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Exponent of Cross-sectional Dependence for Residuals

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Exponent of Cross-sectional Dependence for Residuals

Abstract

In this paper we focus on estimating the degree of cross-sectional dependence in the error terms of a classical panel data regression model. For this purpose we propose an estimator of the exponent of cross-sectional dependence denoted by α ; which is based on the number of non-zero pair-wise cross correlations of these errors. We prove that our estimator, $\tilde{\alpha}$; is consistent and derive the rate at which $\tilde{\alpha}$ approaches its true value. We evaluate the finite sample properties of the proposed estimator by use of a Monte Carlo simulation study. The numerical results are encouraging and supportive of the theoretical findings. Finally, we undertake an empirical investigation of α for the errors of the CAPM model and its Fama-French extensions using 10-year rolling samples from S&P 500 securities over the period Sept 1989 - May 2018.

JEL-Codes: C210, C320.

Keywords: pair-wise correlations, cross-sectional dependence, cross-sectional averages, weak and strong factor models, CAPM and Fama-French factors.

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1 Introduction

Interest in the analysis of cross-sectional dependence applied to households, firms, markets, regional and national economies has become prominent over the past decade, especially so in the aftermath of the latest financial crisis given its effects on the global economy. Researchers in many fields have turned to network theory, spatial and factor models to obtain a better understanding of the extent and nature of such cross dependencies. There are many issues to be considered: how to test for the presence of cross-sectional dependence, how to measure the degree of cross-sectional dependence, how to model cross-sectional dependence, and how to carry out counterfactual exercises under alternative network formations or market interconnections. Many of these topics are the subject of ongoing research. In this paper we focus on measuring cross-sectional dependence.

Bailey, Kapetanios, and Pesaran (2016, BKP hereafter) give a thorough account of the rationale and motivation behind the need for determining the extent of cross-sectional dependence, be it in finance, micro or macroeconomics. They focus on the asymptotic behaviour of the variance of the cross section average of the observations on a double array of random variables, say x_{it} , indexed by $i = 1, 2, \dots, N$ and $t = 1, 2, \dots, T$, over space and time. In particular, they analyse the rate at which this variance tends to zero and show that it depends on the degree or exponent of cross-sectional dependence which they denote by α . They explore a factor model setting as a vehicle for characterising strong and semi-strong covariance structures as defined in Chudik et al. (2011). They relate these to the degree of pervasiveness of factors in unobserved factor models often used in the literature to model cross-sectional dependence.

In this paper we build on BKP and extend the analysis in two respects. First, we consider a more generic setting which does not require a common factor representation and holds more generally for both moderate to sizable cross-sectional dependence. We achieve this by directly considering the significance of individual pair-wise correlations, and do not concern ourselves with the factors that might underlie these pair-wise correlations. Second, we consider estimating the exponent of cross-sectional dependence, α , of the residuals obtained from a panel data regression model.

We propose a new estimator of α based on the number of statistically significant pair-wise correlations of the residuals from the panel regression under consideration. To establish the statistical significance of the correlation coefficients we adopt the thresholding multiple testing (MT) estimator proposed by Bailey et al. (2014), BPS. Other thresholding estimators can also be used. See, for example, Bickel and Levina (2008) or Karoui (2008) and Cai and Liu (2011) or Fan et al. (2013). The MT testing procedure advanced by BPS has the advantage that it directly considers the statistical significance of the correlation coefficient which is invariant to scales. Other thresholding procedures focus on the sample covariances and resort to cross validation to identify the threshold. Bickel and Levina (2008) use universal thresholding, namely comparing all the sample covariances to the same threshold value, whilst Cai and Liu (2011) propose an ‘adaptive’ thresholding procedure that allows for differing thresholds across the different pairs of sample covariances. Other contributions to this literature include the work of Huang et al. (2006), Rothman et al. (2009), Cai and Zhou (2011) and Cai and Zhou (2012), Wang and

Zhou (2010), and Fan et al. (2011).¹ All these contributions apply the thresholding procedure to sample covariances and do not apply to the residuals from a panel regression model that concerns us in this paper. It is also important to bear in mind that when estimating α we do not assume that the underlying error covariance matrix is sparse, as is assumed in the literature on regularization of the sample covariance. Our objective is to estimate the degree of sparsity of the covariance matrix rather than assume sparsity for the purpose of consistent estimation of the covariance matrix or its inverse. What matters for estimation of α is to ensure that all non-zero entries of the correlation matrix are correctly identified.

We establish consistency of our estimator under the assumptions of exogeneity of regressors and symmetry of the error distribution. We also explain how the derivations can be extended to the case when weakly exogenous variables are present, as for example in a dynamic panel data setting. The proposed estimator is simple to compute and is shown to perform well in small samples, for a variety of correlation matrices, irrespective of whether the cross correlations are generated from a multi-factor structure or specified by a given correlation matrix with a specified degree of sparsity. This is especially the case as compared to basing the estimation of α on the largest eigenvalue of the correlation matrix, which performs particularly poorly. The rate of convergence of our preferred estimator is complex and depends on an interplay of the cross-sectional and time dimensions, N and T . The Monte Carlo results also show that the error in estimating α is smaller for values of α close to unity, which is likely to be of greater interest in practice. The problem of making inference about the value of α raises additional technical difficulties and will not be addressed in this paper. In practice bootstrap techniques can be used to obtain confidence bounds around our proposed estimator. Finally, we provide an empirical application investigating the degree of inter-linkages between financial variables using the Standard & Poor's 500 index. We present 10-year rolling estimates of α applied to excess returns on securities included in the S&P 500 data set as well as α estimates applied to the residuals obtained from the CAPM and its Fama-French extensions used extensively in the finance literature.

The rest of the paper is organised as follows: Section 2 discusses alternative characterisations of α , the exponent of cross-sectional dependence, and the conditions under which these measures are equivalent as $N \rightarrow \infty$. Section 3 sets up the panel data model and discusses its underlying assumptions. Section 4 proposes the estimator of α in terms of the number of statistically significant non-zero pair-wise correlations of the residuals. Section 5 presents the main theoretical results of the paper for the a static panel data model with strictly exogenous regressors. Extensions to dynamic panels or panels with weakly exogenous regressors are discussed in the sub-section 5.2. Section 6 presents a detailed Monte Carlo simulation study. The empirical application is discussed in Section 7. Finally, Section 8 concludes. Proofs of all theoretical results are provided in the Appendix.

¹Shrinkage procedures have also been proposed in the literature for regularization of covariance matrices. See, for example, Stein (1956), Ledoit and Wolf (2003) and Ledoit and Wolf (2004). However, the shrinkage procedure does not set any elements of the covariance matrix to zero, and is not suitable for estimation of α which builds on the support recovery properties of the estimated covariance matrix.

2 Degrees of cross-sectional dependence: alternative measures

Our analysis focuses on the covariance matrix of $\boldsymbol{\varepsilon}_t = (\varepsilon_{1t}, \varepsilon_{2t}, \dots, \varepsilon_{Nt})'$, where $\boldsymbol{\varepsilon}_t$ is the $N \times 1$ vector of errors from a panel data regression model. Let $\boldsymbol{\Sigma}_N = E(\boldsymbol{\varepsilon}_t \boldsymbol{\varepsilon}_t') = (\sigma_{ij})$, and denote its largest eigenvalue by $\lambda_{\max}(\boldsymbol{\Sigma}_N) > 0$. The errors ε_{it} are said to be strongly cross-sectionally correlated, if $\lambda_{\max}(\boldsymbol{\Sigma}_N) = \Theta(N)$, where Θ denotes exact order of magnitude, and they are said to be weakly cross-sectionally correlated, if $\lambda_{\max}(\boldsymbol{\Sigma}_N)$ is bounded in N . All intermediate cases can be parameterized in terms of the exponent α_λ , such that

$$\lambda_{\max}(\boldsymbol{\Sigma}_N) = \Theta(N^{\alpha_\lambda}). \quad (1)$$

The weak and strong cross dependence cases then relate to $\alpha_\lambda \rightarrow 0$ and $\alpha_\lambda \rightarrow 1$, respectively.

Suppose now that the cross dependence of ε_{it} is characterized by the following approximate multiple-factor error process

$$\varepsilon_{it} = \beta_i' \mathbf{f}_t + u_{it}, \quad (2)$$

where \mathbf{f}_t is the $m \times 1$ vector of unobserved common factors with zero means, and $\beta_i = (\beta_{i1}, \beta_{i2}, \dots, \beta_{im})'$ is the associated $m \times 1$ vector of factor loadings, and u_{it} is the idiosyncratic component assumed to have mean zero and the covariance matrix $\mathbf{V} = E(\mathbf{u}_t \mathbf{u}_t')$, where $\mathbf{u}_t = (u_{1t}, u_{2t}, \dots, u_{Nt})'$. Then

$$\boldsymbol{\Sigma}_N = E(\boldsymbol{\varepsilon}_t \boldsymbol{\varepsilon}_t') = \mathbf{B}\mathbf{B}' + \mathbf{V},$$

where $\mathbf{B} = (\beta_1, \beta_2, \dots, \beta_N)'$, and without loss of generality we have set $E(\mathbf{f}_t \mathbf{f}_t') = \mathbf{I}_m$. To identify the factor component from the idiosyncratic component we assume that $\lambda_{\max}(\mathbf{V}) = O(1)$, but allow the factor loadings to satisfy the condition

$$\mathbf{B}'\mathbf{B} = \sum_{i=1}^N \beta_i \beta_i' = \Theta(N^{\alpha_\beta}), \quad \alpha_\beta > 0, \quad (3)$$

where α_β measures the degree to which the factors are pervasive, in the sense that they have non-zero effects on the individual errors, ε_{it} . In what follows we refer to α_β as the exponent of factor loadings. In the standard approximate factor models it is assumed that $\alpha_\beta = 1$, whilst in practice, where the possibility of weak factors can not be ruled out, α_β could be a parameter of interest to be estimated.

To see how α_β and α_λ are related note that

$$\lambda_{\max}(\boldsymbol{\Sigma}_N) \leq \|\mathbf{B}\mathbf{B}' + \mathbf{V}\|_1 \leq \|\mathbf{B}\|_1 \|\mathbf{B}\|_\infty + \|\mathbf{V}\|_1,$$

where $\|\mathbf{B}\|_1$ and $\|\mathbf{B}\|_\infty$ are column and row norms of \mathbf{B} , respectively. To ensure that $\text{Var}(\varepsilon_{it})$ is bounded we must have $\|\mathbf{B}\|_\infty < K$. Also to ensure that $\lambda_{\max}(\mathbf{V}) = O(1)$, we must have $\|\mathbf{V}\|_1 < K$. Therefore, the rate at which $\lambda_{\max}(\boldsymbol{\Sigma}_N)$ can rise with N is controlled by

$$\|\mathbf{B}\|_1 = \max_{1 \leq j \leq N} \sum_{i=1}^N |\beta_{ij}|. \quad (4)$$

Setting $\sum_{i=1}^N |\beta_{ij}| = \Theta(N^{\alpha_{\beta_j}})$, for $j = 1, 2, \dots, m$, then $\|\mathbf{B}\|_1 = \Theta(N^{\alpha_\beta})$, where $\alpha_\beta = \max_j(\alpha_{\beta_j})$. Then

$$\lambda_{\max}(\boldsymbol{\Sigma}_N) = \Theta(N^{\alpha_\beta}) + O(1).$$

To distinguish the effects of the factor component from those of the idiosyncratic component we must have $\alpha_\beta > 0$. Comparing this result with (1) establishes that $\alpha_\lambda \rightarrow \alpha_\beta > 0$, as $N \rightarrow \infty$.

The above analysis suggests two alternative ways of estimating α_λ . A direct procedure would be to base the estimate of α_λ on $\lambda_{\max}(\boldsymbol{\Sigma}_N)$ and set

$$\lambda_{\max}(\boldsymbol{\Sigma}_N) = O(N^{\alpha_\lambda}) = \kappa N^{\alpha_\lambda},$$

where κ is a constant independent of N . Then

$$\alpha_\lambda = \frac{\ln[\lambda_{\max}(\boldsymbol{\Sigma}_N)]}{\ln(N)} - \frac{\ln(\kappa)}{\ln(N)}. \quad (5)$$

In order to identify α_λ , as $N \rightarrow \infty$, we set $\kappa = 1$, so that (5) becomes

$$\alpha_\lambda = \frac{\ln[\lambda_{\max}(\boldsymbol{\Sigma}_N)]}{\ln(N)}. \quad (6)$$

In this form, the value of α_λ is susceptible to the scaling of elements in $\boldsymbol{\varepsilon}_t$. For this reason we focus our attention rather on the corresponding correlation matrix $\mathbf{R}_N = (\rho_{ij})$ given by

$$\mathbf{R}_N = \mathbf{D}_N^{-1/2} \boldsymbol{\Sigma}_N \mathbf{D}_N^{-1/2},$$

where

$$\mathbf{D}_N = \text{diag}(\sigma_{ii}, i = 1, 2, \dots, N). \quad (7)$$

Hence, (6) finally becomes

$$\alpha_\lambda = \frac{\ln[\lambda_{\max}(\mathbf{R}_N)]}{\ln(N)}, \quad (8)$$

and α_λ has fixed bounds at zero and unity, as $N \rightarrow \infty$.

Developing a theory based on the maximum eigenvalue of the correlation matrix \mathbf{R}_N can be challenging. To avoid some of the technical problems involved in estimating $\lambda_{\max}(\mathbf{R}_N)$, and noting that $\text{Var}(\bar{\varepsilon}_t) \leq N^{-1} \lambda_{\max}(\boldsymbol{\Sigma}_N)$, BKP propose basing the estimation of α_λ on $\text{Var}(\bar{\varepsilon}_t)$, where $\bar{\varepsilon}_t = N^{-1} \sum_{i=1}^N \varepsilon_{it}$. In the case where ε_{it} has a factor representation, BKP show that $\text{Var}(\bar{\varepsilon}_t) = O\left[\max\left(N^{2(\alpha_\beta-1)}, N^{-1}\right)\right]$, which reduces to $\text{Std}(\bar{\varepsilon}_t) = O\left(N^{(\alpha_\beta-1)}\right)$, if $2(\alpha_\beta - 1) > -1$, or if $\alpha_\beta > 1/2$. This means that at least $N^{1/2}$ of the factor loadings must have non-zero values for $\boldsymbol{\Sigma}_N$ to differ sufficiently from a diagonal $\boldsymbol{\Sigma}_N$.

In this paper we consider an alternative estimation strategy that does not require ε_{it} to have a factor representation. Since

$$\lambda_{\max}(\mathbf{R}_N) \leq \|\mathbf{R}_N\|_1 = \max_{1 \leq j \leq N} \sum_{i=1}^N |\rho_{ij}|,$$

we focus directly on estimation of ρ_{ij} and distinguish between values of ρ_{ij} that are close to zero and those that are significantly different from zero, and measure the exponent of cross-sectional dependence in terms of the number of significant (non-zero) cross-correlation coefficients. Specifically, we define α such that $M_N = N^{2\alpha}$ where M is the number of non-zero

elements of \mathbf{R}_N which can be written equivalently as $M_N = \boldsymbol{\tau}'_N \boldsymbol{\Delta}_N \boldsymbol{\tau}_N$, where $\boldsymbol{\tau}_N$ is an $N \times 1$ vector of ones and $\boldsymbol{\Delta}_N = (\delta_{ij})$ is an $N \times N$ matrix of population correlation indicators with typical elements given by

$$\delta_{ij} = I(\rho_{ij} \neq 0), \quad i, j = 1, 2, \dots, N,$$

in which $I(\mathcal{A})$ is equal to unity if \mathcal{A} is true and zero otherwise. Note that by construction $\delta_{ii} = 1$. Hence,

$$\alpha = \frac{\ln(M_N)}{\ln(N^2)} = \frac{\ln(\boldsymbol{\tau}'_N \boldsymbol{\Delta}_N \boldsymbol{\tau}_N)}{\ln N^2}. \quad (9)$$

Cross-sectional independence refers to the case when $\mathbf{R}_N = \mathbf{I}_N$ and $\alpha = 1/2$, while the case of cross-sectional strong dependence corresponds to all pair-wise correlation coefficients being non-zero such that $\alpha = 1$. Note that by construction $1/2 \leq \alpha \leq 1$, with $\alpha = 1/2$ arising when $\boldsymbol{\Delta}_N = \mathbf{I}_N$, and $\alpha = 1$ if $\rho_{ij} \neq 0$ for all i and j .

Other exponents of cross-sectional dependence can be defined by focussing only on the off-diagonal elements of \mathbf{R}_N and consider the following exponent of cross-sectional dependence:²

$$\alpha' = \frac{\ln[\boldsymbol{\tau}'_N (\boldsymbol{\Delta}_N - \mathbf{I}_N) \boldsymbol{\tau}_N]}{\ln N(N-1)},$$

assuming that $\boldsymbol{\Delta}_N \neq \mathbf{I}_N$. Unlike α the above measure is not defined if $\mathbf{R}_N = \mathbf{I}_N$. The two measures coincide, namely $\alpha = \alpha' = 1$, if $\rho_{ij} \neq 0$ for all i and j . In cases where ε_{it} have a multi-factor error representation given by (2), the largest exponent of the factor loadings is given by $\alpha_\beta > 0$. It then readily follows that $\boldsymbol{\tau}'_N (\boldsymbol{\Delta}_N - \mathbf{I}_N) \boldsymbol{\tau}_N = N^{2\alpha_\beta} - N^{\alpha_\beta} + N$, where $(N^{2\alpha_\beta} - N^{\alpha_\beta}) = N^{\alpha_\beta} (N^{\alpha_\beta} - 1)$ is the total number of off-diagonal non-zero pair-wise cross correlations of the errors and N corresponds to the additional diagonal elements which are non-zero. In such a multi-factor error set up we have

$$\alpha = \frac{\ln(\boldsymbol{\tau}'_N \boldsymbol{\Delta}_N \boldsymbol{\tau}_N)}{\ln N^2} = \frac{\ln(N^{2\alpha_\beta} - N^{\alpha_\beta} + N)}{\ln N^2},$$

and

$$\alpha' = \frac{\ln[\boldsymbol{\tau}'_N (\boldsymbol{\Delta}_N - \mathbf{I}_N) \boldsymbol{\tau}_N]}{\ln N(N-1)} = \frac{\ln(N^{2\alpha_\beta} - N^{\alpha_\beta})}{\ln N(N-1)}.$$

Recall that we must have $\alpha_\beta > 1/2$ for factors to be distinguishable from the idiosyncratic components. It is then easily seen that α and $\alpha' \rightarrow \alpha_\beta$ as $N \rightarrow \infty$. However, the two measures could differ if N is not sufficiently large. In finite samples α' can be written in terms of α by first solving the quadratic equation

$$N^{2\alpha_\beta} + (N - N^{\alpha_\beta}) = N^{2\alpha}, \quad (10)$$

²One can also consider only the distinct off-diagonal elements of \mathbf{R}_N and define α as

$$\begin{aligned} \alpha' &= \frac{\ln[\frac{1}{2} \boldsymbol{\tau}'_N (\boldsymbol{\Delta}_N - \mathbf{I}_N) \boldsymbol{\tau}_N]}{\ln[\frac{1}{2} N(N-1)]} = \frac{\ln[\boldsymbol{\tau}'_N (\boldsymbol{\Delta}_N - \mathbf{I}_N) \boldsymbol{\tau}_N] - \ln(2)}{\ln[N(N-1)] - \ln(2)} \\ &= \frac{\alpha - \frac{\ln(2)}{\ln[N(N-1)]}}{1 - \frac{\ln(2)}{\ln[N(N-1)]}} \rightarrow \alpha, \text{ as } N \rightarrow \infty. \end{aligned}$$

for α_β , namely

$$\alpha_\beta = \frac{\ln \left(1 + \sqrt{1 - 4(N - N^{2\alpha})} \right) - \ln 2}{\ln N}. \quad (11)$$

Since $[N^{\alpha_\beta}]$ can only take positive or zero values the second root of (10) is clearly redundant. In what follows, we focus on α since it is defined even if $\mathbf{R}_N = \mathbf{I}_N$, and α is suitably scaled to lie in the range $(1/2, 1]$.

3 Panel data model

Consider the panel data regression model

$$y_{it} = \gamma_i' \mathbf{x}_{it} + \varepsilon_{it}, \text{ for } i = 1, 2, \dots, N; t = 1, 2, \dots, T, \quad (12)$$

where \mathbf{x}_{it} is a $k \times 1$ vector of observed regressors, γ_i is the associated vector of coefficients, and ε_{it} are the model's errors. We are interested in estimating the exponent of the cross-sectional dependence of the errors, ε_{it} , defined by (9). First, we obtain residuals e_{it} computed as

$$e_{it} = y_{it} - \mathbf{x}_{it}' \hat{\gamma}_i = y_{it} - \mathbf{x}_{it}' (\mathbf{X}_i' \mathbf{X}_i)^{-1} \mathbf{X}_i' \mathbf{y}_i, \quad (13)$$

where \mathbf{X}_i is the $T \times k$ matrix of observations on the regressors for the i^{th} unit, and \mathbf{y}_i is the $T \times 1$ vector of observations on the dependent variable of the i^{th} unit. We assume that the regressors are strictly exogenous.

We define the standardized errors, ξ_{it} , and the associated standardized residuals, z_{it} , as

$$\xi_{it} = \frac{\varepsilon_{it}}{(T^{-1} \boldsymbol{\varepsilon}_i' \mathbf{M}_i \boldsymbol{\varepsilon}_i)^{1/2}}, \quad (14)$$

$$z_{it} = \frac{e_{it}}{(T^{-1} \mathbf{e}_i' \mathbf{M}_i \mathbf{e}_i)^{1/2}} = \frac{e_{it}}{(T^{-1} \mathbf{e}_i' \mathbf{e}_i)^{1/2}}, \quad (15)$$

where $\boldsymbol{\varepsilon}_i = (\varepsilon_{i1}, \varepsilon_{i2}, \dots, \varepsilon_{iT})'$, $\mathbf{e}_i = (e_{i1}, e_{i2}, \dots, e_{iT})'$, and $\mathbf{M}_i = \mathbf{I}_T - \mathbf{X}_i (\mathbf{X}_i' \mathbf{X}_i)^{-1} \mathbf{X}_i'$. Note that (since $\mathbf{e}_i = \mathbf{M}_i \boldsymbol{\varepsilon}_i$)

$$z_{it} = \frac{\varepsilon_{it} + \mathbf{x}_{it}' (\gamma_i - \hat{\gamma}_i)}{(T^{-1} \boldsymbol{\varepsilon}_i' \mathbf{M}_i \boldsymbol{\varepsilon}_i)^{1/2}} = \xi_{it} - \mathbf{a}_{it}' \boldsymbol{\xi}_i, \quad (16)$$

where

$$\mathbf{a}_{it} = \mathbf{X}_i (\mathbf{X}_i' \mathbf{X}_i)^{-1} \mathbf{x}_{it},$$

and $\boldsymbol{\xi}_i = (\xi_{i1}, \xi_{i2}, \dots, \xi_{iT})'$.

Further, in what follows we assume that the error terms are symmetrically distributed.

Assumption 1 *Conditional on \mathbf{X}_i , the errors of the panel data model, (12), (a) ε_{it} are symmetrically distributed with zero means and variances $0 < c < \sigma_i^2 < K < \infty$, (b) ε_{it} are serially independent, (c) ε_{it} and ε_{jt} are distributed independently if $E(\varepsilon_{it} \varepsilon_{jt}) = 0$, for all $i \neq j$.*

Under the above assumption, and using (14) it readily follows that

$$E(\boldsymbol{\xi}_i | \mathbf{X}_i) = \mathbf{0}, \quad (17)$$

and

$$E(\xi_i \xi_j' | \mathbf{X}_i, \mathbf{X}_j) = \rho_{ij} \mathbf{I}_T. \quad (18)$$

Our main analysis will condition on the observed regressors. Remark 2 will discuss an unconditional version of our results. For the observed regressors, we make the following assumption:

Assumption 2 *The $k \times 1$ vector of regressors \mathbf{x}_{it} in (12) is bounded: $\sup_{i,t} \|\mathbf{x}_{it}\| < \infty$. Further, for some T_0 , we have*

$$\inf_{T > T_0, i} \lambda_{\min} \left(\frac{\mathbf{X}_i' \mathbf{X}_i}{T} \right) > 0.$$

Under Assumption 1 it readily follows that

$$E(\mathbf{a}'_{it} \xi_i | \mathbf{X}_i) = \mathbf{x}'_{it} (\mathbf{X}_i' \mathbf{X}_i)^{-1} \mathbf{X}_i' E(\xi_i | \mathbf{X}_i) = 0,$$

where \mathbf{x}'_{it} is the t^{th} row of $\mathbf{X}_i = (\mathbf{x}_{i1}, \mathbf{x}_{i2}, \dots, \mathbf{x}_{iT})'$. Hence,

$$E(z_{it} | \mathbf{X}_i) = 0, \quad (19)$$

which in turn implies that z_{it} is a martingale difference process with respect to the non-decreasing information set, $\Omega_{it} = (\mathbf{x}_{i1}, \mathbf{x}_{i2}, \dots, \mathbf{x}_{it})$. By construction $\Omega_{iT} \equiv \mathbf{X}_i$. Indeed, since Ω_{it} is a subset of Ω_{iT} , then by the chain rule of conditional expectations we have

$$E(z_{it} | \Omega_{it}) = E[E(z_{it} | \Omega_{iT}) | \Omega_{it}].$$

But in view of (19), $E(z_{it} | \Omega_{iT}) = 0$, and it also follows that $E(z_{it} | \Omega_{it}) = 0$, for all i and t .

We also require the following assumption that sets a lower bound condition on the non-zero values of the pair-wise correlations.

Assumption 3 *Let $\rho_{\min} = \inf_{i,j} (|\rho_{ij}| \mid \rho_{ij} \neq 0)$. Then,*

$$\lim_{T \rightarrow \infty} \frac{\ln(T)}{\sqrt{T} \rho_{\min}} = 0. \quad (20)$$

Remark 1 *This assumption is needed for successful recovery of non-zero pair-wise correlations, and is weaker than requiring $\rho_{\min} > 0$, as it allows ρ_{\min} to tend to zero with N or T or both, so long as its rate of decline is slower than $\ln(T)/\sqrt{T}$.*

4 Consistent estimation of α

Consider the sample estimate of the pair-wise correlation coefficients of the residuals from units i and j ,

$$\hat{\rho}_{ij} = \frac{T^{-1} \mathbf{e}'_i \mathbf{e}_j}{(T^{-1} \mathbf{e}'_i \mathbf{e}_i)^{1/2} (T^{-1} \mathbf{e}'_j \mathbf{e}_j)^{1/2}}, \quad (21)$$

where $\mathbf{e}_i = (e_{i1}, e_{i2}, \dots, e_{iT})'$, e_{it} is defined by (13), and by construction the sample mean of e_{it} is exactly zero. We can re-write (21) equivalently as

$$\hat{\rho}_{ij} = T^{-1} \sum_{t=1}^T z_{it} z_{jt}, \quad (22)$$

where z_{it} is defined by (15). In order to identify whether the pair-wise correlation coefficients $\hat{\rho}_{ij}$ are significantly different from zero we follow Bailey et al. (2014) and apply the multiple testing estimator associated with $\hat{\rho}_{ij}$. This is defined by

$$\tilde{\rho}_{ij} = \hat{\rho}_{ij} I \left(|\hat{\rho}_{ij}| > \frac{c_p(n, \delta)}{\sqrt{T}} \right), \quad (23)$$

where

$$c_p(n, \delta) = \Phi^{-1} \left(1 - \frac{p/2}{n^\delta} \right), n = \frac{1}{2}N(N-1), \quad (24)$$

n is the number of tests carried out, p is the nominal size of the individual test, which can be set to 1%, 5% or 10%, $\Phi^{-1}(\cdot)$ is the inverse of the standard normal distribution function, and δ is a tuning parameter to be set *a priori*. This thresholding method is based on the notion that for each unit (i, j) pairs we carry out a total of $\frac{1}{2}N(N-1)$ individual tests of the null hypothesis that $\rho_{ij} = 0$ where $j \neq i$, $i, j = 1, 2, \dots, N$. Such tests can result in spurious outcomes especially when N is larger than T . The critical value function, $c_p(n, \delta)$, is therefore adjusted using parameter δ to take account of the effects of the multiple testing procedure for the estimation of α . It is important to bear in mind that the multiple testing problem encountered here differs from the standard one studied in the literature by Bonferroni (1935), Holm (1979) and others. Our focus here is on identifying the range of values for δ such that α can be consistently estimated, rather than controlling the overall size of the multiple tests being carried out.

Accordingly, we propose to estimate α by

$$\tilde{\alpha} = \frac{\ln \left(\boldsymbol{\tau}' \tilde{\boldsymbol{\Delta}} \boldsymbol{\tau} \right)}{2 \ln N}, \quad (25)$$

where $\tilde{\boldsymbol{\Delta}} = (\tilde{\delta}_{ij})$, with

$$\begin{aligned} \tilde{\delta}_{ij} &= I(\tilde{\rho}_{ij} \neq 0) = I \left(|\hat{\rho}_{ij}| > \frac{c_p(n, \delta)}{\sqrt{T}} \right), \text{ for } i \neq j \\ &= 1 \text{ for } i = j. \end{aligned}$$

5 Theoretical Derivations

5.1 Main Results

To establish that $\tilde{\alpha}$ converges to α , in addition to Assumptions 1, 2 and 3, we also require the following additional technical sub-exponential assumption:

Assumption 4 *There exist sufficiently large positive constants $C_0, C_1, s > 0$, such that*

$$\sup_{i,t} \Pr (|\varepsilon_{it}| > \alpha) \leq C_0 \exp(-C_1 \alpha^s), \text{ for all } \alpha > 0. \quad (26)$$

The rate of convergence of $\tilde{\alpha}$ to α is given in Theorem 1 below:

Theorem 1 *Consider the panel data regression model (12) and suppose that Assumptions 1-4 hold. Let $\tilde{\alpha}$ and δ be defined by (25), (23), and (24). Then, conditional on the observed \mathbf{x}_{it} , as $N, T \rightarrow \infty$, and if, for some $d > 0$, $N = O(T^d) = o(\exp(T))$,*

$$\begin{aligned} 2(\ln N)(\tilde{\alpha} - \alpha) = & O(N^{2(1-\alpha-\varkappa\delta)}) + O(N^{2(1-\alpha)} \exp(-C_0 T^{C_1})) + O(N^{-\alpha}) + O(N^{1-2\alpha}) = \\ & (27) \\ & O(T^{2d(1-\alpha-\varkappa\delta)}) + O(\exp[2d(1-\alpha)\ln(T) - C_0 T^{C_1}]) + O(T^{-d\alpha}) + O(T^{d(1-2\alpha)}) \end{aligned}$$

for any $0 < \varkappa < 1$, and some $C_0, C_1 > 0$.

As long as δ is set large enough ($\delta > 1 - \alpha$), the first term on the RHS of (27) can be made sufficiently small.

Remark 2 *If we do not wish to condition on the observed \mathbf{x}_{it} , one could obtain the result of Theorem 1, unconditionally, if it is assumed that the regressors satisfy the following sub-exponential condition for some $s > 0$,*

$$\sup_{i,t} \Pr (\|\mathbf{x}_{it}\| > \alpha) \leq C_0 \exp(-C_1 \alpha^s), \text{ for all } \alpha > 0, \quad (28)$$

and if for some T_0 , $\left(\frac{\mathbf{x}'_i \mathbf{x}_i}{T}\right)^{-1}$ exists for all $T > T_0$. Under these conditions on \mathbf{x}_{it} (which replace Assumption 2), we can then use Lemma A6 of Chudik et al. (2018) to establish probability bounds on $\left(\frac{\xi'_i \mathbf{x}_i}{T}\right) \left(\frac{\mathbf{x}'_i \mathbf{x}_i}{T}\right)^{-1} \left(\frac{\mathbf{x}'_i \xi_i}{T}\right)$, and to show that, for some $C_0, C_1 > 0$, and $0 < \pi < 1$,

$$\sup_{i,j} \Pr \left(\left| \sum_{t=1}^T z_{it} z_{jt} \right| > \sqrt{T} c_p(n, \delta) \right) \leq \sup_{i,j} \Pr \left(\left| \sum_{t=1}^T \xi_{it} \xi_{jt} \right| > (1 - \pi) \sqrt{T} c_p(n, \delta) \right) + \exp(-C_0 T^{C_1}).$$

5.2 Extension to panels with weakly exogenous regressors

In the case of panels with lagged dependent variables, the use of OLS residuals for estimation of α could still be justifiable so long as T is sufficiently large, such that the time series bias in the estimated residuals is not too large. This is supported by the Monte Carlo evidence provided for dynamic panels below.

However, the mathematical proofs provided above will not be applicable to the OLS residuals if the panel regression model, (12), contains weakly exogenous regressors, such as lagged values of y_{it} . An alternative approach which avoids some of the technical issues associated with the use of OLS residuals would be to base the estimation of α on the recursive residuals. Specifically, one could consider the recursive residuals defined by

$$\check{\varepsilon}_{it} = y_{it} - \check{\gamma}'_{i,t-1} \mathbf{x}_{it},$$

where

$$\check{\gamma}_{i,t-1} = \left(\sum_{\tau=1}^{t-1} \mathbf{x}_{i\tau} \mathbf{x}'_{i\tau} \right)^{-1} \left(\sum_{\tau=1}^{t-1} \mathbf{x}_{i\tau} y_{i\tau} \right).$$

Then the pair-wise correlations based on these recursive residuals are given by

$$\check{\rho}_{ij} = (T-h)^{-1} \sum_{t=h}^T \check{z}_{it} \check{z}_{jt},$$

where

$$\check{z}_{it} = \frac{\check{e}_{it}}{\check{\sigma}_{it}},$$

and

$$\check{\sigma}_{it}^2 = \frac{1}{t} \sum_{\tau=1}^t \check{e}_{i\tau}^2.$$

Here h is the size of the training period, which needs to be set by the researcher. It is then easily seen that under the cross-sectional independence, $\check{z}_{it} \check{z}_{jt}$ is a martingale process with respect to $\Omega_{ii,t-1}$, where $\Omega_{ii,t-1} = (y_{i\tau}, x_{i\tau}; \text{ for } \tau = t-1, t-2, \dots, 1)$. This and other related results then allow us to apply the mathematical analysis of the previous sections to the recursive residuals, after suitable adjustments. The main open question is what critical value to use when checking the significance of $\check{\rho}_{ij}$, and hence the threshold value in the determination of the indicators δ_{ij} defined above. This issue will not be pursued in this paper.

6 Monte Carlo Simulations

We investigate the small sample properties of our proposed estimator of α , defined by (25), using a number of different simulation designs, allowing for dynamics as well as non-Gaussian errors. We consider the following relatively general dynamic panel data model

$$y_{it} = a_i + \vartheta_i y_{i,t-1} + \gamma_i x_{it} + \varepsilon_{it}, \text{ for } i = 1, 2, \dots, N; t = 2, 3, \dots, T, \quad (29)$$

with exogenous, but serially correlated, regressors:

$$x_{it} = \rho_{ix} x_{i,t-1} + (1 - \rho_{ix}^2)^{1/2} \nu_{it}, \text{ for } i = 1, 2, \dots, N; t = -50, -49, \dots, 0, \dots, T,$$

where $\rho_{ix} \sim IIDU(0.0, 0.95)$, and $\nu_{it} \sim IIDN(0, 1)$, with $x_{i,-50} = \nu_{i,-50}$ for $i = 1, 2, \dots, N$. The first 50 observations are disregarded for all units to minimize the effects of the initial values on the observations used in the estimation.

We consider the following cases: (i) a static panel data model, where $\vartheta_i = 0$, $a_i \sim IIDN(1, 1)$, and $\gamma_i \sim IIDN(1, 1)$, for $i = 1, 2, \dots, N$; and (ii) a dynamic panel data model with exogenous regressors, where $\vartheta_i \sim IIDU(0.0, 0.95)$, $\gamma_i \sim IIDN(1, 1)$, and $a_i \sim IIDN(1, 1)$, for $i = 1, 2, \dots, N$.³ Our estimator is robust to possible correlations between the fixed effects, α_i and the regressors, x_{it} .

³We also consider (iii) a dynamic panel data model with no exogenous regressors, where $\vartheta_i \sim IIDU(0.0, 0.95)$ and $\gamma_i = 0$, for $i = 1, 2, \dots, N$. Simulation results for case (iii) are similar to those for cases (i) and (ii) and are available in the online supplement, Tables S1a-S1d.

We consider two different designs for generating the errors, ε_{it} , both with the same exponent of cross-sectional dependence, α :

Design 1 We draw $N \times 1$ vector $\mathbf{b}_N = (b_1, b_2, \dots, b_N)'$ as *Uniform* (0.7, 0.9) for the first $N_b (\leq N)$ elements, where $N_b = \lceil N^{\alpha_\beta} \rceil$ and set the remaining elements to zero. Then, we construct the correlation matrix \mathbf{R}_N given by

$$\mathbf{R}_N = \mathbf{I}_N + \mathbf{b}_N \mathbf{b}'_N - \mathbf{B}_N^2, \quad (30)$$

where $\mathbf{B}_N = \text{diag}(\mathbf{b}_N)$ is an $N \times N$ diagonal matrix with its i^{th} diagonal element given by b_i . The degree of sparseness of \mathbf{R}_N is determined by the choice of α_β . If $\alpha_\beta = 0$ then $\mathbf{R}_N = \mathbf{I}_N$ and $\alpha = 1/2$, while if $\alpha_\beta = 1$ then all elements of \mathbf{R}_N will be non-zero and we have $\alpha = 1$. For all intermediate values of α_β , \mathbf{R}_N will have a total of $\lceil N^{\alpha_\beta} (N^{\alpha_\beta} - 1) + N \rceil$ non-zero elements. The exact relationship between α and α_β is given by (11). Further, we generate the variances of ε_{it} as $\sigma_{ii} \sim \text{IID } 0.5 [1 + 0.5\chi^2(2)]$, for $i = 1, 2, \dots, N$, and set $\mathbf{D}_N = \text{diag}(\sigma_{ii}, i = 1, 2, \dots, N)$. We now generate ε_{it} so that its correlation matrix is equal to \mathbf{R}_N . To this end we first obtain matrix \mathbf{P}_N as the Cholesky factor of \mathbf{R}_N , and then set $\mathbf{W}_N = \mathbf{D}_N^{1/2} \mathbf{P}_N = (w_{ij})$, where w_{ij} is the (i, j) element of \mathbf{W}_N . We then generate ε_{it} as

$$\varepsilon_{it} = \sum_{j=1}^N w_{ij} u_{jt}, \quad i = 1, 2, \dots, N; \quad t = 1, 2, \dots, T, \quad (31)$$

where u_{jt} are *IID* draws from Gaussian or non-Gaussian distributions, to be specified below.⁴

Design 2 The second design closely follows the set up in BKP and employs the two-factor specification given by

$$\varepsilon_{it} = \beta_{i1} f_{1t} + \beta_{i2} f_{2t} + u_{it}, \quad \text{for } i = 1, 2, \dots, N; \quad t = 1, 2, \dots, T, \quad (32)$$

where $f_{jt} \sim \text{IIDN}(0, 1)$, $j = 1, 2$ and $u_{it} \sim \text{IIDN}(0, 1)$ for $i = 1, 2, \dots, N$. With regard to the factor loadings, we generate them as follows:

$$\begin{aligned} \beta_{i1} &= \dot{v}_{i1}, \quad \text{for } i = 1, 2, \dots, \lceil N^{\alpha_{\beta 1}} \rceil \\ \beta_{i1} &= 0, \quad \text{for } i = \lceil N^{\alpha_{\beta 1}} \rceil + 1, \lceil N^{\alpha_{\beta 1}} \rceil + 2, \dots, N \\ \beta_{i2} &= \dot{v}_{i2}, \quad \text{for } i = 1, 2, \dots, \lceil N^{\alpha_{\beta 2}} \rceil, \\ \beta_{i2} &= 0, \quad \text{for } i = \lceil N^{\alpha_{\beta 2}} \rceil + 1, \lceil N^{\alpha_{\beta 2}} \rceil + 2, \dots, N, \end{aligned} \quad (33)$$

where β_{i2} are then randomised across i to achieve independence from β_{i1} . The loadings are generated as $\dot{v}_{ij} \sim \text{IIDU}(\mu_{v_j} - 0.2, \mu_{v_j} + 0.2)$, for $j = 1, 2$. We examine the case where $\alpha_{\beta 2} < \alpha_{\beta 1} = \alpha_\beta$ and consider values of α_β and $\alpha_{\beta 2}$ such that $\alpha_{\beta 2} = \frac{2\alpha_\beta}{3}$. We set $\mu_{v_2} = 0.71$ and $\mu_{v_1} = \sqrt{\mu_v^2 - N^{2(\alpha_{\beta 2} - \alpha_\beta)} \mu_{v_2}^2}$ such that $\mu_{v_1}^2 + \mu_{v_2}^2 = \mu_v^2 = 0.75$ - see BKP for further details. Both μ_{v_1} and μ_{v_2} are chosen such that $\mu_{v_j} \neq 0$, $j = 1, 2$, without $\mu'_{v_j} s$ being too distant from zero either. As before, the exact relationship between α and α_β is given by (11).

⁴Note that $\varepsilon_t = \mathbf{W}_N \mathbf{u}_t$, and $\text{Var}(\varepsilon_t) = \mathbf{W}_N \mathbf{W}'_N = \mathbf{D}_N^{1/2} \mathbf{P}_N \mathbf{P}'_N \mathbf{D}_N^{1/2} = \mathbf{D}_N^{1/2} \mathbf{R}_N \mathbf{D}_N^{1/2}$, as required.

In both designs, we examine two cases for the innovations u_{jt} : (i) Gaussian, where $u_{jt} \sim IIDN(0, 1)$ for $j = 1, 2, \dots, N$; (ii) non-Gaussian, where u_{jt} follows a multivariate t-distribution with v degrees of freedom. This is achieved by generating u_{jt} as

$$u_{jt} = \left(\frac{v-2}{\chi_{v,t}^2} \right)^{1/2} \tilde{v}_{jt}, \text{ for } j = 1, 2, \dots, N,$$

where $\tilde{v}_{jt} \sim IIDN(0, 1)$ and $\chi_{v,t}^2$ is a chi-squared random variate with $v = 8$ degrees of freedom.

For the estimation of α , we first compute the OLS residuals

$$e_{it} = y_{it} - \hat{a}_i - \hat{\vartheta}_i y_{i,t-1} - \hat{\gamma}_i x_{it}, \quad i = 1, 2, \dots, N; \quad t = 2, 3, \dots, T,$$

where \hat{a}_i , $\hat{\vartheta}_i$, and $\hat{\gamma}_i$ are the OLS estimators of the regressions of y_{it} on an intercept, $y_{i,t-1}$ and x_{it} ; computed using the observations $t = 2, 3, \dots, T$, for each i . In the case of the static regressions where $\vartheta_i = 0$, the residuals are computed from regressions of y_{it} on an intercept and x_{it} using the observations $t = 1, 2, \dots, T$. The sample covariance matrix, $\hat{\Sigma}_N = (\hat{\sigma}_{ij})$, is then computed as

$$\hat{\sigma}_{ij} = T^{-1} \sum_{t=2}^T e_{it} e_{jt}, \quad \text{for } i, j = 1, 2, \dots, N,$$

and diagonal elements $\hat{\sigma}_{ii}$ collected in $\hat{D}_N = \text{diag}(\hat{\sigma}_{ii}, i = 1, 2, \dots, N)$. The corresponding sample correlation matrix is then given by

$$\hat{R}_N = \hat{D}_N^{-1/2} \hat{\Sigma}_N \hat{D}_N^{-1/2}.$$

We evaluate the bias and RMSE of the exponent of cross-sectional dependence $\tilde{\alpha}$ computed as in (25) with the critical value, $c_p(n, \delta)$, given by (24). For p and δ we consider the values $p = \{0.05, 0.10\}$ and $\delta = \{1/2, 1/3\}$.⁵ Further, we compute the bias-corrected version of the exponent of cross-sectional dependence estimator developed in BKP, $\hat{\alpha}$, and compare its performance with that of $\tilde{\alpha}$.⁶ However, it is important to bear in mind that BKP provide theoretical justification for their estimator only in the case of demeaned observations, namely $x_{it} - \bar{x}_i$, and do not consider residuals from panel regressions as we do in this paper. As a by-product, this paper also provides Monte Carlo evidence on the properties of the estimator, $\hat{\alpha}$, when applied to residuals from panel regressions.

For all experiments we consider the values of $\alpha = 0.55, 0.60, \dots, 0.90, 0.95, 1.00$, and the sample sizes $N, T = 100, 200, 500$, and carry out all experiments with $R = 2,000$ replications. The values of a_i , ϑ_i and γ_i are drawn randomly in each replication.

⁵The value of $\delta = 1/4$ was also considered. Results are in line with those for $\delta = 1/3$ and are available in the online supplement, Tables S2a-S2d.

⁶Recall that $\hat{\alpha}$ corresponds to the most robust bias-adjusted estimator of the exponent of cross-sectional dependence considered in Bailey et al. (2016) and allows for both serial correlation in the factors and weak cross-sectional dependence in the error terms. It is given by

$$\hat{\alpha} = \hat{\alpha}(\hat{\mu}_v^2) = 1 + \frac{1}{2} \frac{\ln(\hat{\sigma}_x^2)}{\ln(N)} - \frac{\ln(\hat{\mu}_v^2)}{2 \ln(N)} - \frac{\hat{c}_N}{2 [N \ln(N)] \hat{\sigma}_x^2}, \quad (34)$$

where $\hat{\sigma}_x^2$, $\hat{\mu}_v^2$ and \hat{c}_N are consistent estimators of σ_x^2 , μ_v^2 , and c_N - see BKP for further details. We use four principal components when estimating \hat{c}_N .

6.1 Small sample results

First, we consider the small sample performance of our proposed estimator, $\tilde{\alpha}$, and investigate its robustness to different choices of p and δ , that govern the critical value, $c_p(n, \delta)$, used in estimating it. The results for Gaussian errors are provided in Tables A1a to A2b, and the results for non-Gaussian errors are reported in Tables A3a to A4b. Each table reports bias and root mean squared error (RMSE) of $\tilde{\alpha}$ computed using the residuals from either static or dynamic panel data regressions. Tables A1a and A1b give the results for static and dynamic panels, respectively, when the cross-sectional dependence in the errors are generated according to Design 1, whilst the same results for Design 2 are summarized in Tables A2a and A2b. Similarly, the results in Tables A3a and A3b give bias and RMSE of $\tilde{\alpha}$ for static and dynamic panels when the errors are non-Gaussian and the cross correlations are generated according to Design 1, whilst the same results for Design 2 are provided in Tables A4a and A4b. In each of these tables, the left panels give bias and RMSE for $p = 0.05$, and the right panels for $p = 0.10$, whilst the top panels give the results for $\delta = 1/2$, and the bottom panels for $\delta = 1/3$. Specifically, each Table gives four sets of results for the combinations (p, δ) , with $p = 0.05, 0.10$ and $\delta = 1/3, 1/2$.

Comparing the left and right panels of the tables, it is clear that $\tilde{\alpha}$ is robust to the choice of p , irrespective of the value of δ , and for all N and T combinations. Observing that $n = N(N - 1)/2$ is quite large even for moderate values of N , the effective p-value of the underlying individual tests is given by $2p/n^\delta$, which is likely to be dominated by the choice of δ as compared to p . Therefore, the test outcomes are more likely to be robust to the choice of p as compared to δ .

Turning to the choice of δ , comparing the results reported in top and bottom panels of the tables, we note that for all N and T combinations the choice of $\delta = 1/2$ produces smaller bias and RMSE as compared to $\delta = 1/3$ for values of α close to $1/2$ ($\alpha \leq 0.75$). The reverse is true when considering values of α close to unity ($\alpha > 0.80$). Again, this is consistent with the result of Theorem 1 which requires δ to be larger than $1 - \alpha$. Hence, for $\alpha \rightarrow 1/2$ setting $\delta = 1/2$ is more appropriate, while as $\alpha \rightarrow 1$ values of δ below $1/2$ are more appropriate. In cases where there is no priori information regarding the range in which the true value of α might fall, the simulation results suggest setting δ to its upper bound value of $\delta = 1/2$.

Overall, irrespective of whether we consider static or dynamic panel regressions, with Gaussian or non-Gaussian errors, the tabulated results show that the small sample performance of $\tilde{\alpha}$ improves as the true exponent of cross-sectional dependence, α , rises from 0.55 towards 1.0, uniformly over N and T combinations. This finding holds for both Designs, although $\tilde{\alpha}$ generally performs better when Design 1 is used to generate the error cross-sectional dependence. Further, both bias and RMSE of $\tilde{\alpha}$ diminish as N rises for all values of T considered. These results are in line with our main theoretical findings as set out in Theorem 1. It is also interesting to note that under Design 1, the bias and RMSE of $\tilde{\alpha}$ are particularly small for values of α in the range of $0.9 - 1$, even if we consider dynamic panels with non-Gaussian errors. For example, for $T = 100$ and $N = 500$, $p = 0.05$, $\delta = 1/3$, the bias and RMSE of estimating $\alpha = 0.95$ by $\tilde{\alpha}$, in the case of dynamic panels with non-Gaussian errors are -0.00008 and 0.00067 , respectively. (see Table A3b)

Tables A2a and A2b summarize the results for static and dynamic panel data models, respectively, when the error cross-sectional dependence is generated by Design 2 (the two-factor structure). Compared with Tables A1a and A1b, both bias and RMSE are more sizeable across the range of α when $T = 100$. However, as T increases, the performance of $\tilde{\alpha}$ improves for all values of α , especially when α approaches unity, as to be expected. Perhaps, the signal-to-noise ratio implied by (32) becomes somewhat distorted when the T dimension is short and adversely affects the accuracy of the multiple testing procedure used to identify the non-zero elements of the error correlation matrix, \mathbf{R}_N . To verify this conjecture, we repeated the same experiments attaching a scaling parameter of $\varsigma = \sqrt{1/2}$ to u_{it} in (32), in line with the simulation setup in BKP. Performance of $\tilde{\alpha}$ is much improved in this case and comparable to those shown in Tables A1a and A1b, even when T is small.⁷ Our conclusions regarding the robustness of $\tilde{\alpha}$ to the choice of p and δ arrived at under Design 1 continue to hold for Design 2.

We now consider the small sample performance of $\tilde{\alpha}$ relative to that of $\hat{\alpha}$, the estimator of α proposed in BKP. $\hat{\alpha}$ is a biased-corrected estimator of α based on the standard deviation of the cross-sectional average of the residuals. As noted earlier, the asymptotic properties of $\hat{\alpha}$ are established only for demeaned observations, but it is conjectured that these asymptotic properties are likely to hold even if $\hat{\alpha}$ is computed using residuals from panel regressions. For comparison we consider bias and RMSE of $\tilde{\alpha}$ computed using $p = 0.05$ and $\delta = 1/2$, and note that similar results are obtained for other choices of p and δ . Table B1 compares the resulting bias and RMSE of the two α estimators when applied to residuals obtained from a static (top panel) and dynamic panel data models (bottom panel). These results refer to Design 1 with Gaussian errors. Both estimators perform well for all values of α , and irrespective of whether the panel regressions are static or dynamic. This is particularly so for values of $\alpha > 0.8$. In comparative terms, $\tilde{\alpha}$ outperforms $\hat{\alpha}$ on average, for all values of α , and all N and T combinations. The superior performance $\tilde{\alpha}$ is more pronounced when $\alpha \leq 0.75$ uniformly over N and T . The results for Design 2 (where the cross-sectional dependence of the errors are generated using a two-factor specification) are summarized in Table B2 which has the same format as Table B1. In the case of these experiments, $\hat{\alpha}$ (the estimator proposed by BKP) performs better than $\tilde{\alpha}$ when T is small ($T = 100$), but the bias and RMSE of $\tilde{\alpha}$ becomes more comparable to $\hat{\alpha}$ as both N and T rise. As noted above, scaling u_{it} by ς in (32) eliminates this relative outperformance of $\hat{\alpha}$.⁸

Finally, Table C displays bias and RMSE results for estimates of the exponent of cross-sectional dependence, given by (8), and computed using the maximum eigenvalue of the correlation matrices $\hat{\mathbf{R}}_N$ derived from the residuals from a static or dynamic panel data model with Gaussian errors generated under Designs 1 or 2. It is clear that all eigenvalue based estimates of α perform rather poorly even for large values of N and T , and even for values of α close to 1.

Overall, we can conclude that using multiple testing for identifying non-zero elements of $\hat{\mathbf{R}}_N$ when computing $\tilde{\alpha}$ is computationally attractive, has sound theoretical properties and has comparable performance to the exponent of cross-sectional dependence, $\hat{\alpha}$, developed in BKP.

⁷These results are available in the online supplement - see Table S3.

⁸Comparison of $\tilde{\alpha}$ and $\hat{\alpha}$ when errors are non-Gaussian are available in the online supplement - see Tables S4a-S4b.

7 Empirical Application: identifying the weak factor component of CAPM

In their paper BKP investigate the extent to which excess returns on the Standard & Poor's 500 (S&P 500) securities are interconnected through the market factor by computing rolling estimates of α , the degree of cross-sectional dependence of S&P 500 securities. According to asset pricing theories such as the capital asset pricing model (CAPM) of Sharpe (1964) and Lintner (1965), and arbitrage pricing (APT) of Ross (1976), such estimates of α should be close to unity at all times. This is because both CAPM and APT assume security returns have a common factor representations with at least one strong common factor, with the idiosyncratic component being weakly correlated - see also the approximate factor model due to Chamberlain (1983). Such a factor structure implies that all individual stock returns are significantly affected by the common factor(s) and in consequence they are all pair-wise correlated with varying degrees.

The subsequent analysis in BKP reveals that a disconnect between some asset returns and the market factor does occur particularly at times of stock market bubbles and crashes where these asset returns could be driven by non-fundamentals. In this paper, we focus on the exponent of cross-sectional dependence of the residuals obtained from different versions of the CAPM model, and provide rolling estimates of the exponent of the cross-sectional dependence of the errors from CAPM and related APT models. This is important since under CAPM, after allowing for the market factor, the errors can not be cross-sectionally strongly correlated. It is therefore of interest to see if this is in fact true at all times, or if there are episodes where market factors are not sufficient to capture all the significant interdependencies that might exist across the security returns.

We update the BKP analysis and consider monthly excess returns of the securities included in the S&P 500 index over the period from September 1989 to May 2018. We obtain estimates of α using rolling samples of 120 months (10 years) to capture possible time variations in the degree of cross-sectional dependence.⁹ Since the composition of the S&P 500 index changes over time, we compiled returns on all 500 securities at the end of each month and included in our analysis only those securities that had at least 10 years of data in the month under consideration. On average, we ended up with 442 securities at the end of each month for the 10-year rolling samples. The one-month US treasury bill rate was chosen as the risk free rate (r_{ft}), and excess returns were computed as $\tilde{r}_{it} = r_{it} - r_{ft}$, where r_{it} is the monthly return on the i^{th} security in the sample inclusive of dividend payments (if any).¹⁰

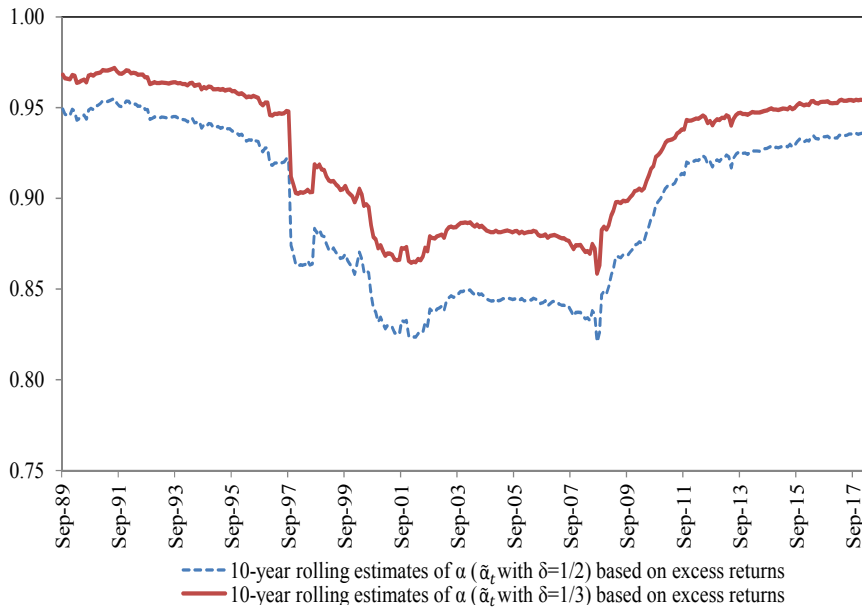
First, following BKP, we estimated α for excess security returns to see the extent to which securities in S&P 500 index are fully interconnected at all times. As noted above, under CAPM we would expect estimates of α to be close to unity. To this end we used our proposed estimator, $\tilde{\alpha}$, defined by (25), and computed 10-year rolling estimates $\tilde{\alpha}_t$, for $t =$ September 1989 to May 2018 - a total of 345 estimates - with $p = 0.05$. To check the robustness of the estimates to the

⁹We also consider rolling samples of size 60 months (5 years). Results for this setting are shown in the online supplement - Figures 3 and 4.

¹⁰For further details of data sources and definitions see Pesaran and Yamagata (2012).

choice of δ we computed the rolling estimates for $\delta = 1/2$ and $1/3$.¹¹ The resultant estimates are shown in Figure 1. As expected, estimates of $\tilde{\alpha}_t$, for $\delta = 1/2$, lie below those of $\tilde{\alpha}_t$, using $\delta = 1/3$, but the series track each other very closely. Also the quantitative differences between the two estimates are not that large. Specifically, all the 345 rolling estimates $\tilde{\alpha}_t$ ($\delta = 1/2$) fall in the interval 0.82 to 0.96, whilst the corresponding estimates $\tilde{\alpha}_t$ ($\delta = 1/3$) all lie in the range 0.86 – 0.97. These estimates show a high degree of inter-linkages across individual securities, and are very close to unity at the start and at the end of the sample, with important departures from unity in between. Considering $\tilde{\alpha}_t$ ($\delta = 1/2$), it recorded lows around 0.86 in 1998 before recovering temporarily and falling further to 0.84. This episode coincides with the Russian and the Long-Term Capital crises of 1998 which originated in bond markets but rapidly transmitted through international equity markets. The $\tilde{\alpha}$ estimates remained low, falling to 0.82 – 0.83, around the turn of the century which saw the burst of the Dotcom bubble and 9/11 terrorist attacks in the US. A slight rise in $\tilde{\alpha}$ to about 0.85 is evident during 2003 – 2007, before a new low of 0.82 was recorded in August 2008, around the time of the sub-prime mortgage crisis in the US, and the ensuing global financial meltdown and economic recession. The estimates of $\tilde{\alpha}$ gradually increase to pre-1997 levels of 0.92 – 0.93 in 2011 and have since remained in this range to the end of our sample, May 2018.

Figure 1: 10-year rolling estimates of the exponent of cross-sectional correlation ($\tilde{\alpha}_t$) of S&P 500 securities’ excess returns



We now turn our attention to the exponent of cross-sectional correlations of the error terms of the CAPM model, and two well known extensions using additional Fama and French factors. Specifically, the first regression is the usual CAPM one-factor representation given by

$$r_{it} - r_{ft} = a_i + \beta_i (r_{mt} - r_{ft}) + u_{1i,t}, \text{ for } i = 1, 2, \dots, N, \quad (35)$$

¹¹For the remaining parameters in (25) we set $p = 0.05$ and $n = N_t(N_t - 1)/2$, where N_t is the number of securities in a given 10-year rolling window ($t = 1, 2, \dots, 345$).

where r_{mt} is the market return computed as the value-weighted returns on all NYSE, AMEX, and NASDAQ stocks. The second and third regressions assume the following extensions to (35) proposed by Fama and French (2004):

$$r_{it} - r_{ft} = a_i + \beta_{1i}(r_{mt} - r_{ft}) + \beta_{2i}smb_t + u_{2i,t}, \text{ for } i = 1, 2, \dots, N \quad (36)$$

and

$$r_{it} - r_{ft} = a_i + \beta_i(r_{mt} - r_{ft}) + \beta_{2i}smb_t + \beta_{3i}hml_t + u_{3i,t}, \text{ for } i = 1, 2, \dots, N, \quad (37)$$

where smb_t stands for average return on the three small portfolios minus the average return on the three big portfolios formed by size, while hml_t refers to the average return on securities with high book value to market value ratio minus the average return of securities with low book value to market value ratio.¹²

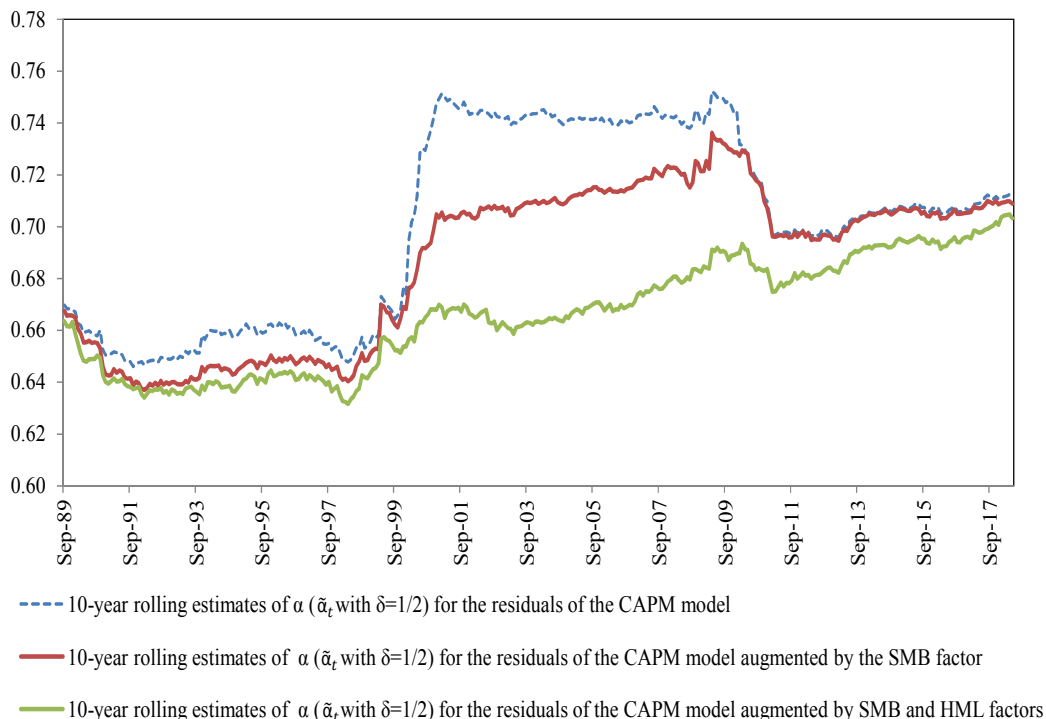
As noted previously, under CAPM we would expect the errors, $u_{1i,t}$, to be cross-sectionally weakly correlated, with α_{u_1} to be close to 1/2. But this need not be the case in reality. In fact the introduction of FF factors, smb_t and hml_t , could be viewed as an attempt to ensure cross-sectionally weakly correlated errors for the augmented CAPM model. It is therefore of interest to consider the estimates of α for the errors, $u_{1i,t}$, $u_{2i,t}$, and $u_{3i,t}$, and see if they are close to 1/2 as required by the theory. To this end, we compute 10-year rolling estimates of α based on the pair-wise correlations of the OLS residuals $\hat{u}_{1i,t}$, $\hat{u}_{2i,t}$ and $\hat{u}_{3i,t}$ in the panel regressions (35), (36) and (37), respectively. These estimates denoted by $\tilde{\alpha}_{\hat{u}_{jt}}$, $j = 1, 2, 3$ for $t =$ September 1989 to May 2018, are shown in Figure 2.¹³

As expected, estimates of α based on the residuals are smaller compared to the estimates obtained for the securities themselves (as depicted in Figure 1). It is also interesting that all the three estimates $\tilde{\alpha}_{\hat{u}_{1t}}$, $\tilde{\alpha}_{\hat{u}_{2t}}$ and $\tilde{\alpha}_{\hat{u}_{3t}}$ are closely clustered over the two sub-periods September 1989 to September 1997, and February 2011 to May 2018, suggesting that the standard CAPM model provides an adequate characterisation of the cross-sectional correlations of securities, and the additional FF factors are not required in these sub-periods. It is also worth noting that, over these two sub-periods, estimates of α fall in the narrow range of 0.63-0.71 which are sufficiently small and support CAPM as an adequate model for characterising cross-correlations of S&P 500 security returns. In contrast, the estimates $\tilde{\alpha}_{\hat{u}_{1t}}$, $\tilde{\alpha}_{\hat{u}_{2t}}$ and $\tilde{\alpha}_{\hat{u}_{3t}}$ tend to diverge over the period from October 1997 to January 2011, and more importantly they all start to rise sharply, suggesting important departures from the basic CAPM model. Using only the market factor, as in (35), results in $\tilde{\alpha}_{\hat{u}_{1t}}$ jumping to levels around 0.74 – 0.76. Adding smb_t to (35) reduces the α estimates of the resulting residuals to 0.69 – 0.73, suggesting that the size portfolio does have some influence on individual security returns during this period. Adding the second FF factor (as in (37)), further reduces the estimates of α to the range 0.66 – 0.68. These results are also in line with the sharp drop in the estimates of α we reported for the excess returns during the period 1998 – 2010, and provide further evidence in favour of the argument that the presence of factors other than the market factor, namely smb_t and hml_t , tend to become relevant during periods of financial crises and turmoils.

¹²For further details of data sources and definitions see Pesaran and Yamagata (2012).

¹³We set $p = 0.05$ and $\delta = 1/2$ when estimating α for the residuals, since *a priori* we would expect the true value of α for the errors of CAPM models to be close to 1/2. See the discussion in Section 6.

Figure 2: 10-year rolling estimates of the exponent of cross-sectional correlation ($\tilde{\alpha}_t$) of residuals from CAPM and its two Fama-French extensions



Notes: CAPM model includes excess market returns, CAPM model augmented by SMB includes excess market returns and small minus big (SMB) firm returns, and CAPM model augmented by SMB and HML includes excess market returns, small minus big (SMB) firm returns and high minus low (HML) firm returns as regressors in (35), (36) and (37), respectively.

8 Conclusions

Cross-sectional dependence and the extent to which it occurs in large multivariate data sets is of great interest for a variety of economic, econometric and financial analyses. Such analyses vary widely. Examples include the effects of idiosyncratic shocks on aggregate macroeconomic variables, the extent to which financial risk can be diversified by investing in disparate assets or asset classes and the performance of standard estimators such as principal components when applied to data sets with unknown collinearity structures. A common characteristic of such analyses is the need to quantify cross-sectional dependence especially when it is prevalent enough to materially affect the outcome of the analysis.

In this paper we generalise the work of Bailey et al. (2016) by proposing a method of measuring the extent of inter-connections in the residuals of large panel data sets in terms of a single parameter. We refer to this as the exponent of cross-sectional dependence of the residuals. We show that this exponent can be used to characterize the degree of sparsity of correlation matrices, or the prevalence of factors in a multi-factor representations routinely used in economic and financial analysis. We propose a simple consistent estimator of the cross-

sectional exponent and derive the rate at which it approaches its true value.

A detailed Monte Carlo study suggests that the proposed estimator has desirable small sample properties especially when $\alpha > 3/4$. We apply our measure to the widely analysed Standard & Poor's 500 data set. We find that for individual securities in S&P 500 index, the 10-year rolling estimates of cross-sectional exponents are sufficiently close to unity over the two sub-periods 1989 – 1997 and 2011 – 2018, but not during the intervening period 1998 – 2010, when markets have been subject to a number of financial turmoils, starting with the LTCM crisis and Dotcom bubble, and ending with the credit crunch of 2007 – 2008. These results carry over when we consider the cross-sectional dependence of errors from the CAPM model and its multi-factor extensions using Fama-French factors. Estimates of α based on the residuals from the CAPM model lend support to CAPM during the sub-periods 1989 – 1997 and 2011 – 2018, but not when we consider the period 1998 – 2010.

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Appendices

Appendix I: Statement of Lemmas

Lemma 1 Consider the panel data regression model (12) and suppose that Assumptions 1-4 hold. Then,

$$\sup_{i,j} \Pr \left(\left| \sum_{t=1}^T z_{it} z_{jt} - E(z_{it} z_{jt} | \Omega_{i,j,t}) \right| > \sqrt{T} c_p(n, \delta) \right) \leq \sup_{i,j} \Pr \left(\left| \sum_{t=1}^T \xi_{it} \xi_{jt} - E(\xi_{is} \xi_{js} | \Omega_{i,j,t}) \right| > (1 - \pi) \sqrt{T} c_p(n, \delta) \right) + \exp(-C_0 T^{C_1}),$$

for some $C_0, C_1 > 0$, where $z_{it} = \frac{e_{it}}{(T^{-1} \mathbf{e}'_i \mathbf{M}_i \mathbf{e}_i)^{1/2}}$.

Lemma 2 Let

$$W_{NT}^0 = N^{-2\alpha} \sum_{i \neq j} I \left(\left| T^{-1} \sum_{t=1}^T z_{it} z_{jt} \right| > \frac{c_p(n, \delta)}{\sqrt{T}} \mid \rho_{ij} = 0 \right)$$

where $z_{it} = \frac{e_{it}}{(T^{-1} \mathbf{e}'_i \mathbf{M}_i \mathbf{e}_i)^{1/2}}$. Under Assumptions 1-4,

$$E(W_{NT}^0) = O(N^{2(1-\alpha-\varkappa\delta)}) + O(N^{2(1-\alpha)} \exp(-C_0 T^{C_1})),$$

for any $0 < \varkappa < 1$, and some $C_0, C_1 > 0$, where δ is defined below (23).

Lemma 3 Let

$$W_{NT}^1 = N^{-2\alpha} \sum_{i \neq j} I \left(\left| T^{-1} \sum_{t=1}^T z_{it} z_{jt} \right| \leq \frac{c_p(n, \delta)}{\sqrt{T}} \mid \rho_{ij} \neq 0 \right)$$

Under Assumptions 1-4, and as long as $N = o(\exp(T))$, $\inf_{i,j} |E(\hat{\rho}_{ij})| > 0$,

$$E(W_{NT}^1) = O(\exp(-C_0 T^{C_1})),$$

for some $C_0, C_1 > 0$, where δ is defined below (23) and $z_{it} = \frac{e_{it}}{(T^{-1} \mathbf{e}'_i \mathbf{M}_i \mathbf{e}_i)^{1/2}}$.

Appendix II: Proofs of Lemmas

Proof of Lemma 1

Recall $a_{i,t} = (a_{i,t1}, \dots, a_{i,tT})' = \mathbf{X}_i (\mathbf{X}'_i \mathbf{X}_i)^{-1} \mathbf{x}_{it}$. Then, using (15),

$$\begin{aligned} \sum_{t=1}^T z_{it} z_{jt} &= \sum_{t=1}^T \left(\xi_{it} + \sum_{s=1}^T a_{i,ts} \xi_{is} \right) \left(\xi_{jt} + \sum_{s=1}^T a_{j,ts} \xi_{js} \right) = \\ &= \sum_{t=1}^T \xi_{it} \xi_{jt} + \sum_{t=1}^T \xi_{it} \left(\sum_{s=1}^T a_{j,ts} \xi_{js} \right) + \\ &+ \sum_{t=1}^T \xi_{jt} \left(\sum_{s=1}^T a_{i,ts} \xi_{is} \right) + \\ &+ \sum_{t=1}^T \left(\sum_{s=1}^T a_{i,ts} \xi_{is} \right) \left(\sum_{s'=1}^T a_{j,ts'} \xi_{js'} \right) \\ &= \sum_{i=1}^4 A_i. \end{aligned}$$

We focus on A_4 . A_2 and A_3 can be treated similarly. We have

$$\begin{aligned} \sum_{t=1}^T \left(\sum_{s=1}^T a_{i,ts} \xi_{is} \right) \left(\sum_{s'=1}^T a_{j,ts'} \xi_{js'} \right) &= \sum_{s=1}^T \sum_{s'=1}^T \xi_{is} \xi_{js'} \sum_{t=1}^T a_{i,ts} a_{j,ts'} \\ \sum_{s=1}^T \sum_{s'=1}^T \xi_{is} \xi_{js'} &= \sum_{s=1}^T \xi_{is} \xi_{js} + \sum_{s=1}^T \xi_{is} \left(\sum_{s \neq s', s'=1}^T \xi_{js'} \right) \end{aligned}$$

Further,

$$\begin{aligned} \sum_{s=1}^T \sum_{s'=1}^T \xi_{is} \xi_{js'} \sum_{t=1}^T a_{i,ts} a_{j,ts'} &= \sum_{s=1}^T \xi_{is} \xi_{js} \left(\sum_{t=1}^T a_{i,ts} a_{j,ts'} \right) + \\ &\quad \sum_{s=1}^T \xi_{is} \left(\sum_{s \neq s', s'=1}^T \xi_{js'} \right) \left(\sum_{t=1}^T a_{i,ts} a_{j,ts'} \right). \end{aligned}$$

Define

$$a_{T,ij,ss'} = T \sum_{t=1}^T a_{i,ts} a_{j,ts'}.$$

By Assumption 2, $\sup_{T,i,j,s,s'} a_{T,ij,ss'} < K < \infty$. Further, define

$$\tilde{\xi}_{1,ij,s} = \xi_{is} \xi_{js} a_{T,ij,ss'}$$

and

$$\tilde{\xi}_{2,ij,s} = \xi_{is} \left(\frac{1}{\sqrt{T}} \sum_{s \neq s', s'=1}^T \xi_{js'} \right) a_{T,ij,ss'}.$$

Then,

$$\begin{aligned} \sum_{s=1}^T \xi_{is} \xi_{js} \left(\sum_{t=1}^T a_{i,ts} a_{j,ts'} \right) + \sum_{s=1}^T \xi_{is} \left(\sum_{s \neq s', s'=1}^T \xi_{js'} \right) \left(\sum_{t=1}^T a_{i,ts} a_{j,ts'} \right) &= \\ \frac{1}{T} \sum_{s=1}^T \tilde{\xi}_{1,ij,s} + \frac{1}{\sqrt{T}} \sum_{s=1}^T \tilde{\xi}_{2,ij,s}. \end{aligned}$$

It can be easily seen that $\tilde{\xi}_{1,ij,s} - E(\tilde{\xi}_{1,ij,s} | \Omega_{i,j,t})$ and $\tilde{\xi}_{2,ij,s} - E(\tilde{\xi}_{2,ij,s} | \Omega_{i,j,t})$ are martingale difference series with finite variances, and that if $\rho_{ij} = 0$ then $E(\tilde{\xi}_{1,ij,s} | \Omega_{i,j,t}) = E(\tilde{\xi}_{2,ij,s} | \Omega_{i,j,t}) = 0$. Define $c_{ij,t} = E(z_{it} z_{jt} | \Omega_{i,j,t})$. Then, using Lemma A3 of Chudik et al. (2018), it easily follows that

$$\begin{aligned} \sup_{i,j} \Pr \left(\left| \sum_{t=1}^T z_{it} z_{jt} - c_{ij,t} \right| > \sqrt{T} c_p(n, \delta) \right) &\leq \sup_{i,j} \Pr \left(\left| \sum_{t=1}^T \xi_{it} \xi_{jt} - E(\xi_{is} \xi_{js} | \Omega_{i,j,t}) \right| > (1 - \pi) \sqrt{T} c_p(n, \delta) \right) \\ &\quad + \exp(-C_0 T^{C_1}), \end{aligned}$$

for some $C_0, C_1 > 0$.

Proof of Lemma 2

Note that W_{NT}^0 can be written as

$$W_{NT}^0 = N^{-2\alpha} \sum_{i \neq j} I \left(\left| T^{-1} \sum_{t=1}^T z_{it} z_{jt} \right| > \frac{c_p(n, \delta)}{\sqrt{T}} \mid \rho_{ij} = 0 \right)$$

Since by assumption ξ_{it} and ξ_{jt} are distributed independently when $\rho_{ij} = 0$, it also follows that

$$\begin{aligned} E(z_{it} z_{jt} | \Omega_{i,j,t}) &= E(\xi_{it} \xi_{jt} | \Omega_{i,j,t}) + E(\xi_{it} q_{jt} | \Omega_{i,j,t}) + E(\xi_{jt} q_{it} | \Omega_{i,j,t}) + E(q_{it} q_{jt} | \Omega_{i,j,t}) \\ &= 0, \end{aligned}$$

and hence $\{z_{it}z_{jt}, \Omega_{i,j,t}\}$, is a zero mean martingale difference sequence, where $\Omega_{i,j,t} = \Omega_{i,t} \cup \Omega_{j,t}$. Then, we note that $z_{it}z_{jt}$ is a normalised process since $z_{it} = \frac{\epsilon_{it}}{(T^{-1}\mathbf{e}'_i\mathbf{e}_i)^{1/2}}$. Then, noting Lemma 1, using Chudik et al. (2018), and, in particular, their Lemmas A3 (which provides a martingale difference exponential probability inequality under (26)), A4 (which handles exponential probability tails for products of random variables) and A9 (which handles the normalisation by $(T^{-1}\mathbf{e}'_i\mathbf{e}_i)^{1/2}$), we have for any $0 < \pi < 1$, any bounded sequence, $d_T > 0$, and some $C_0, C_1 > 0$,

$$\sup_{i,j} \Pr \left(\left| \sum_{t=1}^T z_{it}z_{jt} \right| > \sqrt{T}c_p(n, \delta) \mid \rho_{ij} = 0 \right) \leq \exp \left[-\frac{(1-\pi)^2 c_p(n, \delta)^2}{2(1+d_T)} \right] + \exp(-C_0 T^{C_1}). \quad (38)$$

Note that $\frac{(1-\pi)^2}{1+d_T} < 1$, but can be made arbitrarily close to 1, and that, by Lemma A2 of Chudik et al. (2018), $\exp[-bc_p^2(n, \delta)] = O(n^{-2b\delta})$. Then, for any $0 < \varkappa < 1$, and some $C_0, C_1 > 0$,

$$\sup_{i,j} \Pr \left(\left| \sum_{t=1}^T z_{it}z_{jt} \right| > \sqrt{T}c_p(n, \delta) \mid \rho_{ij} = 0 \right) = O(n^{-\varkappa\delta}) + O(\exp(-C_0 T^{C_1})) = O(N^{-2\varkappa\delta}) + O(\exp(-C_0 T^{C_1})).$$

Then, for some $C_0, C_1 > 0$, any $0 < \varkappa < 1$, and if $N = O(T^d)$

$$\begin{aligned} E(W_{NT}^0) &= N^{-2\alpha} \sum_{i \neq j}^N \Pr \left(\left| T^{-1} \sum_{t=1}^T z_{it}z_{jt} \right| > \frac{c_p(n, \delta)}{\sqrt{T}} \mid \rho_{ij} = 0 \right) = O(N^{2(1-\alpha-\varkappa\delta)}) + O(N^{2(1-\alpha)} \exp(-C_0 T^{C_1})) = \\ &= O(T^{2d(1-\alpha-\varkappa\delta)}) + O(\exp[2d(1-\alpha)\ln(T) - C_0 T^{C_1}]). \end{aligned}$$

Proof of Lemma 3

We need to derive $h_{N,T}$ in

$$\Pr \left[\left| \sum_{t=1}^T z_{it}z_{jt} \right| \leq \sqrt{T}c_p(n, \delta) \mid \rho_{ij} \neq 0 \right] = O(h_{N,T}).$$

Let $c_{ij,t} = E(z_{it}z_{jt} \mid \Omega_{i,j,t})$, and note the inequality

$$\Pr(|X + B| \leq C) \leq \Pr(|X| > |B| - C),$$

where X is a random variable, B and C are constants, and $|B| \geq C > 0$. Then, for some $0 < \pi < 1$

$$\begin{aligned} &\Pr \left[\left| \sum_{t=1}^T (z_{it}z_{jt} - c_{ij,t}) + c_{ij,t} - E(c_{ij,t}) + E(c_{ij,t}) \right| \leq \sqrt{T}c_p(n, \delta) \mid \rho_{ij} \neq 0 \right] \\ &\leq \Pr \left[\left| \sum_{t=1}^T (z_{it}z_{jt} - c_{ij,t}) + c_{ij,t} - E(c_{ij,t}) \right| > \left| \sum_{t=1}^T E(c_{ij,t}) \right| - \sqrt{T}c_p(n, \delta) \mid \rho_{ij} \neq 0 \right] \leq \\ &\leq \Pr \left[\left| \sum_{t=1}^T (z_{it}z_{jt} - c_{ij,t}) \right| > (1-\pi) \left[\left| \sum_{t=1}^T E(c_{ij,t}) \right| - \sqrt{T}c_p(n, \delta) \right] \mid \rho_{ij} \neq 0 \right] + \end{aligned} \quad (39)$$

$$\Pr \left[\left| \sum_{t=1}^T c_{ij,t} - E(c_{ij,t}) \right| > \pi \left[\left| \sum_{t=1}^T E(c_{ij,t}) \right| - \sqrt{T}c_p(n, \delta) \right] \mid \rho_{ij} \neq 0 \right] \quad (40)$$

But, by (28), (40) is bounded by $\exp(-C_0 T^{C_1})$ for some $C_0, C_1 > 0$. We consider

$$\Pr \left[\left| \sum_{t=1}^T (z_{it}z_{jt} - c_{ij,t}) \right| > (1-\pi) \left[\left| \sum_{t=1}^T E(c_{ij,t}) \right| - \sqrt{T}c_p(n, \delta) \right] \mid \rho_{ij} \neq 0 \right]$$

But, by (20) of Assumption 1, $\lim_{N,T \rightarrow \infty} \frac{\sqrt{T}c_p(n, \delta)}{\sum_{t=1}^T E(c_{ij,t})} = 0$. Therefore, using again Lemma 1 and (38) of Lemma 2, we have

$$\sup_{i,j} \Pr \left[\left| \sum_{t=1}^T (z_{it}z_{jt} - c_{ij,t}) \right| > (1-\pi) \left[\left| \sum_{t=1}^T E(c_{ij,t}) \right| - \sqrt{T}c_p(n, \delta) \right] \mid \rho_{ij} \neq 0 \right] \leq \exp(-CT),$$

for some $C > 0$, proving the result.

Appendix III: Proof of Theorem 1

We prove that $\tilde{\alpha}$ converges to α under our assumptions of exogenous regressors and symmetrically distributed errors. We first note that

$$\begin{aligned}\tilde{\alpha} &= \frac{1}{2} \frac{\ln \left(\frac{\boldsymbol{\tau}' \tilde{\boldsymbol{\Delta}} \boldsymbol{\tau}}{N^{2\alpha}} \right)}{\ln N} = \frac{1}{2} \frac{\ln \left(\frac{\boldsymbol{\tau}' \tilde{\boldsymbol{\Delta}} \boldsymbol{\tau}}{N^{2\alpha}} \right)}{\ln N} + \alpha, \\ (\ln N) (\tilde{\alpha} - \alpha) &= \frac{1}{2} \ln \left(\frac{\boldsymbol{\tau}' \tilde{\boldsymbol{\Delta}} \boldsymbol{\tau}}{N^{2\alpha}} - 1 + 1 \right),\end{aligned}\quad (41)$$

and

$$2(\ln N) (\tilde{\alpha} - \alpha) = \frac{\boldsymbol{\tau}' \tilde{\boldsymbol{\Delta}} \boldsymbol{\tau}}{N^{2\alpha}} - 1 + O \left(\left[\frac{\boldsymbol{\tau}' \tilde{\boldsymbol{\Delta}} \boldsymbol{\tau}}{N^{2\alpha}} - 1 \right]^2 \right).$$

Further

$$N^{-2\alpha} \boldsymbol{\tau}' \tilde{\boldsymbol{\Delta}} \boldsymbol{\tau} = N^{-2\alpha} \sum_{i \neq j}^N I \left(|\hat{\rho}_{ij}| > \frac{c_p(n, \delta)}{\sqrt{T}} \right),$$

and

$$\begin{aligned}N^{-2\alpha} \boldsymbol{\tau}' \tilde{\boldsymbol{\Delta}} \boldsymbol{\tau} - 1 &= N^{-2\alpha} \sum_{i \neq j}^N I \left(|\hat{\rho}_{ij}| > \frac{c_p(n, \delta)}{\sqrt{T}} \mid \rho_{ij} = 0 \right) \\ &\quad + \left(N^{-2\alpha} \sum_{i \neq j}^N I \left(|\hat{\rho}_{ij}| > \frac{c_p(n, \delta)}{\sqrt{T}} \mid \rho_{ij} \neq 0 \right) - 1 \right) \\ &= W_{NT}^0 + W_{NT}^1 + O(N^{1-2\alpha}).\end{aligned}\quad (42)$$

We now have

$$\begin{aligned}N^{-2\alpha} \sum_{i \neq j}^N I \left(|\hat{\rho}_{ij}| > \frac{c_p(n, \delta)}{\sqrt{T}} \mid \rho_{ij} \neq 0 \right) - 1 \\ &= N^{-2\alpha} (N^{2\alpha} - N^\alpha) - 1 - N^{-2\alpha} \sum_{i \neq j}^N I \left(|\hat{\rho}_{ij}| \leq \frac{c_p(n, \delta)}{\sqrt{T}} \mid \rho_{ij} \neq 0 \right) \\ &= -N^{-2\alpha} \sum_{i \neq j}^N I \left(|\hat{\rho}_{ij}| \leq \frac{c_p(n, \delta)}{\sqrt{T}} \mid \rho_{ij} \neq 0 \right) - N^{-\alpha}.\end{aligned}$$

Hence,

$$\begin{aligned}N^{-2\alpha} \boldsymbol{\tau}' \tilde{\boldsymbol{\Delta}} \boldsymbol{\tau} - 1 &= N^{-2\alpha} \sum_{i \neq j}^N I \left(|\hat{\rho}_{ij}| > \frac{c_p(n, \delta)}{\sqrt{T}} \mid \rho_{ij} = 0 \right) - \\ &\quad N^{-2\alpha} \sum_{i \neq j}^N I \left(|\hat{\rho}_{ij}| \leq \frac{c_p(n, \delta)}{\sqrt{T}} \mid \rho_{ij} \neq 0 \right) - N^{-\alpha},\end{aligned}$$

which we write more compactly as

$$N^{-2\alpha} \boldsymbol{\tau}' \tilde{\boldsymbol{\Delta}} \boldsymbol{\tau} - 1 = W_{NT}^0 - \tilde{W}_{NT}^1 + O_p(N^{-\alpha}),$$

where

$$\begin{aligned}W_{NT}^0 &= N^{-2\alpha} \sum_{i \neq j}^N I \left(|\hat{\rho}_{ij}| > \frac{c_p(n, \delta)}{\sqrt{T}} \mid \rho_{ij} = 0 \right), \\ \tilde{W}_{NT}^1 &= N^{-2\alpha} \sum_{i \neq j}^N I \left(|\hat{\rho}_{ij}| \leq \frac{c_p(n, \delta)}{\sqrt{T}} \mid \rho_{ij} \neq 0 \right).\end{aligned}$$

It is worth noting that $E |W_{NT}^0| = E (W_{NT}^0)$, and $E |\tilde{W}_{NT}^1| = E (\tilde{W}_{NT}^1)$, and hence,

$$E \left| N^{-2\alpha} \boldsymbol{\tau}' \tilde{\boldsymbol{\Delta}} \boldsymbol{\tau} - 1 \right| \leq E (W_{NT}^0) + E (\tilde{W}_{NT}^1) + O(N^{-\alpha}) + O(N^{1-2\alpha}).$$

Lemmas 2-3 provide bounds for $E (W_{NT}^0)$ and $E (\tilde{W}_{NT}^1)$ proving the result.

Table A1a: Bias and RMSE ($\times 100$) for the $\tilde{\alpha}$ estimate of the cross-sectional exponent of the errors from a static panel data model
 Cross correlations are generated using Design 1 with Gaussian errors

α		0.55	0.60	0.65	0.70	0.75	0.80	0.85	0.90	0.95	1.00	
		$c_p(n, \delta)$ with $n = N(N-1)/2$ and $p = 0.10$										
		$\delta = 1/2$										
T	N	Bias										
100	100	0.204	-0.609	-0.584	-0.355	-0.289	-0.001	-0.259	-0.279	-0.094	-0.016	100
	200	0.066	-0.426	-0.156	-0.302	-0.254	-0.101	-0.165	-0.080	-0.058	-0.025	100
	500	0.090	-0.120	-0.162	-0.144	-0.149	-0.035	-0.042	-0.046	-0.050	-0.041	200
200	100	0.243	-0.584	-0.557	-0.328	-0.267	0.017	-0.240	-0.262	-0.078	0.000	200
	200	0.118	-0.380	-0.120	-0.271	-0.226	-0.074	-0.139	-0.057	-0.035	0.000	200
	500	0.158	-0.064	-0.114	-0.101	-0.106	0.007	-0.001	-0.006	-0.011	0.000	500
500	100	0.277	-0.556	-0.549	-0.325	-0.261	0.019	-0.239	-0.262	-0.078	0.000	500
	200	0.157	-0.367	-0.110	-0.265	-0.223	-0.073	-0.138	-0.056	-0.035	0.000	200
	500	0.189	-0.047	-0.105	-0.097	-0.104	0.009	0.000	-0.006	-0.011	0.000	500
T	N	RMSE										
100	100	0.311	0.633	0.596	0.364	0.296	0.051	0.264	0.282	0.102	0.041	100
	200	0.145	0.438	0.174	0.308	0.261	0.124	0.171	0.093	0.070	0.053	200
	500	0.118	0.137	0.177	0.157	0.165	0.073	0.078	0.076	0.077	0.079	500
200	100	0.347	0.611	0.571	0.338	0.272	0.042	0.242	0.263	0.079	0.000	200
	200	0.183	0.391	0.133	0.274	0.228	0.076	0.140	0.057	0.035	0.000	200
	500	0.173	0.075	0.116	0.102	0.107	0.010	0.004	0.007	0.011	0.000	500
500	100	0.374	0.584	0.563	0.337	0.268	0.043	0.240	0.263	0.078	0.000	500
	200	0.218	0.379	0.126	0.268	0.225	0.075	0.138	0.057	0.035	0.000	200
	500	0.203	0.063	0.108	0.098	0.104	0.011	0.004	0.007	0.011	0.000	500
T	N	Bias										
100	100	1.507	0.288	-0.013	-0.003	-0.072	0.127	-0.177	-0.232	-0.070	-0.005	100
	200	1.433	0.445	0.354	0.011	-0.066	0.010	-0.097	-0.038	-0.032	-0.006	200
	500	1.533	0.715	0.314	0.123	0.008	0.061	0.021	-0.001	-0.015	-0.009	500
200	100	1.585	0.330	0.018	0.030	-0.058	0.141	-0.169	-0.226	-0.065	0.000	200
	200	1.531	0.529	0.403	0.040	-0.046	0.024	-0.087	-0.031	-0.026	0.000	200
	500	1.684	0.813	0.370	0.161	0.032	0.078	0.034	0.009	-0.006	0.000	500
500	100	1.627	0.386	0.041	0.035	-0.044	0.144	-0.167	-0.225	-0.064	0.000	500
	200	1.613	0.564	0.432	0.054	-0.039	0.028	-0.083	-0.030	-0.025	0.000	200
	500	1.783	0.867	0.399	0.173	0.040	0.082	0.037	0.010	-0.005	0.000	500
T	N	RMSE										
100	100	1.571	0.448	0.248	0.170	0.143	0.153	0.187	0.236	0.075	0.015	100
	200	1.461	0.490	0.378	0.098	0.093	0.053	0.102	0.046	0.036	0.016	200
	500	1.541	0.722	0.321	0.133	0.037	0.067	0.032	0.019	0.023	0.022	500
200	100	1.652	0.475	0.255	0.180	0.137	0.167	0.181	0.230	0.068	0.000	200
	200	1.559	0.570	0.426	0.109	0.082	0.052	0.092	0.037	0.028	0.000	200
	500	1.692	0.820	0.377	0.167	0.044	0.081	0.036	0.013	0.007	0.000	500
500	100	1.693	0.524	0.254	0.191	0.138	0.170	0.179	0.229	0.068	0.000	500
	200	1.641	0.604	0.456	0.116	0.080	0.057	0.090	0.037	0.028	0.000	200
	500	1.790	0.874	0.405	0.180	0.051	0.085	0.040	0.014	0.007	0.000	500

Notes: Parameters of the static panel data model, (29), are generated as: $a_i \sim IIDN(1, 1)$, $\rho_{ix} \sim U(0.0, 0.95)$, $\vartheta_i = 0$ and $\gamma_i \sim IIDN(1, 1)$, for $i = 1, 2, \dots, N$. Gaussian errors are generated as $u_{it} \sim IIDN(0, 1)$ in (31). Design 1 assumes $b_i \sim U(0.7, 0.9)$ for the first $N_b (\leq N)$ elements of vector \mathbf{b}_N , $i = 1, 2, \dots, N$, in the construction of the correlation matrix of the errors, $\mathbf{R}_N = \mathbf{I}_N + \mathbf{b}_N \mathbf{b}'_N - \mathbf{B}_N^2$ given by (30), where $\mathbf{B}_N = \text{diag}(\mathbf{b}_N)$. $c_p(n, \delta)$ corresponds to the critical value used in the multiple testing procedure shown in (23). The number of replications is set to $R = 2000$.

Table A1b: Bias and RMSE ($\times 100$) for the $\tilde{\alpha}$ estimate of the cross-sectional exponent of the errors from a dynamic panel data model with

exogenous regressors

Cross correlations are generated using Design 1 with Gaussian errors

α		0.55	0.60	0.65	0.70	0.75	0.80	0.85	0.90	0.95	1.00	
		$c_p(n, \delta)$ with $n = N(N-1)/2$ and $p = 0.10$										
		$\delta = 1/2$										
T	N	Bias						RMSE				
100	100	0.198	-0.621	-0.594	-0.363	-0.295	-0.007	-0.261	-0.284	-0.099	-0.021	100
200	200	0.064	-0.432	-0.161	-0.309	-0.262	-0.108	-0.173	-0.088	-0.070	-0.033	200
500	500	0.084	-0.123	-0.169	-0.158	-0.046	-0.054	-0.059	-0.066	-0.049	500	
200	100	0.255	-0.573	-0.561	-0.330	-0.266	0.018	-0.240	-0.263	-0.078	0.000	200
200	200	0.123	-0.381	-0.118	-0.272	-0.225	-0.074	-0.139	-0.056	-0.035	0.000	200
500	500	0.156	-0.063	-0.113	-0.101	-0.106	0.007	-0.001	-0.007	-0.011	0.000	500
500	100	0.281	-0.550	-0.550	-0.322	-0.261	0.020	-0.240	-0.261	-0.078	0.000	500
200	200	0.153	-0.366	-0.108	-0.266	-0.222	-0.072	-0.137	-0.056	-0.035	0.000	200
500	500	0.185	-0.049	-0.104	-0.097	-0.104	0.009	0.000	-0.006	-0.011	0.000	500
T	N	Bias						RMSE				
100	100	0.302	0.644	0.607	0.375	0.305	0.056	0.266	0.289	0.108	0.045	100
200	200	0.153	0.444	0.180	0.316	0.272	0.126	0.183	0.106	0.094	0.064	200
500	500	0.117	0.144	0.185	0.179	0.174	0.102	0.102	0.107	0.090	0.090	500
200	100	0.354	0.600	0.573	0.339	0.272	0.043	0.241	0.264	0.079	0.000	200
200	200	0.190	0.392	0.131	0.274	0.226	0.076	0.139	0.057	0.035	0.000	200
500	500	0.169	0.074	0.116	0.102	0.107	0.010	0.004	0.007	0.011	0.000	500
500	100	0.375	0.580	0.563	0.332	0.267	0.045	0.241	0.262	0.079	0.000	500
200	200	0.213	0.378	0.123	0.270	0.224	0.074	0.138	0.057	0.035	0.000	200
500	500	0.198	0.065	0.107	0.098	0.104	0.011	0.004	0.007	0.011	0.000	500
T	N	Bias						RMSE				
100	100	1.509	0.279	-0.026	-0.005	-0.074	0.125	-0.177	-0.233	-0.071	-0.006	100
200	200	1.431	0.450	0.359	0.009	-0.069	0.009	-0.099	-0.040	-0.035	-0.008	200
500	500	1.532	0.716	0.311	0.119	0.007	0.059	0.018	-0.004	-0.019	-0.011	500
200	100	1.569	0.341	0.017	0.027	-0.054	0.142	-0.168	-0.227	-0.065	0.000	200
200	200	1.552	0.529	0.407	0.042	-0.045	0.026	-0.085	-0.030	-0.026	0.000	200
500	500	1.686	0.819	0.371	0.158	0.032	0.078	0.034	0.010	-0.006	0.000	500
500	100	1.634	0.384	0.040	0.040	-0.040	0.146	-0.167	-0.223	-0.064	0.000	500
200	200	1.609	0.563	0.432	0.057	-0.038	0.029	-0.083	-0.030	-0.025	0.000	200
500	500	1.780	0.864	0.401	0.174	0.041	0.082	0.036	0.010	-0.005	0.000	500
T	N	Bias						RMSE				
100	100	1.576	0.441	0.242	0.165	0.144	0.150	0.187	0.238	0.076	0.015	100
200	200	1.459	0.493	0.383	0.095	0.098	0.050	0.105	0.049	0.043	0.020	200
500	500	1.541	0.724	0.319	0.130	0.036	0.069	0.037	0.029	0.033	0.026	500
200	100	1.632	0.486	0.247	0.175	0.135	0.169	0.180	0.230	0.069	0.000	200
200	200	1.579	0.568	0.430	0.109	0.084	0.053	0.091	0.037	0.028	0.000	200
500	500	1.694	0.826	0.377	0.165	0.045	0.081	0.037	0.013	0.007	0.000	500
500	100	1.700	0.529	0.262	0.182	0.134	0.171	0.179	0.228	0.068	0.000	500
200	200	1.636	0.602	0.454	0.116	0.081	0.056	0.090	0.037	0.028	0.000	200
500	500	1.787	0.871	0.407	0.180	0.052	0.085	0.039	0.014	0.007	0.000	500

Notes: Parameters of the dynamic panel data model, (29), are generated as: $a_i \sim IIDN(1, 1)$, $\rho_{ix} \sim U(0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0)$ and $\gamma_i \sim IIDN(1, 1)$, for $i = 1, 2, \dots, N$. Gaussian errors are generated as $u_{it} \sim IIDN(0, 1)$ in (31). Design 1 assumes $b_i \sim U(0, 7, 0, 9)$ for the first $N_b (\leq N)$ elements of vector \mathbf{b}_N , $i = 1, 2, \dots, N$, in the construction of the correlation matrix of the errors, $\mathbf{R}_N = \mathbf{I}_N + \mathbf{b}_N \mathbf{b}'_N - \mathbf{B}_N^2$ given by (30), where $\mathbf{B}_N = \text{diag}(\mathbf{b}_N)$. $c_p(n, \delta)$ corresponds to the critical value used in the multiple testing procedure shown in (23). The number of replications is set to $R = 2000$.

Table A2a: Bias and RMSE ($\times 100$) for the $\tilde{\alpha}$ estimate of the cross-sectional exponent of the errors from a static panel data model

		Cross correlations are generated using Design 2 with Gaussian errors											
α		0.55	0.60	0.65	0.70	0.75	0.80	0.85	0.90	0.95	1.00		
		$c_p(n, \delta)$ with $n = N(N-1)/2$ and $p = 0.10$											
		$\delta = 1/2$											
T	N	Bias											
100	100	-0.880	-2.378	-2.922	-3.026	-3.232	-3.131	-3.509	-3.648	-3.576	-3.598	100	100
200	100	-1.221	-2.480	-2.800	-3.267	-3.464	-3.434	-3.595	-3.591	-3.658	-3.714	200	100
500	100	-1.501	-2.658	-3.266	-3.581	-3.780	-3.779	-3.859	-3.934	-4.011	-4.057	500	100
200	100	0.705	-0.090	-0.160	0.120	0.009	0.205	-0.145	-0.299	-0.242	-0.304	200	100
200	200	0.375	-0.068	0.134	-0.086	-0.146	-0.063	-0.191	-0.206	-0.269	-0.326	200	200
500	100	0.219	-0.024	-0.086	-0.158	-0.206	-0.153	-0.212	-0.267	-0.325	-0.368	500	100
500	100	1.043	0.295	0.247	0.563	0.433	0.625	0.264	0.084	0.114	-0.001	500	100
200	200	0.683	0.313	0.550	0.326	0.254	0.329	0.203	0.161	0.087	-0.001	200	200
500	200	0.530	0.355	0.331	0.249	0.205	0.246	0.176	0.117	0.056	-0.001	500	200
T	N	RMSE											
100	100	1.252	2.688	3.301	3.469	3.699	3.623	3.974	4.094	4.029	4.051	100	100
200	100	1.455	2.748	3.167	3.655	3.868	3.859	4.002	4.009	4.070	4.130	200	100
500	100	1.662	2.899	3.580	3.938	4.153	4.176	4.260	4.336	4.411	4.456	500	100
200	100	0.803	0.377	0.369	0.346	0.298	0.354	0.312	0.405	0.359	0.399	200	100
200	200	0.457	0.291	0.307	0.274	0.296	0.251	0.315	0.318	0.367	0.415	200	200
500	100	0.303	0.241	0.262	0.299	0.328	0.295	0.326	0.366	0.418	0.456	500	100
500	100	1.068	0.337	0.272	0.571	0.442	0.631	0.279	0.116	0.136	0.002	500	100
200	200	0.697	0.325	0.553	0.329	0.257	0.332	0.208	0.166	0.093	0.001	200	200
500	200	0.534	0.357	0.332	0.250	0.206	0.246	0.177	0.118	0.057	0.001	500	200
T	N	RMSE											
100	100	1.121	-0.410	-1.023	-1.161	-1.437	-1.372	-1.789	-1.971	-1.926	-1.985	100	100
200	100	0.926	-0.377	-0.770	-1.305	-1.554	-1.571	-1.758	-1.790	-1.871	-1.940	200	100
500	100	0.905	-0.329	-1.008	-1.398	-1.634	-1.672	-1.775	-1.859	-1.939	-1.991	500	100
200	100	1.120	0.962	0.590	0.671	0.425	0.538	0.134	-0.064	-0.038	-0.131	200	100
200	200	1.900	0.990	0.855	0.438	0.246	0.250	0.080	0.027	-0.056	-0.130	200	200
500	100	1.870	1.040	0.622	0.337	0.173	0.156	0.059	-0.017	-0.085	-0.135	500	100
500	100	2.305	1.160	0.797	0.898	0.641	0.748	0.338	0.123	0.131	0.000	500	100
200	200	2.073	1.186	1.057	0.633	0.434	0.431	0.261	0.191	0.099	0.000	200	200
500	200	2.060	1.231	0.815	0.514	0.348	0.322	0.216	0.137	0.064	0.000	500	200
T	N	RMSE											
100	100	1.349	0.991	1.458	1.608	1.841	1.798	2.153	2.306	2.267	2.325	100	100
200	100	1.102	0.867	1.210	1.653	1.883	1.906	2.061	2.097	2.170	2.241	200	100
500	100	1.017	0.766	1.325	1.693	1.916	1.966	2.063	2.144	2.220	2.272	500	100
200	100	2.169	1.026	0.649	0.711	0.469	0.568	0.216	0.175	0.153	0.187	200	100
200	200	1.922	1.014	0.873	0.463	0.284	0.281	0.155	0.125	0.135	0.181	200	200
500	100	1.878	1.049	0.633	0.357	0.208	0.192	0.124	0.111	0.144	0.180	500	100
500	100	2.345	1.200	0.824	0.910	0.651	0.755	0.350	0.149	0.150	0.001	500	100
200	200	2.091	1.200	1.063	0.638	0.437	0.434	0.266	0.195	0.105	0.000	200	200
500	200	2.065	1.234	0.817	0.515	0.349	0.322	0.217	0.138	0.065	0.000	500	200

Notes: Parameters of the static panel data model, (29), are generated as: $a_i \sim IIDN(1, 1)$, $\rho_{ix} \sim U(0, 0.95)$, $\vartheta_i = 0$ and $\gamma_i \sim IIDN(1, 1)$, for $i = 1, 2, \dots, N$. Design 2 assumes a two-factor model with $[N_{\alpha\beta}]$ and $[N_{\alpha\beta_2}]$ non-zero loadings for the first and second factor, respectively. We set: $\alpha_{\beta 2} = 2\alpha_{\beta}/3$, where α_{β} relates to α under (11), f_{jt} and $u_{it} \sim IIDN(0, 1)$, $v_{ij} \sim IIDU(\mu_{v_j} - 0.2, \mu_{v_j} + 0.2)$, $j = 1, 2$, $\mu_v = 0.87$, $\mu_{v_2} = 0.71$, $\mu_{v_1} = \sqrt{\mu_v^2 - N^2(\alpha_{\beta 2} - \alpha_{\beta})\mu_{v_2}^2}$, in (32) and (33). The number of replications is set to $R = 2000$.

Table A3a: Bias and RMSE ($\times 100$) for the $\tilde{\alpha}$ estimate of the cross-sectional exponent of the errors from a static panel data model

Cross correlations are generated using Design 1 with non-Gaussian errors											
$c_p(n, \delta)$ with $n = N(N-1)/2$ and $p = 0.10$											
$\delta = 1/2$											
Bias											
RMSE											
$\delta = 1/3$											
Bias											
RMSE											
$\delta = 1/2$											
Bias											
RMSE											
$\delta = 1/3$											
Bias											
RMSE											
α	0.55	0.60	0.65	0.70	0.75	0.80	0.85	0.90	0.95	1.00	
	$c_p(n, \delta)$ with $n = N(N-1)/2$ and $p = 0.05$										
	$\delta = 1/2$										
	Bias										
	RMSE										
	$\delta = 1/2$										
	Bias										
	RMSE										
	$\delta = 1/3$										
	Bias										
	RMSE										
	$\delta = 1/2$										
	Bias										
	RMSE										
	$\delta = 1/3$										
	Bias										
	RMSE										

Notes: Parameters of the static panel data model, (29), are generated as: $a_i \sim IIDN(1, 1)$, $\rho_{ix} \sim U(0, 0.95)$, $\vartheta_i = 0$ and $\gamma_i \sim IIDN(1, 1)$, for $i = 1, 2, \dots, N$. Non-Gaussian errors are generated as $u_{it} = \left(\frac{v_{it} - 2}{x_{v, it}^2}\right)^{1/2} \tilde{v}_{it}$, for $i = 1, 2, \dots, N$, $\tilde{v}_{it} \sim IIDN(0, 1)$ and $x_{v, it}^2$ is a chi-squared random variate with $v = 8$ degrees of freedom, in (31). Design 1 assumes $b_i \sim U(0.7, 0.9)$ for the first $N_b (\leq N)$ elements of vector \mathbf{b}_N , $i = 1, 2, \dots, N$, in the construction of the correlation matrix of the errors, $\mathbf{R}_N = \mathbf{I}_N + \mathbf{b}_N \mathbf{b}'_N - \mathbf{B}'_N \mathbf{B}_N$ given by (30), where $\mathbf{B}_N = \text{diag}(\mathbf{b}_N)$. $c_p(n, \delta)$ corresponds to the critical value used in the multiple testing procedure shown in (23). The number of replications is set to $R = 2000$.

Table A4a: Bias and RMSE ($\times 100$) for the $\tilde{\alpha}$ estimate of the cross-sectional exponent of the errors from a static panel data model

		Cross correlations are generated using Design 2 with non-Gaussian errors											
		$c_p(n, \delta)$ with $n = N(N-1)/2$ and $p = 0.10$											
		$\delta = 1/2$											
		Bias					RMSE						
α	n	0.55	0.60	0.65	0.70	0.75	0.80	0.85	0.90	0.95	1.00	T	N
	100	1.116	-0.949	-2.057	-2.527	-3.024	-3.083	-3.551	-3.745	-3.713	-3.754	100	100
	200	0.933	-1.004	-1.930	-2.775	-3.210	-3.357	-3.631	-3.693	-3.777	-3.844	200	200
	500	0.898	-1.048	-2.261	-3.008	-3.456	-3.615	-3.782	-3.901	-3.998	-4.063	500	500
	200	2.764	1.265	0.642	0.535	0.221	0.270	-0.159	-0.346	-0.318	-0.394	200	100
	200	2.655	1.332	0.886	0.307	0.020	-0.036	-0.226	-0.283	-0.361	-0.430	200	200
	500	2.623	1.359	0.641	0.190	-0.076	-0.138	-0.258	-0.342	-0.412	-0.460	500	500
	500	3.337	1.824	1.180	1.096	0.741	0.794	0.349	0.121	0.125	-0.002	500	100
	200	3.163	1.851	1.417	0.822	0.525	0.467	0.270	0.187	0.094	-0.002	200	200
	500	3.186	1.896	1.174	0.697	0.431	0.353	0.223	0.135	0.060	-0.002	500	500
		$\delta = 1/3$											
		Bias					RMSE						
α	n	0.55	0.60	0.65	0.70	0.75	0.80	0.85	0.90	0.95	1.00	T	N
	100	2.735	0.497	-0.797	-1.407	-2.004	-2.134	-2.645	-2.879	-2.879	-2.869	100	100
	200	2.503	0.356	-0.758	-1.738	-2.272	-2.485	-2.792	-2.879	-2.979	-3.055	200	200
	500	2.404	0.238	-1.163	-2.048	-2.582	-2.795	-2.993	-3.126	-3.229	-3.296	500	500
	200	4.186	2.322	1.376	1.039	0.575	0.530	0.042	-0.187	-0.184	-0.281	200	100
	200	4.035	2.311	1.535	0.751	0.324	0.185	-0.053	-0.141	-0.235	-0.315	200	200
	500	3.925	2.236	1.209	0.558	0.175	0.043	-0.112	-0.215	-0.294	-0.346	500	500
	500	4.715	2.806	1.821	1.481	0.974	0.926	0.424	0.159	0.140	-0.001	500	100
	200	4.521	2.764	1.968	1.155	0.717	0.569	0.324	0.213	0.104	-0.001	200	200
	500	4.510	2.739	1.667	0.971	0.576	0.426	0.258	0.151	0.065	-0.001	500	500
		$\delta = 1/3$											
		Bias					RMSE						
α	n	0.55	0.60	0.65	0.70	0.75	0.80	0.85	0.90	0.95	1.00	T	N
	100	7.902	4.856	2.736	1.406	0.297	-0.215	-0.957	-1.367	-1.467	-1.611	100	100
	200	8.402	5.243	3.068	1.291	0.150	-0.479	-1.025	-1.275	-1.466	-1.596	200	200
	500	9.147	5.730	3.112	1.241	0.017	-0.644	-1.095	-1.363	-1.534	-1.633	500	500
	200	8.911	5.987	3.991	2.793	1.746	1.286	0.558	0.157	0.049	-0.125	200	100
	200	9.459	6.401	4.326	2.928	1.534	0.950	0.446	0.192	0.005	-0.131	200	200
	500	10.131	6.809	4.298	2.520	1.378	0.770	0.351	0.108	-0.043	-0.133	500	500
	500	9.412	6.412	4.345	3.101	2.002	1.517	0.765	0.332	0.204	0.000	500	100
	200	9.926	6.788	4.642	2.883	1.753	1.145	0.628	0.355	0.155	0.000	200	200
	500	10.730	7.297	4.680	2.806	1.607	0.960	0.519	0.265	0.104	0.000	500	500
		$\delta = 1/3$											
		Bias					RMSE						
α	n	0.55	0.60	0.65	0.70	0.75	0.80	0.85	0.90	0.95	1.00	T	N
	100	8.224	5.216	3.119	1.827	1.034	0.949	1.335	1.668	1.775	1.915	100	100
	200	8.758	5.623	3.433	1.738	0.952	0.995	1.359	1.581	1.763	1.890	200	200
	500	9.551	6.175	3.594	1.836	1.054	1.131	1.433	1.665	1.827	1.926	500	500
	200	9.141	6.221	4.193	2.934	1.851	1.344	0.614	0.229	0.145	0.182	200	100
	200	9.728	6.666	4.539	2.801	1.653	1.015	0.489	0.229	0.115	0.180	200	200
	500	10.412	7.093	4.552	2.724	1.518	0.844	0.392	0.157	0.125	0.186	500	100
	500	9.550	6.551	4.462	3.182	2.062	1.551	0.793	0.357	0.220	0.007	500	200
	200	10.081	6.940	4.764	2.980	1.820	1.181	0.648	0.365	0.160	0.001	200	200
	500	10.902	7.473	4.843	2.941	1.704	1.014	0.547	0.276	0.107	0.001	500	500

Notes: Parameters of the static panel data model, (29), are generated as: $a_i \sim IIDN(1, 1)$, $\rho_{ix} \sim U(0, 0, 0.95)$, $\vartheta_i = 0$ and $\gamma_i \sim IIDN(1, 1)$, for $i = 1, 2, \dots, N$. Design 2 assumes a two-factor model with $[N_{\alpha_\beta}]$ and $[N_{\alpha_{\beta 2}}]$ non-zero loadings for the first and second factor, respectively. We set: $\alpha_{\beta 2} = 2\alpha_\beta/3$, where α_β relates to α under (11) and $f_{jt} \sim IIDN(0, 1)$. Non-Gaussian errors are generated as: $v_{it} = \left(\frac{v_{i,t}}{\chi_{v,t}}\right)^{1/2} \tilde{v}_{it}$, for $i = 1, 2, \dots, N$, $\tilde{v}_{it} \sim IIDN(0, 1)$ and $\chi_{v,t}^2$ is a chi-squared random variate with $v = 8$ degrees of freedom, in (31). $v_{ij} \sim IIDU(\mu_{v_j} - 0.2, \mu_{v_j} + 0.2)$, $j = 1, 2$, $\mu_v = 0.87$, $\mu_{v_2} = 0.71$, $\mu_{v_1} = \sqrt{\mu_{v_2}^2 - N^2(\alpha_{\beta 2} - \alpha_\beta)^2 \mu_{v_2}^2}$, in (32) and (33). The number of replications is set to $R = 2000$.

Table A4b: Bias and RMSE ($\times 100$) for the $\hat{\alpha}$ estimate of the cross-sectional exponent of the errors from a dynamic panel data model with exogenous regressors

Cross correlations are generated using Design 2 with non-Gaussian errors														
$c_p(n, \delta)$ with $n = N(N-1)/2$ and $p = 0.10$														
$\delta = 1/2$														
Bias					RMSE									
T	N	T	N	α	0.55	0.60	0.65	0.70	0.75	0.80	0.85	0.90	0.95	1.00
100	100	100	100	100	2.662	0.365	-0.939	-1.597	-2.202	-2.358	-2.883	-3.124	-3.111	-3.162
200	200	200	200	200	2.380	0.191	-0.959	-1.989	-2.541	-2.774	-3.072	-3.159	-3.261	-3.337
500	500	500	500	500	2.163	-0.004	-1.413	-2.297	-2.832	-3.055	-3.256	-3.387	-3.488	-3.560
200	100	200	100	200	4.150	2.294	1.352	1.013	0.542	0.501	0.014	-0.217	-0.216	-0.307
200	200	200	200	200	3.950	2.241	1.482	0.701	0.292	0.157	-0.086	-0.166	-0.263	-0.340
500	500	500	500	500	3.869	2.188	1.178	0.531	0.155	0.025	-0.130	-0.233	-0.312	-0.365
100	100	500	100	100	4.812	2.886	1.868	1.505	0.989	0.927	0.426	0.164	0.139	-0.001
200	200	200	200	200	4.557	2.796	1.992	1.173	0.727	0.575	0.326	0.216	0.102	-0.001
500	500	500	500	500	4.489	2.722	1.651	0.959	0.567	0.420	0.256	0.149	0.065	-0.001
$\delta = 1/3$														
Bias					RMSE									
T	N	T	N	α	0.55	0.60	0.65	0.70	0.75	0.80	0.85	0.90	0.95	1.00
100	100	100	100	100	3.358	1.765	1.797	2.176	2.676	2.840	3.307	3.537	3.540	3.591
200	200	200	200	200	3.285	1.877	1.861	2.502	2.967	3.189	3.469	3.562	3.661	3.737
500	500	500	500	500	3.321	2.061	2.232	2.785	3.223	3.432	3.628	3.757	3.858	3.930
200	100	200	100	200	4.533	2.687	1.675	1.210	0.707	0.603	0.287	0.347	0.347	0.408
200	200	200	200	200	4.421	2.684	1.786	0.970	0.526	0.343	0.275	0.304	0.374	0.436
500	500	500	500	500	4.458	2.730	1.632	0.920	0.512	0.328	0.291	0.341	0.406	0.455
100	100	500	100	100	5.051	3.105	2.027	1.583	1.037	0.947	0.444	0.189	0.157	0.004
200	200	200	200	200	4.914	3.122	2.207	1.335	0.828	0.619	0.349	0.225	0.109	0.005
500	500	500	500	500	4.792	2.967	1.819	1.063	0.621	0.439	0.263	0.151	0.067	0.002
$\delta = 1/3$														
Bias					RMSE									
T	N	T	N	α	0.55	0.60	0.65	0.70	0.75	0.80	0.85	0.90	0.95	1.00
100	100	100	100	100	7.868	4.785	2.655	1.301	0.186	-0.344	-1.112	-1.516	-1.617	-1.751
200	200	200	200	200	8.272	5.102	2.925	1.133	-0.017	-0.647	-1.190	-1.440	-1.632	-1.760
500	500	500	500	500	8.905	5.513	2.917	1.072	-0.136	-0.790	-1.239	-1.502	-1.669	-1.771
200	100	200	100	200	8.879	5.963	3.979	2.784	1.732	1.277	0.545	0.142	0.031	-0.136
200	200	200	200	200	9.350	6.310	4.262	2.570	1.498	0.925	0.425	0.179	-0.009	-0.142
500	500	500	500	500	10.062	6.749	4.255	2.487	1.356	0.756	0.342	0.100	-0.050	-0.141
100	100	500	100	100	9.532	6.514	4.416	3.144	2.029	1.530	0.772	0.340	0.205	0.000
200	200	200	200	200	9.941	6.805	4.663	2.897	1.765	1.152	0.632	0.360	0.154	0.000
500	500	500	500	500	10.705	7.276	4.661	2.793	1.598	0.953	0.515	0.262	0.103	0.000
$\delta = 1/3$														
Bias					RMSE									
T	N	T	N	α	0.55	0.60	0.65	0.70	0.75	0.80	0.85	0.90	0.95	1.00
100	100	100	100	100	8.200	5.156	3.058	1.759	1.041	1.045	1.500	1.839	1.944	2.073
200	200	200	200	200	8.643	5.511	3.336	1.690	1.036	1.148	1.527	1.750	1.927	2.055
500	500	500	500	500	9.316	5.974	3.429	1.730	1.062	1.215	1.540	1.773	1.933	2.036
200	100	200	100	200	9.121	6.208	4.186	2.924	1.834	1.330	0.594	0.222	0.153	0.198
200	200	200	200	200	9.613	6.569	4.472	2.744	1.622	0.994	0.475	0.225	0.132	0.198
500	500	500	500	500	10.346	7.040	4.522	2.710	1.521	0.855	0.404	0.160	0.126	0.191
100	100	500	100	100	9.685	6.665	4.542	3.226	2.085	1.559	0.795	0.361	0.220	0.002
200	200	200	200	200	10.131	6.996	4.821	3.029	1.860	1.206	0.662	0.373	0.161	0.003
500	500	500	500	500	10.863	7.433	4.799	2.899	1.667	0.988	0.531	0.268	0.105	0.001

Notes: Parameters of the dynamic panel data model, (29), are generated as: $a_i \sim IIDN(1, 1)$, $\rho_x \sim U(0, 0.95)$, $\vartheta_i \sim U(0, 0.95)$ and $\gamma_i \sim IIDN(1, 1)$, for $i = 1, 2, \dots, N$. Design 2 assumes a two-factor model with $[N_{\alpha\beta}]$ and $[N_{\alpha\beta 2}]$ non-zero loadings for the first and second factor, respectively. We set: $\alpha_{\beta 2} = 2\alpha_\beta/3$, where α_β relates to α under (11) and $f_{jt} \sim IIDN(0, 1)$. Non-Gaussian errors are generated as: $u_{it} = \left(\frac{\chi_{v,t}^2}{\chi_{v,t}^2}\right)^{1/2} \tilde{v}_{it}$, for $i = 1, 2, \dots, N$, $\tilde{v}_{it} \sim IIDN(0, 1)$ and $\chi_{v,t}^2$ is a chi-squared random variate with $v=8$ degrees of freedom, in (31). $v_{ij} \sim IIDU(\mu_{v_j} - 0.2, \mu_{v_j} + 0.2)$, $j = 1, 2$, $\mu_v = 0.87$, $\mu_{v_2} = 0.71$, $\mu_{v_1} = \sqrt{\mu_v^2 - N^{2(\alpha_{\beta 2} - \alpha_\beta)} \mu_{v_2}^2}$, in (32) and (33). The number of replications is set to $R = 2000$.

Table B2: Comparison of Bias and RMSE ($\times 100$) for the $\tilde{\alpha}$ and $\hat{\alpha}$ of the cross-sectional exponent of the errors of a static and dynamic panel data model with exogenous regressors

		Cross correlations are generated using Design 2 with Gaussian errors																						
		Bias						RMSE																
α		0.55	0.60	0.65	0.70	0.75	0.80	0.85	0.90	0.95	1.00													
Static model: $\vartheta_i = 0$, for $i = 1, 2, \dots, N$																								
T	N	$\tilde{\alpha}$																						
		T	N	T	N	T	N	T	N	T	N	T	N	$\hat{\alpha}$										
100	100	-0.880	-2.378	-2.922	-3.026	-3.232	-3.131	-3.509	-3.648	-3.576	-3.598	100	100	1.252	2.688	3.301	3.469	3.699	3.623	3.974	4.094	4.029	4.051	
	200	-1.221	-2.480	-2.800	-3.267	-3.464	-3.434	-3.595	-3.591	-3.658	-3.714	200	200	1.455	2.748	3.167	3.655	3.868	3.859	4.002	4.009	4.070	4.130	
	500	-1.501	-2.658	-3.266	-3.581	-3.780	-3.779	-3.859	-3.934	-4.011	-4.057	500	500	1.662	2.899	3.580	3.938	4.153	4.176	4.260	4.336	4.411	4.456	
	200	100	0.705	-0.090	-0.160	0.009	0.205	-0.145	-0.299	-0.242	-0.304	200	100	0.803	0.377	0.369	0.346	0.298	0.354	0.312	0.405	0.359	0.399	
	200	0.375	-0.068	0.134	-0.086	-0.146	-0.063	-0.191	-0.206	-0.269	-0.326	200	200	0.457	0.291	0.307	0.274	0.296	0.251	0.315	0.318	0.367	0.415	
	500	0.219	-0.024	-0.086	-0.158	-0.206	-0.153	-0.212	-0.267	-0.325	-0.368	500	500	0.303	0.241	0.262	0.299	0.328	0.295	0.326	0.366	0.418	0.456	
	500	100	1.043	0.295	0.247	0.563	0.433	0.625	0.264	0.084	0.114	-0.001	500	100	1.068	0.337	0.272	0.571	0.442	0.631	0.279	0.116	0.136	0.002
	200	0.683	0.313	0.550	0.326	0.254	0.329	0.203	0.161	0.087	-0.001	200	200	0.697	0.325	0.553	0.329	0.257	0.332	0.208	0.166	0.093	0.001	
	500	0.530	0.355	0.331	0.249	0.205	0.246	0.176	0.117	0.056	-0.001	500	500	0.534	0.357	0.332	0.250	0.206	0.246	0.177	0.118	0.057	0.001	
Dynamic model with exogenous regressors: $\vartheta_i \sim U(0.0, 0.95)$ for $i = 1, 2, \dots, N$																								
T	N	$\tilde{\alpha}$																						
		T	N	T	N	T	N	T	N	T	N	T	N	$\hat{\alpha}$										
100	100	-1.016	-2.571	-3.183	-3.319	-3.568	-3.479	-3.838	-3.998	-3.922	-3.952	100	100	1.360	2.876	3.549	3.762	4.031	3.981	4.301	4.455	4.378	4.411	
	200	-1.320	-2.642	-3.032	-3.558	-3.745	-3.763	-3.918	-3.925	-4.003	-4.057	200	200	1.536	2.902	3.398	3.954	4.170	4.208	4.358	4.367	4.445	4.497	
	500	-1.610	-2.849	-3.517	-3.890	-4.107	-4.124	-4.222	-4.304	-4.388	-4.439	500	500	1.769	3.101	3.849	4.273	4.512	4.554	4.659	4.740	4.822	4.870	
	200	100	0.689	-0.109	-0.179	0.088	-0.030	0.166	-0.181	-0.329	-0.272	-0.336	200	100	0.794	0.393	0.393	0.369	0.326	0.359	0.355	0.440	0.398	0.446
	200	0.357	-0.089	0.115	-0.120	-0.177	-0.095	-0.227	-0.240	-0.301	-0.360	200	200	0.448	0.292	0.301	0.306	0.330	0.284	0.354	0.355	0.401	0.453	
	500	0.209	-0.033	-0.095	-0.170	-0.215	-0.165	-0.225	-0.281	-0.337	-0.380	500	500	0.295	0.245	0.274	0.306	0.329	0.296	0.335	0.376	0.422	0.463	
	500	100	1.040	0.294	0.248	0.565	0.436	0.627	0.081	0.115	-0.001	500	100	1.065	0.332	0.272	0.573	0.443	0.634	0.276	0.118	0.135	0.002	
	200	0.685	0.313	0.548	0.324	0.253	0.330	0.202	0.162	0.086	-0.001	200	200	0.698	0.326	0.551	0.328	0.257	0.332	0.207	0.166	0.092	0.001	
	500	0.527	0.353	0.330	0.249	0.204	0.246	0.176	0.117	0.055	-0.001	500	500	0.531	0.355	0.331	0.250	0.205	0.246	0.176	0.118	0.057	0.001	
Dynamic model with exogenous regressors: $\vartheta_i \sim U(0.0, 0.95)$ for $i = 1, 2, \dots, N$																								
T	N	$\tilde{\alpha}$																						
		T	N	T	N	T	N	T	N	T	N	T	N	$\hat{\alpha}$										
100	100	1.092	0.325	0.312	0.674	0.496	0.664	0.291	0.130	0.187	0.105	100	100	2.274	1.834	1.492	1.393	1.047	0.958	0.597	0.386	0.289	0.105	
	200	0.157	-0.008	0.432	0.286	0.251	0.341	0.222	0.201	0.148	0.094	200	200	1.470	1.363	1.163	0.888	0.687	0.586	0.418	0.308	0.201	0.094	
	500	-0.160	-0.033	0.162	0.193	0.188	0.257	0.204	0.157	0.116	0.081	500	500	1.110	0.978	0.772	0.587	0.457	0.399	0.298	0.215	0.142	0.081	
	200	100	2.791	1.573	1.132	1.355	0.832	0.878	0.414	0.168	0.187	0.051	200	100	3.395	2.377	1.890	1.911	1.280	1.161	0.708	0.414	0.305	0.051
	200	0.881	0.432	0.679	0.416	0.290	0.345	0.211	0.175	0.114	0.046	200	200	1.463	1.254	1.197	0.864	0.623	0.542	0.372	0.261	0.162	0.046	
	500	0.261	0.128	0.240	0.206	0.181	0.237	0.180	0.133	0.081	0.040	500	500	0.801	0.733	0.630	0.473	0.369	0.332	0.244	0.174	0.101	0.040	
	500	100	7.388	4.904	3.529	3.455	2.334	1.980	1.188	0.579	0.377	0.019	500	100	7.714	5.231	3.849	3.696	2.620	2.222	1.448	0.829	0.527	0.019
	200	2.603	2.107	2.046	1.437	0.883	0.723	0.459	0.266	0.143	0.018	200	200	2.932	2.460	2.343	1.753	1.176	0.939	0.642	0.370	0.209	0.018	
	500	0.636	0.531	0.644	0.397	0.292	0.274	0.183	0.121	0.063	0.016	500	500	0.955	0.904	0.935	0.620	0.472	0.363	0.240	0.156	0.084	0.016	

Notes: Remaining parameters of the panel data model, (29), are generated as: $a_i \sim IIDN(1, 1)$, $\rho_{ix} \sim U(0.0, 0.95)$ and $\gamma_i \sim IIDN(1, 1)$, for $i = 1, 2, \dots, N$. Design 2 assumes a two-factor model with $[N_{\alpha_\beta}]$ and $[N_{\alpha_{\beta 2}}]$ non-zero loadings for the first and second factor, respectively. We set: $\alpha_{\beta 2} = 2\alpha_\beta/3$, where α_β relates to α under (11), f_{jt} and $u_{it} \sim IIDN(0, 1)$, $v_{ij} \sim IIDU(\mu_{v_j} - 0.2, \mu_{v_j} + 0.2)$, $j = 1, 2$, $\mu_{v_1} = 0.87$, $\mu_{v_2} = 0.71$, $\mu_{v_3} = \sqrt{\mu_{v_2}^2 - N^{2(\alpha_{\beta 2} - \alpha_\beta)} \mu_{v_2}^2}$, in (32) and (33). $\tilde{\alpha}$ is computed using $c_p(n, \delta)$ with $n = N(N-1)/2$, $\delta = 1/2$ and $p = 0.05$ in the multiple testing procedure shown in (23). $\hat{\alpha}$ corresponds to the most robust estimator of the exponent of cross-sectional dependence considered in Bailey et al. (2016) and allows for both serial correlation in the factors and weak cross-sectional dependence in the error terms. We use four principal components when estimating \hat{c}_N in the expression for $\hat{\alpha}$. The number of replications is set to $R = 2000$.

An Online Supplement for
Exponent of Cross-sectional Dependence for Residuals

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This online supplement provides additional Monte Carlo and empirical results.

Appendix A

Additional Monte Carlo results

The Monte Carlo results provided in the tables below are based on the designs set out in Section 6 of the paper.

Table S1c: Bias and RMSE ($\times 100$) for the $\tilde{\alpha}$ estimate of the cross-sectional exponent of the errors from a dynamic panel data model with no exogenous regressors

Cross correlations are generated using Design 1 with non-Gaussian errors											
$\rho_p(n, \delta)$ with $n = N(N-1)/2$ and $p = 0.10$											
$\delta = 1/2$											
Bias											
RMSE											
$\delta = 1/3$											
Bias											
RMSE											
α	0.55	0.60	0.65	0.70	0.75	0.80	0.85	0.90	0.95	1.00	1.00
T	100	200	500	100	200	500	100	200	500	100	200
N	100	200	500	100	200	500	100	200	500	100	200
	3.623	1.816	1.052	0.642	0.327	0.364	-0.053	-0.184	-0.067	-0.032	-0.032
	3.396	1.840	1.242	0.523	0.214	0.166	-0.021	-0.026	-0.052	-0.044	-0.044
	3.226	1.802	0.952	0.483	0.183	0.140	0.026	-0.027	-0.061	-0.064	-0.064
	4.226	2.262	1.302	0.854	0.432	0.441	0.005	-0.130	-0.028	0.000	0.000
	4.016	2.242	1.454	0.718	0.363	0.235	0.042	0.027	-0.003	0.000	0.000
	3.961	2.317	1.259	0.692	0.334	0.248	0.114	0.048	0.007	0.000	0.000
	4.588	2.462	1.491	0.942	0.531	0.475	0.038	-0.117	-0.021	0.000	0.000
	4.403	2.567	1.713	0.824	0.427	0.286	0.057	0.032	0.001	0.000	0.000
	4.461	2.648	1.446	0.820	0.396	0.261	0.123	0.050	0.008	0.000	0.000
T	100	200	500	100	200	500	100	200	500	100	200
N	100	200	500	100	200	500	100	200	500	100	200
	4.183	2.465	1.734	0.993	0.735	0.551	0.272	0.235	0.107	0.069	0.069
	4.041	2.563	1.736	0.981	0.541	0.472	0.292	0.125	0.093	0.082	0.082
	3.922	2.464	1.469	0.917	0.449	0.351	0.189	0.131	0.118	0.120	0.120
	4.762	2.789	1.759	1.229	0.623	0.574	0.231	0.207	0.068	0.001	0.001
	4.487	2.766	1.810	1.066	0.733	0.335	0.279	0.094	0.051	0.000	0.000
	4.453	2.797	1.594	1.028	0.609	0.428	0.217	0.127	0.035	0.002	0.002
	4.904	2.765	1.847	1.171	0.702	0.565	0.252	0.187	0.092	0.000	0.000
	4.717	3.015	2.048	1.102	0.629	0.394	0.148	0.079	0.040	0.000	0.000
	4.807	3.004	1.711	1.117	0.616	0.346	0.165	0.075	0.022	0.000	0.000
T	100	200	500	100	200	500	100	200	500	100	200
N	100	200	500	100	200	500	100	200	500	100	200
	8.467	5.599	3.744	2.451	1.502	1.085	0.384	0.059	0.037	-0.011	-0.011
	8.831	5.954	4.051	2.362	1.354	0.849	0.378	0.182	0.044	-0.014	-0.014
	9.373	6.256	3.921	2.329	1.289	0.763	0.363	0.151	0.033	-0.019	-0.019
	9.129	6.167	4.082	2.744	1.642	1.179	0.444	0.106	0.064	0.000	0.000
	9.564	6.471	4.342	2.652	1.565	0.926	0.431	0.219	0.068	0.000	0.000
	10.253	6.974	4.408	2.656	1.488	0.881	0.438	0.198	0.061	0.000	0.000
	9.611	6.456	4.367	2.898	1.812	1.239	0.507	0.129	0.076	0.000	0.000
	10.076	6.945	4.752	2.843	1.701	1.021	0.473	0.229	0.076	0.000	0.000
	10.861	7.477	4.736	2.913	1.630	0.929	0.466	0.208	0.064	0.000	0.000
T	100	200	500	100	200	500	100	200	500	100	200
N	100	200	500	100	200	500	100	200	500	100	200
	8.816	5.980	4.152	2.713	1.766	1.264	0.571	0.247	0.118	0.030	0.030
	9.190	6.355	4.388	2.661	1.583	1.056	0.575	0.266	0.091	0.031	0.031
	9.726	6.639	4.271	2.646	1.508	0.950	0.469	0.222	0.076	0.046	0.046
	9.439	6.473	4.362	2.994	1.789	1.310	0.580	0.273	0.128	0.000	0.000
	9.824	6.760	4.580	2.885	1.800	1.023	0.584	0.275	0.110	0.000	0.000
	10.504	7.252	4.640	2.901	1.696	1.043	0.529	0.271	0.088	0.001	0.001
	9.797	6.629	4.575	3.057	1.941	1.330	0.631	0.263	0.155	0.000	0.000
	10.241	7.171	4.960	3.014	1.841	1.112	0.535	0.266	0.106	0.000	0.000
	11.026	7.665	4.905	3.110	1.787	1.015	0.513	0.236	0.078	0.000	0.000

Notes: Parameters of the dynamic panel data model, (29), are generated as: $a_i \sim IIDN(1, 1)$, $\rho_{ix} \sim U(0, 0, 0.95)$, $\vartheta_i \sim U(0, 0, 0.95)$ and $\gamma_i = 0$, for $i = 1, 2, \dots, N$. Non-Gaussian errors are generated as $u_{it} = \left(\frac{v_{it}}{\chi_{v,t}^2}\right)^{1/2} \tilde{v}_{it}$, for $i = 1, 2, \dots, N$, $\tilde{v}_{it} \sim IIDN(0, 1)$ and $\chi_{v,t}^2$ is a chi-squared random variate with $v=8$ degrees of freedom, in (31). Design 1 assumes $b_i \sim U(0.7, 0.9)$ for the first $N_b (\leq N)$ elements of vector \mathbf{b}_N , $i = 1, 2, \dots, N$, in the construction of the correlation matrix of the errors, $\mathbf{R}_N = \mathbf{I}_N + \mathbf{b}_N \mathbf{b}'_N - \mathbf{B}_N^2$ given by (30), where $\mathbf{B}_N = \text{diag}(\mathbf{b}_N)$. $c_p(n, \delta)$ corresponds to the critical value used in the multiple testing procedure shown in (23). The number of replications is set to $R = 2000$.

Table S1d: Bias and RMSE ($\times 100$) for the $\tilde{\alpha}$ estimate of the cross-sectional exponent of the errors from a dynamic panel data model with no exogenous regressors

Cross correlations are generated using Design 2 with non-Gaussian errors											
		$c_p(n, \delta)$ with $n = N(N-1)/2$ and $p = 0.10$									
		$\delta = 1/2$									
		Bias									
T	N	0.55	0.60	0.65	0.70	0.75	0.80	0.85	0.90	0.95	1.00
100	100	1.186	-0.898	-2.047	-2.570	-3.052	-3.125	-3.605	-3.814	-3.780	-3.820
200	100	0.953	-1.004	-1.983	-2.865	-3.320	-3.493	-3.754	-3.805	-3.909	-3.970
500	100	0.712	-1.198	-2.406	-3.140	-3.569	-3.722	-3.886	-4.008	-4.101	-4.159
200	100	2.816	1.279	0.648	0.531	0.198	0.241	-0.187	-0.370	-0.345	-0.424
200	200	2.495	1.215	0.809	0.255	-0.007	-0.049	-0.242	-0.291	-0.368	-0.434
500	100	2.592	1.330	0.625	0.174	-0.084	-0.147	-0.266	-0.350	-0.419	-0.469
500	100	3.405	1.873	1.221	1.125	0.752	0.798	0.355	0.124	0.127	-0.002
200	200	3.125	1.824	1.398	0.813	0.519	0.463	0.266	0.186	0.094	-0.002
500	200	3.153	1.876	1.159	0.686	0.425	0.350	0.222	0.134	0.059	-0.002
		RMSE									
T	N	2.409	2.073	2.718	3.131	3.556	3.628	4.059	4.266	4.243	4.287
100	100	2.287	2.057	2.599	3.322	3.753	3.940	4.197	4.254	4.358	4.418
200	100	2.169	2.100	2.899	3.554	3.972	4.144	4.319	4.442	4.538	4.595
200	100	3.281	1.792	1.110	0.820	0.509	0.451	0.401	0.507	0.487	0.548
200	200	2.867	1.563	1.026	0.508	0.349	0.321	0.404	0.434	0.493	0.549
500	100	3.184	1.857	1.057	0.575	0.378	0.334	0.392	0.459	0.520	0.566
500	100	3.764	2.240	1.498	1.264	0.840	0.829	0.385	0.159	0.146	0.009
200	200	3.402	2.053	1.517	0.892	0.561	0.477	0.275	0.192	0.100	0.004
500	200	3.518	2.167	1.357	0.815	0.500	0.378	0.233	0.138	0.061	0.005
		RMSE									
T	N	5.161	2.533	0.852	-0.095	-0.901	-1.189	-1.810	-2.115	-2.149	-2.246
100	100	5.538	2.858	1.192	-0.184	-1.004	-1.417	-1.820	-1.972	-2.122	-2.218
500	100	6.093	3.222	1.169	-0.209	-1.067	-1.483	-1.794	-1.989	-2.117	-2.193
200	100	6.311	3.936	2.505	1.789	1.068	0.846	0.263	-0.033	-0.078	-0.206
200	200	6.569	4.157	2.760	1.563	0.856	0.535	0.187	0.036	-0.095	-0.195
500	100	7.481	4.754	2.859	1.586	0.812	0.442	0.159	-0.012	-0.124	-0.195
500	100	6.848	4.402	2.908	2.167	1.397	1.164	0.563	0.229	0.167	-0.001
200	200	7.225	4.701	3.206	1.933	1.170	0.818	0.453	0.274	0.126	0.000
500	200	8.051	5.221	3.235	1.890	1.079	0.681	0.381	0.204	0.084	-0.001
		RMSE									
T	N	5.711	3.248	1.868	1.395	1.569	1.708	2.192	2.470	2.513	2.614
100	100	6.071	3.497	1.959	1.290	1.534	1.825	2.174	2.317	2.464	2.558
500	100	6.602	3.810	1.944	1.247	1.526	1.840	2.125	2.313	2.441	2.517
200	100	6.638	4.256	2.771	1.960	1.195	0.911	0.363	0.211	0.214	0.286
200	200	6.831	4.394	2.923	1.685	0.941	0.585	0.264	0.173	0.198	0.266
500	100	7.822	5.079	3.131	1.793	0.953	0.521	0.229	0.139	0.193	0.252
500	100	7.100	4.652	3.115	2.299	1.489	1.206	0.598	0.260	0.185	0.007
200	200	7.419	4.878	3.331	2.026	1.227	0.844	0.467	0.281	0.132	0.001
500	200	8.267	5.427	3.405	2.015	1.160	0.721	0.400	0.211	0.086	0.002

Notes: Parameters of the dynamic panel data model, (29), are generated as: $a_i \sim IIDN(1, 1)$, $\rho_{ix} \sim U(0.0, 0.95)$, $\vartheta_i \sim U(0.0, 0.95)$ and $\gamma_i = 0$, for $i = 1, 2, \dots, N$. Design 2 assumes a two-factor model with $[N_{\alpha_\beta}]$ and $[N_{\alpha_{\beta 2}}]$ non-zero loadings for the first and second factor, respectively. We set: $\alpha_{\beta 2} = 2\alpha_\beta/3$, where α_β relates to α under (11) and $f_{jt} \sim IIDN(0, 1)$. Non-Gaussian errors are generated as: $u_{it} = \left(\frac{v-2}{\chi_{v,t}^2}\right)^{1/2} \tilde{v}_{it}$, for $i = 1, 2, \dots, N$, $\tilde{v}_{it} \sim IIDN(0, 1)$ and $\chi_{v,t}^2$ is a chi-squared random variate with $v=8$ degrees of freedom, in (31). $v_{ij} \sim IIDU(\mu_{v_j} - 0.2, \mu_{v_j} + 0.2)$, $j = 1, 2$, $\mu_{v_1} = 0.87$, $\mu_{v_2} = 0.71$, $\mu_{v_1} = \sqrt{\mu_{v_1}^2 - N^{2(\alpha_{\beta 2} - \alpha_\beta)} \mu_{v_2}^2}$, in (32) and (33). The number of replications is set to $R = 2000$.

Table S2d: Bias and RMSE ($\times 100$) for the $\tilde{\alpha}$ estimate of the cross-sectional exponent of the errors from a dynamic panel data model with exogenous regressors

Cross correlations are generated using Design 2 with Gaussian and non-Gaussian errors											
$c_p(n, \delta)$ with $n = N(N-1)/2$, $\delta = 1/4$ and $p = 0.10$											
Gaussian errors											
Bias											
T	N	0.55	0.60	0.65	0.70	0.75	0.80	0.85	0.90	0.95	1.00
100	100	2.937	1.072	0.123	-0.244	-0.710	-0.755	-1.231	-1.471	-1.460	-1.555
200	100	3.160	1.347	0.512	-0.315	-0.740	-0.908	-1.162	-1.250	-1.366	-1.452
500	100	3.620	1.662	0.436	-0.331	-0.784	-0.957	-1.142	-1.268	-1.368	-1.435
200	100	3.709	2.103	1.368	1.185	0.759	0.755	0.280	0.042	0.032	-0.090
200	200	3.807	2.290	1.669	0.956	0.577	0.459	0.217	0.120	0.011	-0.084
500	100	4.264	2.572	1.549	0.886	0.504	0.358	0.190	0.077	-0.010	-0.075
500	100	3.840	2.250	1.340	0.921	0.314	0.431	0.172	0.152	0.000	0.000
200	200	3.955	2.440	1.818	1.099	0.709	0.587	0.345	0.233	0.114	0.000
500	200	4.407	2.704	1.675	1.001	0.614	0.461	0.287	0.170	0.075	0.000
RMSE											
T	N	5.820	3.447	2.011	1.305	0.777	0.712	0.880	1.119	1.162	1.287
100	100	3.200	1.471	0.874	0.844	1.110	1.247	1.450	1.523	1.632	1.711
500	100	3.635	1.730	0.757	0.787	1.094	1.248	1.412	1.524	1.616	1.681
200	100	3.747	2.142	1.403	1.210	0.783	0.774	0.320	0.141	0.126	0.138
200	200	3.823	2.305	1.679	0.967	0.589	0.471	0.240	0.150	0.088	0.120
500	100	4.268	2.575	1.552	0.890	0.510	0.364	0.202	0.103	0.068	0.103
500	100	3.877	2.283	1.535	1.354	0.932	0.921	0.443	0.196	0.170	0.000
200	200	3.969	2.453	1.826	1.105	0.714	0.590	0.349	0.237	0.120	0.000
500	200	4.411	2.707	1.677	1.002	0.615	0.462	0.287	0.171	0.077	0.000
Non-Gaussian errors											
T	N	11.542	7.952	5.212	3.270	1.685	0.788	-0.211	-0.790	-1.015	-1.228
100	100	8.884	5.619	3.330	1.443	0.217	-0.321	-1.092	-1.499	-1.603	-1.738
500	100	10.209	6.610	3.777	1.710	0.329	-0.446	-0.967	-1.272	-1.459	-1.571
200	100	8.946	6.018	4.020	2.811	1.750	1.289	0.553	0.147	0.033	-0.135
200	200	9.923	6.769	4.592	2.799	1.646	1.015	0.479	0.211	0.010	-0.131
500	100	11.279	7.734	4.975	2.969	1.652	0.926	0.437	0.154	-0.018	-0.120
500	100	9.601	6.570	4.457	3.173	2.047	1.540	0.778	0.343	0.206	0.000
200	200	10.511	7.264	4.991	3.122	1.907	1.233	0.676	0.381	0.162	0.000
500	200	11.910	8.257	5.382	3.272	1.884	1.108	0.593	0.296	0.115	0.000
RMSE											
T	N	11.763	8.187	5.442	3.474	1.917	1.104	0.780	1.106	1.301	1.495
100	100	9.230	5.995	3.701	1.910	1.037	1.023	1.387	1.625	1.812	1.946
500	100	10.559	7.000	4.194	2.191	1.077	0.965	1.273	1.531	1.709	1.821
200	100	9.187	6.261	4.226	2.951	1.852	1.342	0.602	0.225	0.152	0.196
200	200	10.170	7.014	4.794	2.967	1.767	1.083	0.526	0.249	0.124	0.184
500	100	11.526	7.991	5.217	3.175	1.807	1.020	0.493	0.195	0.101	0.166
500	100	9.753	6.720	4.583	3.254	2.104	1.569	0.801	0.364	0.221	0.002
200	200	10.689	7.444	5.143	3.251	2.000	1.287	0.706	0.395	0.169	0.003
500	200	12.047	8.398	5.510	3.374	1.954	1.145	0.610	0.303	0.117	0.001

Notes: Parameters of the dynamic panel data model, (29), are generated as: $a_i \sim IIDN(1, 1)$, $\rho_{ix} \sim U(0, 0, 0.95)$, $\theta_i \sim U(0, 0, 0.95)$ and $\gamma_i \sim IIDN(1, 1)$, for $i = 1, 2, \dots, N$. Design 2 assumes a two-factor model with $[N_{c\beta}]$ and $[N_{c\beta 2}]$ non-zero loadings for the first and second factor, respectively. We set: $\alpha_{\beta 2} = 2\alpha_{\beta 1}/3$, where α_{β} relates to α under (11) and $f_{jt} \sim IIDN(0, 1)$. Gaussian errors are given by: $u_{it} \sim IIDN(0, 1)$ and non Gaussian errors are generated as: $u_{it} = \left(\frac{v-2}{\chi^2_{v,t}}\right)^{1/2} \tilde{v}_{it}$, for $i = 1, 2, \dots, N$, $\tilde{v}_{it} \sim IIDN(0, 1)$ and $\chi^2_{v,t}$ is a chi-squared random variate with $v=8$ degrees of freedom, in (31). $v_{ij} \sim IIDU(\mu_{v_j} - 0.2, \mu_{v_j} + 0.2)$, $j = 1, 2$, $\mu_v = 0.87$, $\mu_{v_2} = 0.71$, $\mu_{v_1} = \sqrt{\mu_v^2 - N^{2(\alpha_{\beta 2} - \alpha_{\beta})} \mu_{v_2}^2}$, in (32) and (33). The number of replications is set to $R = 2000$.

Appendix B

Additional Empirical results

The figures below provide the estimates displayed in Figures 1 and 2 but using the shorter 5-year rolling samples.

Figure 3: 5-year rolling estimates of the exponent of cross-sectional correlation ($\tilde{\alpha}_t$) of S&P 500 securities' excess returns

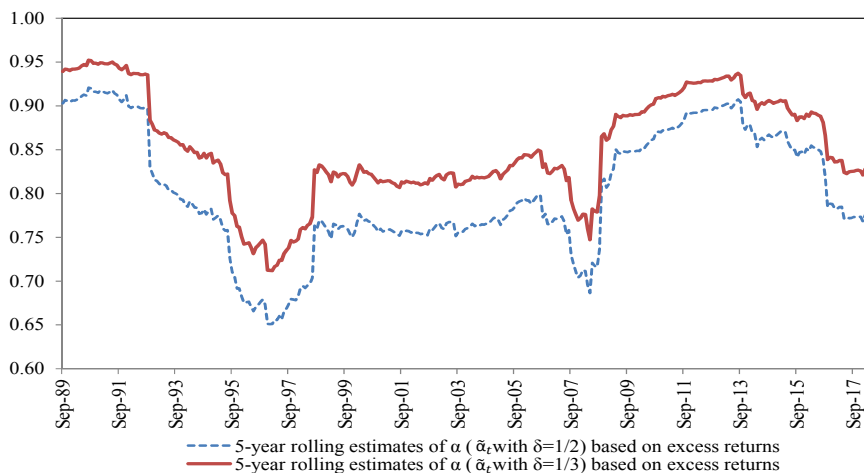
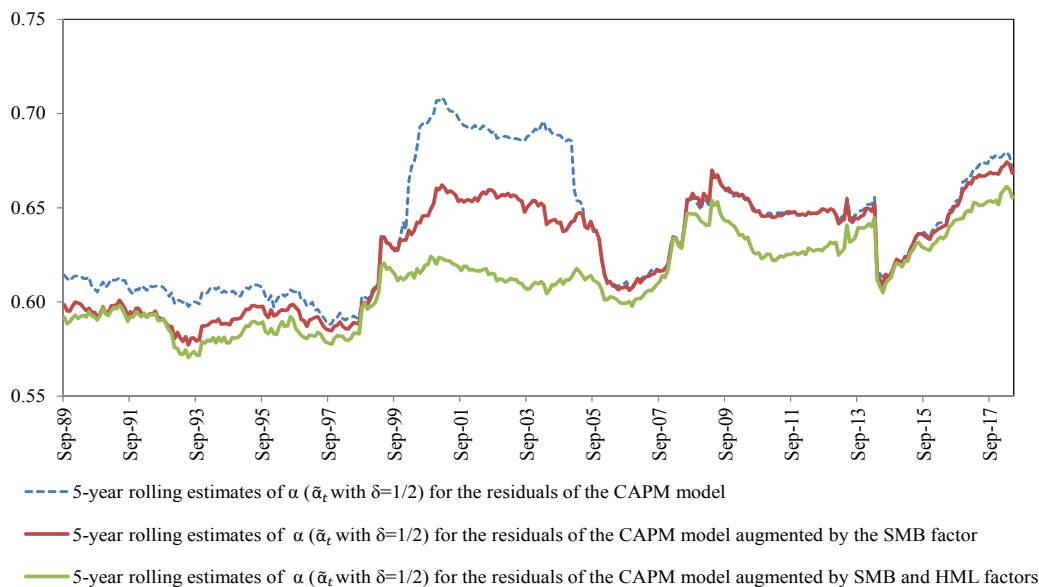


Figure 4: 5-year rolling estimates of the exponent of cross-sectional correlation ($\tilde{\alpha}_t$) of residuals from CAPM and its two Fama-French extensions



Notes: CAPM model includes excess market returns, CAPM model augmented by SMB includes excess market returns and small minus big (SMB) firm returns, and CAPM model augmented by SMB and HML includes excess market returns, small minus big (SMB) firm returns and high minus low (HML) firm returns as regressors in (35), (36) and (37), respectively.