \rightarrow CP\phi \end{array} \right\} \vdash \boxed{B\phi \rightarrow BB\phi}$$

*Proof.*

1.  $B\phi \rightarrow CB\phi$  (Proposition 6)
2.  $CB\phi \rightarrow BB\phi$  (Proposition 5)
3.  $B\phi \rightarrow BB\phi$ . (1, 2, PL)

□

**Remark 6.** Note that, in order to obtain positive introspection of belief we do not need to postulate positive nor negative introspection of certainty.

**Proposition 10.**

$$\mathbb{L} + \left\{ \begin{array}{l} (CP_1) \quad C\phi \rightarrow P\phi \\ (CP_2) \quad C\phi \rightarrow \neg P\neg\phi \\ (CP_3) \quad \neg P\phi \rightarrow C\neg P\phi \\ (CP_4) \quad P\phi \rightarrow CP\phi \end{array} \right\} \vdash \boxed{\neg B\phi \rightarrow B\neg B\phi}$$

*Proof.*

1.  $\neg B\phi \rightarrow C\neg B\phi$  (Remark 5)
2.  $C\neg B\phi \rightarrow B\neg B\phi$  (Proposition 5)
3.  $\neg B\phi \rightarrow B\neg B\phi$ . (1, 2, PL)

□

**Remark 7.** Note that, in order to obtain negative introspection of belief we need to postulate positive introspection of certainty.

**Proposition 11.**

$$\mathbb{L} + \left\{ \begin{array}{l} (4_C) \quad C\phi \rightarrow CC\phi \\ (CP_3) \quad \neg P\phi \rightarrow C\neg P\phi \\ (CP_4) \quad P\phi \rightarrow CP\phi \end{array} \right\} \vdash \boxed{BB\phi \rightarrow B\phi}$$

*Proof.*

1.  $\neg B\phi \rightarrow C\neg B\phi$  (Proposition 7)
2.  $\neg C\neg B\phi \rightarrow B\phi$  (1, PL)
3.  $BB\phi \leftrightarrow (\neg C\neg B\phi \wedge PB\phi \wedge \neg P\neg B\phi)$  (DefB)
4.  $BB\phi \rightarrow \neg C\neg B\phi$  (3, PL)
5.  $BB\phi \rightarrow B\phi$ . (2, 4, PL)

□



**Proposition 12.**

$$\mathbb{L} + \left\{ \begin{array}{l} (CP_1) \quad C\phi \rightarrow P\phi \\ (CP_3) \quad \neg P\phi \rightarrow C\neg P\phi \\ (CP_4) \quad P\phi \rightarrow CP\phi \end{array} \right\} \vdash \boxed{B\neg B\phi \rightarrow \neg B\phi}$$

*Proof.*

1.  $B\neg B\phi \leftrightarrow (\neg CB\phi \wedge P\neg B\phi \wedge \neg PB\phi)$  (DefB)
2.  $\neg B\neg B\phi \leftrightarrow (CB\phi \vee \neg P\neg B\phi \vee PB\phi)$  (1, PL)
3.  $B\phi \rightarrow CB\phi$  (Proposition 6)
4.  $CB\phi \rightarrow (CB\phi \vee \neg P\neg B\phi \vee PB\phi)$  (tautology)
5.  $B\phi \rightarrow \neg B\neg B\phi$  (3, 4, 2, PL)
6.  $B\neg B\phi \rightarrow \neg B\phi$ . (5, PL)

□

**Remark 8.** In virtue of Remark 4, the schema  $B\neg B\phi \rightarrow \neg B\phi$  is also a theorem of the logic obtained by replacing Axiom (CP<sub>1</sub>) with Axiom (5<sub>C</sub>) in Proposition 12, that is,

$$\mathbb{L} + \left\{ \begin{array}{l} (5_C) \quad \neg C\phi \rightarrow C\neg C\phi \\ (CP_3) \quad \neg P\phi \rightarrow C\neg P\phi \\ (CP_4) \quad P\phi \rightarrow CP\phi \end{array} \right\} \vdash \boxed{B\neg B\phi \rightarrow \neg B\phi}$$

## 6 Semantics for certainty and plausibility

**Definition 6.1.** A frame is a triple  $\langle \Omega, C, \mathcal{P} \rangle$  where

- $\Omega$  is a set of states or possible worlds; the subsets of  $\Omega$  will be called *events* or *propositions*.
- $C \subseteq \Omega \times \Omega$  is a binary relation on  $\Omega$ , representing “certainty”. The interpretation of  $\omega C \omega'$  is that if the true, or actual, state is  $\omega$  then the agent cannot rule out the possibility that the true state is  $\omega'$ . We denote by  $C(\omega) = \{\omega' \in \Omega : \omega C \omega'\}$  the set of states that the agent cannot rule out at state  $\omega$ .

- $\mathcal{P} : \Omega \rightarrow 2^{2^\Omega}$  is a “plausibility” function that associates with every state a collection of events. The interpretation of  $E \in \mathcal{P}(\omega)$  is that at state  $\omega$  the agent “considers event  $E$  plausible”. The function  $\mathcal{P}$  is known in modal logic as a *neighborhood function* (see, for example, [Pacuit \(2017\)](#)).

For example, suppose that  $\Omega = \{\omega_1, \omega_2, \dots, \omega_7\}$ ,  $C(\omega_1) = \{\omega_4, \omega_5, \omega_6, \omega_7\}$  and  $\mathcal{P}(\omega_1) = \{\{\omega_4, \omega_5\}, \Omega\}$ . Then, if the true state is  $\omega_1$ , the agent (erroneously) rules out the possibility that the state is  $\omega_1$  as well as (correctly) the possibility that the true state is either  $\omega_2$  or  $\omega_3$ ; moreover – although she does not rule out the possibility that the true state is either  $\omega_6$  or  $\omega_7$  – she judges these two states as implausible and thus dismisses them as serious possibilities. Furthermore, since  $\{\omega_4, \omega_5\} \in \mathcal{P}(\omega_1)$  and  $(\Omega \setminus \{\omega_4, \omega_5\}) \notin \mathcal{P}(\omega_1)$  then we can say that – at state  $\omega_1$  – the agent *believes* that the true state is either  $\omega_4$  or  $\omega_5$ . This is the semantic counterpart of Definition 2.1 in Section 2 (see the validation rule (Bval) below).

The connection between syntax and semantics is given by the notion of model. Given a frame  $\mathbb{F} = \langle \Omega, C, \mathcal{P} \rangle$ , a *model based on*  $\mathbb{F}$  is obtained by adding a *valuation*  $V : \text{At} \rightarrow 2^\Omega$  which associates with every atomic proposition  $p \in \text{At}$  the set of states at which  $p$  is true. The truth of an arbitrary formula at a state is then defined inductively as follows ( $\omega \models \phi$  denotes that formula  $\phi$  is true at state  $\omega$ ;  $\|\phi\|$  denotes the truth set of  $\phi$ , that is,  $\|\phi\| = \{\omega \in \Omega : \omega \models \phi\}$ ):

- if  $q$  is an atomic proposition,  $\omega \models q$  if and only if  $\omega \in V(q)$
- $\omega \models \neg\phi$  if and only if  $\omega \not\models \phi$
- $\omega \models \phi \vee \psi$  if and only if either  $\omega \models \phi$  or  $\omega \models \psi$  (or both)
- $\omega \models C\phi$  if and only if  $C(\omega) \subseteq \|\phi\|$
- $\omega \models P\phi$  if and only if  $\|\phi\| \in \mathcal{P}(\omega)$ .

It follows that  $\omega \models \neg C\neg\phi$  if and only if  $C(\omega) \cap \|\phi\| \neq \emptyset$  and  $\omega \models \neg P\neg\phi$  if and only if  $(\Omega \setminus \|\phi\|) \notin \mathcal{P}(\omega)$  (since  $\|\neg\phi\| = (\Omega \setminus \|\phi\|)$ ).

With Axiom (DefB) in mind we can add the following validation rule for belief:

$\omega \models B\phi$  if and only if:

1.  $C(\omega) \cap \|\phi\| \neq \emptyset$ ,
  2.  $\|\phi\| \in \mathcal{P}(\omega)$  and
  3.  $(\Omega \setminus \|\phi\|) \notin \mathcal{P}(\omega)$ .
- (Bval)

We say that a formula  $\phi$  is *valid in a model* if  $\omega \models \phi$  for all  $\omega \in \Omega$ , that is, if  $\phi$  is true at every state in that model. A formula  $\phi$  is *valid in a frame* if it is valid in every model based on that frame. Finally, we say that a property of frames is *characterized by* (or *characterizes*) an axiom if (1) the axiom is valid in any frame that satisfies the property and, conversely, (2) if a frame violates the property then there is a model based on that frame such that the axiom is not valid in that model.

Next we point out the semantic properties that characterize the axioms considered in Sections 3.1 and 3.2.

**Remark 9.** *Concerning the certainty operator, it is well known that*

1. *Axiom ( $D_C$ ) (Consistency:  $C\phi \rightarrow \neg C\neg\phi$ ) is characterized by seriality of the relation  $C$ :  $\forall \omega \in \Omega$ ,*

$$C(\omega) \neq \emptyset. \quad (F_{D_C})$$

2. *Axiom ( $A_C$ ) (Positive Introspection:  $C\phi \rightarrow CC\phi$ ) is characterized by transitivity of the relation  $C$ :  $\forall \omega, \omega' \in \Omega$ ,*

$$\text{if } \omega' \in C(\omega) \text{ then } C(\omega') \subseteq C(\omega). \quad (F_{A_C})$$

3. *Axiom ( $5_C$ ) (Negative Introspection:  $\neg C\phi \rightarrow C\neg C\phi$ ) is characterized by euclideaness of the relation  $C$ :  $\forall \omega, \omega' \in \Omega$ ,*

$$\text{if } \omega' \in C(\omega) \text{ then } C(\omega) \subseteq C(\omega'). \quad (F_{5_C})$$

**Proposition 13.** *Concerning the interaction axioms between certainty and plausibility,*

1. *Axiom ( $CP_1$ ) ( $C\phi \rightarrow P\phi$ ) is characterized by the following property:*

$$\forall \omega \in \Omega, \forall E \in 2^\Omega, \quad \text{if } C(\omega) \subseteq E \text{ then } E \in \mathcal{P}(\omega). \quad (F_{CP_1})$$

2. *Axiom ( $CP_2$ ) ( $C\phi \rightarrow \neg P\neg\phi$ ) is characterized by the following property:*

$$\forall \omega \in \Omega, \forall E \in 2^\Omega, \quad \text{if } C(\omega) \subseteq E \text{ then } (\Omega \setminus E) \notin \mathcal{P}(\omega). \quad (F_{CP_2})$$

3. *Axiom ( $CP_3$ ) ( $\neg P\phi \rightarrow C\neg P\phi$ ) is characterized by the following property:*

$$\forall \omega, \omega' \in \Omega, \quad \text{if } \omega' \in C(\omega) \text{ then } \mathcal{P}(\omega') \subseteq \mathcal{P}(\omega). \quad (F_{CP_3})$$

4. Axiom (CP<sub>4</sub>) ( $P\phi \rightarrow CP\phi$ ) is characterized by the following property:  
 $\forall \omega, \omega' \in \Omega,$   
 if  $\omega' \in C(\omega)$  then  $\mathcal{P}(\omega) \subseteq \mathcal{P}(\omega')$ . (F<sub>CP<sub>4</sub></sub>)

*Proof.*

1. Fix an arbitrary model based on a frame that satisfies the property that,  $\forall \omega \in \Omega, \forall E \in 2^\Omega$ , if  $C(\omega) \subseteq E$  then  $E \in \mathcal{P}(\omega)$ . Fix an arbitrary  $\omega \in \Omega$  and an arbitrary formula  $\phi$  and suppose that  $\omega \models C\phi$ , that is,  $C(\omega) \subseteq \|\phi\|$ . Then, by the assumed property,  $\|\phi\| \in \mathcal{P}(\omega)$ , that is,  $\omega \models P\phi$ . Conversely, fix a frame that violates the property, that is, there is an  $\omega \in \Omega$  and an  $E \in 2^\Omega$  such that  $C(\omega) \subseteq E$  and  $E \notin \mathcal{P}(\omega)$ . Let  $q$  be an atomic formula and construct a model where  $V(q) = E$ . Then  $\omega \models Cq$  and, since  $\|q\| \notin \mathcal{P}(\omega)$ ,  $\omega \not\models Pq$ , so that Axiom (CP<sub>1</sub>) is not valid in this model.
2. Fix an arbitrary model based on a frame that satisfies the property that,  $\forall \omega \in \Omega, \forall E \in 2^\Omega$ , if  $C(\omega) \subseteq E$  then  $(\Omega \setminus E) \notin \mathcal{P}(\omega)$ . Fix an arbitrary  $\omega \in \Omega$  and an arbitrary formula  $\phi$  and suppose that  $\omega \models C\phi$ , that is,  $C(\omega) \subseteq \|\phi\|$ . Then, by the assumed property,  $(\Omega \setminus \|\phi\|) \notin \mathcal{P}(\omega)$ , that is,  $\omega \not\models P\neg\phi$  or, equivalently,  $\omega \models \neg P\neg\phi$ . Conversely, fix a frame that violates the property, that is, there is an  $\omega \in \Omega$  and an  $E \in 2^\Omega$  such that  $C(\omega) \subseteq E$  and  $(\Omega \setminus E) \in \mathcal{P}(\omega)$ . Let  $q$  be an atomic formula and construct a model where  $V(q) = E$ . Then  $\omega \models Cq$  and, since  $\|\neg q\| = (\Omega \setminus E) \in \mathcal{P}(\omega)$ ,  $\omega \models P\neg q$  (equivalently,  $\omega \not\models \neg P\neg q$ ), so that Axiom (CP<sub>2</sub>) is not valid in this model.
3. Fix an arbitrary model based on a frame that satisfies the property that,  $\forall \omega, \omega' \in \Omega$ , if  $\omega' \in C(\omega)$  then  $\mathcal{P}(\omega') \subseteq \mathcal{P}(\omega)$ . Fix an arbitrary  $\omega \in \Omega$  and an arbitrary formula  $\phi$  and suppose that  $\omega \models \neg P\phi$ , that is,  $\|\phi\| \notin \mathcal{P}(\omega)$ . Fix an arbitrary  $\omega' \in C(\omega)$ . Then, by the assumed property,  $\|\phi\| \notin \mathcal{P}(\omega')$ , so that  $\omega' \models \neg P\phi$  and thus, since  $\omega' \in C(\omega)$  was chosen arbitrarily,  $\omega \models C\neg P\phi$ . Conversely, fix a frame that violates the property, that is, there exist  $\omega, \omega' \in \Omega$  and  $E \in 2^\Omega$  such that (a)  $\omega' \in C(\omega)$ , (b)  $E \in \mathcal{P}(\omega')$  and (c)  $E \notin \mathcal{P}(\omega)$ . Let  $q$  be an atomic formula and construct a model where  $V(q) = E$ . Then, by (c),  $\omega \models \neg Pq$  while, by (b),  $\omega' \models Pq$  so that, since  $\omega' \in C(\omega)$ ,  $\omega \not\models C\neg Pq$ , thus invalidating axiom (CP<sub>3</sub>).
4. Fix an arbitrary model based on a frame that satisfies the property that,  $\forall \omega, \omega' \in \Omega$ , if  $\omega' \in C(\omega)$  then  $\mathcal{P}(\omega) \subseteq \mathcal{P}(\omega')$ . Fix an arbitrary  $\omega \in \Omega$  and an arbitrary formula  $\phi$  and suppose that  $\omega \models P\phi$ , that is,  $\|\phi\| \in \mathcal{P}(\omega)$ . Fix an arbitrary  $\omega' \in C(\omega)$ . Then, by the assumed property,  $\|\phi\| \in \mathcal{P}(\omega')$ , so

that  $\omega' \models P\phi$  and thus, since  $\omega' \in C(\omega)$  was chosen arbitrarily,  $\omega \models CP\phi$ . Conversely, fix a frame that violates the property, that is, there exist  $\omega, \omega' \in \Omega$  and  $E \in 2^\Omega$  such that (a)  $\omega' \in C(\omega)$ , (b)  $E \in \mathcal{P}(\omega)$  and (c)  $E \notin \mathcal{P}(\omega')$ . Let  $q$  be an atomic formula and construct a model where  $V(q) = E$ . Then, by (b),  $\omega \models Pq$  while, by (c),  $\omega' \not\models Pq$  so that, since  $\omega' \in C(\omega)$ ,  $\omega \not\models CPq$ , thus invalidating axiom  $(CP_4)$ .

□

A logic  $\mathcal{S}$  is said to be *sound with respect to a class of frames* if every theorem of  $\mathcal{S}$  is valid in every frame of that class. Since Axiom  $(K_C)$  is valid on all the frames considered here and the rules of inference  $(MP)$ ,  $(Nec_C)$  and  $(R_E^P)$  are validity preserving,<sup>11</sup> logic  $\mathbb{L}_0$  is sound with respect to the class of all frames. Adding the validation rule  $(Bval)$  for the belief operator, it follows that logic  $\mathbb{L}$  is sound with respect to the class of all frames. The following proposition then follows from Remark 9 and Proposition 13.

**Proposition 14.** *Let  $\{\phi_1, \dots, \phi_n\}$  ( $n \geq 0$ ) be any collection of axioms from the set  $\{D_C, 4_C, 5_C, CP_1, CP_2, CP_3, CP_4\}$ . Then (under the validation rule  $(Bval)$  for the belief operator  $B$ )  $\mathbb{L} + \{\phi_1, \dots, \phi_n\}$  is sound with respect to the class of frames that satisfy the collection of properties  $\{F_1, \dots, F_n\}$  where, for every  $i = 1, \dots, n$ ,  $F_i$  is the property that characterizes axiom  $\phi_i$  (see Remark 9 and Proposition 13).*

For example, the logic  $\mathbb{L} + \{5_C, CP_3, CP_4\}$  is sound with respect to the class of frames where  $C$  is euclidean and the following property holds:  $\forall \omega, \omega' \in \Omega$ , if  $\omega' \in C(\omega)$  then  $\mathcal{P}(\omega') = \mathcal{P}(\omega)$ .

**Remark 10.** *In virtue of Proposition 14, to show that  $\mathbb{L} + \{\phi_1, \dots, \phi_n\}$  is a minimal extension of  $\mathbb{L}$  that yields axiom  $\phi$  as a theorem, it is sufficient to show that, for every  $i = 1, \dots, n$ ,  $\phi$  is not valid in the class of frames that satisfy the collection of properties  $\{F_1, \dots, F_{i-1}, F_{i+1}, \dots, F_n\}$ .*

For example, let us show that logic  $\mathbb{L} + \{CP_1, CP_2\}$  is a minimal extension of  $\mathbb{L}$  that yields the theorem  $C\phi \rightarrow B\phi$  (see Proposition 5). First of all, by Proposition 14, this logic is sound with respect to the class of frames where properties  $(F_{CP_1})$  and  $(F_{CP_2})$  are satisfied. In both of the following examples we take  $\Omega = \{\alpha, \beta, \gamma\}$ .

---

<sup>11</sup>Proof that  $(R_E^P)$  is validity preserving. Consider an arbitrary model and suppose that  $\phi \leftrightarrow \psi$  is valid, that is,  $\|\phi \leftrightarrow \psi\| = \Omega$ . Since  $\|\phi \leftrightarrow \psi\| = ((\Omega \setminus \|\phi\|) \cup \|\psi\|) \cap (\|\phi\| \cup (\Omega \setminus \|\psi\|))$ , we have that  $((\Omega \setminus \|\phi\|) \cup \|\psi\|) = \Omega$ , that is,  $\|\phi\| \subseteq \|\psi\|$  and  $(\|\phi\| \cup (\Omega \setminus \|\psi\|)) = \Omega$ , that is,  $\|\psi\| \subseteq \|\phi\|$ . Thus  $\|\phi\| = \|\psi\|$ . Fix an arbitrary  $\omega \in \Omega$ . Then  $\omega \models P\phi$  if and only if  $\|\phi\| \in \mathcal{P}(\omega)$  if and only if  $\|\psi\| \in \mathcal{P}(\omega)$  if and only if  $\omega \models P\psi$ , so that  $\omega \models (P\phi \leftrightarrow P\psi)$ . Thus  $\|P\phi \leftrightarrow P\psi\| = \Omega$ .

1. Consider the following frame:  $C(\alpha) = C(\beta) = C(\gamma) = \{\beta, \gamma\}$ ,<sup>12</sup>  $\mathcal{P}(\alpha) = \mathcal{P}(\beta) = \mathcal{P}(\gamma) = \{\{\alpha\}, \{\beta, \gamma\}, \Omega\}$ . This frame satisfies Property  $(F_{CP_1})$  (thus it validates the logic  $\mathbb{L} + \{CP_1\}$ ), but fails Property  $(F_{CP_2})$ , since  $C(\alpha) \subseteq \{\beta, \gamma\}$  and yet  $\Omega \setminus \{\beta, \gamma\} = \{\alpha\} \in \mathcal{P}(\alpha)$ . Let  $p$  be an atomic formula and consider a model where  $V(p) = \{\beta, \gamma\}$ . Then  $\alpha \models Cp$  but  $\alpha \not\models Bp$  since  $\alpha \not\models \neg P\neg p$  (because  $\|\neg p\| = \Omega \setminus \|\!p\| = \{\alpha\} \in \mathcal{P}(\alpha)$ , so that  $\alpha \models P\neg p$ ).
2. Consider the following frame:  $C(\alpha) = C(\beta) = C(\gamma) = \{\beta\}$ ,<sup>13</sup>  $\mathcal{P}(\alpha) = \mathcal{P}(\beta) = \mathcal{P}(\gamma) = \{\{\beta\}, \Omega\}$ . This frame satisfies Property  $(F_{CP_2})$  (thus it validates the logic  $\mathbb{L} + \{CP_2\}$ ), but fails Property  $(F_{CP_1})$ , since  $C(\alpha) = \{\beta\} \subseteq \{\beta, \gamma\}$  and yet  $\{\beta, \gamma\} \notin \mathcal{P}(\alpha)$ . Let  $p$  be an atomic formula and consider a model where  $V(p) = \{\beta, \gamma\}$ . Then  $\alpha \models Cp$  but  $\alpha \not\models Bp$  since  $\alpha \not\models Pp$  (because  $\|\!p\| = \{\beta, \gamma\} \notin \mathcal{P}(\alpha)$ ).

Since, as the above example shows, proving that an extension of logic  $\mathbb{L}$  is a minimal extension that yields a particular theorem is rather laborious, the proof of the following proposition is relegated to the Appendix.

**Proposition 15.** *The extensions of logic  $\mathbb{L}$  considered in Propositions 5-12 are minimal extensions for the corresponding theorems.*

In the next section we consider the circumstances under which belief satisfies conjunction properties.

## 7 Conjunction properties of belief

In this section we consider the following conjunction properties of belief:

$$(M_B) \quad B(\phi \wedge \psi) \rightarrow (B\phi \wedge B\psi)$$

$$(C_B^{weak}) \quad (B\phi \wedge B\psi \wedge \neg C\neg(\phi \wedge \psi)) \rightarrow B(\phi \wedge \psi)$$

As the following example shows, none of the extensions of logic  $\mathbb{L}$  considered above yields  $(M_B)$  as a theorem. The frame considered in the following example

<sup>12</sup>Note that  $C$  is transitive and euclidean.

<sup>13</sup>Note that  $C$  is transitive and euclidean.

validates the largest extension of  $\mathbb{L}$  considered so far, namely the logic

$$\mathbb{L} + \left\{ \begin{array}{l} \text{(D}_C\text{)} \quad C\phi \rightarrow \neg C\neg\phi \\ \text{(4}_C\text{)} \quad C\phi \rightarrow CC\phi \\ \text{(5}_C\text{)} \quad \neg C\phi \rightarrow C\neg C\phi \\ \text{(CP}_1\text{)} \quad C\phi \rightarrow P\phi \\ \text{(CP}_2\text{)} \quad C\phi \rightarrow \neg P\neg\phi \\ \text{(CP}_3\text{)} \quad \neg P\phi \rightarrow C\neg P\phi \\ \text{(CP}_4\text{)} \quad P\phi \rightarrow CP\phi \end{array} \right.$$

**Example 1** (violation of  $M_B$ ). Consider the following frame:  $\Omega = \{\alpha, \beta, \gamma\}$ ,  $C(\alpha) = C(\beta) = C(\gamma) = \Omega$ ,  $\mathcal{P}(\alpha) = \mathcal{P}(\beta) = \mathcal{P}(\gamma) = \{\{\beta\}, \Omega\}$ .<sup>14</sup> Let  $p$  and  $q$  be atomic propositions and construct a model where  $V(p) = \{\alpha, \beta\}$  and  $V(q) = \{\beta, \gamma\}$ . Then  $\|p \wedge q\| = \{\beta\}$  and  $\beta \models B(p \wedge q)$  (since  $C(\beta) \cap \|p \wedge q\| = \{\beta\} \neq \emptyset$ ,  $\|p \wedge q\| \in \mathcal{P}(\beta)$  and  $(\Omega \setminus \|p \wedge q\|) = \{\alpha, \gamma\} \notin \mathcal{P}(\beta)$ ); however  $\beta \not\models Bp$  because  $\beta \not\models Pp$  (since  $\|p\| = \{\alpha, \beta\} \notin \mathcal{P}(\beta)$ ); we also have that, for the same reason,  $\beta \not\models Bq$ .

The reason why the frame considered in Example 1 fails to validate  $(M_B)$  is that, although event  $\{\beta\}$  is considered plausible, no superset of it (with the exception of  $\Omega$ ) is considered plausible. Such a possibility might be viewed as incompatible with an intuitive notion of plausibility. Thus one might want to impose the following restriction on frames ('MON' stands for 'Monotonicity'):

$$\forall \omega \in \Omega, \forall E, F \in 2^\Omega, \text{ if } E \subseteq F \text{ and } E \in \mathcal{P}(\omega) \text{ then } F \in \mathcal{P}(\omega). \quad (\text{MON})$$

Since  $E \subseteq F$  is equivalent to  $(\Omega \setminus E) \cup F = \Omega$ , the frames that satisfy (MON) are such that the following rule of inference is validity preserving:<sup>15</sup>

$$\frac{\phi \rightarrow \psi}{P\phi \rightarrow P\psi}. \quad (\text{R}_K^P)$$

Thus we have the following result.

<sup>14</sup> $C$  is serial, transitive and euclidean and Properties  $F_{CP_1}$ ,  $F_{CP_2}$ ,  $F_{CP_3}$ , and  $F_{CP_4}$  are all satisfied. Note that Property  $F_{CP_1}$  only requires that  $\Omega \in \mathcal{P}(\omega)$ ,  $\forall \omega \in \Omega$ , and Property  $F_{CP_2}$  only requires that  $\emptyset \notin \mathcal{P}(\omega)$ ,  $\forall \omega \in \Omega$ .

<sup>15</sup>*Proof.* Consider an arbitrary model based on a frame that satisfies (MON). Suppose that  $\phi \rightarrow \psi$  is valid, that is,  $\|\phi \rightarrow \psi\| = \Omega$  or, equivalently,  $\|\neg\phi \vee \psi\| = \|\neg\phi\| \cup \|\psi\| = (\Omega \setminus \|\phi\|) \cup \|\psi\| = \Omega$ . Then  $\|\phi\| \subseteq \|\psi\|$ . Fix an arbitrary  $\omega \in \Omega$  and suppose that  $\omega \models P\phi$ , that is,  $\|\phi\| \in \mathcal{P}(\omega)$ . Then, by (MON),  $\|\psi\| \in \mathcal{P}(\omega)$  and thus  $\omega \models P\psi$ .

**Proposition 16.** Let  $\mathbb{L} + R_K^P$  be the logic obtained by adding the rule of inference ( $R_K^P$ ) to  $\mathbb{L}$ . Then  $\mathbb{L} + R_K^P$  is sound with respect to the class of frames that satisfy (MON).

The next proposition shows that  $(M_B)$  is a theorem of  $\mathbb{L} + R_K^P$ .

**Proposition 17.**

$$\mathbb{L} + R_K^P \vdash \boxed{B(\phi \wedge \psi) \rightarrow (B\phi \wedge B\psi)}.$$

*Proof.*

1.  $(\phi \wedge \psi) \rightarrow \phi$  (tautology)
2.  $P(\phi \wedge \psi) \rightarrow P\phi$  (1, Rule  $R_K^P$ )
3.  $\neg\phi \rightarrow \neg(\phi \wedge \psi)$  (tautology)
4.  $P\neg\phi \rightarrow P\neg(\phi \wedge \psi)$  (3, Rule  $R_K^P$ )
5.  $\neg P\neg(\phi \wedge \psi) \rightarrow \neg P\neg\phi$  (4, PL)
6.  $(\phi \wedge \psi) \rightarrow \psi$  (tautology)
7.  $P(\phi \wedge \psi) \rightarrow P\psi$  (6, Rule  $R_K^P$ )
8.  $\neg\psi \rightarrow \neg(\phi \wedge \psi)$  (tautology)
9.  $P\neg\psi \rightarrow P\neg(\phi \wedge \psi)$  (8, Rule  $R_K^P$ )
10.  $\neg P\neg(\phi \wedge \psi) \rightarrow \neg P\neg\psi$  (9, PL)
11.  $\neg C\neg(\phi \wedge \psi) \rightarrow (\neg C\neg\phi \wedge \neg C\neg\psi)$  (Remark 3)
12.  $B(\phi \wedge \psi) \leftrightarrow (\neg C\neg(\phi \wedge \psi) \wedge P(\phi \wedge \psi) \wedge \neg P\neg(\phi \wedge \psi))$  (DefB)
13.  $(\neg C\neg(\phi \wedge \psi) \wedge P(\phi \wedge \psi) \wedge \neg P\neg(\phi \wedge \psi))$   
 $\rightarrow (\neg C\neg\phi \wedge P\phi \wedge \neg P\neg\phi \wedge \neg C\neg\psi \wedge P\psi \wedge \neg P\neg\psi)$  (2, 5, 7, 10, 11, PL)
14.  $(\neg C\neg\phi \wedge P\phi \wedge \neg P\neg\phi \wedge \neg C\neg\psi \wedge P\psi \wedge \neg P\neg\psi)$   
 $\leftrightarrow (B\phi \wedge B\psi)$  (DefB)
15.  $B(\phi \wedge \psi) \rightarrow (B\phi \wedge B\psi)$ . (12, 13, 14, PL)

□



The following example shows that  $(C_B^{weak})$  is not a theorem of  $\mathbb{L} + R_K^P$  (or any extension of it). Indeed, the frame considered in Example 2 validates the logic

$$\mathbb{L} + R_K^P + \left\{ \begin{array}{l} (D_C) \quad C\phi \rightarrow \neg C\neg\phi \\ (4_C) \quad C\phi \rightarrow CC\phi \\ (5_C) \quad \neg C\phi \rightarrow C\neg C\phi \\ (CP_1) \quad C\phi \rightarrow P\phi \\ (CP_2) \quad C\phi \rightarrow \neg P\neg\phi \\ (CP_3) \quad \neg P\phi \rightarrow C\neg P\phi \\ (CP_4) \quad P\phi \rightarrow CP\phi \end{array} \right.$$

**Example 2** (violation of  $C_B^{weak}$ ). Consider the following frame:  $\Omega = \{\alpha, \beta, \gamma\}$ ,  $C(\alpha) = C(\beta) = C(\gamma) = \Omega$ ,  $\mathcal{P}(\alpha) = \mathcal{P}(\beta) = \mathcal{P}(\gamma) = \{\{\alpha, \beta\}, \{\beta, \gamma\}, \Omega\}$ .<sup>16</sup> Let  $p$  and  $q$  be atomic propositions and construct a model where  $V(p) = \{\alpha, \beta\}$  and  $V(q) = \{\beta, \gamma\}$ . Then  $\beta \models Bp \wedge Bq \wedge \neg C\neg(p \wedge q)$  (since (1)  $C(\beta) \cap \|p\| = \{\alpha, \beta\} \neq \emptyset$ ,  $\|p\| \in \mathcal{P}(\beta)$ ,  $(\Omega \setminus \|p\|) = \{\gamma\} \notin \mathcal{P}(\beta)$ ), (2)  $C(\beta) \cap \|q\| = \{\beta, \gamma\} \neq \emptyset$ ,  $\|q\| \in \mathcal{P}(\beta)$ ,  $(\Omega \setminus \|q\|) = \{\alpha\} \notin \mathcal{P}(\beta)$  and (3)  $C(\beta) \cap \|p \wedge q\| = \{\beta\} \neq \emptyset$ ); however  $\beta \not\models P(p \wedge q)$  since  $\|p \wedge q\| = \{\beta\} \notin \mathcal{P}(\beta)$ .

In order to obtain  $(C_B^{weak})$  as a theorem we need to add the following two axioms:

$$\neg P\phi \rightarrow P\neg\phi \quad (P_1)$$

$$(\neg P\phi \wedge \neg P\psi) \rightarrow \neg P(\phi \vee \psi) \quad (P_2)$$

Axiom (P<sub>1</sub>) says that if  $\phi$  is not plausible then its negation must be plausible and Axiom (P<sub>2</sub>) says that if neither  $\phi$  nor  $\psi$  are plausible then their disjunction is not plausible either.

The proof of the following proposition is straightforward and is omitted.

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<sup>16</sup> $C$  is serial, transitive and euclidean and Properties  $F_{CP_1}$ ,  $F_{CP_2}$ ,  $F_{CP_3}$ , and  $F_{CP_4}$  are all satisfied.

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**Proposition 18.**

1. Axiom (P<sub>1</sub>) ( $\neg P\phi \rightarrow P\neg\phi$ ) is characterized by the following property:  
 $\forall\omega \in \Omega, \forall E \in 2^\Omega,$

$$\text{if } E \notin \mathcal{P}(\omega) \text{ then } (\Omega \setminus E) \in \mathcal{P}(\omega). \quad (F_{P_1})$$

2. Axiom (P<sub>2</sub>) ( $(\neg P\phi \wedge \neg P\psi) \rightarrow \neg P(\phi \vee \psi)$ ) is characterized by the following property:  $\forall\omega \in \Omega, \forall E, F \in 2^\Omega,$

$$\text{if } E \notin \mathcal{P}(\omega) \text{ and } F \notin \mathcal{P}(\omega) \text{ then } (E \cup F) \notin \mathcal{P}(\omega). \quad (F_{P_2})$$

The following result is a consequence of Propositions 14 and 18.

**Proposition 19.** Let  $\{\phi_1, \dots, \phi_n\}$  ( $n \geq 0$ ) be any collection of axioms from the set  $\{D_C, 4_C, 5_C, CP_1, CP_2, CP_3, CP_4, P_1, P_2\}$ . Then

1.  $\mathbb{L} + \{\phi_1, \dots, \phi_n\}$  is sound with respect to the class of frames that satisfy the collection of properties  $\{F_1, \dots, F_n\}$  where, for every  $i = 1, \dots, n$ ,  $F_i$  is the property that characterizes axiom  $\phi_i$ .
2.  $\mathbb{L} + R_K^P + \{\phi_1, \dots, \phi_n\}$  is sound with respect to the class of frames that satisfy Property (MON) as well as the collection of properties  $\{F_1, \dots, F_n\}$  where, for every  $i = 1, \dots, n$ ,  $F_i$  is the property that characterizes axiom  $\phi_i$ .

We can now show that, by adding Axioms (P<sub>1</sub>) and (P<sub>2</sub>) to logic  $\mathbb{L}$ , one obtains ( $C_B^{\text{weak}}$ ) as a theorem. First we need the following lemma.

**Lemma 1.**

$$\mathbb{L}_0 + \left\{ (P_2) \quad (\neg P\phi \wedge \neg P\psi) \rightarrow \neg P(\phi \vee \psi) \right\} \vdash \boxed{(\neg P\neg\phi \wedge \neg P\neg\psi) \rightarrow \neg P\neg(\phi \wedge \psi)}$$

*Proof.*

1.  $(\neg P\neg\phi \wedge \neg P\neg\psi) \rightarrow \neg P(\neg\phi \vee \neg\psi)$  (Axiom P<sub>2</sub>)
2.  $(\neg\phi \vee \neg\psi) \leftrightarrow \neg(\phi \wedge \psi)$  (tautology)
3.  $P(\neg\phi \vee \neg\psi) \leftrightarrow P\neg(\phi \wedge \psi)$  (2, R<sub>E</sub><sup>P</sup>)
4.  $\neg P(\neg\phi \vee \neg\psi) \leftrightarrow \neg P\neg(\phi \wedge \psi)$  (3, PL)
5.  $(\neg P\neg\phi \wedge \neg P\neg\psi) \rightarrow \neg P\neg(\phi \wedge \psi)$ . (1, 4, PL)

□

**Proposition 20.**

$$\mathbb{L}+\left\{ \begin{array}{l} (P_1) \quad \neg P\phi \rightarrow P\neg\phi \\ (P_2) \quad (\neg P\phi \wedge \neg P\psi) \rightarrow \neg P(\phi \vee \psi) \end{array} \right\} \vdash \boxed{B\phi \wedge B\psi \wedge \neg C\neg(\phi \wedge \psi) \rightarrow B(\phi \wedge \psi)}$$

*Proof.* <sup>17</sup>

1.  $B\phi \leftrightarrow (\neg C\neg\phi \wedge P\phi \wedge \neg P\neg\phi)$  (DefB)
2.  $B\psi \leftrightarrow (\neg C\neg\psi \wedge P\psi \wedge \neg P\neg\psi)$  (DefB)
3.  $(B\phi \wedge B\psi) \rightarrow (\neg P\neg\phi \wedge \neg P\neg\psi)$  (1, 2, PL)
4.  $(\neg P\neg\phi \wedge \neg P\neg\psi) \rightarrow \neg P\neg(\phi \wedge \psi)$  (Lemma 1)
5.  $(B\phi \wedge B\psi) \rightarrow \neg P\neg(\phi \wedge \psi)$  (3, 4, PL)
6.  $\neg P\neg(\phi \wedge \psi) \rightarrow P(\phi \wedge \psi)$  (Axiom P<sub>1</sub>)
7.  $(B\phi \wedge B\psi) \rightarrow (\neg P\neg(\phi \wedge \psi) \wedge P(\phi \wedge \psi))$  (5, 6, PL)
8.  $(B\phi \wedge B\psi \wedge \neg C\neg(\phi \wedge \psi)) \rightarrow$   
 $\rightarrow (\neg C\neg(\phi \wedge \psi) \wedge P(\phi \wedge \psi) \wedge \neg P\neg(\phi \wedge \psi))$  (7, PL)
9.  $B(\phi \wedge \psi) \leftrightarrow (\neg C\neg(\phi \wedge \psi) \wedge P(\phi \wedge \psi) \wedge \neg P\neg(\phi \wedge \psi))$  (DefB)
10.  $(B\phi \wedge B\psi \wedge \neg C\neg(\phi \wedge \psi)) \rightarrow B(\phi \wedge \psi)$ . (11, 12, PL)

□

## 8 Special cases

In this section we consider some of the many possible interpretations of plausibility (and certainty).

### 8.1 Plausibility as positive probability

A commonly used semantic structure in game theory (see, for example, [Aumann \(1976; 1999\)](#), [Battigalli and Bonanno \(1999\)](#)) consists of:

<sup>17</sup>Step 6 in the following proof is a shortcut; the full proof of that step is as follows:

1.  $\neg P\neg(\phi \wedge \psi) \rightarrow P\neg\neg(\phi \wedge \psi)$  (Axiom P<sub>1</sub>)
2.  $\neg\neg(\phi \wedge \psi) \leftrightarrow (\phi \wedge \psi)$  (tautology)
3.  $P\neg\neg(\phi \wedge \psi) \leftrightarrow P(\phi \wedge \psi)$  (2, Rule R<sub>E</sub><sup>2</sup>)
4.  $\neg P\neg(\phi \wedge \psi) \rightarrow P(\phi \wedge \psi)$ . (1, 3, PL)

1. An equivalence relation  $\approx$  on a finite<sup>18</sup> set of states  $\Omega$ , representing the agent's possible states of knowledge; the interpretation of  $\omega \approx \omega'$  is that at state  $\omega$  the agent considers  $\omega'$  possible. Letting  $[\omega]$  be the equivalence class that contains state  $\omega$  (that is,  $[\omega] = \{\omega' \in \Omega : \omega \approx \omega'\}$ ), we have that the collection  $\{[\omega] : \omega \in \Omega\}$  is a partition of  $\Omega$ .
2. A probability distribution  $\mu : \Omega \rightarrow [0, 1]$  (thus  $\sum_{\omega \in \Omega} \mu(\omega) = 1$ ) such that,  $\forall \omega \in \Omega$ ,  $Supp(\mu) \cap [\omega] \neq \emptyset$ , where  $Supp(\mu) = \{\omega \in \Omega : \mu(\omega) > 0\}$  denotes the support of  $\mu$ .<sup>19</sup> For every state  $\omega$ , let  $\mu_{[\omega]}$  be the probability distribution obtained by conditioning  $\mu$  on  $[\omega]$ , that is,

$$\mu_{[\omega]}(\omega') = \begin{cases} \frac{\mu(\omega')}{\sum_{x \in [\omega]} \mu(x)} & \text{if } \omega \approx \omega' \\ 0 & \text{otherwise.} \end{cases}$$

The probability distribution  $\mu_{[\omega]}$  is meant to represent the probabilistic assessment of the agent when her state of knowledge is given by the cell  $[\omega]$  of her partition.

Within this framework one can define an event  $E \in 2^\Omega$  to be plausible at state  $\omega$  if it has positive conditional probability, that is, one can define

$$\mathcal{P}(\omega) = \{E \in 2^\Omega : Supp(\mu_{[\omega]}) \cap E \neq \emptyset\}.$$

Then the Monotonicity property (MON) is satisfied and so are the properties that characterize Axioms (P<sub>1</sub>) and (P<sub>2</sub>) (see Proposition 18). Furthermore, in any model based on such a framework, for every formula  $\|\phi\|$  and every state  $\omega$ ,

$$\omega \models (P\phi \wedge \neg P\neg\phi) \quad \text{if and only if} \quad Supp(\mu_{[\omega]}) \subseteq \|\phi\|$$

that is, if  $\|\phi\|$  has conditional probability 1:  $\sum_{\omega' \in \|\phi\|} \mu_{[\omega]}(\omega') = 1$ . It follows from

the validation rule for belief (Bval) that, *in these structures, believing  $\phi$  coincides with assigning conditional probability 1 to  $\phi$* .<sup>20</sup> If one then identifies the certainty

<sup>18</sup>For simplicity we restrict attention to the case where the set of states is finite.

<sup>19</sup>Equivalently, one can postulate, for every state  $\omega$ , a probability distribution  $\mu_\omega$  on  $\Omega$  such that:

(1) if  $\omega \approx \omega'$  then  $\mu_\omega = \mu_{\omega'}$  and (2)  $Supp(\mu_\omega) \subseteq [\omega]$ .

<sup>20</sup>Dodd (2017) argues that [in a probabilistic setting] this ought to be the correct notion of belief:

I argue that believing that  $p$  implies having a credence of 1 in  $p$ . This is true because the belief that  $p$  involves representing  $p$  as being the case, representing  $p$  as being the case involves not allowing for the possibility of not- $p$ , while having a credence that's greater than 0 in not- $p$  involves regarding not- $p$  as a possibility.

relation  $C$  with the equivalence relation  $\approx$  then this class of frames validates all the axioms considered in Sections 3.1 and 3.2, as well as Axioms  $(P_1)$  and  $(P_2)$  and the rule of inference  $(R_K^P)$ . Hence the logic validated by this class of frameworks is the largest extension of  $\mathbb{L}$  considered in this paper, namely<sup>21</sup>

$$\mathbb{L} + R_K^P + \{D_C, 4_C, 5_C, CP_1, CP_2, CP_3, CP_4, P_1, P_2\}.$$

Hence all the schemata considered in Propositions 5-20 are theorems of this logic and thus are valid in this class of frames.

## 8.2 Cautious plausibility

While in the case considered in the previous section it is possible for the agent to state “I believe, but I am not certain, that  $\phi$ ” we now consider an interpretation of plausibility where such statements would not be valid. Consider the class of frames where,  $\forall \omega \in \Omega$ ,

$$\mathcal{P}(\omega) = \{E \in 2^\Omega : C(\omega) \subseteq E\}.$$

This class of frames validates Axioms  $(CP_1)$  and  $(CP_2)$  as well as the rule of inference  $(R_K^P)$ . It also validates the following axiom:

$$P\phi \rightarrow C\phi. \tag{P_3}$$

Thus the corresponding logic will be

$$\mathbb{L} + R_K^P + \{CP_1, CP_2, P_3\} + S$$

where  $S$  is the (possibly empty) collection of axioms from the set  $\{D_C, 4_C, 5_C\}$  that are postulated for certainty. Since  $\mathbb{L} + \{CP_1, CP_2, P_3\} \vdash B\phi \leftrightarrow C\phi$ , in this class of frames one obtains the *reduction of belief to certainty*.

## 8.3 Plausibility based on a preference order

A common approach in computer science and philosophy is to postulate a preference order  $\succeq$  on the set of worlds, capturing the perceived relative likelihood of worlds:  $\omega \succeq \omega'$  is interpreted as “state  $\omega$  is at least as likely as state

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<sup>21</sup>Because of reflexivity of the equivalence relation, also the Truth Axiom  $(C\phi \rightarrow \phi)$  holds, so that certainty can be interpreted as knowledge. However, the Truth Axiom (reflexivity of  $C$ ) played no role in any of our results and thus one could replace the equivalence relation with a serial, transitive and euclidean relation.

$\omega'$ ." Various authors (see, for example, [Boutilier \(1992\)](#), [Goldszmidt and Pearl \(1992\)](#), [Katsuno and Mendelzon \(1991\)](#), [Spohn \(1988\)](#)) have then interpreted "the agent believes  $\phi$ " as " $\phi$  is true in the most preferred worlds among those that the agent considers possible". To obtain this interpretation of belief within our approach, let  $Best_{\succeq}C(\omega)$  be the set of most preferred worlds among the ones that are possible according to  $C$  at state  $\omega$ :

$$Best_{\succeq}C(\omega) = \{\omega' \in C(\omega) : \omega' \succeq x, \forall x \in C(\omega)\}.$$

Define  $\mathcal{P}$  as follows:  $\forall \omega \in \Omega, \forall E \in 2^\Omega$ ,

$$E \in \mathcal{P}(\omega) \text{ if and only if } E \cap Best_{\succeq}C(\omega) \neq \emptyset$$

that is,  $E$  is judged to be plausible at state  $\omega$  if it contains at least one of the most plausible states within the set  $C(\omega)$ . Then this class of frames validates Axioms  $(CP_1)$ ,  $(CP_2)$ ,  $(P_1)$  and  $(P_2)$  as well as the rule of inference  $(R_K^P)$ . Thus the corresponding logic will be

$$\mathbb{L} + R_K^P + \{CP_1, CP_2, P_1, P_2\} + S$$

where  $S$  is the (possibly empty) collection of axioms from the set  $\{D_C, 4_C, 5_C\}$  that are postulated for certainty. *If one postulates consistency of certainty (Axiom  $D_C$ ) then one obtains the interpretation of belief as "true at the most preferred among the possible worlds", that is, in every model based on such a framework,  $\forall \omega \in \Omega$  and for every formula  $\phi$ ,*

$$\omega \models B\phi \text{ if and only if } Best_{\succeq}C(\omega) \subseteq \|\phi\|.$$

#### 8.4 Plausibility as truth at 100s% of the accessible worlds

Another interpretation of " $\phi$  is plausible" is in terms of the fraction of the accessible worlds at which  $\phi$  is true. Let  $s$  be a positive number between 0 and 1:  $s \in (0, 1]$ . Then " $\phi$  is plausible" is interpreted as " $\phi$  is true at, at least, the fraction  $s$  of accessible worlds", that is, assuming that  $\Omega$  is finite and letting  $\#F$  denote the number of elements in event  $F \subseteq \Omega$ ,

$$\mathcal{P}(\omega) = \left\{ E \in 2^\Omega : \frac{\#(E \cap C(\omega))}{\#C(\omega)} \geq s \right\}.$$

We can distinguish three cases.

1.  $s \in (0, \frac{1}{2}]$ . In this case we have that at state  $\omega$  the agent believes  $\phi$  if and only if  $\phi$  is true at more than  $100(1 - s)\%$  of the accessible states, that is,

$$\omega \models B\phi \text{ if and only if } \frac{\#\{(\|\phi\| \cap C(\omega))\}}{\#C(\omega)} > 1 - s.$$

A special case of this is  $s = \frac{1}{2}$ , in which case  $\phi$  is believed if it is true at the majority of accessible worlds.

The logic validated by this class of frames is

$$\mathbb{L} + R_K^P + \{CP_1, CP_2, P_1\} + S$$

where  $S$  is the (possibly empty) collection of axioms from the set  $\{D_C, 4_C, 5_C\}$  that are postulated for certainty. Note that Axiom  $(P_2)$  is not valid in this frames and neither is  $(P_3)$ .

2.  $s \in (\frac{1}{2}, 1)$ . In this case we have that the agent believes  $\phi$  if and only if the agent consider  $\phi$  plausible, that is, we have the *reduction of belief to plausibility*:  $\omega \models B\phi$  if and only  $\omega \models P\phi$ . As in this previous case, the logic validated by this class of frames is

$$\mathbb{L} + R_K^P + \{CP_1, CP_2, P_1\} + S$$

where  $S$  is the (possibly empty) collection of axioms from the set  $\{D_C, 4_C, 5_C\}$  that are postulated for certainty.

3.  $s = 1$ . In this case Axiom  $(P_3)$  is valid and thus the logic validated by this class of frames is

$$\mathbb{L} + R_K^P + \{CP_1, CP_2, P_1, P_3\} + S$$

(where  $S$  is the, possibly empty, collection of axioms from the set  $\{D_C, 4_C, 5_C\}$  that are postulated for certainty) and we get the *reduction of belief to certainty*:  $\omega \models B\phi$  if and only  $\omega \models C\phi$ .

The case where  $s$  is 1 or close to 1 is an interesting case since it allows one to capture phenomena such as the “lottery paradox”.<sup>22</sup>

<sup>22</sup>See, for example, [https://en.wikipedia.org/wiki/Lottery\\_paradox](https://en.wikipedia.org/wiki/Lottery_paradox)

## 9 Related Literature

As mentioned in the Introduction, there is a long tradition in philosophy and computer science where the agent is modeled as having two epistemic levels, typically called knowledge and belief. Both notions are taken to be primitives and their relationship is explored either from a syntactic or from a semantic point of view. Our approach has not been along these lines, since we do not take belief as primitive but derive it from the two notions of certainty and plausibility. While certainty is modeled in the standard way, by means of a normal syntactic operator (or, semantically, as a binary Kripke relation), plausibility is treated as a more general concept: syntactically it is only required to be a classical operator and semantically it is modeled by means of a neighborhood function.<sup>23</sup>

Our approach is closer to the second strand in the literature mentioned in the Introduction, where belief is not a primitive but a derived notion: it is derived from knowledge and/or plausibility. In some papers the agent is said to believe  $\phi$  if he/she knows  $\phi$  to be true in the most plausible worlds.<sup>24</sup> We showed in Section 8.3 that belief as truth in the most plausible worlds is a special case of our framework.<sup>25</sup>

Perhaps the closest paper to ours is [Friedman and Halpern \(1997\)](#). The authors start with two primitive notions: knowledge and plausibility. Knowledge is modeled semantically by a reflexive, transitive and euclidean Kripke relation (and syntactically by an S5 operator), while plausibility is defined as a *plausibility measure* which allows one to compare any two events in terms of their relative plausibility; thus their framework has more structure than ours since it allows one to make assertions of the form “even  $E$  is more plausible than event  $F$ ”, denoted by  $Pl(E) \geq Pl(F)$ . Syntactically, the plausibility measure is represented by a binary modal operator  $\hookrightarrow$  and the interpretation of  $\phi \hookrightarrow \psi$  is “according to the agent,  $\phi$  typically implies  $\psi$ ”; the validation rule is as follows:

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<sup>23</sup>Along somewhat similar lines, [Balbiani et al. \(2018\)](#) investigate a logic that distinguishes the concept of “explicit belief” from the concept of “background knowledge”. They use a relational semantics for background knowledge and a neighbourhood semantics for explicit beliefs and discuss axioms that express the relationship between the two concepts. The two concepts of background knowledge and explicit belief are taken to be primitive notions.

<sup>24</sup>For example, [Moses and Shoham \(1993\)](#). However, in that paper plausibility is not defined by an ordering but in terms of a formula, which can be thought of identifying the most plausible worlds.

<sup>25</sup>In other papers (e.g. [Lamarre and Shoham \(1994\)](#)) plausibility is the only primitive and is used to define both knowledge and belief. Other authors (e.g. [Boutilier \(1992\)](#), [Goldszmidt and Pearl \(1992\)](#)) postulate a preference ordering over possible worlds to characterize formulas of the form “after learning  $\psi$ , the agent believes  $\phi$ ”; this approach is linked to the sizeable literature on the AGM theory of belief revision (for a survey see [Fermé and Hansson \(2011; 2018\)](#)).

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$\omega \models \phi \leftrightarrow \psi$  if either  $Pl(\|\phi\|) = \perp$  or  $Pl(\|\phi \wedge \psi\|) > Pl(\|\phi \wedge \neg\psi\|)$ . The authors then define belief as follows: the agent believes  $\phi$  if and only if he knows that  $\phi$  is more plausible than  $\neg\phi$ :  $B\phi \leftrightarrow K(true \leftrightarrow \phi)$ . As we did in Sections 4-7, the authors then consider a number of properties defining the interaction between knowledge and plausibility and study how these properties are translated into properties of belief.<sup>26</sup> It is clear that our framework is “lighter” than theirs, since we model plausibility as a unary modal operator, which is not even required to be a normal operator (that is, it does not correspond to a binary Kripke relation) and we do not impose the S5 logic on the certainty operator. The less structure one imposes, the clearer it is to grasp what is really necessary in order to obtain “desirable” properties of belief (such as positive and negative introspection).<sup>27</sup>

## 10 Conclusion

From the two primitive notions of certainty and plausibility we obtained belief as a derived notion, interpreted as “maximally plausible possibility”: the agent believes  $\phi$  if (1) she cannot rule out  $\phi$  (that is, it is not the case that she is certain that  $\neg\phi$ ), (2) she judges  $\phi$  to be plausible and (3) she does not judge  $\neg\phi$  to be plausible. We then considered interaction properties between certainty and plausibility and studied how these properties translate into properties of belief (positive and negative introspection, their converses, conjunction, etc.). The purpose was to identify minimal logics that would yield the desired properties of belief. In order to do so we started with a basic logic where certainty is modeled as a normal operator and plausibility as a weaker operator (a classical operator) and then added as few axioms as possible (concerning certainty and the interaction between certainty and plausibility) to obtain various properties of beliefs. The analysis was carried out syntactically, but in Section 6 we introduced the semantics in order to prove (in the Appendix) the minimality of the various logics considered. The semantics for certainty was specified in terms of a standard binary Kripke relation, while plausibility was represented by a neighborhood function. In Section 8 we considered a number of possible interpretations of plausibility (thereby establishing a link to the existing literature) and identified the minimal logic associated with each interpretation. In future

<sup>26</sup>Note that the analysis in [Friedman and Halpern \(1997\)](#) goes beyond this, because they also introduce time and use the extended framework to incorporate belief revision and belief update.

<sup>27</sup>Indeed we showed that no interaction properties (between certainty and plausibility) are needed at all to obtain consistency of beliefs.

work we plan to apply (a multi-agent version of) the framework introduced in this paper to a qualitative analysis of game theory.

## A Proof of Proposition 15

The proof for Proposition 5 was given in Section 6. Thus we only need to give a proof for Propositions 6-12.

**Proof of minimality for Proposition 6.** We want to prove that the schema

$$B\phi \rightarrow CB\phi \text{ is not a theorem of any sub-logic of } \mathbb{L} + \left\{ \begin{array}{l} (\text{CP}_1) \quad C\phi \rightarrow P\phi \\ (\text{CP}_3) \quad \neg P\phi \rightarrow C\neg P\phi \\ (\text{CP}_4) \quad P\phi \rightarrow CP\phi \end{array} \right\}.$$

$\setminus \text{CP}_1$  Let  $\Omega = \{\alpha, \beta\}, C(\alpha) = \{\alpha, \beta\}, C(\beta) = \{\beta\}, \mathcal{P}(\alpha) = \mathcal{P}(\beta) = \{\{\alpha\}, \Omega\}$ . This frame satisfies Properties  $(F_{\text{CP}_3})$  and  $(F_{\text{CP}_4})$  (and thus validates Axioms  $(\text{CP}_3)$  and  $(\text{CP}_4)$ ) but violates Property  $(F_{\text{CP}_1})$  since  $C(\beta) \subseteq \{\beta\}$  but  $\{\beta\} \notin \mathcal{P}(\beta)$ .<sup>28</sup> Let  $p$  be an atomic formula and construct a model based on this frame where  $\|p\| = \{\alpha\}$ . Then  $\|Bp\| = \{\alpha\}$  ( $\beta \not\models Bp$  since  $C(\beta) \cap \|p\| = \emptyset$ ) and thus  $\alpha \models Bp$  but  $\alpha \not\models CBp$ .

$\setminus \text{CP}_3$  Let  $\Omega = \{\alpha, \beta\}, C(\alpha) = C(\beta) = \{\beta\}, \mathcal{P}(\alpha) = \{\{\beta\}, \Omega\}, \mathcal{P}(\beta) = \{\{\alpha\}, \{\beta\}, \Omega\}$ . This frame satisfies Properties  $(F_{\text{CP}_1})$  and  $(F_{\text{CP}_4})$  (and thus validates Axioms  $(\text{CP}_1)$  and  $(\text{CP}_4)$ ) but violates Property  $(F_{\text{CP}_3})$  since  $\beta \in C(\alpha)$  but  $\mathcal{P}(\beta) \not\subseteq \mathcal{P}(\alpha)$ . Let  $p$  be an atomic formula and construct a model based on this frame where  $\|p\| = \{\beta\}$ . Then  $\|Bp\| = \{\alpha\}$  ( $\beta \not\models Bp$  since  $\Omega \setminus \|p\| = \{\alpha\} \in \mathcal{P}(\beta)$ ) and thus  $\beta \not\models \neg P\neg p$  so that  $\alpha \models Bp$  but  $\alpha \not\models CBp$ .

$\setminus \text{CP}_4$  Let  $\Omega = \{\alpha, \beta, \gamma\}, C(\alpha) = C(\beta) = C(\gamma) = \{\beta, \gamma\}, \mathcal{P}(\alpha) = \{\{\beta\}, \{\beta, \gamma\}, \Omega\}, \mathcal{P}(\beta) = \mathcal{P}(\gamma) = \{\{\beta, \gamma\}, \Omega\}$ . This frame satisfies Properties  $(F_{\text{CP}_1})$  and  $(F_{\text{CP}_3})$  (and thus validates Axioms  $(\text{CP}_1)$  and  $(\text{CP}_3)$ ) but violates Property  $(F_{\text{CP}_4})$  since  $\beta \in C(\alpha)$  but  $\mathcal{P}(\alpha) \not\subseteq \mathcal{P}(\beta)$ . Let  $p$  be an atomic formula and construct a model based on this frame where  $\|p\| = \{\beta\}$ . Then  $\|Bp\| = \{\alpha\}$  ( $\beta \not\models Bp$  since  $\|p\| \notin \mathcal{P}(\beta)$ ) and thus  $\beta \not\models Pp$  and the same is true of  $\gamma$ ) so that  $\alpha \models Bp$  but  $\alpha \not\models CBp$ .  $\square$

<sup>28</sup>Note that  $C$  is transitive but not euclidean. Indeed, as implied by Remark 4, it cannot be euclidean. Note also that the proof given for Proposition 6 is also a proof of minimality for the logic of Remark 4 since in the remaining two cases  $C$  is in fact euclidean.

**Proof of minimality for Proposition 7.** We want to prove that the schema  $\neg B\phi \rightarrow C\neg B\phi$  is not a theorem of any sub-logic of  $\mathbb{L} + \left\{ \begin{array}{l} (4_C) \quad C\phi \rightarrow CC\phi \\ (CP_3) \quad \neg P\phi \rightarrow C\neg P\phi \\ (CP_4) \quad P\phi \rightarrow CP\phi \end{array} \right\}$ .

$\setminus 4_C$  Let  $\Omega = \{\alpha, \beta, \gamma\}, C(\alpha) = \{\beta\}, C(\beta) = C(\gamma) = \{\beta, \gamma\}, \mathcal{P}(\alpha) = \mathcal{P}(\beta) = \mathcal{P}(\gamma) = \{\{\gamma\}, \Omega\}$ . This frame satisfies Properties  $(F_{CP_3})$  and  $(F_{CP_4})$  (and thus validates Axioms  $(CP_3)$  and  $(CP_4)$ ) but  $C$  is not transitive (although it is euclidean). Let  $p$  be an atomic formula and construct a model based on this frame where  $\|p\| = \{\gamma\}$ . Then  $\|Bp\| = \{\beta, \gamma\}$  ( $\alpha \not\models Bp$  because  $C(\alpha) \cap \|p\| = \emptyset$ ).<sup>29</sup> Thus  $\alpha \models \neg Bp$  but  $\alpha \not\models C\neg Bp$ .

$\setminus CP_3$  Let  $\Omega = \{\alpha, \beta\}, C(\alpha) = C(\beta) = \{\beta\}, \mathcal{P}(\alpha) = \{\Omega\}, \mathcal{P}(\beta) = \{\{\beta\}, \Omega\}$ . This frame satisfies transitivity of  $C$  and Property  $(F_{CP_4})$  (and thus validates Axioms  $(4_C)$  and  $(CP_4)$ ) but violates Property  $(F_{CP_3})$  since  $\beta \in C(\alpha)$  but  $\mathcal{P}(\beta) \not\subseteq \mathcal{P}(\alpha)$ . Let  $p$  be an atomic formula and construct a model based on this frame where  $\|p\| = \{\beta\}$ . Then  $\|Bp\| = \{\beta\}$  ( $\alpha \not\models Bp$  since  $\|p\| \notin \mathcal{P}(\alpha)$ ) and thus  $\alpha \models \neg Bp$  but  $\alpha \not\models C\neg Bp$ .

$\setminus CP_4$  Let  $\Omega = \{\alpha, \beta\}, C(\alpha) = C(\beta) = \{\beta\}, \mathcal{P}(\alpha) = \{\{\alpha\}, \{\beta\}, \Omega\}, \mathcal{P}(\beta) = \{\{\beta\}, \Omega\}$ . This frame satisfies transitivity of  $C$  and Property  $(F_{CP_3})$  (and thus validates Axioms  $(4_C)$  and  $(CP_3)$ ) but violates Property  $(F_{CP_4})$  since  $\beta \in C(\alpha)$  but  $\mathcal{P}(\alpha) \not\subseteq \mathcal{P}(\beta)$ . Let  $p$  be an atomic formula and construct a model based on this frame where  $\|p\| = \{\beta\}$ . Then  $\|Bp\| = \{\beta\}$  ( $\alpha \not\models Bp$  since  $\Omega \setminus \|p\| \in \mathcal{P}(\alpha)$  so that  $\alpha \not\models \neg P\neg p$ ). Thus  $\alpha \models \neg Bp$  but  $\alpha \not\models C\neg Bp$ .  $\square$

**Proof of minimality for Proposition 8.** We want to prove that the schema  $BC\phi \rightarrow B\phi$  is not a theorem of any sub-logic of  $\mathbb{L} + \left\{ \begin{array}{l} (5_C) \quad \neg C\phi \rightarrow C\neg C\phi \\ (CP_1) \quad C\phi \rightarrow P\phi \\ (CP_2) \quad C\phi \rightarrow \neg P\neg\phi \end{array} \right\}$ .

$\setminus 5_C$  Let  $\Omega = \{\alpha, \beta, \gamma\}, C(\alpha) = \Omega, C(\beta) = C(\gamma) = \{\gamma\}, \mathcal{P}(\alpha) = \{\{\beta, \gamma\}, \Omega\}, \mathcal{P}(\beta) = \mathcal{P}(\gamma) = \{\{\gamma\}, \{\alpha, \gamma\}, \{\beta, \gamma\}, \Omega\}$ . This frame satisfies Properties  $(F_{CP_1})$  and  $(F_{CP_2})$  (and thus validates Axioms  $(CP_1)$  and  $(CP_2)$ ) but  $C$  is not euclidean (although it is transitive). Let  $p$  be an atomic formula and construct a model based on this frame where  $\|p\| = \{\gamma\}$ . Then  $\|Cp\| = \{\beta, \gamma\}$  and thus  $\alpha \models BCp$  but  $\alpha \not\models Bp$  because  $\|p\| \notin \mathcal{P}(\alpha)$ .

<sup>29</sup>We have that  $\beta \models Bp$  because  $C(\beta) \cap \|p\| = \{\gamma\} \neq \emptyset$ ,  $\|p\| \in \mathcal{P}(\beta)$  and  $\Omega \setminus \|p\| = \{\alpha, \beta\} \notin \mathcal{P}(\beta)$ . Similarly for  $\gamma$ .

- \CP<sub>1</sub> Let  $\Omega = \{\alpha, \beta\}, C(\alpha) = C(\beta) = \{\beta\}, \mathcal{P}(\alpha) = \{\Omega\}, \mathcal{P}(\beta) = \{\{\beta\}, \Omega\}$ . This frame satisfies euclideaness of  $C$  and Property (F<sub>CP<sub>2</sub></sub>) (and thus validates Axioms (5<sub>C</sub>) and (CP<sub>2</sub>)) but violates Property (F<sub>CP<sub>1</sub></sub>) since  $C(\alpha) \subseteq \{\beta\}$  but  $\{\beta\} \notin \mathcal{P}(\alpha)$ . Let  $p$  be an atomic formula and construct a model based on this frame where  $\|p\| = \{\beta\}$ . Then  $\|Cp\| = \Omega$  and thus  $\alpha \models BCp$  but  $\alpha \not\models Bp$  since  $\|p\| \notin \mathcal{P}(\alpha)$ .
- \CP<sub>2</sub> Let  $\Omega = \{\alpha, \beta, \gamma\}, C(\alpha) = C(\beta) = C(\gamma) = \{\beta, \gamma\}, \mathcal{P}(\alpha) = \{\{\alpha\}, \{\beta, \gamma\}, \Omega\}, \mathcal{P}(\beta) = \mathcal{P}(\gamma) = \{\{\beta, \gamma\}, \Omega\}$ . This frame satisfies euclideaness of  $C$  and Property (F<sub>CP<sub>1</sub></sub>) (and thus validates Axioms (5<sub>C</sub>) and (CP<sub>1</sub>)) but violates Property (F<sub>CP<sub>2</sub></sub>) since  $C(\alpha) \subseteq \{\beta, \gamma\}$  but  $\Omega \setminus \{\beta, \gamma\} = \{\alpha\} \in \mathcal{P}(\alpha)$ . Let  $p$  be an atomic formula and construct a model based on this frame where  $\|p\| = \{\beta, \gamma\}$ . Then  $\|Cp\| = \Omega$  and thus  $\alpha \models BCp$  but  $\alpha \not\models Bp$  since  $\alpha \not\models \neg P\neg p$  (because  $\|\neg p\| = \{\alpha\} \in \mathcal{P}(\alpha)$ ).  $\square$

**Proof of minimality for Proposition 9.** We want to prove that the schema

$$B\phi \rightarrow BB\phi \text{ is not a theorem of any sub-logic of } \mathbb{L} + \left\{ \begin{array}{l} (\text{CP}_1) \quad C\phi \rightarrow P\phi \\ (\text{CP}_2) \quad C\phi \rightarrow \neg P\neg\phi \\ (\text{CP}_3) \quad \neg P\phi \rightarrow C\neg P\phi \\ (\text{CP}_4) \quad P\phi \rightarrow CP\phi \end{array} \right\}.$$

- \CP<sub>1</sub> Let  $\Omega = \{\alpha, \beta, \gamma\}, C(\alpha) = C(\beta) = \{\beta\}, C(\gamma) = \{\gamma\}, \mathcal{P}(\alpha) = \mathcal{P}(\beta) = \{\{\beta\}, \Omega\}, \mathcal{P}(\gamma) = \{\{\gamma\}, \{\alpha, \gamma\}, \{\beta, \gamma\}, \Omega\}$ . This frame satisfies Properties (F<sub>CP<sub>2</sub></sub>), (F<sub>CP<sub>3</sub></sub>) and (F<sub>CP<sub>4</sub></sub>) (and thus validates Axioms (CP<sub>2</sub>), (CP<sub>3</sub>) and (CP<sub>4</sub>); note also that  $C$  is euclidean and thus validates Axiom (5<sub>C</sub>)) but violates Property (F<sub>CP<sub>1</sub></sub>) since  $C(\alpha) \subseteq \{\beta, \gamma\}$  but  $\{\beta, \gamma\} \notin \mathcal{P}(\alpha)$ . Let  $p$  be an atomic formula and construct a model based on this frame where  $\|p\| = \{\beta\}$ . Then  $\|Bp\| = \{\alpha, \beta\}$  so that  $\alpha \models Bp$  but  $\alpha \not\models BBp$  since  $\|Bp\| \notin \mathcal{P}(\alpha)$ .
- \CP<sub>2</sub> Let  $\Omega = \{\alpha, \beta, \gamma\}, C(\alpha) = C(\beta) = \{\beta\}, C(\gamma) = \{\gamma\}, \mathcal{P}(\alpha) = \mathcal{P}(\beta) = \{\{\beta\}, \{\gamma\}, \{\alpha, \beta\}, \{\beta, \gamma\}, \Omega\}, \mathcal{P}(\gamma) = \{\{\gamma\}, \{\alpha, \gamma\}, \{\beta, \gamma\}, \Omega\}$ . This frame satisfies Properties (F<sub>CP<sub>1</sub></sub>), (F<sub>CP<sub>3</sub></sub>) and (F<sub>CP<sub>4</sub></sub>) (and thus validates Axioms (CP<sub>1</sub>), (CP<sub>3</sub>) and (CP<sub>4</sub>); note also that  $C$  is transitive and euclidean) but violates Property (F<sub>CP<sub>2</sub></sub>) since  $C(\alpha) \subseteq \{\alpha, \beta\}$  and  $\Omega \setminus \{\alpha, \beta\} = \{\gamma\} \in \mathcal{P}(\alpha)$ . Let  $p$  be an atomic formula and construct a model based on this frame where  $\|p\| = \{\beta\}$ . Then  $\|Bp\| = \{\alpha, \beta\}$  so that  $\alpha \models Bp$  but  $\alpha \not\models BBp$  since  $\alpha \not\models \neg P\neg Bp$  (because  $\|\neg Bp\| \in \mathcal{P}(\alpha)$ ).
- \CP<sub>3</sub> Let  $\Omega = \{\alpha, \beta, \gamma, \delta\}, C(\alpha) = \{\alpha, \beta\}, C(\beta) = C(\gamma) = C(\delta) = \{\beta, \gamma, \delta\}, \mathcal{P}(\alpha) = \{\{\alpha, \beta\}, \{\alpha, \beta, \gamma\}, \{\alpha, \beta, \delta\}, \Omega\}, \mathcal{P}(\beta) = \mathcal{P}(\gamma) = \mathcal{P}(\delta) =$

$\{\{\alpha, \beta\}, \{\gamma, \delta\}, \{\alpha, \beta, \gamma\}, \{\alpha, \beta, \delta\}, \{\beta, \gamma, \delta\}, \Omega\}$ . This frame satisfies Properties  $(F_{CP_1})$ ,  $(F_{CP_2})$  and  $(F_{CP_4})$  (and thus validates Axioms  $(CP_1)$ ,  $(CP_2)$  and  $(CP_4)$ ) but violates Property  $(F_{CP_3})$  since  $\beta \in C(\alpha)$  but  $\mathcal{P}(\beta) \not\subseteq \mathcal{P}(\alpha)$ . Let  $p$  be an atomic formula and construct a model based on this frame where  $\|p\| = \{\alpha, \beta\}$ . Then  $\|Bp\| = \{\alpha\}$  ( $\beta \notin \|Bp\|$  because  $\Omega \setminus \|p\| = \{\gamma, \delta\} \in \mathcal{P}(\beta)$  and the same is true of  $\gamma$  and  $\delta$ ) so that  $\alpha \models Bp$  but  $\alpha \not\models BBp$  since  $\|Bp\| \notin \mathcal{P}(\alpha)$ .

$\setminus CP_4$  Let  $\Omega = \{\alpha, \beta, \gamma\}$ ,  $C(\alpha) = C(\beta) = C(\gamma) = \{\beta, \gamma\}$  (note that  $C$  is transitive and euclidean),  $\mathcal{P}(\alpha) = \{\{\beta\}, \{\beta, \gamma\}, \Omega\}$ ,  $\mathcal{P}(\beta) = \mathcal{P}(\gamma) = \{\{\beta, \gamma\}, \Omega\}$ . This frame satisfies Properties  $(F_{CP_1})$ ,  $(F_{CP_2})$  and  $(F_{CP_3})$  (and thus validates Axioms  $(CP_1)$ ,  $(CP_2)$  and  $(CP_3)$ ) but violates Property  $(F_{CP_4})$  since  $\beta \in C(\alpha)$  but  $\mathcal{P}(\alpha) \not\subseteq \mathcal{P}(\beta)$ . Let  $p$  be an atomic formula and construct a model based on this frame where  $\|p\| = \{\beta\}$ . Then  $\|Bp\| = \{\alpha\}$  ( $\beta \notin \|Bp\|$  because  $\|p\| \notin \mathcal{P}(\beta)$  and the same is true of  $\gamma$ ) so that  $\alpha \models Bp$  but  $\alpha \not\models BBp$  since  $C(\alpha) \cap \|Bp\| = \emptyset$ .  $\square$

**Proof of minimality for Proposition 10.** We want to prove that the schema

$\neg B\phi \rightarrow B\neg B\phi$  is not a theorem of any sub-logic of  $\mathbb{L}+$   $\left\{ \begin{array}{l} (CP_1) \quad C\phi \rightarrow P\phi \\ (CP_2) \quad C\phi \rightarrow \neg P\neg\phi \\ (CP_3) \quad \neg P\phi \rightarrow C\neg P\phi \\ (CP_4) \quad P\phi \rightarrow CP\phi \end{array} \right\}$ .

$\setminus CP_1$  Let  $\Omega = \{\alpha, \beta, \gamma\}$ ,  $C(\alpha) = C(\beta) = \{\beta\}$ ,  $C(\gamma) = \{\gamma\}$ ,  $\mathcal{P}(\alpha) = \mathcal{P}(\beta) = \{\Omega\}$ ,  $\mathcal{P}(\gamma) = \{\{\beta, \gamma\}, \Omega\}$ . This frame satisfies Properties  $(F_{CP_2})$ ,  $(F_{CP_3})$  and  $(F_{CP_4})$  (and thus validates Axioms  $(CP_2)$ ,  $(CP_3)$  and  $(CP_4)$ ) and  $C$  is transitive (so that Axiom  $(4_C)$  is valid; note also that  $C$  is euclidean and thus also Axiom  $(5_C)$  is valid) but violates Property  $(F_{CP_1})$  since  $C(\alpha) \subseteq \{\beta\}$  but  $\{\beta\} \notin \mathcal{P}(\alpha)$ . Let  $p$  be an atomic formula and construct a model based on this frame where  $\|p\| = \{\beta, \gamma\}$ . Then  $\|Bp\| = \{\gamma\}$  ( $\alpha \notin \|Bp\|$  because  $\|p\| \notin \mathcal{P}(\alpha)$  and the same is true of  $\beta$ ). Thus  $\alpha \models \neg Bp$  but  $\alpha \not\models B\neg Bp$  because  $\|B\neg Bp\| = \Omega \setminus \|Bp\| = \{\alpha, \beta\}$  and  $\{\alpha, \beta\} \notin \mathcal{P}(\alpha)$ .

$\setminus CP_2$  Let  $\Omega = \{\alpha, \beta, \gamma\}$ ,  $C(\alpha) = C(\beta) = \{\beta\}$ ,  $C(\gamma) = \{\gamma\}$ ,  $\mathcal{P}(\alpha) = \mathcal{P}(\beta) = \{\{\alpha\}, \{\beta\}, \{\gamma\}, \{\alpha, \beta\}, \{\beta, \gamma\}, \Omega\}$ ,  $\mathcal{P}(\gamma) = \{\{\gamma\}, \{\alpha, \gamma\}, \{\beta, \gamma\}, \Omega\}$ . This frame satisfies Properties  $(F_{CP_1})$ ,  $(F_{CP_3})$  and  $(F_{CP_4})$  (and thus validates Axioms  $(CP_1)$ ,  $(CP_3)$  and  $(CP_4)$ ) and  $C$  is transitive (so that Axiom  $(4_C)$  is valid; note also that  $C$  is euclidean and thus also Axiom  $(5_C)$  is valid) but violates Property  $(F_{CP_2})$  since  $C(\alpha) \subseteq \{\beta, \gamma\}$  and  $\Omega \setminus \{\beta, \gamma\} = \{\alpha\} \in \mathcal{P}(\alpha)$ . Let  $p$  be an atomic formula and construct a model based on this frame where  $\|p\| = \{\beta, \gamma\}$ . Then  $\|Bp\| = \{\gamma\}$  ( $\alpha \notin \|Bp\|$  because  $\Omega \setminus \|p\| \in \mathcal{P}(\alpha)$ )

and the same is true of  $\beta$ ). Thus  $\alpha \models \neg Bp$  but  $\alpha \not\models B\neg Bp$  because  $\|\neg Bp\| = \Omega \setminus \|Bp\| = \{\alpha, \beta\}$  and  $\Omega \setminus \|\neg Bp\| = \{\gamma\} \in \mathcal{P}(\alpha)$ .

\CP<sub>3</sub> Let  $\Omega = \{\alpha, \beta, \gamma\}, C(\alpha) = C(\beta) = C(\gamma) = \{\beta, \gamma\}, \mathcal{P}(\alpha) = \{\{\beta, \gamma\}, \Omega\}, \mathcal{P}(\beta) = \mathcal{P}(\gamma) = \{\{\alpha, \beta\}, \{\beta, \gamma\}, \Omega\}$ . This frame satisfies Properties (F<sub>CP<sub>1</sub></sub>), (F<sub>CP<sub>2</sub></sub>) and (F<sub>CP<sub>4</sub></sub>) (and thus validates Axioms (CP<sub>1</sub>), (CP<sub>2</sub>) and (CP<sub>4</sub>)) and  $C$  is transitive (so that Axiom (4<sub>C</sub>) is valid; note also that  $C$  is euclidean and thus also Axiom (5<sub>C</sub>) is valid) but violates Property (F<sub>CP<sub>3</sub></sub>) since  $\beta \in C(\alpha)$  but  $\mathcal{P}(\beta) \not\subseteq \mathcal{P}(\alpha)$ . Let  $p$  be an atomic formula and construct a model based on this frame where  $\|p\| = \{\alpha, \beta\}$ . Then  $\|Bp\| = \{\beta, \gamma\}$  ( $\alpha \not\models Bp$  since  $\|p\| \notin \mathcal{P}(\alpha)$ ). Thus  $\alpha \models \neg Bp$  but  $\alpha \not\models B\neg Bp$  because  $\|\neg Bp\| = \Omega \setminus \|Bp\| = \{\alpha\}$  and  $C(\alpha) \cap \{\alpha\} = \emptyset$ .

\CP<sub>4</sub> Let  $\Omega = \{\alpha, \beta, \gamma\}, C(\alpha) = C(\beta) = C(\gamma) = \{\beta, \gamma\}, \mathcal{P}(\alpha) = \{\{\gamma\}, \{\alpha, \beta\}, \{\beta, \gamma\}, \Omega\}, \mathcal{P}(\beta) = \mathcal{P}(\gamma) = \{\{\alpha, \beta\}, \{\beta, \gamma\}, \Omega\}$ . This frame satisfies Properties (F<sub>CP<sub>1</sub></sub>), (F<sub>CP<sub>2</sub></sub>) and (F<sub>CP<sub>3</sub></sub>) (and thus validates Axioms (CP<sub>1</sub>), (CP<sub>2</sub>) and (CP<sub>3</sub>)) and  $C$  is transitive (so that Axiom (4<sub>C</sub>) is valid; note also that  $C$  is euclidean and thus also Axiom (5<sub>C</sub>) is valid) but violates Property (F<sub>CP<sub>4</sub></sub>) since  $\beta \in C(\alpha)$  but  $\mathcal{P}(\alpha) \not\subseteq \mathcal{P}(\beta)$ . Let  $p$  be an atomic formula and construct a model based on this frame where  $\|p\| = \{\alpha, \beta\}$ . Then  $\|Bp\| = \{\beta, \gamma\}$  ( $\alpha \not\models Bp$  since  $\Omega \setminus \|p\| = \{\gamma\} \in \mathcal{P}(\alpha)$ ). Thus  $\alpha \models \neg Bp$  but  $\alpha \not\models B\neg Bp$  because  $\|\neg Bp\| = \Omega \setminus \|Bp\| = \{\alpha\}$  and  $C(\alpha) \cap \{\alpha\} = \emptyset$ .  $\square$

**Proof of minimality for Proposition 11.** We want to prove that the schema

$BB\phi \rightarrow B\phi$  is not a theorem of any sub-logic of  $\mathbb{L} + \left\{ \begin{array}{l} (4_C) \quad C\phi \rightarrow CC\phi \\ (CP_3) \quad \neg P\phi \rightarrow C\neg P\phi \\ (CP_4) \quad P\phi \rightarrow CP\phi \end{array} \right\}$ .

\4<sub>C</sub> Let  $\Omega = \{\alpha, \beta, \gamma\}, C(\alpha) = \{\beta\}, C(\beta) = C(\gamma) = \{\gamma\}, \mathcal{P}(\alpha) = \mathcal{P}(\beta) = \mathcal{P}(\gamma) = \{\{\gamma\}, \{\alpha, \gamma\}, \{\beta, \gamma\}, \Omega\}$ . This frame violates transitivity of  $C$  but satisfies Properties (F<sub>CP<sub>3</sub></sub>) and (F<sub>CP<sub>4</sub></sub>) (and thus validates Axioms (CP<sub>3</sub>) and (CP<sub>4</sub>)). Let  $p$  be an atomic formula and construct a model based on this frame where  $\|p\| = \{\alpha, \gamma\}$ . Then  $\|Bp\| = \{\beta, \gamma\}$  ( $\alpha \notin \|Bp\|$  because  $C(\alpha) \cap \|p\| = \emptyset$ ) so that  $\alpha \models BBp$  but  $\alpha \not\models Bp$ .

\CP<sub>3</sub> Let  $\Omega = \{\alpha, \beta, \gamma\}, C(\alpha) = \{\beta, \gamma\}, C(\beta) = C(\gamma) = \{\gamma\}, \mathcal{P}(\alpha) = \{\{\beta, \gamma\}, \Omega\}, \mathcal{P}(\beta) = \mathcal{P}(\gamma) = \{\{\gamma\}, \{\alpha, \gamma\}, \{\beta, \gamma\}, \Omega\}$ . This frame satisfies transitivity of  $C$  and Property (F<sub>CP<sub>4</sub></sub>) (and thus validates Axioms (4<sub>C</sub>) and (CP<sub>4</sub>)) but violates Property (CP<sub>3</sub>) since  $\beta \in \mathcal{P}(\alpha)$  but  $\mathcal{P}(\beta) \not\subseteq \mathcal{P}(\alpha)$ . Let  $p$  be an atomic formula and construct a model based on this frame where  $\|p\| = \{\alpha, \gamma\}$ . Then  $\|Bp\| = \{\beta, \gamma\}$  so that  $\alpha \models BBp$  but  $\alpha \not\models Bp$  since  $\|p\| \notin \mathcal{P}(\alpha)$ .

\CP<sub>4</sub> Let  $\Omega = \{\alpha, \beta, \gamma\}$ ,  $C(\alpha) = \{\beta, \gamma\}$ ,  $C(\beta) = C(\gamma) = \{\gamma\}$ ,  $\mathcal{P}(\alpha) = \{\{\beta\}, \{\alpha, \gamma\}, \{\beta, \gamma\}, \Omega\}$ ,  $\mathcal{P}(\beta) = \mathcal{P}(\gamma) = \{\{\alpha, \gamma\}, \{\beta, \gamma\}, \Omega\}$ . This frame satisfies transitivity of  $C$  and Property (F<sub>CP<sub>3</sub></sub>) (and thus validates Axioms (4<sub>C</sub>) and (CP<sub>3</sub>)) but violates Property (CP<sub>4</sub>) since  $\beta \in \mathcal{P}(\alpha)$  but  $\mathcal{P}(\alpha) \not\subseteq \mathcal{P}(\beta)$ . Let  $p$  be an atomic formula and construct a model based on this frame where  $\|p\| = \{\alpha, \gamma\}$ . Then  $\|Bp\| = \{\beta, \gamma\}$  ( $\alpha \notin \|Bp\|$  because  $\Omega \setminus \|p\| = \{\beta\} \in \mathcal{P}(\alpha)$ ) so that  $\alpha \models BBp$  but  $\alpha \not\models Bp$ .

**Proof of minimality for Proposition 12.** We want to prove that the schema

$B\neg B\phi \rightarrow \neg B\phi$  is not a theorem of any sub-logic of  $\mathbb{L} + \left\{ \begin{array}{l} (\text{CP}_1) \quad C\phi \rightarrow P\phi \\ (\text{CP}_3) \quad \neg P\phi \rightarrow C\neg P\phi \\ (\text{CP}_4) \quad P\phi \rightarrow CP\phi \end{array} \right\}$ .

\CP<sub>1</sub> Let  $\Omega = \{\alpha, \beta, \gamma\}$ ,  $C(\alpha) = \{\beta, \gamma\}$ ,  $C(\beta) = C(\gamma) = \{\gamma\}$  (note that  $C$  is transitive but not euclidean<sup>30</sup>),  $\mathcal{P}(\alpha) = \mathcal{P}(\beta) = \mathcal{P}(\gamma) = \{\{\alpha, \beta\}, \{\beta, \gamma\}, \Omega\}$ . This frame satisfies Properties (F<sub>CP<sub>3</sub></sub>) and (F<sub>CP<sub>4</sub></sub>) (and thus validates Axioms (CP<sub>3</sub>) and (CP<sub>4</sub>)) but violates Property (F<sub>CP<sub>1</sub></sub>) since  $C(\beta) \subseteq \{\gamma\}$  but  $\{\gamma\} \notin \mathcal{P}(\beta)$ . Let  $p$  be an atomic formula and construct a model based on this frame where  $\|p\| = \{\alpha, \beta\}$ . Then  $\|Bp\| = \{\alpha\}$  ( $\beta \notin \|Bp\|$  because  $C(\beta) \cap \|p\| = \emptyset$  and the same is true of  $\gamma$ ) so that  $\|\neg Bp\| = \Omega \setminus \|Bp\| = \{\beta, \gamma\}$  and thus  $\alpha \models B\neg Bp$  but  $\alpha \not\models \neg Bp$ .

\CP<sub>3</sub> Let  $\Omega = \{\alpha, \beta, \gamma\}$ ,  $C(\alpha) = C(\beta) = C(\gamma) = \{\beta, \gamma\}$  (note that  $C$  is transitive and euclidean),  $\mathcal{P}(\alpha) = \{\{\alpha, \beta\}, \{\beta, \gamma\}, \Omega\}$ ,  $\mathcal{P}(\beta) = \mathcal{P}(\gamma) = \{\{\gamma\}, \{\alpha, \beta\}, \{\beta, \gamma\}, \Omega\}$ . This frame satisfies Properties (F<sub>CP<sub>1</sub></sub>) and (F<sub>CP<sub>4</sub></sub>) (and thus validates Axioms (CP<sub>1</sub>) and (CP<sub>4</sub>)) but violates Property (F<sub>CP<sub>3</sub></sub>) since  $\beta \in C(\alpha)$  but  $\mathcal{P}(\beta) \not\subseteq \mathcal{P}(\alpha)$ . Let  $p$  be an atomic formula and construct a model based on this frame where  $\|p\| = \{\alpha, \beta\}$ . Then  $\|Bp\| = \{\alpha\}$  ( $\beta \notin \|Bp\|$  because  $\Omega \setminus \|p\| = \{\gamma\} \in \mathcal{P}(\beta)$  and the same is true of  $\gamma$ ) so that  $\|\neg Bp\| = \{\beta, \gamma\}$  and thus  $\alpha \models B\neg Bp$  but  $\alpha \not\models \neg Bp$ .

\CP<sub>4</sub> Let  $\Omega = \{\alpha, \beta, \gamma\}$ ,  $C(\alpha) = C(\beta) = C(\gamma) = \{\beta, \gamma\}$  (note that  $C$  is transitive and euclidean),  $\mathcal{P}(\alpha) = \{\{\alpha, \beta\}, \{\beta, \gamma\}, \Omega\}$ ,  $\mathcal{P}(\beta) = \mathcal{P}(\gamma) = \{\{\beta, \gamma\}, \Omega\}$ . This frame satisfies Properties (F<sub>CP<sub>1</sub></sub>) and (F<sub>CP<sub>3</sub></sub>) (and thus validates Axioms (CP<sub>1</sub>) and (CP<sub>3</sub>)) but violates Property (F<sub>CP<sub>4</sub></sub>) since  $\beta \in C(\alpha)$  but  $\mathcal{P}(\alpha) \not\subseteq \mathcal{P}(\beta)$ . Let  $p$  be an atomic formula and construct a model based on this frame where  $\|p\| = \{\alpha, \beta\}$ . Then  $\|Bp\| = \{\alpha\}$  ( $\beta \notin \|Bp\|$  because  $\|p\| \notin \mathcal{P}(\beta)$  and the same is true of  $\gamma$ ) so that  $\|\neg Bp\| = \{\beta, \gamma\}$  and thus  $\alpha \models B\neg Bp$  but  $\alpha \not\models \neg Bp$ .

□

<sup>30</sup>Indeed, it cannot be euclidean because of Remark 8.

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